Comparing Payoff Distributions: First- and Second-Order Stochastic Dominance

EconS 501, a handout on the comparison of payoff distributions in Chapter 5

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Up to this point in Chapter 5, we compared individuals: for a given lottery, we asked how much a more risk-averse individual would pay to avoid it. This handout turns the question around and compares lotteries: for a given class of individuals, when can we say that one lottery is better than another without knowing each individual's utility function? We study two answers:

- First-order stochastic dominance (FOSD): one lottery yields unambiguously higher payoffs than another, so every individual who prefers more money to less prefers it.
- Second-order stochastic dominance (SOSD): two lotteries have the same mean, but one of them is unambiguously less risky, so every risk-averse individual prefers it.

Each concept is defined, characterized with cumulative distribution functions (CDFs), and illustrated with discrete and continuous examples. The last section presents three stronger orderings (hazard rate, reverse hazard rate, and likelihood ratio) that are common in auctions, contracts, and signaling games.

Throughout, F and G denote the CDFs of two monetary lotteries with support in an interval $[a, b]$, so that $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$. When the lotteries have densities, we write them as f and g. The expected utility of lottery F is $\int_a^b u(x)\, dF(x)$, where u is the individual's Bernoulli utility function. A useful expression for the mean of F, obtained by integrating by parts, is

$$\mathbb{E}_F[x] = \int_a^b x\, dF(x) = [xF(x)]_a^b - \int_a^b F(x)\, dx = b - \int_a^b F(x)\, dx. \tag{1}$$

In words, the mean is the upper end of the support minus the area under the CDF, so a lower CDF means a higher mean. When payoffs are nonnegative but the support is unbounded, as in Example 2 below, the same steps yield

$$\mathbb{E}_F[x] = \int_0^\infty [1 - F(x)]\, dx,$$

and all the results in this handout continue to hold.

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1 First-order stochastic dominance

1.1 Definition

Distribution F first-order stochastically dominates distribution G ("F FOSD G") if every individual who prefers more money to less weakly prefers F to G, that is,

$$\int_a^b u(x)\,dF(x) \geq \int_a^b u(x)\,dG(x) \quad \text{for every weakly increasing function } u : \mathbb{R} \to \mathbb{R}. \tag{2}$$

The definition does not restrict the curvature of u: the individual can be risk averse, risk neutral, or risk loving.

1.2 The CDF test

Result 1 (CDF test for FOSD). F FOSD G if and only if

$$F(x) \leq G(x) \quad \text{for every } x, \text{ or, equivalently,} \quad 1 - F(x) \geq 1 - G(x) \quad \text{for every } x. \tag{3}$$

This test is due to Hadar and Russell (1969) and Hanoch and Levy (1969), and Mas-Colell, Whinston, and Green (1995, Section 6.D) present it together with the test for SOSD. The second form of (3) gives the intuition: for every amount of money x, the probability of receiving more than x is higher under F than under G. Graphically, the CDF of the dominant lottery lies (weakly) below the CDF of the dominated lottery everywhere.

Why the test works. For a differentiable u, integrating by parts,

$$\begin{aligned}\int_a^b u(x)\,dF(x) - \int_a^b u(x)\,dG(x) &= \Big[u(x)\big(F(x) - G(x)\big)\Big]_a^b - \int_a^b u'(x)\big[F(x) - G(x)\big]dx \\ &= \int_a^b u'(x)\big[G(x) - F(x)\big]dx, \end{aligned} \tag{4}$$

where the bracket vanishes because $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$. We can now check both directions:

- If $F(x) \leq G(x)$ for every x, every term $u'(x)[G(x) - F(x)]$ in (4) is nonnegative, since $u'(x) \geq 0$. Hence F yields a weakly higher expected utility for every weakly increasing u.
- If, instead, $F(x_0) > G(x_0)$ at some x_0, consider the step function $u(x) = 1$ if $x > x_0$ and $u(x) = 0$ otherwise, which is weakly increasing. Its expected utility is the probability of receiving more than x_0, which equals $1 - F(x_0)$ under F and $1 - G(x_0)$ under G. Since $1 - F(x_0) < 1 - G(x_0)$, this individual strictly prefers G, so F does not FOSD G.

Two consequences.

- FOSD implies a weakly higher mean. Setting $u(x) = x$ in (2) yields $\mathbb{E}_F[x] \geq \mathbb{E}_G[x]$; alternatively, from (1), the area under F is smaller than the area under G. The converse does not hold, as Example 5 shows.

- FOSD can be understood as "adding a nonnegative amount." If F FOSD G, we can build a lottery with CDF F by first drawing a payoff from G and then adding a nonnegative (possibly random) amount to it. For instance, shifting every payoff of G up by $c > 0$ yields $F(x) = G(x-c) \leq G(x)$.

Figure 1 depicts the CDF test for two continuous distributions on $[0, 1]$: the uniform distribution, $G(x) = x$, and $F(x) = x^2$ (Example 3 below). The CDF F lies below G at every interior x, so F assigns more probability to high payoffs. The vertical distance between the curves at any x, $G(x) - F(x)$, is the extra probability that F assigns to payoffs above x.

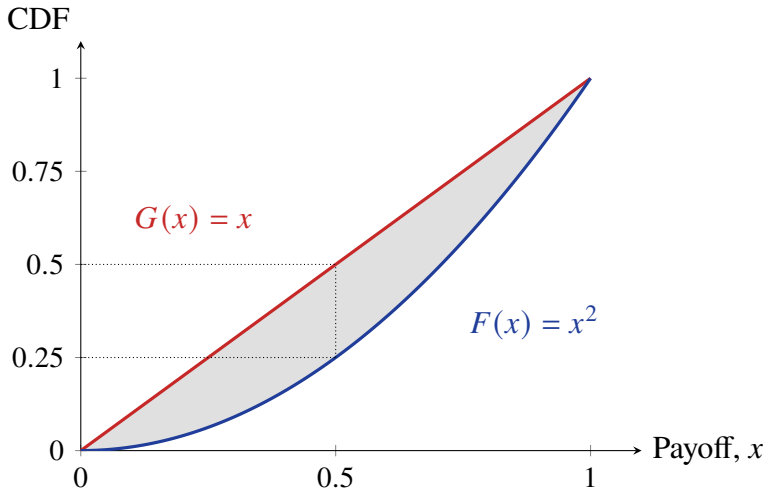


Figure 1. F FOSD G: the CDF of F lies below that of G.

1.3 Examples

Example 1 (Discrete lotteries). Lotteries F and G pay \$0, \$10, or \$20 with the following probabilities:

	\$0	\$10	\$20	Mean
Probabilities under F	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	12.5
Probabilities under G	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	7.5

Table 1. Two discrete lotteries.

Accumulating the probabilities from the left, the CDFs are step functions:

- For $x \in [0, 10)$: $F(x) = \frac{1}{4}$ and $G(x) = \frac{1}{2}$.
- For $x \in [10, 20)$: $F(x) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ and $G(x) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.
- For $x \geq 20$: $F(x) = G(x) = 1$.

Hence $F(x) \leq G(x)$ at every x, and F FOSD G. Figure 2 depicts both step functions. Intuitively, F is obtained from G by moving probability $\frac{1}{4}$ from \$0 to \$20, a pure improvement. As a check,

an individual with $u(x) = \sqrt{x}$ obtains

$$\mathbb{E}_F[u] = \frac{1}{4}\sqrt{0} + \frac{1}{4}\sqrt{10} + \frac{1}{2}\sqrt{20} \approx 3.027 \qquad \text{and} \qquad \mathbb{E}_G[u] = \frac{1}{2}\sqrt{0} + \frac{1}{4}\sqrt{10} + \frac{1}{4}\sqrt{20} \approx 1.909,$$

and a risk lover with $u(x) = x^2$ obtains

$$\mathbb{E}_F[u] = \frac{1}{4}(100) + \frac{1}{2}(400) = 225 \qquad \text{and} \qquad \mathbb{E}_G[u] = \frac{1}{4}(100) + \frac{1}{4}(400) = 125.$$

Both individuals prefer F, as Result 1 predicts.

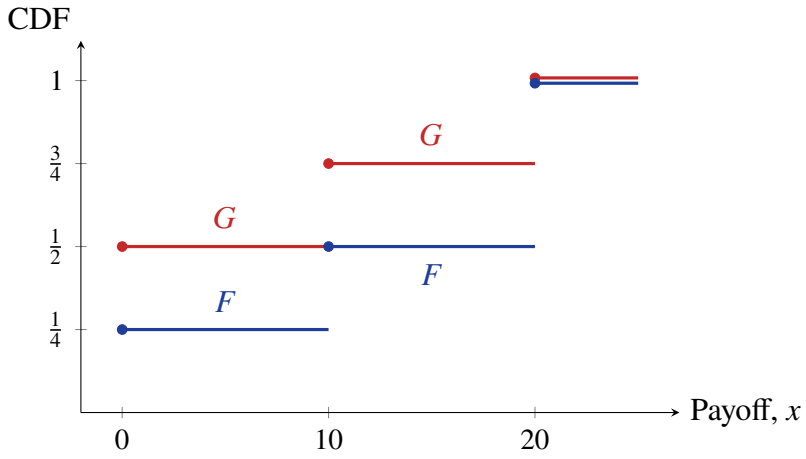


Figure 2. Example 1: the step CDF of F lies below that of G.

Example 2 (Exponential distributions). Let the payoff be exponentially distributed, $F_\lambda(x) = 1 - e^{-\lambda x}$ for $x \geq 0$, with mean $\frac{1}{\lambda}$. For two parameters $\lambda_1 < \lambda_2$, we have $e^{-\lambda_1 x} \geq e^{-\lambda_2 x}$, so

$$F_{\lambda_1}(x) = 1 - e^{-\lambda_1 x} \leq 1 - e^{-\lambda_2 x} = F_{\lambda_2}(x) \quad \text{for every } x \geq 0.$$

Therefore, a lower λ yields FOSD. Figure 3 depicts $\lambda_1 = 1$ (mean 1) and $\lambda_2 = 2$ (mean $\frac{1}{2}$). The probability of receiving more than $x = 1$, for instance, is $e^{-1} \approx 0.368$ under F_1 but only $e^{-2} \approx 0.135$ under F_2.

Example 3 (Power distributions). Consider the family $F_\alpha(x) = x^\alpha$ on $[0, 1]$, with $\alpha > 0$, which nests the uniform distribution when $\alpha = 1$. For any $x \in (0, 1)$, the function x^α decreases in α, since

$$\frac{\partial x^\alpha}{\partial \alpha} = x^\alpha \ln x < 0,$$

where the sign follows from $\ln x < 0$ for $x < 1$. Therefore, a higher α yields FOSD. The density of F_α, $f_\alpha(x) = \alpha x^{\alpha-1}$, places more weight on high payoffs as α increases, and its mean is

$$\mathbb{E}_\alpha[x] = \int_0^1 x \cdot \alpha x^{\alpha-1}\, dx = \alpha \int_0^1 x^\alpha\, dx = \frac{\alpha}{\alpha + 1},$$

which increases in α. Figure 1 depicts the case $\alpha = 2$ against the uniform distribution ($\alpha = 1$); their means are $\frac{2}{3}$ and $\frac{1}{2}$, respectively.

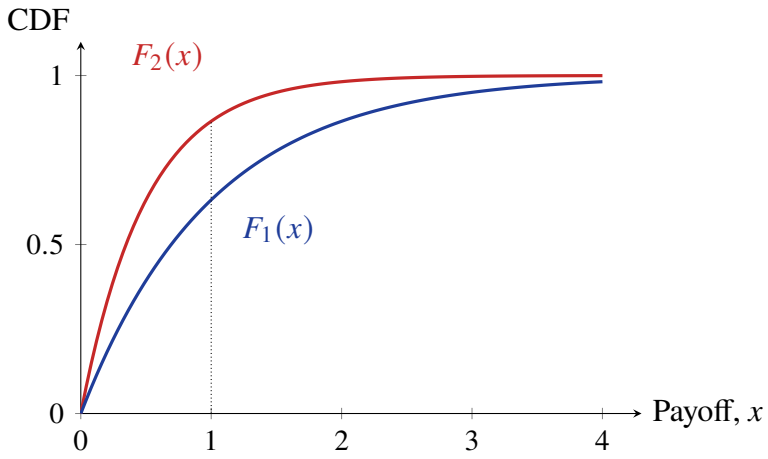


Figure 3. Example 2: exponential CDFs with $\lambda = 1$ and $\lambda = 2$.

Example 4 (More likely to win a prize). A lottery pays a prize $z > 0$ with probability p and \$0 otherwise, so its CDF equals $1 - p$ for $x \in [0, z)$ and 1 for $x \geq z$. If $p_1 > p_2$, the CDF of the first lottery, $1 - p_1$, lies below that of the second, $1 - p_2$, on $[0, z)$, and both equal 1 afterwards. Hence the lottery with the higher winning probability FOSD the other. Every individual who likes money prefers this lottery, regardless of her attitude towards risk, since her expected utility

$$(1 - p)\, u(0) + p\, u(z) = u(0) + p\, [u(z) - u(0)]$$

is weakly increasing in p when $u(z) \geq u(0)$.

1.4 When FOSD fails: crossing CDFs

Example 5 (A higher mean is not enough). Compare lottery F, which pays \$4 for sure, with lottery G, which pays \$0 or \$10 with probability $\frac{1}{2}$ each. Lottery G has the higher mean, 5 versus 4. However, as Figure 4 shows, the CDFs cross:

- For $x \in [0, 4)$: $G(x) = \frac{1}{2} > 0 = F(x)$, since only G can pay \$0.
- For $x \in [4, 10)$: $G(x) = \frac{1}{2} < 1 = F(x)$, since only G can pay more than \$4.

Neither CDF lies below the other at every x, so neither lottery FOSD the other. Individuals who like money can then disagree:

- A risk-neutral individual, $u(x) = x$, prefers G, since $5 > 4$.
- An individual with $u(x) = \sqrt{x}$ prefers F, since $\sqrt{4} = 2 > \frac{1}{2}\sqrt{10} \approx 1.581$.

Since FOSD requires every weakly increasing u to agree, one individual who prefers the lottery with the lower mean is enough to rule it out.

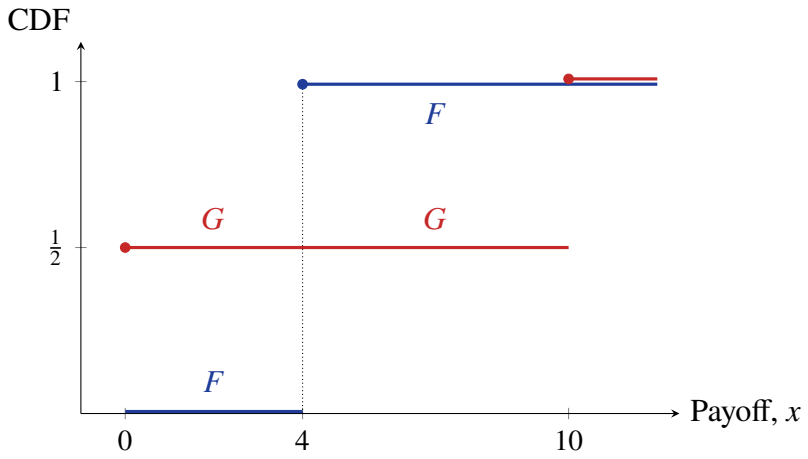


Figure 4. Example 5: the CDFs cross, so neither lottery FOSD the other.

2 Second-order stochastic dominance

2.1 Definition

We now focus on the riskiness of a lottery, as opposed to the level of its payoffs. To separate the two, we compare lotteries with the same mean, $\mathbb{E}_F[x] = \mathbb{E}_G[x]$. Distribution F second-order stochastically dominates distribution G ("F SOSD G") if every risk-averse individual weakly prefers F to G, that is,

$$\int_a^b u(x)\,dF(x) \geq \int_a^b u(x)\,dG(x) \quad \text{for every weakly increasing and concave function } u. \tag{5}$$

We then say that G is riskier than F. Compared with FOSD, definition (5) restricts attention to a smaller set of individuals, those with concave utility functions, so it is a weaker requirement: more pairs of lotteries can be ranked.

2.2 The integrated-CDF test

Define the area under each CDF up to payoff x,

$$I_F(x) \equiv \int_a^x F(t)\,dt \qquad \text{and} \qquad I_G(x) \equiv \int_a^x G(t)\,dt.$$

From (1), equal means imply $I_F(b) = I_G(b)$.

Result 2 (Integrated-CDF test for SOSD). Let F and G have the same mean. Then F SOSD G if and only if

$$I_F(x) = \int_a^x F(t)\,dt \leq \int_a^x G(t)\,dt = I_G(x) \quad \text{for every } x \in [a, b]. \tag{6}$$

Condition (6) allows the CDFs to cross, unlike the FOSD test, but the area under G, accumulated from the left, must never fall behind the area under F. Intuitively, the riskier lottery G places more probability on low payoffs first; it compensates later with more probability on high payoffs, which keeps the mean unchanged.

Why the test works. Let $D(x) \equiv I_G(x) - I_F(x)$, so that $D(a) = 0$, $D(b) = 0$ (equal means), and $D'(x) = G(x) - F(x)$. For a twice differentiable u, starting from (4) and integrating by parts a second time,

$$\int_a^b u(x)\,dF(x) - \int_a^b u(x)\,dG(x) = \int_a^b u'(x)D'(x)\,dx = \Big[u'(x)D(x)\Big]_a^b - \int_a^b u''(x)D(x)\,dx$$
$$= -\int_a^b u''(x)D(x)\,dx, \tag{7}$$

where the bracket vanishes because $D(a) = D(b) = 0$.

- If $D(x) \geq 0$ for every x, which is condition (6), and $u''(x) \leq 0$, the right side of (7) is nonnegative, so every risk-averse individual weakly prefers F.
- If, instead, $D(x_0) < 0$ at some x_0, consider $u(x) = \min\{x, x_0\}$, which is weakly increasing and concave. This individual values every payoff above x_0 as x_0. Integrating by parts as in (1), her expected utility under F is

  $$\int_a^{x_0} x\,dF(x) + x_0\,[1 - F(x_0)] = [x_0F(x_0) - I_F(x_0)] + x_0 - x_0F(x_0) = x_0 - I_F(x_0),$$

  and, similarly, $x_0 - I_G(x_0)$ under G. The difference is $I_G(x_0) - I_F(x_0) = D(x_0) < 0$, so this risk-averse individual strictly prefers G.

Three consequences.

- Since the means are equal, the condition in (5) also holds for every concave u, increasing or not, because (7) does not use the sign of $u'(x)$. In particular, $u(x) = -x^2$ yields $\mathbb{E}_F[x^2] \leq \mathbb{E}_G[x^2]$, so the dominant lottery has a weakly lower variance. The converse does not hold, as Example 10 shows.
- Risk lovers rank the two lotteries in the opposite way: if u is convex, then $u''(x) \geq 0$ in (7) and G is weakly preferred.
- Risk-neutral individuals, $u''(x) = 0$, are indifferent, since only the mean matters to them.

2.3 Mean-preserving spreads

The most common way to make a lottery riskier is a mean-preserving spread (MPS): distribution G is an MPS of F if a payoff from G can be generated by drawing a payoff x from F and then adding noise ε with zero mean conditional on x, $\mathbb{E}[\varepsilon \mid x] = 0$. The noise spreads probability over a larger set of payoffs without changing the mean. Rothschild and Stiglitz (1970) show that, for lotteries with the same mean, the following three statements are equivalent:

- G is an MPS of F;
- every risk-averse individual weakly prefers F to G (F SOSD G);
- condition (6) holds.

The intuition for the second statement follows from Jensen's inequality: for a concave u, adding zero-mean noise ε to a given payoff x yields

$$\mathbb{E}\left[u(x+\varepsilon)\,|\,x\right] \leq u\left(\mathbb{E}\left[x+\varepsilon\,|\,x\right]\right) = u(x),$$

and taking expectations over x gives $\mathbb{E}_G[u] \leq \mathbb{E}_F[u]$.

2.4 Examples

Example 6 (A sure amount and a fair bet). Lottery F pays \$5 for sure, and lottery G pays \$0 or \$10 with probability $\frac{1}{2}$ each, so both have mean 5. Lottery G is an MPS of F, since it adds noise $\varepsilon = \pm 5$, with probability $\frac{1}{2}$ each, to the sure payoff of \$5. Checking condition (6):

- For $x \in [0,5]$: $F(t) = 0$ and $G(t) = \frac{1}{2}$ on $[0,x)$, so $I_F(x) = 0$ and $I_G(x) = \frac{x}{2}$, and $D(x) = \frac{x}{2} \geq 0$.
- For $x \in [5,10]$: $F(t) = 1$ on $[5,x)$, so $I_F(x) = x - 5$, while $I_G(x) = \frac{x}{2}$. Then $D(x) = \frac{x}{2} - (x-5) = 5 - \frac{x}{2} \geq 0$, which reaches zero at $x = 10$, as equal means require.

Hence F SOSD G: every risk-averse individual prefers the sure \$5, which is the familiar consequence of Jensen's inequality,

$$u(5) = u\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 10\right) \geq \frac{1}{2}u(0) + \frac{1}{2}u(10).$$

Example 7 (A discrete MPS, from the slides). Lottery F pays \$2 or \$3 with probability $\frac{1}{2}$ each, and lottery G pays \$1, \$2, \$3, or \$4 with probability $\frac{1}{4}$ each. Both have mean 2.5, and G is an MPS of F: start from F, replace the payoff \$2 with a fair coin flip between \$1 and \$3, and replace the payoff \$3 with a fair coin flip between \$2 and \$4. Each replacement adds noise of ± 1 with zero mean, and the resulting payoffs \$1, \$3, \$2, and \$4 each occur with probability $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$, which is lottery G.

The CDFs cross, so neither lottery FOSD the other: $G(x) = \frac{1}{4} > 0 = F(x)$ for $x \in [1,2)$, and $G(x) = \frac{3}{4} < 1 = F(x)$ for $x \in [3,4)$; on $[2,3)$ both equal $\frac{1}{2}$. Integrating the difference $G - F$ from the left (Figure 5, right panel):

- On $[1,2]$: $D(x) = \frac{1}{4}(x-1)$, which grows from 0 to $\frac{1}{4}$.
- On $[2,3]$: $G = F$, so $D(x)$ stays at $\frac{1}{4}$.
- On $[3,4]$: $G - F = -\frac{1}{4}$, so $D(x) = \frac{1}{4} - \frac{1}{4}(x-3)$, which falls from $\frac{1}{4}$ to 0.

Since $D(x) \geq 0$ everywhere, F SOSD G. In the left panel of Figure 5, the shaded area where G lies above F (on $[1,2)$) equals the shaded area where F lies above G (on $[3,4)$); this is the graphical

counterpart of equal means. The variances are

$$\mathrm{Var}_F = \frac{1}{2}\left(-\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{4} \qquad \text{and} \qquad \mathrm{Var}_G = \frac{1}{4}\left[\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2\right] = \frac{5}{4},$$

and, with $u(x) = \sqrt{x}$, we obtain

$$\mathbb{E}_F[u] = \frac{\sqrt{2}+\sqrt{3}}{2} \approx 1.573 \qquad \text{and} \qquad \mathbb{E}_G[u] = \frac{1+\sqrt{2}+\sqrt{3}+2}{4} \approx 1.537,$$

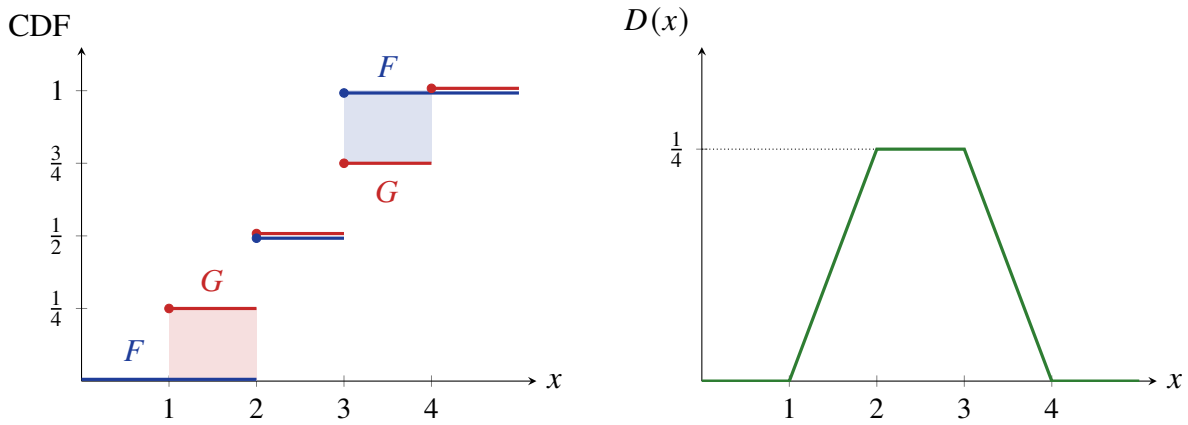


Figure 5. Example 7. Left: the CDFs cross, and the two shaded areas are equal. Right: $D(x) = I_G(x) - I_F(x) \geq 0$.

Example 8 (Uniform distributions with the same center). Let F be uniform on $[4, 6]$ and G be uniform on $[0, 10]$, so both have mean 5. Their CDFs are $F(x) = \frac{x-4}{2}$ on $[4, 6]$ and $G(x) = \frac{x}{10}$ on $[0, 10]$, and they cross once at the common mean, $x = 5$ (Figure 6):

- For $x < 5$, $G(x) > F(x)$: the wide distribution G places more probability on low payoffs.
- For $x > 5$, $G(x) < F(x)$: it also places more probability on high payoffs.

The area between the CDFs to the left of $x = 5$ equals

$$\int_0^4 \frac{x}{10}\,dx + \int_4^5 \left(\frac{x}{10} - \frac{x-4}{2}\right) dx = \frac{16}{20} + \int_4^5 \left(2 - \frac{2x}{5}\right) dx = \frac{4}{5} + \frac{1}{5} = 1,$$

and, by symmetry, the area to the right of $x = 5$ is also 1. Hence $D(x)$ increases until $x = 5$, where it reaches its maximum of 1, and then decreases back to zero at $x = 10$, so $D(x) \geq 0$ and F SOSD G. This is a general pattern: when two CDFs with the same mean cross only once, the one that starts below (the one placing less probability on low payoffs) SOSD the other. Similarly, a uniform distribution on $[5 - s, 5 + s]$ becomes riskier in the SOSD sense as its half-width s increases, since any two such distributions have the same mean and their CDFs cross once, at $x = 5$.

Example 9 (Diversification). Lottery G pays \$0 or \$10 with probability $\frac{1}{2}$ each. An investor can instead split her money equally between two independent copies of this lottery, so she receives the average of two independent draws. This lottery, F, pays

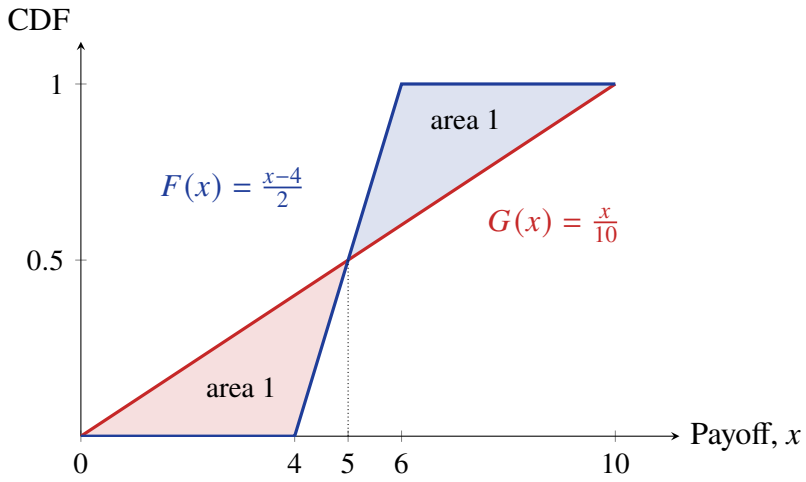


Figure 6. Example 8: single-crossing CDFs with the same mean; the two shaded areas are equal.

- \$0 if both draws are \$0, with probability $\frac{1}{4}$;
- \$5 if one draw is \$0 and the other is \$10, with probability $\frac{1}{2}$;
- \$10 if both draws are \$10, with probability $\frac{1}{4}$.

Both lotteries have mean 5, and G is an MPS of F: starting from F, replace the payoff \$5 with a fair coin flip between \$0 and \$10. Therefore, F SOSD G, and every risk-averse investor prefers to diversify. With $u(x) = \sqrt{x}$, for instance,

$$\mathbb{E}_F[u] = \frac{1}{4}\sqrt{0} + \frac{1}{2}\sqrt{5} + \frac{1}{4}\sqrt{10} \approx 1.909 \qquad \text{and} \qquad \mathbb{E}_G[u] = \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{10} \approx 1.581.$$

The variance falls from $\frac{1}{2}(25) + \frac{1}{2}(25) = 25$ under G to $\frac{1}{4}(25) + \frac{1}{4}(25) = 12.5$ under F.

Example 10 (A lower variance is not enough). Consider an individual with wealth \$10 who faces one of two lotteries over gains and losses, both with mean zero:

- Lottery A loses \$9 with probability 0.1 and gains \$1 with probability 0.9. Its variance is $0.1(81) + 0.9(1) = 9$.
- Lottery B loses or gains \$4 with probability $\frac{1}{2}$ each. Its variance is 16.

Lottery A has the lower variance, but it carries a small probability of a large loss (downside risk). Two risk-averse individuals rank these lotteries differently:

- With $u(x) = \ln x$, where x is final wealth, her expected utility is

  $$\mathbb{E}_A[u] = 0.1 \ln 1 + 0.9 \ln 11 \approx 2.158 \qquad \text{and} \qquad \mathbb{E}_B[u] = \frac{1}{2}\ln 6 + \frac{1}{2}\ln 14 \approx 2.215,$$

  so she prefers B, the lottery with the higher variance.
- With $u(x) = -e^{-0.1x}$, after dividing by the common positive factor $e^{-0.1(10)}$, her expected utility is

  $$-\left[0.1e^{0.9} + 0.9e^{-0.1}\right] \approx -1.060 \text{ under } A \qquad \text{and} \qquad -\frac{1}{2}\left[e^{0.4} + e^{-0.4}\right] \approx -1.081 \text{ under } B,$$

so she prefers A.

Hence neither lottery SOSD the other. The integrated-CDF test confirms it. Working with the gain $y \in [-9, 4]$ rather than final wealth, let F_A and F_B denote the CDFs of the two lotteries, $I_A(y)$ and $I_B(y)$ the areas under them from -9 to y, and $D(y) = I_B(y) - I_A(y)$:

- For $y \in [-9, -4)$: only A has accumulated probability, $F_A(y) = 0.1$ and $F_B(y) = 0$, so $D(-4) = -0.1(5) = -0.5 < 0$. Hence A does not SOSD B, which would require $D \geq 0$ everywhere.
- For $y \in [-4, 1)$: $F_B(y) - F_A(y) = 0.5 - 0.1 = 0.4$, so $D(1) = -0.5 + 0.4(5) = 1.5 > 0$. Hence B does not SOSD A, which would require $D \leq 0$ everywhere.
- For $y \in [1, 4]$: $F_B(y) - F_A(y) = 0.5 - 1 = -0.5$, so $D(4) = 1.5 - 0.5(3) = 0$, as equal means require.

The individual with log utility dislikes the large loss of lottery A more than the larger spread of lottery B, since $\ln x$ becomes very steep at low wealth levels. Variance treats losses and gains symmetrically and does not capture this concern; SOSD does, because it considers every concave utility function.

Example 11 (Elementary increase in risk). Distribution G is an elementary increase in risk (EIR) of F if G takes all the probability that F assigns to an interval $[x', x'']$ and moves it to the endpoints x' and x'', in proportions that preserve the mean. Let F pay \$2, \$4, or \$6 with probability $\frac{1}{3}$ each (mean 4). Taking the interval $[3, 5]$, F assigns probability $\frac{1}{3}$ to it (at \$4). Moving this probability to the endpoints, with $\frac{1}{6}$ at \$3 and $\frac{1}{6}$ at \$5, keeps the mean of this part at 4, so we obtain lottery G, which pays \$2, \$3, \$5, and \$6 with probabilities $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{6}$, and $\frac{1}{3}$. Its mean and variance are

$$\mathbb{E}_G[x] = \frac{2}{3} + \frac{3}{6} + \frac{5}{6} + \frac{6}{3} = 4 \qquad \text{and} \qquad \mathrm{Var}_G = \frac{1}{3}(4) + \frac{1}{6}(1) + \frac{1}{6}(1) + \frac{1}{3}(4) = 3,$$

so the variance rises from $\mathrm{Var}_F = \frac{1}{3}(4) + \frac{1}{3}(0) + \frac{1}{3}(4) = \frac{8}{3}$ to 3.

Every EIR is an MPS, since each payoff in the interval is replaced by a fair spread to the endpoints, but the converse does not hold. In Example 7, for instance, G is an MPS of F, yet it is not an EIR of F. An EIR adds at most two new payoffs (the endpoints) and removes every payoff strictly inside the interval. Creating both \$1 and \$4 thus requires the interval $[1, 4]$, which removes \$2 and \$3 and yields \$1 and \$4 with probability $\frac{1}{2}$ each, not the four equally likely payoffs of G.

2.5 How FOSD and SOSD are related

FOSD compares the level of payoffs, while SOSD (as defined above, with equal means) compares their dispersion, so the two concepts answer different questions. Two remarks connect them:

- If F FOSD G and $F \neq G$, then F has a strictly higher mean, so the two lotteries cannot be compared under our definition of SOSD.
- Some textbooks drop the equal-mean requirement and say that F SOSD G whenever condition (6) holds. Under that broader definition, (7) gains the term $u'(b)D(b) \geq 0$, and FOSD implies

SOSD: if $F(x) \leq G(x)$ at every x, the areas satisfy $I_F(x) \leq I_G(x)$ at every x. Intuitively, a lottery that every individual who likes money prefers is also preferred by every risk-averse individual who likes money, since the latter group is a subset of the former.

3 Stronger orderings: hazard rate, reverse hazard rate, and likelihood ratio

In auctions, contracts, and signaling games, FOSD is often not enough to sign the comparative statics, and models assume one of three stronger orderings; Shaked and Shanthikumar (2007) provide a complete treatment. Throughout this section, F and G have densities f and g.

3.1 Hazard rate dominance

The hazard rate of F is

$$h_F(x) = \frac{f(x)}{1 - F(x)}. \tag{8}$$

If x measures time, $h_F(x)$ is the rate at which an event (a machine stops working, a worker finds a job) happens at time x, given that it has not happened before x. Lottery F dominates G in the hazard rate order if $h_F(x) \leq h_G(x)$ for every x: conditional on reaching any payoff x, the payoff under F is less likely to stop there.

Result 3 (Hazard rate dominance implies FOSD). If $h_F(x) \leq h_G(x)$ for every x, then F FOSD G.

To see why, notice that, by the chain rule,

$$\frac{d}{dx} \ln [1 - F(x)] = \frac{-f(x)}{1 - F(x)} = -h_F(x).$$

Integrating from a to x, and using $1 - F(a) = 1$ and $\ln 1 = 0$, we obtain $\ln [1 - F(x)] = -\int_a^x h_F(t)\, dt$, that is,

$$1 - F(x) = \exp\left(-\int_a^x h_F(t)\, dt\right). \tag{9}$$

The same expression holds for G. If $h_F \leq h_G$ at every point, the integral in (9) is smaller under F, so $1 - F(x) \geq 1 - G(x)$ at every x, which is the FOSD condition (3).

3.2 Reverse hazard rate dominance

The reverse hazard rate of F is

$$r_F(x) = \frac{f(x)}{F(x)}, \tag{10}$$

which measures how likely the payoff is to be near x, conditional on the payoff being at most x. Lottery F dominates G in the reverse hazard rate order if $r_F(x) \geq r_G(x)$ for every x. Since $r_F(x) =$

$\frac{d}{dx} \ln F(x)$, integrating from x to b, and using $F(b) = 1$ and $\ln 1 = 0$, yields $-\ln F(x) = \int_x^b r_F(t)\, dt$, that is,

$$F(x) = \exp\left(-\int_x^b r_F(t)\, dt\right),$$

so $r_F \geq r_G$ at every point implies $F(x) \leq G(x)$ at every x, which is again FOSD.

3.3 Likelihood ratio dominance

Lottery F dominates G in the likelihood ratio order if the ratio $\frac{f(x)}{g(x)}$ is weakly increasing in x. In words, observing a higher payoff makes it relatively more likely that the payoff was drawn from F than from G. This property is the monotone likelihood ratio property (MLRP) that appears in moral hazard and signaling models.

Result 4 (Likelihood ratio dominance implies hazard rate and reverse hazard rate dominance). If $\frac{f(x)}{g(x)}$ is weakly increasing, then $h_F(x) \leq h_G(x)$ and $r_F(x) \geq r_G(x)$ for every x.

For the hazard rate, consider any $t \geq x$. Since the ratio $\frac{f}{g}$ is weakly increasing,

$$\frac{f(t)}{g(t)} \geq \frac{f(x)}{g(x)}, \qquad \text{or, equivalently,} \qquad f(t)g(x) \geq f(x)g(t).$$

Integrating both sides over $t \in [x, b]$, and using $\int_x^b f(t)\, dt = 1 - F(x)$ and $\int_x^b g(t)\, dt = 1 - G(x)$,

$$g(x)\,[1 - F(x)] \geq f(x)\,[1 - G(x)], \qquad \text{that is,} \qquad \frac{f(x)}{1 - F(x)} \leq \frac{g(x)}{1 - G(x)}.$$

The argument for the reverse hazard rate is symmetric: for any $t \leq x$, we have $f(x)g(t) \geq f(t)g(x)$, and integrating over $t \in [a, x]$ yields

$$f(x)G(x) \geq g(x)F(x), \qquad \text{that is,} \qquad \frac{f(x)}{F(x)} \geq \frac{g(x)}{G(x)}.$$

Combining Results 3 and 4:

- Likelihood ratio dominance implies hazard rate dominance, which implies FOSD.
- Likelihood ratio dominance implies reverse hazard rate dominance, which implies FOSD.

None of these implications can be reversed in general; Example 14 shows it for the step from hazard rate dominance to FOSD.

3.4 Examples

Example 12 (Power distributions, continued). For $F(x) = x^2$ and $G(x) = x$ on $[0, 1]$ (Figure 1), the densities are $f(x) = 2x$ and $g(x) = 1$. All three orderings hold:

- Likelihood ratio: $\frac{f(x)}{g(x)} = 2x$ increases in x.

- Hazard rate: the two hazard rates are

$$h_F(x) = \frac{2x}{1-x^2} = \frac{2x}{(1-x)(1+x)} \qquad \text{and} \qquad h_G(x) = \frac{1}{1-x}.$$

  Multiplying both by $1 - x > 0$, $h_F(x) \leq h_G(x)$ if and only if $\frac{2x}{1+x} \leq 1$, that is, $x \leq 1$, which holds on the whole support.

- Reverse hazard rate: $r_F(x) = \frac{2x}{x^2} = \frac{2}{x} \geq \frac{1}{x} = r_G(x)$.

More generally, within the power family x^α, the likelihood ratio

$$\frac{f_{\alpha_1}(x)}{f_{\alpha_2}(x)} = \frac{\alpha_1 x^{\alpha_1 - 1}}{\alpha_2 x^{\alpha_2 - 1}} = \frac{\alpha_1}{\alpha_2} x^{\alpha_1 - \alpha_2}$$

increases in x whenever $\alpha_1 > \alpha_2$, so a higher α yields all three orderings.

Example 13 (Exponential distributions, continued). The exponential distribution of Example 2 has density $f_\lambda(x) = \lambda e^{-\lambda x}$ and a constant hazard rate,

$$h_\lambda(x) = \frac{\lambda e^{-\lambda x}}{1 - \left(1 - e^{-\lambda x}\right)} = \lambda.$$

In words, the probability that the event happens now does not depend on how long we have waited (the "memoryless" property). With $\lambda_1 < \lambda_2$, we have $h_{\lambda_1} = \lambda_1 < \lambda_2 = h_{\lambda_2}$, so F_{λ_1} dominates F_{λ_2} in the hazard rate order, consistent with the FOSD ranking in Example 2. The likelihood ratio

$$\frac{f_{\lambda_1}(x)}{f_{\lambda_2}(x)} = \frac{\lambda_1 e^{-\lambda_1 x}}{\lambda_2 e^{-\lambda_2 x}} = \frac{\lambda_1}{\lambda_2} e^{(\lambda_2 - \lambda_1)x}$$

increases in x, so likelihood ratio dominance holds as well.

Example 14 (FOSD without hazard rate dominance). Let G be uniform on $[0, 2]$, so $G(x) = \frac{x}{2}$, and let F draw a payoff from the uniform distribution on $[\frac{1}{2}, 1]$ with probability $\frac{1}{2}$, and from the uniform distribution on $[\frac{3}{2}, 2]$ with probability $\frac{1}{2}$. Each half carries probability $\frac{1}{2}$ over an interval of length $\frac{1}{2}$, so the density of F equals 1 on both intervals and 0 elsewhere, and its CDF is

$$F(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}], \\ x - \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1], \\ \frac{1}{2} & \text{if } x \in [1, \frac{3}{2}], \\ x - 1 & \text{if } x \in [\frac{3}{2}, 2]. \end{cases}$$

We can check FOSD interval by interval:

- On $[0, \frac{1}{2}]$: $0 \leq \frac{x}{2}$.
- On $[\frac{1}{2}, 1]$: $x - \frac{1}{2} \leq \frac{x}{2}$ if and only if $x \leq 1$.
- On $[1, \frac{3}{2}]$: $\frac{1}{2} \leq \frac{x}{2}$, since $x \geq 1$.

- On $[\frac{3}{2}, 2]$: $x - 1 \leq \frac{x}{2}$ if and only if $x \leq 2$.

Hence F FOSD G. The means are

$$\mathbb{E}_F[x] = \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{7}{4} = \frac{5}{4} \qquad \text{and} \qquad \mathbb{E}_G[x] = 1,$$

where $\frac{3}{4}$ and $\frac{7}{4}$ are the midpoints of the two intervals over which F is uniform. The hazard rate of G is

$$h_G(x) = \frac{\frac{1}{2}}{1 - \frac{x}{2}} = \frac{1}{2 - x},$$

while the hazard rate of F is

- $h_F(x) = 0 < h_G(x)$ on $[0, \frac{1}{2})$ and on $(1, \frac{3}{2})$, where F has no density;
- on $(\frac{1}{2}, 1)$, where $f(x) = 1$ and $F(x) = x - \frac{1}{2}$,

$$h_F(x) = \frac{1}{1 - \left(x - \frac{1}{2}\right)} = \frac{1}{\frac{3}{2} - x} > \frac{1}{2 - x} = h_G(x);$$

- on $(\frac{3}{2}, 2)$, where $f(x) = 1$ and $F(x) = x - 1$,

$$h_F(x) = \frac{1}{1 - (x - 1)} = \frac{1}{2 - x} = h_G(x).$$

At $x = \frac{3}{4}$, for instance, $h_F(\frac{3}{4}) = \frac{4}{3} \approx 1.33$, while $h_G(\frac{3}{4}) = \frac{4}{5} = 0.8$. Hence F does not dominate G in the hazard rate order, although it does in the FOSD sense. Intuitively, F places half of its probability on the interval $[\frac{1}{2}, 1]$, so a payoff that has reached $\frac{3}{4}$ under F is quite likely to stop before 1; under G, the remaining probability is spread over the whole interval $[\frac{3}{4}, 2]$. Figure 7 depicts both hazard rates.

4 Summary

Table 2 collects the orderings studied in this handout, the test that checks each of them, and the main implications.

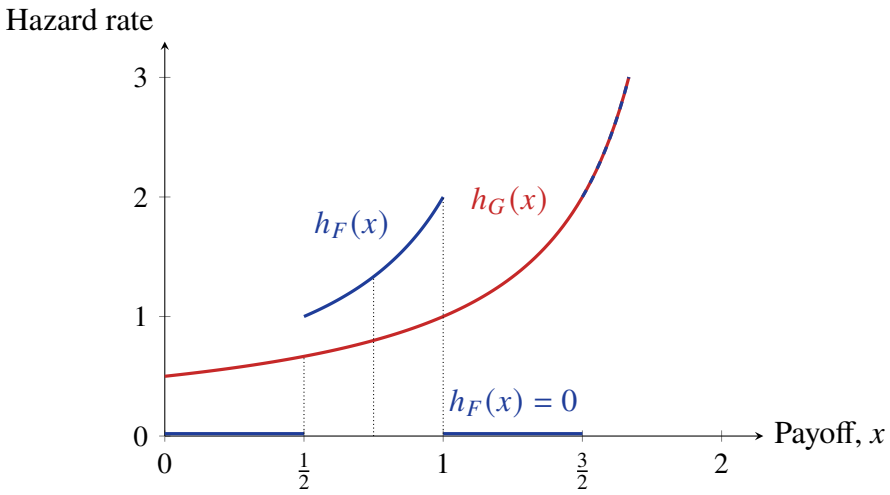


Figure 7. Example 14: $h_F(x) > h_G(x)$ on $(\frac{1}{2}, 1)$, although F FOSD G. The two hazard rates coincide on $(\frac{3}{2}, 2)$.

F dominates G in the...	Test	Who prefers F	Implies
FOSD	$F(x) \leq G(x)$	every u weakly increasing	$\mathbb{E}_F[x] \geq \mathbb{E}_G[x]$
SOSD (same mean)	$\int_a^x F \leq \int_a^x G$	every u weakly increasing and concave	$\text{Var}_F \leq \text{Var}_G$
Hazard rate	$\frac{f}{1-F} \leq \frac{g}{1-G}$		FOSD
Reverse hazard rate	$\frac{f}{F} \geq \frac{g}{G}$		FOSD
Likelihood ratio	$\frac{f}{g}$ weakly increasing		hazard rate and reverse hazard rate

Table 2. Orderings of payoff distributions. Every test must hold at every x.

We can summarize the examples in three points:

- To check FOSD, plot the two CDFs: one must lie below the other everywhere (Examples 1 to 4). If they cross, individuals who like money can disagree, even when one lottery has a higher mean (Example 5).
- To check SOSD between two lotteries with the same mean, plot the area under each CDF from the left: the area under the safer lottery must never exceed the area under the riskier one (Examples 6 to 9). A lower variance is not enough (Example 10).
- Hazard rate, reverse hazard rate, and likelihood ratio dominance are stronger than FOSD (Examples 12 and 13), and FOSD does not imply them (Example 14).

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