EconS 501 - Microeconomic Theory I
Midterm Exam #2 - Answer key

1. **Cost minimization across two plants. [32 points]** A firm can produce a homogeneous good in two plants. Plant A has cost function $C_A(q_A) = q_A^2$ and plant B has cost function $C_B(q_B) = 2q_B^2$. To produce a total output Q, the firm allocates production across the two plants to minimize total cost.

   (a) [12 points] Set up the cost-minimization problem and show that at the optimum the marginal costs of the two plants are equalized. Find the cost-minimizing allocation $q_A(Q)$ and $q_B(Q)$.

   - The firm solves $\min_{q_A,q_B} q_A^2 + 2q_B^2$ subject to $q_A + q_B = Q$. Substituting $q_B = Q - q_A$ and differentiating, or using a Lagrangian, the first-order conditions equate the two marginal costs, $MC_A = MC_B$, that is,

   $$2q_A = 4q_B, \qquad \text{which implies} \qquad q_A = 2q_B.$$

   The second-order condition holds, since the objective $q_A^2 + 2(Q - q_A)^2$ has second derivative $2 + 4 = 6 > 0$. Combining $q_A = 2q_B$ with $q_A + q_B = Q$ gives $3q_B = Q$, so

   $$q_B(Q) = \frac{Q}{3} \qquad \text{and} \qquad q_A(Q) = \frac{2Q}{3}.$$

   The lower-cost plant A produces twice as much as plant B.

   (b) [10 points] Derive the firm's total cost function $C(Q)$ and its marginal cost function $MC(Q)$.

   - Substituting the allocation,

   $$C(Q) = \left(\frac{2Q}{3}\right)^2 + 2\left(\frac{Q}{3}\right)^2 = \frac{4Q^2}{9} + \frac{2Q^2}{9} = \frac{6Q^2}{9} = \frac{2Q^2}{3},$$

   so the marginal cost is $MC(Q) = \frac{dC}{dQ} = \frac{4Q}{3}$. (As a check, $MC = \frac{4Q}{3}$ equals $2q_A = 2 \cdot \frac{2Q}{3} = \frac{4Q}{3}$, the common marginal cost of the two plants.)

   (c) [6 points] If the good sells at a competitive price p, find the firm's supply function $Q(p)$.

   - A competitive firm produces where price equals marginal cost, $p = \frac{4Q}{3}$, so its supply function is

   $$Q(p) = \frac{3p}{4}.$$

   (d) [4 points] Explain the economic principle behind the "equal-marginal-cost" rule for allocating production across plants.

   - If the marginal costs of the two plants differed, the firm could shift one unit of production from the high-marginal-cost plant to the low-marginal-cost plant and reduce total cost without changing total output. Cost minimization therefore requires the marginal costs to be equalized across plants. The firm's

overall marginal cost curve is the *horizontal sum* of the plants' marginal cost curves: for any target marginal cost, each plant supplies the amount at which its own marginal cost reaches that level. In this exercise, at marginal cost MC, plant A produces $q_A = \frac{MC}{2}$ and plant B produces $q_B = \frac{MC}{4}$, so $Q = \frac{3MC}{4}$, which yields $MC(Q) = \frac{4Q}{3}$, as in part (b). This is the same logic (equate marginal costs across sources) that governs cost-effective pollution abatement across firms and optimal dispatch across power plants.

2. **Mean-variance preferences over lotteries. [35 points]** An individual evaluates monetary lotteries according to the mean-variance criterion

$$V(L) = \mu - \frac{\gamma}{2}\sigma^2,$$

where μ and σ^2 denote the mean and variance of the payoffs of lottery L, and $\gamma > 0$ measures her dislike for risk. For every $p \in [0,1]$, let L_p denote the lottery that pays \$4 with probability p and \$0 with probability $1-p$. In particular, L_0 pays \$0 for sure and L_1 pays \$4 for sure.

(a) [6 points] Find the mean μ, the variance σ^2, and the value $V(L_p)$ of lottery L_p.

- The mean of lottery L_p is $\mu = p \cdot 4 + (1-p)\cdot 0 = 4p$. Its variance is

$$\begin{aligned}\sigma^2 &= p\,(4-4p)^2 + (1-p)\,(0-4p)^2 = 16p(1-p)^2 + 16p^2(1-p)\\ &= 16p(1-p)\,[(1-p)+p] = 16p(1-p).\end{aligned}$$

Hence, the value of lottery L_p is

$$V(L_p) = 4p - \frac{\gamma}{2}16p(1-p) = 4p\,[1 - 2\gamma(1-p)]\,.$$

(b) [14 points] *Independence axiom.* Let $\gamma = 1$. Show that the individual is indifferent between lotteries L_0 and $L_{\frac{1}{2}}$. Then, mix each of these two lotteries with lottery L_1, using weight $\alpha = \frac{1}{2}$, that is, consider the compound lotteries $\frac{1}{2}L_0 + \frac{1}{2}L_1$ and $\frac{1}{2}L_{\frac{1}{2}} + \frac{1}{2}L_1$. Which compound lottery does she prefer? Do her preferences satisfy the independence axiom? Can they be represented by an expected utility function?

- *Indifference.* Evaluating $V(L_p)$ from part (a) at $\gamma = 1$, we obtain $V(L_0) = 0$ and

$$V(L_{\frac{1}{2}}) = 4 \cdot \frac{1}{2}\left[1 - 2\left(1 - \frac{1}{2}\right)\right] = 2 \cdot 0 = 0,$$

so $L_0 \sim L_{\frac{1}{2}}$. Intuitively, the coin toss between \$0 and \$4 has mean 2 and variance 4, and the variance penalty $\frac{1}{2}\cdot 4 = 2$ offsets the mean.

- *Compound lotteries.* The first compound lottery, $\frac{1}{2}L_0 + \frac{1}{2}L_1$, pays \$4 with probability $\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}$, so its reduced lottery is $L_{\frac{1}{2}}$, with value $V(L_{\frac{1}{2}}) = 0$. The second compound lottery, $\frac{1}{2}L_{\frac{1}{2}} + \frac{1}{2}L_1$, pays \$4 with probability $\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 1 = \frac{3}{4}$, so its reduced lottery is $L_{\frac{3}{4}}$, with value

$$V(L_{\frac{3}{4}}) = 4 \cdot \frac{3}{4}\left[1 - 2\left(1 - \frac{3}{4}\right)\right] = 3 \cdot \frac{1}{2} = \frac{3}{2}.$$

Hence, she strictly prefers the second compound lottery, $\frac{1}{2}L_{\frac{1}{2}} + \frac{1}{2}L_1 \succ \frac{1}{2}L_0 + \frac{1}{2}L_1$.

- *Independence axiom.* The independence axiom requires that, for all lotteries L, L', L'' and every $\alpha \in (0,1)$, $L \succsim L'$ if and only if $\alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$. Applying it to $L = L_0$, $L' = L_{\frac{1}{2}}$, $L'' = L_1$, and $\alpha = \frac{1}{2}$, indifference $L_0 \sim L_{\frac{1}{2}}$ should carry over to the two compound lotteries. Since it does not, her preferences violate the independence axiom.
- *Expected utility.* Preferences that admit an expected utility representation satisfy the independence axiom. Since these preferences violate it, they cannot be represented by an expected utility function. The source of the violation is the variance: it is not linear in probabilities, as $\sigma^2 = 16p(1-p)$ shows, whereas expected utility is.

(c) [15 points] *First-order stochastic dominance.* Show that lottery L_p first-order stochastically dominates L_0 for every $p \in (0,1]$. Then, for a general $\gamma > 0$, find the values of p for which the individual strictly prefers L_0 to L_p. For which values of γ does the individual violate first-order stochastic dominance?

- *Dominance.* The cumulative distribution function of L_0 is $F_0(x) = 1$ for all $x \geq 0$, while that of L_p is

$$F_p(x) = \begin{cases} 1-p & \text{if } 0 \leq x < 4, \text{ and} \\ 1 & \text{if } x \geq 4. \end{cases}$$

  Hence, $F_p(x) \leq F_0(x)$ for all x, with strict inequality for $x \in [0,4)$ when $p > 0$. Therefore, L_p first-order stochastically dominates L_0: every individual who prefers more money to less should prefer L_p to L_0.
- *Preference for the dominated lottery.* Since $V(L_0) = 0$, the individual strictly prefers L_0 to L_p if and only if $V(L_p) < 0$, that is,

$$4p\left[1 - 2\gamma(1-p)\right] < 0.$$

  For $p > 0$, this inequality holds if and only if $1 - 2\gamma(1-p) < 0$. Solving for p,

$$p < \widehat{p}(\gamma) \equiv 1 - \frac{1}{2\gamma}.$$

- *Cutoffs on γ.* The cutoff $\widehat{p}(\gamma)$ is positive if and only if $\gamma > \frac{1}{2}$. Therefore:
  - If $\gamma \leq \frac{1}{2}$, then $\widehat{p}(\gamma) \leq 0$, so $V(L_p) > 0 = V(L_0)$ for every $p \in (0,1]$: the individual respects first-order stochastic dominance in this family of lotteries.
  - If $\gamma > \frac{1}{2}$, the individual strictly prefers the dominated lottery L_0 to L_p for every $p \in (0, \widehat{p}(\gamma))$, violating first-order stochastic dominance. The cutoff $\widehat{p}(\gamma)$ increases in γ and approaches 1 as $\gamma \to \infty$, so the region of violations expands as the individual dislikes risk more.

  As a check, at $\gamma = 1$ the cutoff is $\widehat{p}(1) = \frac{1}{2}$, which is consistent with the indifference $L_0 \sim L_{\frac{1}{2}}$ found in part (b). Figure 1 depicts cutoff $\widehat{p}(\gamma)$ and shades the region where the individual prefers the dominated lottery L_0.

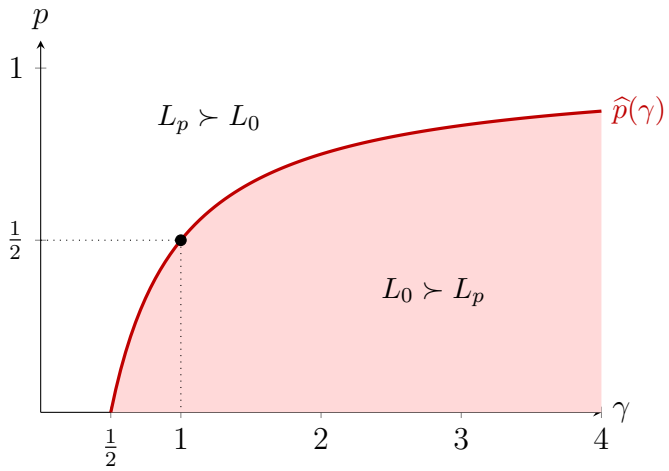


Figure 1. Region where the individual prefers the dominated lottery L_0.

3. **Buying and selling a lottery ticket. [33 points]** An individual is an expected utility maximizer with Bernoulli utility function $u(x) = \sqrt{x}$, where x denotes her final wealth, and her initial wealth is $w = 36$. Consider a lottery ticket X that pays 64 with probability $\frac{1}{2}$ and 0 with probability $\frac{1}{2}$.

   (a) [7 points] Find the expected value of the lottery ticket. Compute the individual's Arrow-Pratt coefficient of absolute risk aversion, $r_A(x)$, and determine whether it increases, decreases, or remains constant in wealth.

   - The expected value of the ticket is $E[X] = \frac{1}{2} \cdot 64 + \frac{1}{2} \cdot 0 = 32$.
   - The first- and second-order derivatives of the Bernoulli utility function are

   $$u'(x) = \frac{1}{2\sqrt{x}} > 0 \qquad \text{and} \qquad u''(x) = -\frac{1}{4x^{3/2}} < 0,$$

   so the individual is risk averse. The coefficient of absolute risk aversion is

   $$r_A(x) = -\frac{u''(x)}{u'(x)} = \frac{\frac{1}{4x^{3/2}}}{\frac{1}{2\sqrt{x}}} = \frac{1}{2x},$$

   which decreases in wealth: the individual exhibits decreasing absolute risk aversion (DARA). As she becomes richer, she is less averse to a given risk.

   (b) [9 points] *Selling price.* Suppose that the individual owns the lottery ticket. Find the minimum price s at which she would be willing to sell it. Find the risk premium she associates with the ticket, and interpret.

   - If she keeps the ticket, her final wealth is $36 + 64 = 100$ with probability $\frac{1}{2}$ and 36 with probability $\frac{1}{2}$, so her expected utility is

   $$\frac{1}{2}\sqrt{100} + \frac{1}{2}\sqrt{36} = \frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 6 = 8.$$

   If she sells the ticket at price s, her final wealth is $36 + s$ for sure. Her minimum selling price solves $\sqrt{36 + s} = 8$, that is, $36 + s = 64$, so

   $$s = 28.$$

Equivalently, her certainty equivalent of the lottery $w + X$ is 64, and $s = 64 - 36$.

- The risk premium is the difference between the expected value of the ticket and her selling price, $E[X] - s = 32 - 28 = 4$. It measures how much expected wealth she is willing to give up to avoid bearing the risk: she would sell the ticket for 28, although it is worth 32 on average, since she is risk averse.

(c) [10 points] *Buying price.* Suppose now that the individual does not own the lottery ticket. Find the maximum price b that she would be willing to pay for it. Compare b and s, and explain your results.

- If she does not buy the ticket, her utility is $\sqrt{36} = 6$. If she buys it at price b, her final wealth is $100 - b$ with probability $\frac{1}{2}$ and $36 - b$ with probability $\frac{1}{2}$. Her maximum buying price solves

$$\frac{1}{2}\sqrt{100 - b} + \frac{1}{2}\sqrt{36 - b} = 6.$$

- To solve this equation, let $A \equiv \sqrt{100 - b}$ and $B \equiv \sqrt{36 - b}$. The equation says that $A + B = 12$. In addition, $A^2 - B^2 = (100 - b) - (36 - b) = 64$, and since $A^2 - B^2 = (A + B)(A - B)$, we obtain $A - B = \frac{64}{12} = \frac{16}{3}$. Adding $A + B = 12$ and $A - B = \frac{16}{3}$ yields $A = \frac{26}{3}$ and, hence, $B = \frac{10}{3}$. Therefore,

$$b = 36 - B^2 = 36 - \frac{100}{9} = \frac{224}{9} \simeq 24.89.$$

- *Comparison.* Her buying price is lower than her selling price, $b \simeq 24.89 < s = 28$, and both are below the expected value of the ticket, 32, since she is risk averse. The gap between b and s is a wealth effect: when she owns the ticket, she is richer (her wealth includes an asset worth 32 on average), and with DARA preferences, a richer individual is less averse to the risk of the ticket, so she values it more. When she must pay for the ticket, her wealth in each state is lower, which makes her more risk averse and reduces her valuation.

(d) [7 points] Assume now that the individual has constant absolute risk aversion, with Bernoulli utility function $u(x) = -e^{-ax}$, where $a > 0$. Show that, for any initial wealth w and any lottery ticket X, her buying and selling prices coincide, $b = s$. Explain.

- *Selling price.* The selling price solves $E\left[-e^{-a(w+X)}\right] = -e^{-a(w+s)}$. Since $e^{-a(w+X)} = e^{-aw}e^{-aX}$, we can cancel e^{-aw} on both sides, obtaining $E\left[e^{-aX}\right] = e^{-as}$, that is,

$$s = -\frac{1}{a}\ln E\left[e^{-aX}\right].$$

- *Buying price.* The buying price solves $E\left[-e^{-a(w-b+X)}\right] = -e^{-aw}$. Canceling e^{-aw}, we obtain $e^{ab}E\left[e^{-aX}\right] = 1$, that is,

$$b = -\frac{1}{a}\ln E\left[e^{-aX}\right].$$

Hence, $b = s$, and neither price depends on initial wealth w.

- *Intuition.* With constant absolute risk aversion, $r_A(x) = a$ for every wealth level, so changes in wealth do not affect the individual's attitude toward the risk of the ticket. Owning the ticket or paying for it changes her wealth, but not her valuation of the ticket, which eliminates the gap between b and s found in part (c).