Partial Equilibrium: Worked Examples

EconS 501, a handout on partial equilibrium in Chapter 6

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1 Introduction

This handout studies a single competitive market. We hold the prices of all other goods fixed and ask how the price and quantity of one good are determined, how they respond to changes in preferences, technology, and taxes, and whether the resulting allocation is efficient.

The slides of Chapter 6 present each of these steps in general form. Here we work through new examples with specific functions and numbers, and we write out the intermediate algebra that the slides skip. Each example is small enough to solve by hand, and several of them reuse the same market, so you can compare the answers that different tools give.

Two ideas appear throughout. First, with quasilinear utility, the equilibrium conditions for the good under study do not depend on how income is distributed, so we can describe the market with an aggregate demand curve and an aggregate supply curve. Second, the slopes of these two curves, or their elasticities, determine almost every comparative static we compute, from the share of a tax that consumers pay to the size of the deadweight loss.

The handout first defines the competitive equilibrium and studies its existence and uniqueness. It then turns to comparative statics, with a focus on per-unit taxes, their incidence, and the extreme cases of elasticities, and then to welfare and the first welfare theorem. It ends with the Marshall–Hicks laws of derived demand.

2 Competitive equilibrium

Consider a market for good x with I consumers, indexed by i, and J firms, indexed by j, all of whom take the market price p as given. Consumer i has the quasilinear utility function $u_i(m_i, x_i) = m_i + \phi_i(x_i)$, where m_i is the amount of a numeraire good, $\phi_i(0) = 0$, $\phi_i'(x_i) > 0$, and

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$\phi_i''(x_i) < 0$. Firm j has a strictly convex cost function $c_j(q_j)$. We maintain these assumptions throughout the handout, which follows the setup of Mas-Colell, Whinston, and Green (1995, Chapter 10).

2.1 The firm and the consumer

Slides 4–9 derive the first-order conditions of firm j and consumer i, which are necessary and sufficient:

$$p \leq c_j'(q_j), \quad \text{with equality if } q_j > 0, \tag{1}$$

and

$$\phi_i'(x_i) \leq p, \quad \text{with equality if } x_i > 0. \tag{2}$$

The firm produces nothing when $p \leq c_j'(0)$, and otherwise produces the output at which its marginal cost equals the price. The consumer buys nothing when $p \geq \phi_i'(0)$, and otherwise buys until her marginal utility equals the price. Since her budget constraint binds, her endowment of the numeraire and her ownership shares of the firms only affect her consumption of the numeraire, m_i.

Example 1 (Individual supply and demand). Consider a firm with cost function $c(q) = q + \frac{1}{2}q^2$ and a consumer with $\phi(x) = 10\ln(1+x)$.

- The firm. Marginal cost is $c'(q) = 1 + q$, with $c'(0) = 1$ and $c''(q) = 1 > 0$. For $p > 1$, condition (1) holds with equality, $p = 1 + q$, so the supply curve is $q(p) = p - 1$. For $p \leq 1$, the firm produces zero.
- The consumer. Marginal utility is $\phi'(x) = \frac{10}{1+x}$, which is positive and decreasing, with $\phi'(0) = 10$. For $p < 10$, condition (2) holds with equality, $\frac{10}{1+x} = p$, that is, $1 + x = \frac{10}{p}$, so the demand curve is $x(p) = \frac{10}{p} - 1$. For $p \geq 10$, she buys zero.

At $p = 5$, for instance, the firm supplies $q(5) = 4$ units, with marginal cost $1 + 4 = 5$, and the consumer demands $x(5) = \frac{10}{5} - 1 = 1$ unit, with marginal utility $\frac{10}{2} = 5$. Neither decision depends on the consumer's endowment of the numeraire or on her ownership shares, which only affect m_i.

2.2 Aggregation

Aggregate demand $x(p) = \sum_i x_i(p)$ and aggregate supply $q(p) = \sum_j q_j(p)$ are the horizontal sums of the individual curves (slides 11–14). Since each individual curve starts at a different price ($\phi_i'(0)$ for consumers and $c_j'(0)$ for firms), the aggregate curves have kinks at the prices where a new group of agents enters the market.

Example 2 (Horizontal summation). Market M, which we use in several examples below, has three types of consumers and three types of firms:

- 60 consumers of type 1, with $\phi_1(x) = 10\ln(1+x)$; 40 consumers of type 2, with $\phi_2(x) = 20\ln(1+x)$; and 30 consumers of type 3, with $\phi_3(x) = 4\ln(1+x)$.
- 40 firms of type A, with $c_A(q) = q + \frac{1}{2}q^2$; 20 firms of type B, with $c_B(q) = 3q + q^2$; and 10 firms of type C, with $c_C(q) = 6q + \frac{1}{2}q^2$.

Following the steps of Example 1, a consumer with $\phi(x) = \alpha \ln(1+x)$ demands $\frac{\alpha}{p} - 1$ units when $p < \alpha$. Hence the individual demands are $\frac{10}{p} - 1$, $\frac{20}{p} - 1$, and $\frac{4}{p} - 1$, which become positive below $p = 10$, $p = 20$, and $p = 4$, respectively. Multiplying by the number of consumers of each type and adding, aggregate demand is

$$x(p) = \begin{cases} 0 & \text{if } p \geq 20, \\ 40\left(\frac{20}{p} - 1\right) = \frac{800}{p} - 40 & \text{if } 10 \leq p < 20, \\ \frac{800}{p} - 40 + 60\left(\frac{10}{p} - 1\right) = \frac{1400}{p} - 100 & \text{if } 4 \leq p < 10, \\ \frac{1400}{p} - 100 + 30\left(\frac{4}{p} - 1\right) = \frac{1520}{p} - 130 & \text{if } p < 4. \end{cases} \tag{3}$$

On the firm side, marginal costs are $1 + q$, $3 + 2q$, and $6 + q$, so individual supplies are $p - 1$ for $p > 1$, $\frac{p-3}{2}$ for $p > 3$, and $p - 6$ for $p > 6$. Aggregate supply is

$$q(p) = \begin{cases} 0 & \text{if } p \leq 1, \\ 40(p-1) & \text{if } 1 < p \leq 3, \\ 40(p-1) + 20 \cdot \frac{p-3}{2} = 50p - 70 & \text{if } 3 < p \leq 6, \\ 50p - 70 + 10(p-6) = 60p - 130 & \text{if } p > 6. \end{cases} \tag{4}$$

Both aggregate curves are continuous at the kinks. For instance, at $p = 10$, both expressions for demand give $\frac{800}{10} - 40 = 40 = \frac{1400}{10} - 100$, and at $p = 3$, both expressions for supply give 80. Figure 1 depicts the horizontal summation of the demands of one type-1 consumer and one type-2 consumer: above $p = 10$, only the type-2 consumer buys, so the sum coincides with her demand; below $p = 10$, both buy, and the sum, $\frac{30}{p} - 2$, is flatter than either individual curve.

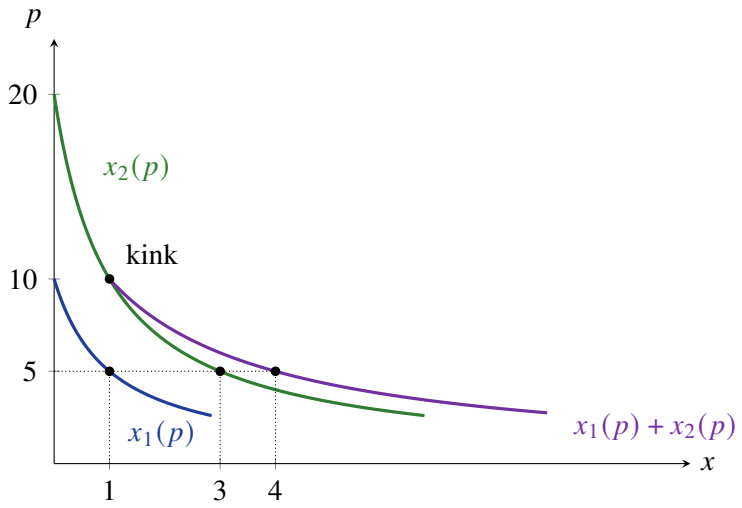


Figure 1: Individual demands of a type-1 consumer, $x_1(p) = \frac{10}{p} - 1$, and a type-2 consumer, $x_2(p) = \frac{20}{p} - 1$, and their horizontal sum. At $p = 5$, the consumers buy 1 and 3 units, so the sum is 4.

2.3 The competitive equilibrium

A price p^* and an allocation $(x_1^*, \ldots, x_I^*, q_1^*, \ldots, q_J^*)$ constitute a competitive equilibrium (CE) if every firm satisfies (1), every consumer satisfies (2), and the market clears, $\sum_i x_i^* = \sum_j q_j^*$ (slide 10). In terms of the aggregate curves, the equilibrium price solves $x(p^*) = q(p^*)$, and every agent then chooses its point on its individual curve at p^*.

Example 3 (Equilibrium in market M). We first guess the segments of (3) and (4) on which the curves cross, and then check the guess.

- First, the guess. At $p = 4$, demand is $\frac{1400}{4} - 100 = 250$ and supply is $50(4) - 70 = 130$, so there is excess demand. At $p = 6$, demand is $\frac{1400}{6} - 100 \approx 133.3$ and supply is $50(6) - 70 = 230$, so there is excess supply. Hence the curves cross in $(4, 6)$, where demand is $\frac{1400}{p} - 100$ and supply is $50p - 70$.
- Second, solve. Setting $\frac{1400}{p} - 100 = 50p - 70$ and multiplying both sides by p,

  $$1400 - 100p = 50p^2 - 70p, \quad \text{that is,} \quad 50p^2 + 30p - 1400 = 0, \quad \text{or} \quad 5p^2 + 3p - 140 = 0.$$

  The positive root is

  $$p^* = \frac{-3 + \sqrt{9 + 4 \cdot 5 \cdot 140}}{2 \cdot 5} = \frac{-3 + \sqrt{2809}}{10} = \frac{-3 + 53}{10} = 5,$$

  which lies in $(4, 6)$, as guessed. The equilibrium quantity is $x^* = \frac{1400}{5} - 100 = 180 = 50(5) - 70$.
- Third, individual choices. Each type-1 consumer buys $\frac{10}{5} - 1 = 1$ unit, and each type-2 consumer buys $\frac{20}{5} - 1 = 3$ units. Each type-A firm produces $5 - 1 = 4$ units, and each type-B firm produces $\frac{5-3}{2} = 1$ unit. Market clearing holds:

  $$60(1) + 40(3) = 180 = 40(4) + 20(1).$$

- Fourth, the inactive agents. Type-3 consumers buy nothing, and condition (2) holds with inequality, $\phi_3'(0) = 4 \leq 5$. Type-C firms produce nothing, and condition (1) holds with inequality, $5 \leq c_C'(0) = 6$.

Figure 2 depicts the equilibrium. The kinks in aggregate demand at $p = 4$ and $p = 10$, and in aggregate supply at $p = 3$ and $p = 6$, are the prices at which a new type of agent becomes active.

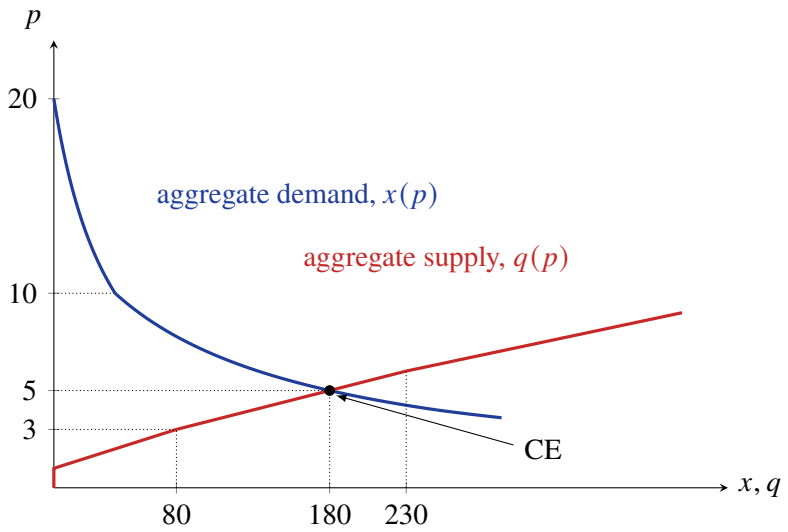


Figure 2: The CE of market M: $p^* = 5$ and $x^* = q^* = 180$. Aggregate supply has kinks at $p = 3$ (type-B firms enter) and $p = 6$ (type-C firms enter); aggregate demand has kinks at $p = 10$ and $p = 4$.

2.4 Existence and uniqueness

The consumer with the highest marginal utility at zero consumption values the first unit at $\max_i \phi_i'(0)$, while the firm with the lowest marginal cost at zero output can produce the first unit at $\min_j c_j'(0)$. Comparing these two numbers determines whether the market is active.

Result 1 (Existence and uniqueness, slides 16–18). Under the assumptions above:

- If $\max_i \phi_i'(0) > \min_j c_j'(0)$, the CE involves positive production and consumption, and the equilibrium price satisfies $\max_i \phi_i'(0) > p^* > \min_j c_j'(0)$.
- If $\max_i \phi_i'(0) \leq \min_j c_j'(0)$, the CE has $x^* = q^* = 0$, and every price between the two values supports it.

In both cases, the equilibrium allocation is unique.

In the first case, uniqueness follows because the excess demand function $z(p) = x(p) - q(p)$ is strictly decreasing between the two values, since demand slopes downward and supply slopes upward (slide 18). In the second case, at every price either demand or supply is zero, so the only quantity that clears the market is zero.

In market M, $\max_i \phi_i'(0) = \phi_2'(0) = 20$ and $\min_j c_j'(0) = c_A'(0) = 1$, so the first case applies, and indeed $20 > p^* = 5 > 1$. In Example 3, $z(p) = \frac{1400}{p} - 100 - (50p - 70)$ on $(4, 6)$, so $z'(p) = -\frac{1400}{p^2} - 50 < 0$, and the crossing at $p^* = 5$ is the only one.

Example 4 (A market that does not open). A new product attracts consumers whose marginal utilities at zero are all at most 6, so aggregate demand is $x(p) = 20(6 - p)$ for $p < 6$ and zero otherwise. The cheapest producer has $c_j'(0) = 8$, and aggregate supply is $q(p) = 20(p - 8)$ for

$p > 8$ and zero otherwise. Since $\max_i \phi_i'(0) = 6 < 8 = \min_j c_j'(0)$, no consumer values the first unit as much as it costs to produce it:

- at any $p < 6$, demand is positive but supply is zero;
- at any $p > 8$, supply is positive but demand is zero;
- at any $p \in [6, 8]$, both are zero.

Hence the CE has $x^* = q^* = 0$, and every $p^* \in [6, 8]$ satisfies the equilibrium conditions. Figure 3 depicts this case.

Suppose now that a cost-saving innovation reduces every firm's marginal cost by 4, so aggregate supply becomes $q(p) = 20(p - 4)$ for $p > 4$. Since $6 > 4$, the market now opens. Setting $20(6 - p) = 20(p - 4)$ gives $120 - 20p = 20p - 80$, that is, $p^* = 5$, and $x^* = 20(6 - 5) = 20$. As Result 1 predicts, $6 > p^* = 5 > 4$.

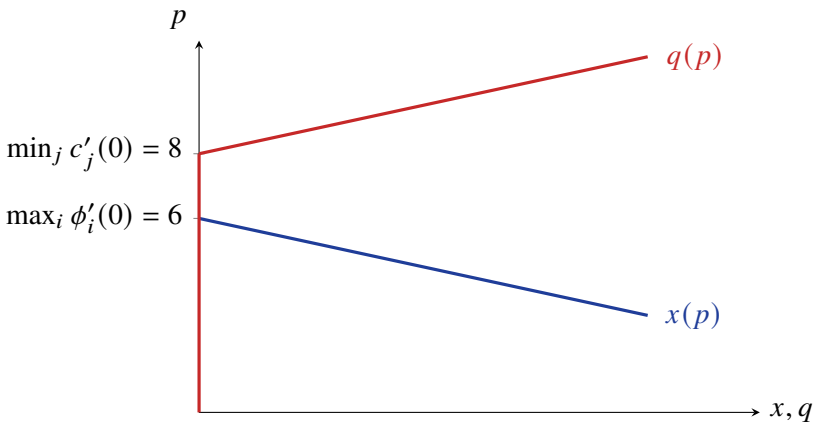


Figure 3: Example 4: since $\max_i \phi_i'(0) < \min_j c_j'(0)$, the demand curve lies below the supply curve, and the CE has $x^* = q^* = 0$. The thick segment of the vertical axis is part of aggregate supply: at prices up to 8, firms produce zero.

3 Comparative statics

3.1 Parameters and the implicit function theorem

Slides 23–29 let consumers' preferences depend on a parameter α, $\phi_i(x_i, \alpha)$, and firms' costs depend on a parameter β, $c_j(q_j, \beta)$, and apply the implicit function theorem (IFT) to the equilibrium conditions: if $F(y, \alpha) = 0$ and $\frac{\partial F}{\partial y} \neq 0$, then

$$\frac{dy}{d\alpha} = -\frac{\frac{\partial F}{\partial \alpha}}{\frac{\partial F}{\partial y}}. \tag{5}$$

In a single market, it is usually simpler to apply (5) to the market-clearing condition written with the aggregate curves, $F(p, \alpha, \beta) = x(p, \alpha) - q(p, \beta) = 0$, than to the full system of $I + J + 1$

equilibrium conditions.

Example 5 (A demand shift, one consumer). Let $\phi(x, \alpha) = \alpha \ln(1+x)$. The consumer's first-order condition is $F(x, \alpha) = \frac{\alpha}{1+x} - p = 0$. Its partial derivatives are

$$\frac{\partial F}{\partial \alpha} = \frac{1}{1+x} \qquad \text{and} \qquad \frac{\partial F}{\partial x} = -\frac{\alpha}{(1+x)^2},$$

so (5) yields

$$\frac{dx}{d\alpha} = -\frac{\frac{1}{1+x}}{-\frac{\alpha}{(1+x)^2}} = \frac{1+x}{\alpha} = \frac{1}{p},$$

where the last equality uses the first-order condition, $\frac{1+x}{\alpha} = \frac{1}{p}$. Differentiating the demand $x = \frac{\alpha}{p} - 1$ directly gives the same answer. At $p = 5$, one more unit of α raises the consumer's demand by $\frac{1}{5} = 0.2$ units.

Example 6 (A demand shift and a cost shift in market M). Near the equilibrium of Example 3, write the valuation of type-2 consumers as α_2 (initially 20) and the marginal cost of type-A firms as $\beta + q$ (initially $\beta = 1$). On the relevant segments, aggregate demand and supply are

$$x(p, \alpha_2) = 60\left(\frac{10}{p} - 1\right) + 40\left(\frac{\alpha_2}{p} - 1\right) = \frac{600 + 40\alpha_2}{p} - 100, \qquad q(p, \beta) = 40(p - \beta) + 10(p - 3).$$

At $p^* = 5$, $\alpha_2 = 20$, and $\beta = 1$, the partial derivatives of $F = x - q$ are

$$\frac{\partial F}{\partial p} = -\frac{600 + 40 \cdot 20}{5^2} - 50 = -56 - 50 = -106, \qquad \frac{\partial F}{\partial \alpha_2} = \frac{40}{5} = 8, \qquad \frac{\partial F}{\partial \beta} = 40.$$

- Demand shift. From (5), $\frac{dp^*}{d\alpha_2} = -\frac{8}{-106} = \frac{4}{53} \approx 0.075$. We can check this result with the closed form: market clearing gives $50p^2 + 30p - (600 + 40\alpha_2) = 0$, so

  $$p^*(\alpha_2) = \frac{-30 + \sqrt{900 + 200(600 + 40\alpha_2)}}{100},$$

  and

  $$\frac{dp^*}{d\alpha_2} = \frac{1}{100} \cdot \frac{8000}{2\sqrt{900 + 200(600 + 40\alpha_2)}} = \frac{40}{\sqrt{280900}} = \frac{40}{530} = \frac{4}{53} \quad \text{at } \alpha_2 = 20.$$

- Cost shift. From (5), $\frac{dp^*}{d\beta} = -\frac{40}{-106} = \frac{20}{53} \approx 0.377$. The price rises, but by much less than the cost increase of the type-A firms, since only some firms are affected and demand responds to the price. Each type-A firm produces $p^* - \beta$, so its output changes by $\frac{20}{53} - 1 = -\frac{33}{53}$, while each type-B firm produces $\frac{p^* - 3}{2}$, so its output increases by $\frac{1}{2} \cdot \frac{20}{53} = \frac{10}{53}$. The firms whose costs did not change expand, but by less than the contraction of the type-A firms: total output changes by $40\left(-\frac{33}{53}\right) + 20 \cdot \frac{10}{53} = -\frac{1120}{53} \approx -21.1$ units.

3.2 Per-unit and ad valorem taxes

As in slide 24, a tax creates a wedge between the price paid by consumers, $\hat{p}_C$, and the price received by firms, which we take as the market price p: a per-unit tax t gives $\hat{p}_C = p + t$, and an ad valorem tax τ gives $\hat{p}_C = p(1+\tau)$. Consumers respond to $\hat{p}_C$ and firms to p, so the equilibrium price solves $x(p+t) = q(p)$ under a per-unit tax, and $x\left(p(1+\tau)\right) = q(p)$ under an ad valorem tax.

Example 7 (Market L, and the two taxes). Market L has demand $x(p) = 90 - 3p$ and supply $q(p) = 2p - 10$ for $p \geq 5$. Without taxes, $90 - 3p = 2p - 10$ gives $p^* = 20$ and $x^* = 30$.

- Per-unit tax. Market clearing is $90 - 3(p+t) = 2p - 10$, that is, $100 - 3t = 5p$, so

$$p^*(t) = 20 - \frac{3}{5}t, \qquad p^*(t) + t = 20 + \frac{2}{5}t, \qquad x^*(t) = 2p^*(t) - 10 = 30 - \frac{6}{5}t. \tag{6}$$

  With $t = 5$, firms receive $p^* = 20 - 3 = 17$, consumers pay $17 + 5 = 22$, and $x^* = 30 - 6 = 24$ units are traded.

- Ad valorem tax. Market clearing is $90 - 3p(1+\tau) = 2p - 10$, so $p^*(\tau) = \frac{100}{5+3\tau}$. With $\tau = \frac{1}{4}$,

$$p^* = \frac{100}{5 + \frac{3}{4}} = \frac{400}{23} \approx 17.39, \qquad p^*(1+\tau) = \frac{400}{23} \cdot \frac{5}{4} = \frac{500}{23} \approx 21.74,$$
$$x^* = 2 \cdot \frac{400}{23} - 10 = \frac{570}{23} \approx 24.78.$$

The ad valorem tax collects $\tau p^* = \frac{100}{23} \approx 4.35$ dollars per unit, and a per-unit tax of $t = \frac{100}{23}$ gives, from (6), $p^* = 20 - \frac{3}{5} \cdot \frac{100}{23} = \frac{400}{23}$, the same equilibrium. In a competitive market, every ad valorem tax has a per-unit tax that produces the same equilibrium, so we focus on per-unit taxes.

3.3 The effect of a per-unit tax

Let $p^*(t)$ denote the equilibrium price received by firms, so that $x\left(p^*(t) + t\right) = q\left(p^*(t)\right)$. Slides 30–33 differentiate this condition with respect to t and obtain

$$p^{*\prime}(t) = -\frac{x'(p^* + t)}{x'(p^* + t) - q'(p^*)}. \tag{7}$$

Writing $|x'| = -x'(p^* + t) > 0$ and $q' = q'(p^*) > 0$, expression (7) becomes

$$p^{*\prime}(t) = -\frac{|x'|}{|x'| + q'} \qquad \text{and} \qquad p^{*\prime}(t) + 1 = \frac{q'}{|x'| + q'}. \tag{8}$$

Hence, a marginal increase in t reduces the price received by firms and increases the price paid by consumers, each by less than the tax increase, and the two changes add up to the tax increase (slides 33–34). The second expression in (8) is called the pass-through rate of the tax: the fraction of a marginal tax increase that is passed on to consumers. The side of the market with the flatter curve (larger $|x'|$ or q') bears the smaller share.

Example 8 (Pass-through in market L). In market L, $|x'| = 3$ and $q' = 2$ at every price, so (8) gives

$$p^{*\prime}(t) = -\frac{3}{3+2} = -\frac{3}{5} \qquad \text{and} \qquad p^{*\prime}(t) + 1 = \frac{2}{3+2} = \frac{2}{5},$$

which coincide with the slopes in (6). For every dollar of tax, firms receive 60 cents less and consumers pay 40 cents more. Consumers bear the smaller share because demand is flatter than supply in the (x, p) plane: a one-dollar increase in the price reduces the quantity demanded by 3 units, while a one-dollar decrease reduces the quantity supplied by only 2 units. Figure 4 depicts the effect of $t = 5$. The tax shifts the demand curve, written as a function of the price received by firms, down by t, from $x(p)$ to $x(p + t)$.

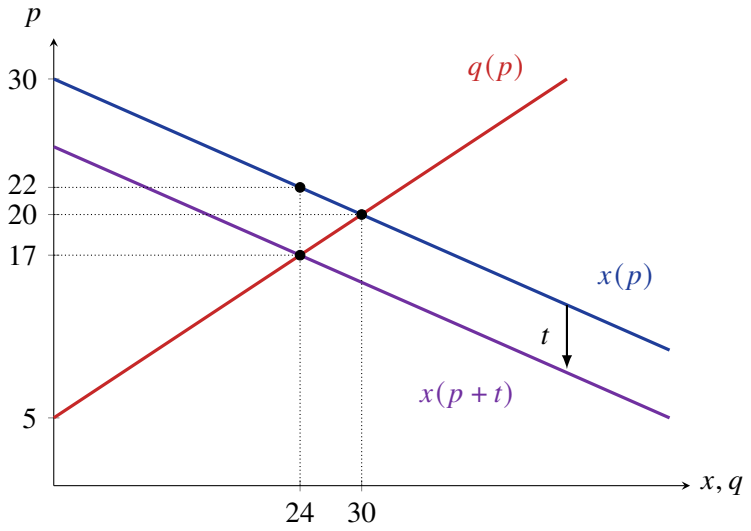


Figure 4: A per-unit tax $t = 5$ in market L. Without the tax, $p^* = 20$ and $x^* = 30$. With the tax, firms receive $p^*(5) = 17$, consumers pay 22, and $x^*(5) = 24$. The arrow shows the downward shift of demand, from $x(p)$ to $x(p + t)$.

3.4 Incidence and elasticities

Expression (8) uses slopes, which depend on the units of measurement. Elasticities do not. At $t = 0$, where $x(p^*) = q(p^*)$, multiply the numerator and the denominator of (7) by $\frac{p^*}{x(p^*)} = \frac{p^*}{q(p^*)}$, and let

$$\varepsilon = x'(p^*)\frac{p^*}{x(p^*)} < 0 \qquad \text{and} \qquad \gamma = q'(p^*)\frac{p^*}{q(p^*)} > 0$$

denote the price elasticities of demand and supply at the equilibrium. Then

$$p^{*\prime}(0) = -\frac{\varepsilon}{\varepsilon - \gamma} \qquad \text{and} \qquad p^{*\prime}(0) + 1 = \frac{\gamma}{\gamma - \varepsilon} = \frac{\gamma}{\gamma + |\varepsilon|}. \tag{9}$$

Result 2 (Incidence of a small tax). Under the assumptions above, the share of a small per-unit tax borne by consumers is $\frac{\gamma}{\gamma+|\varepsilon|}$, and the share borne by firms is $\frac{|\varepsilon|}{\gamma+|\varepsilon|}$. The side of the market with the lower elasticity (in absolute value) bears the larger share.

Example 9 (Elasticities in market L). At $p^* = 20$ and $x^* = 30$,

$$\varepsilon = -3 \cdot \frac{20}{30} = -2 \qquad \text{and} \qquad \gamma = 2 \cdot \frac{20}{30} = \frac{4}{3}.$$

From (9), consumers bear

$$\frac{\gamma}{\gamma + |\varepsilon|} = \frac{\frac{4}{3}}{\frac{4}{3} + 2} = \frac{\frac{4}{3}}{\frac{10}{3}} = \frac{2}{5},$$

the same share we obtained in Example 8 with slopes. Demand is more elastic than supply at the equilibrium ($|\varepsilon| = 2 > \frac{4}{3} = \gamma$), so firms bear the larger share, $\frac{3}{5}$.

3.5 Extreme cases of elasticities

Slides 36–41 use (8) to study the limits. Consumers bear the whole tax when supply is very responsive ($q' \to \infty$) or demand is very unresponsive ($|x'| \to 0$), and firms bear the whole tax when supply is very unresponsive ($q' \to 0$) or demand is very responsive ($|x'| \to \infty$). Example 10 moves between the two limits of supply.

Example 10 (A family of supply curves). Let demand be $x(p) = 60 - 2p$, and consider the family of supply curves $q(p) = 36 + k(p - 12)$, where the slope $k \geq 0$ measures how responsive supply is. Every member of the family passes through the point $(36, 12)$, which is the equilibrium without taxes, since $x(12) = 60 - 24 = 36$. With a per-unit tax t, market clearing is

$$60 - 2(p + t) = 36 + k(p - 12), \quad \text{that is,} \quad 24 + 12k - 2t = (2 + k)p,$$

and, since $24 + 12k = 12(2 + k)$,

$$p^*(t) = 12 - \frac{2t}{2 + k} \qquad \text{and} \qquad p^*(t) + t = 12 + \frac{kt}{2 + k}.$$

Table 1 reports the prices for $t = 3$ and several values of k.

k	Price received by firms	Price paid by consumers	Consumers' share	Quantity
any k	$12 - \frac{6}{2+k}$	$15 - \frac{6}{2+k}$	$\frac{k}{2+k}$	$30 + \frac{12}{2+k}$
0	9	12	0	36
1	10	13	$\frac{1}{3}$	34
2	10.5	13.5	$\frac{1}{2}$	33
8	11.4	14.4	$\frac{4}{5}$	31.2
98	11.94	14.94	$\frac{49}{50}$	30.12
$k \to \infty$	12	15	1	30

Table 1. Example 10 with $t = 3$: as supply becomes more responsive, consumers bear a larger share of the tax.

For instance, with $k = 8$, firms receive $12 - \frac{6}{10} = 11.4$, consumers pay $11.4 + 3 = 14.4$, and the quantity is $60 - 2(14.4) = 31.2$. The consumers' share of the tax is $\frac{k}{2+k}$, which matches the pass-through rate $\frac{q'}{|x'|+q'}$ in (8) with $q' = k$ and $|x'| = 2$. The two limits are depicted in Figure 5:

- With $k = 0$, supply is vertical at $q = 36$. The quantity cannot fall, so the price paid by consumers cannot rise: firms absorb the whole tax, and their price falls from 12 to 9.
- As $k \to \infty$, supply becomes horizontal at $p = 12$. Firms do not sell at any price below 12, so their price cannot fall: consumers pay the whole tax, and their price rises from 12 to 15, which reduces the quantity from 36 to 30.

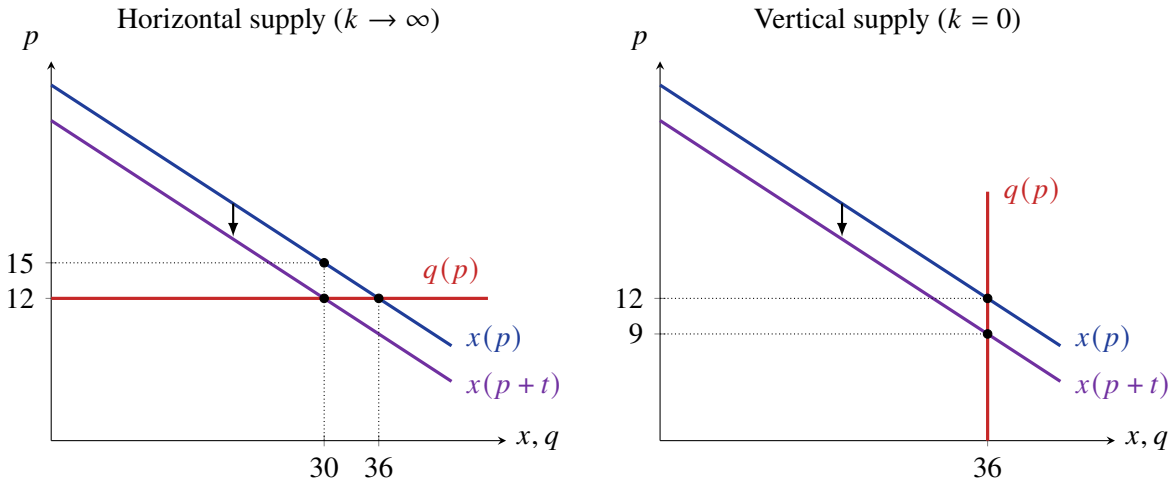


Figure 5: The two limits of Example 10 with $t = 3$. Left: with horizontal supply, firms still receive 12, consumers pay 15, and the quantity falls from 36 to 30. Right: with vertical supply, the quantity stays at 36, consumers still pay 12, and firms receive 9. In both panels, the arrow shows the downward shift of demand by $t = 3$.

3.6 Constant elasticities (Example 6.3 in the slides)

Slides 42–44 let aggregate demand be $x(p) = Ap^{\varepsilon}$, with $A > 0$ and $\varepsilon < 0$, and aggregate supply be $q(p) = ap^{\gamma}$, with $a > 0$ and $\gamma > 0$. The exponents are the elasticities, since $x'(p)\frac{p}{x(p)} = A\varepsilon p^{\varepsilon-1}\frac{p}{Ap^{\varepsilon}} = \varepsilon$, and similarly for supply, so the expressions in (9) hold at $t = 0$ with elasticities that do not depend on the price. The scale parameters A and a do not appear: they affect the equilibrium price and quantity, but not how a small tax is shared.

Example 11 (Two constant-elasticity markets).

- With $\varepsilon = -2$ and $\gamma = \frac{1}{2}$,

$$p^{*\prime}(0) = -\frac{-2}{-2 - \frac{1}{2}} = -\frac{2}{\frac{5}{2}} = -\frac{4}{5} \qquad \text{and} \qquad p^{*\prime}(0) + 1 = \frac{\frac{1}{2}}{\frac{1}{2} + 2} = \frac{1}{5}.$$

  The supply elasticity is one fourth of the absolute demand elasticity, so firms bear four fifths of a small tax.

- With $x(p) = \frac{160}{p}$ ($A = 160$, $\varepsilon = -1$) and $q(p) = 10p$ ($a = 10$, $\gamma = 1$), the equilibrium without taxes solves $\frac{160}{p} = 10p$, so $p^* = 4$ and $x^* = 40$, and (9) gives $p^{*\prime}(0) = -\frac{-1}{-1-1} = -\frac{1}{2}$: a small tax is split equally. In this market, we can also solve for $p^*(t)$ at every t. Market clearing, $\frac{160}{p+t} = 10p$, becomes $p^2 + tp - 16 = 0$, whose positive root is

$$p^*(t) = \frac{-t + \sqrt{t^2 + 64}}{2}, \qquad \text{so} \qquad p^{*\prime}(t) = \frac{1}{2}\left(-1 + \frac{t}{\sqrt{t^2 + 64}}\right).$$

  At $t = 0$, $p^{*\prime}(0) = -\frac{1}{2}$, as expected. At $t = 6$, $\sqrt{36 + 64} = 10$, so firms receive $p^*(6) = \frac{-6+10}{2} = 2$, consumers pay 8, and $x^* = 10(2) = 20 = \frac{160}{8}$. The marginal effect is now $p^{*\prime}(6) = \frac{1}{2}\left(-1 + \frac{6}{10}\right) = -\frac{1}{5}$, which we can confirm with (7), since $x'(8) = -\frac{160}{64} = -\frac{5}{2}$ and $q'(2) = 10$:

$$p^{*\prime}(6) = -\frac{-\frac{5}{2}}{-\frac{5}{2} - 10} = -\frac{\frac{5}{2}}{\frac{25}{2}} = -\frac{1}{5}.$$

The second market shows that (9) describes a small tax introduced at $t = 0$. Once a large tax is in place, demand is evaluated at a higher consumer price, where its slope $|x'|$ is smaller, while supply is evaluated at a lower producer price, where its slope is unchanged. From (8), consumers then bear a larger share of an additional dollar of tax: four fifths at $t = 6$, instead of one half at $t = 0$.

Slides 45–48 take the four limits of (9): firms bear the whole tax when supply is perfectly inelastic ($\gamma \to 0$) or demand is perfectly elastic ($\varepsilon \to -\infty$), and consumers bear it when demand is perfectly inelastic ($\varepsilon \to 0$) or supply is perfectly elastic ($\gamma \to +\infty$). Figure 6 depicts the consumers' share $\frac{\gamma}{\gamma - \varepsilon}$ between these limits, as a function of γ for three demand elasticities.

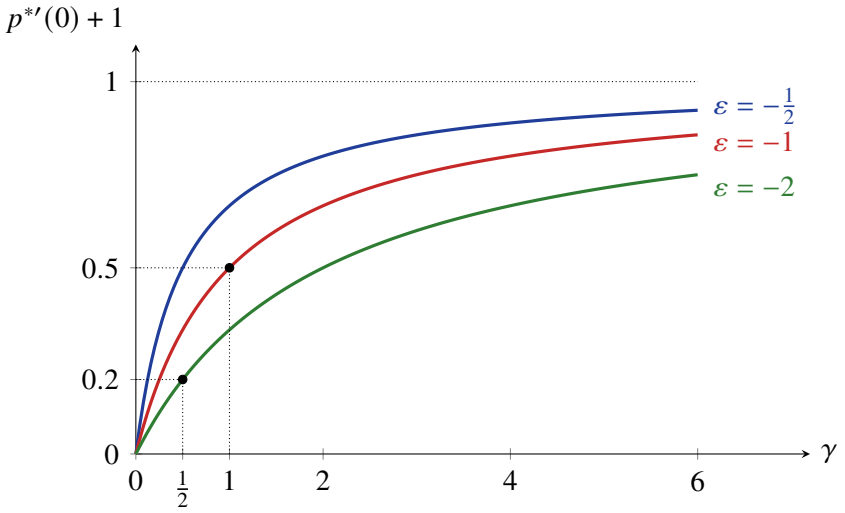


Figure 6: Share of a small per-unit tax borne by consumers, $\frac{\gamma}{\gamma - \varepsilon}$, as a function of the supply elasticity γ. The share increases in γ and approaches 1 as γ grows; for a given γ, it is lower when demand is more elastic. The two dots are the markets of Example 11.

4 Welfare analysis

4.1 Aggregate surplus and its marginal change

With quasilinear utility, slides 50–55 measure welfare with the aggregate surplus

$$S = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j). \tag{10}$$

Transfers of the numeraire across agents (prices paid, taxes collected and returned, dividends) add to one agent's utility what they subtract from another's, so they do not appear in (10). Let $P(x)$ denote the inverse aggregate demand and $C(q)$ the aggregate (industry) cost function, the lowest total cost of producing q units with all firms. When a small change in the quantity traded, dx, is consumed as in a CE at price $P(x)$ and produced at the lowest total cost, every active consumer has $\phi_i'(x_i) = P(x)$ and every active firm has $c_j'(q_j) = C'(q)$, so the slides obtain

$$dS = [P(x) - C'(x)]\, dx. \tag{11}$$

A marginal unit adds its value to consumers, $P(x)$, and subtracts its cost to firms, $C'(x)$.

Example 12 (The industry cost function of market M). Inverting the aggregate supply curve (4) gives the industry marginal cost,

$$C'(q) = \begin{cases} 1 + \frac{q}{40} & \text{if } 0 \le q \le 80, \\ \frac{q+70}{50} & \text{if } 80 < q \le 230, \\ \frac{q+130}{60} & \text{if } q > 230. \end{cases}$$

For instance, from $q = 50p - 70$ on the second segment, $p = \frac{q+70}{50}$. To produce $q = 180$ units at the lowest cost, every active firm must have the same marginal cost, $C'(180) = \frac{250}{50} = 5$, which is the allocation of Example 3 (four units per type-A firm and one unit per type-B firm). The industry cost of 180 units is the area under $C'(q)$,

$$\begin{aligned} C(180) &= \int_0^{80} \left(1 + \frac{s}{40}\right) ds + \int_{80}^{180} \frac{s+70}{50}\, ds \\ &= \left(80 + \frac{80^2}{80}\right) + \left(\frac{180^2 - 80^2}{100} + \frac{70 \cdot 100}{50}\right) = 160 + 400 = 560. \end{aligned}$$

Adding up the costs firm by firm gives the same number: $40\, c_A(4) + 20\, c_B(1) = 40(4+8) + 20(3+1) = 480 + 80 = 560$.

4.2 Total surplus and the first welfare theorem

Let $S(x)$ denote the aggregate surplus when x units are consumed and produced as described above, that is, by the consumers who value them most and at the lowest cost. Integrating (11) from zero

to x (slides 56–58),

$$S(x) = S_0 + \int_0^x [P(s) - C'(s)]\, ds, \tag{12}$$

where $S_0 = -\sum_j c_j(0)$ is the surplus with no production, which is zero without fixed costs. Slides 59–60 show that S is strictly concave and is maximized where $P(x) = C'(x)$, which the CE satisfies, since $P(x^*) = p^* = C'(x^*)$. This is the first welfare theorem: with quasilinear utility, maximizing aggregate surplus is equivalent to Pareto optimality, so every CE is Pareto optimal. The theorem also holds when the market does not open: if $x^* = 0$, then $P(0) \le p^* \le C'(0)$, so $S'(0) \le 0$, and $x^* = 0$ maximizes the aggregate surplus.

The theorem contains three statements: the CE quantity is the surplus-maximizing one; the CE allocates consumption to the consumers who value it most; and it allocates production to the firms that produce it at the lowest cost.

Example 13 (Surplus in market L). In market L, inverse demand is $P(x) = 30 - \frac{x}{3}$ and, inverting $q = 2p - 10$, industry marginal cost is $C'(x) = 5 + \frac{x}{2}$. With no fixed costs, (12) gives

$$S(x) = \int_0^x \left[\left(30 - \frac{s}{3}\right) - \left(5 + \frac{s}{2}\right)\right] ds = \int_0^x \left(25 - \frac{5s}{6}\right) ds = 25x - \frac{5x^2}{12}.$$

Setting $S'(x) = 25 - \frac{5x}{6} = 0$ gives $x = 30$, the CE quantity of Example 7, and $S(30) = 750 - 375 = 375$. At $x = 24$, the quantity traded under the tax of Example 7, surplus is $S(24) = 600 - 240 = 360$, and the marginal unit is worth $S'(24) = 25 - 20 = 5$: consumers value it at $P(24) = 22$, and it costs $C'(24) = 17$ to produce. Figure 7 depicts this computation in two panels. Surplus is zero again at $x = 60$, where the losses from units 31 to 60, each valued below its cost, cancel the gains from the first 30 units.

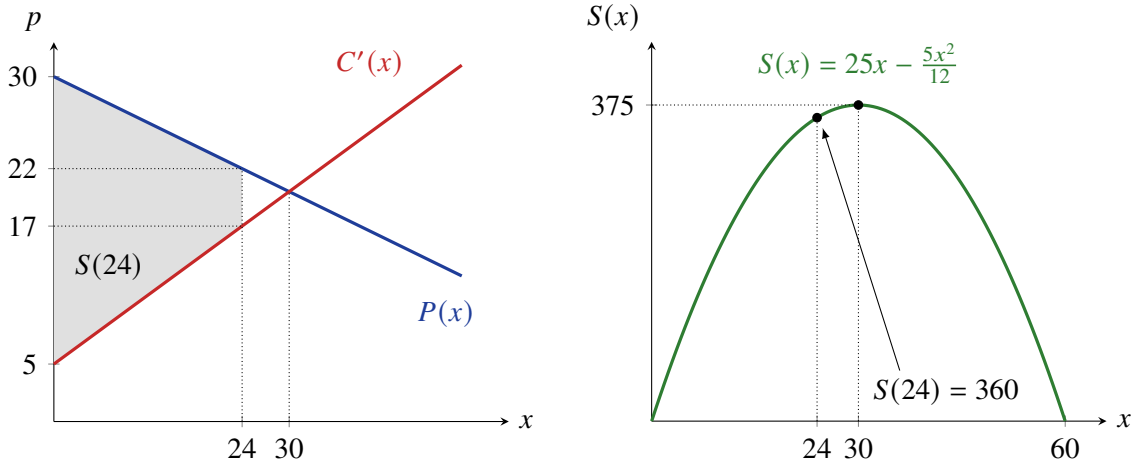


Figure 7: Example 13. Left: the shaded area between $P(x)$ and $C'(x)$ from 0 to 24 is $S(24) = 360$; extending it to $x = 30$ adds the triangle between the curves, worth 15. Right: aggregate surplus is concave and reaches its maximum, 375, at the CE quantity $x^* = 30$.

Example 14 (The CE allocates consumption and production efficiently). In market M, the CE of Example 3 trades 180 units. Along the inverse aggregate demand, each additional unit goes to

the consumers who value it most, so the marginal gross benefit of the s-th unit is $P(s)$. Since the CE allocation of 180 units has every active consumer at $\phi_i'(x_i) = 5 = P(180)$, the gross benefit $\sum_i \phi_i(x_i)$ equals the area under P up to 180. Inverting (3), $P(x) = \frac{800}{x+40}$ for $x \le 40$ and $P(x) = \frac{1400}{x+100}$ for $40 < x \le 250$, so

$$\begin{aligned}\int_0^{180} P(s)\,ds &= \int_0^{40} \frac{800}{s+40}\,ds + \int_{40}^{180} \frac{1400}{s+100}\,ds \\ &= 800\ln\frac{80}{40} + 1400\ln\frac{280}{140} = 2200\ln 2 \approx 1524.9,\end{aligned}$$

which coincides with the sum consumer by consumer, $60\cdot 10\ln 2 + 40\cdot 20\ln 4 = 600\ln 2 + 1600\ln 2 = 2200\ln 2$. With the industry cost from Example 12, the CE surplus is

$$S = 2200\ln 2 - 560 \approx 964.9.$$

Now consider two inefficient ways of trading the same 180 units:

- Equal consumption. If the 180 units are split equally among the 100 consumers of types 1 and 2, each receives 1.8 units, and the gross benefit falls to $60\cdot 10\ln 2.8 + 40\cdot 20\ln 2.8 = 1400\ln 2.8 \approx 1441.5$. Type-1 consumers would now have marginal utility $\frac{10}{2.8} \approx 3.57$, and type-2 consumers $\frac{20}{2.8} \approx 7.14$: moving a unit from a type-1 to a type-2 consumer would raise surplus.
- Equal production. If the 60 type-A and type-B firms produce 3 units each, the cost rises to $40\,c_A(3) + 20\,c_B(3) = 40(3+4.5) + 20(9+9) = 300 + 360 = 660$, compared with 560. Type-A firms would now have marginal cost $1 + 3 = 4$, and type-B firms $3 + 6 = 9$.

With both inefficiencies, surplus would be $1441.5 - 660 \approx 781.5$, about 183 below the CE surplus. Prices avoid both losses: every consumer and every firm faces the same price, so marginal utilities and marginal costs are equalized across agents.

Example 15 (More firms raise surplus). This example is a variant of Example 6.4 in the slides (slides 61–65), in which each firm's marginal cost has a positive intercept. Let demand be $x(p) = a - bp$ and let J identical firms have cost $c(q) = cq + \frac{1}{2}q^2$, so that each firm supplies $p - c$ for $p > c$ and aggregate supply is $J(p-c)$. The market is active (the first case of Result 1) if and only if $\frac{a}{b} > c$, that is, $a - bc > 0$.

- Equilibrium. Setting $a - bp = J(p-c)$ gives

$$p^* = \frac{a + Jc}{b + J}, \qquad x^* = J(p^* - c) = J\cdot\frac{a + Jc - bc - Jc}{b+J} = \frac{J(a-bc)}{b+J}.$$

  The price decreases in J, since $\frac{\partial p^*}{\partial J} = \frac{c(b+J)-(a+Jc)}{(b+J)^2} = -\frac{a-bc}{(b+J)^2} < 0$.

- Surplus. Inverse demand is $P(x) = \frac{a-x}{b}$, and the industry marginal cost is $C'(x) = c + \frac{x}{J}$, since the J firms share output equally. Then

$$S(x^*) = \int_0^{x^*} \left[\frac{a-bc}{b} - \left(\frac{1}{b} + \frac{1}{J}\right)s\right] ds = \frac{a-bc}{b}\,x^* - \frac{b+J}{2bJ}\,(x^*)^2.$$

Substituting x^*, the second term equals $\frac{b+J}{2bJ} \cdot \frac{J^2(a-bc)^2}{(b+J)^2} = \frac{J(a-bc)^2}{2b(b+J)}$, which is half of the first term, $\frac{J(a-bc)^2}{b(b+J)}$. Hence

$$S(x^*) = \frac{J(a-bc)^2}{2b(b+J)}, \qquad \text{with} \qquad \frac{\partial S(x^*)}{\partial J} = \frac{(a-bc)^2}{2b} \cdot \frac{(b+J)-J}{(b+J)^2} = \frac{(a-bc)^2}{2(b+J)^2} > 0.$$

With $a = 120$, $b = 2$, and $c = 10$, we have $a - bc = 100$. With $J = 8$ firms, $p^* = \frac{120+80}{10} = 20$, $x^* = \frac{8 \cdot 100}{10} = 80$, and $S = \frac{8 \cdot 10000}{4 \cdot 10} = 2000$. As a check, consumer surplus is $\frac{1}{2}(60-20) \cdot 80 = 1600$, and each firm earns $(p^* - c)q - \frac{1}{2}q^2 = 10 \cdot 10 - 50 = 50$, so producer surplus is $8 \cdot 50 = 400$, and $1600 + 400 = 2000$. With $J = 18$ firms, $p^* = \frac{300}{20} = 15$, $x^* = \frac{1800}{20} = 90$, and $S = \frac{18 \cdot 10000}{4 \cdot 20} = 2250$. As $J \to \infty$, the price approaches $c = 10$ and surplus approaches $\frac{(a-bc)^2}{2b} = 2500$. Each additional firm lowers the industry marginal cost $C'(x) = c + \frac{x}{J}$, because output is spread across more firms whose marginal costs increase in their own output.

4.3 The welfare cost of a per-unit tax

A per-unit tax drives a wedge between $P(x)$ and $C'(x)$, so the market trades less than the surplus-maximizing quantity. The tax revenue is a transfer from consumers and firms to the government, so it is part of the aggregate surplus; the loss is the surplus of the units that are no longer traded.

Example 16 (Deadweight loss in market L). With $t = 5$, from Example 7, consumers pay 22, firms receive 17, and 24 units are traded. The choke price of demand is 30 and supply starts at $p = 5$, so:

- consumer surplus is $CS = \frac{1}{2}(30-22) \cdot 24 = 96$;
- producer surplus is $PS = \frac{1}{2}(17-5) \cdot 24 = 144$;
- tax revenue is $R = 5 \cdot 24 = 120$;
- the deadweight loss is $DWL = \frac{1}{2} \cdot 5 \cdot (30-24) = 15$.

Without the tax, $CS = \frac{1}{2}(30-20) \cdot 30 = 150$ and $PS = \frac{1}{2}(20-5) \cdot 30 = 225$, so $S = 375$. With the tax, $CS + PS + R = 96 + 144 + 120 = 360 = 375 - 15$, which coincides with $S(24)$ in Example 13. Figure 8 depicts the four areas.

For a general t, (6) gives $x^*(t) = 30 - \frac{6}{5}t$, so the reduction in quantity is $\frac{6}{5}t$ and

$$DWL(t) = \frac{1}{2} \cdot t \cdot \frac{6}{5}t = \frac{3}{5}t^2, \qquad R(t) = t\left(30 - \frac{6}{5}t\right).$$

The deadweight loss grows with the square of the tax: doubling the tax from 5 to 10 multiplies the loss by four, from 15 to $\frac{3}{5} \cdot 100 = 60$. Its derivative at $t = 0$ is zero, so a small tax causes almost no loss, in line with (11): the first units removed are worth almost as much to consumers as they cost to produce. Revenue, instead, is maximized where $R'(t) = 30 - \frac{12}{5}t = 0$, that is, at $t = 12.5$; beyond this tax, the lost quantity reduces revenue more than the higher tax per unit increases it.

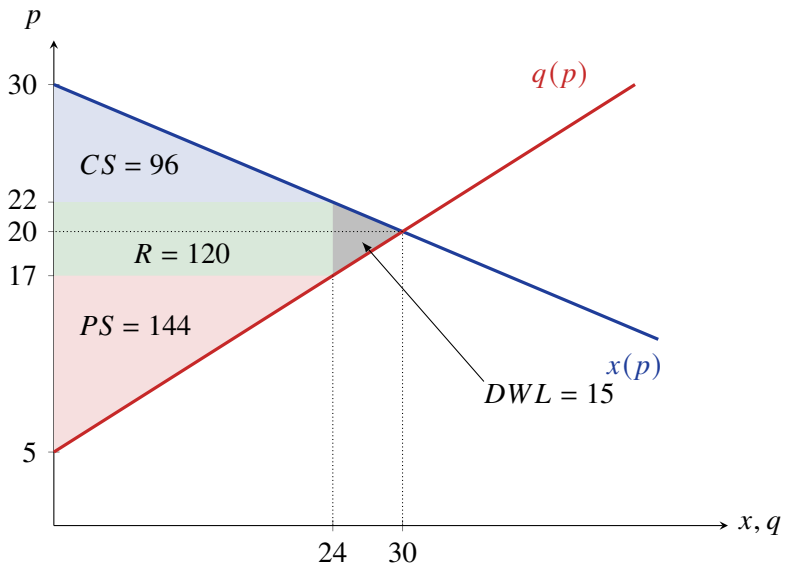


Figure 8: Example 16: a per-unit tax $t = 5$ in market L. Consumers pay 22, firms receive 17, and 24 units are traded. The tax revenue (green rectangle) is a transfer, while the gray triangle between $x = 24$ and $x = 30$ is the deadweight loss.

5 The Marshall–Hicks laws of derived demand

5.1 Setup

Slides 67–73 study the demand for an input, which is derived from the demand for the output that the input helps produce. An industry produces output $q = f(K, L)$ with capital K and labor L, where f has constant returns to scale. Output demand has absolute price elasticity $\varepsilon_{q,p} > 0$, capital is bought in a competitive market at price r, and labor supply $w(L)$ has elasticity $s_{L,w}$, which is negative when labor supply is backward bending. Let $s_{K,r}$ denote the elasticity of the derived demand for capital with respect to its price, σ the elasticity of substitution between capital and labor, and $\theta_K = \frac{rK}{pq}$ and $\theta_L = \frac{wL}{pq}$ the cost shares of capital and labor relative to sales. Under constant returns and zero profits, $pq = rK + wL$, so $\theta_L = 1 - \theta_K$.

The slides present a general formula for $s_{K,r}$, due to Marshall (1920), Hicks (1932), and Allen (1938). A demand for capital becomes more elastic when $s_{K,r}$, which is negative when $s_{L,w} > 0$, decreases. The four laws of derived demand (slide 73) state that, with $s_{L,w} > 0$, this happens as σ, $\varepsilon_{q,p}$, θ_K, or $s_{L,w}$ increases, where the law for θ_K requires $\varepsilon_{q,p} > \sigma$. We focus on the two cases the slides study, in which each firm takes the output price and the wage as given.

5.2 Marshall's presentation: fixed proportions

When inputs cannot be substituted ($\sigma = 0$), we can derive the elasticity directly. Each unit of output uses a_K units of capital and a_L units of labor, so $K = a_K q$ and $L = a_L q$, and competition

sets the output price equal to the unit cost, $p = a_K r + a_L w$. Consider a small percentage increase in r, written $d \ln r$, and trace its effects:

- First, the output price. Differentiating $p = a_K r + a_L w$ and dividing by p,

$$d \ln p = \frac{a_K r}{p}\, d \ln r + \frac{a_L w}{p}\, d \ln w = \theta_K\, d \ln r + \theta_L\, d \ln w,$$

  where $\frac{a_K r}{p} = \frac{rK}{pq} = \theta_K$ and, similarly, $\frac{a_L w}{p} = \theta_L$.
- Second, output and inputs. Output demand gives $d \ln q = -\varepsilon_{q,p}\, d \ln p$, and fixed proportions give $d \ln K = d \ln L = d \ln q$.
- Third, the wage. Along the labor supply curve, $d \ln L = s_{L,w}\, d \ln w$, so $d \ln w = \frac{d \ln q}{s_{L,w}}$.

Substituting the first and third expressions into the second,

$$d \ln q = -\varepsilon_{q,p} \left(\theta_K\, d \ln r + \frac{\theta_L}{s_{L,w}}\, d \ln q \right), \quad \text{that is,} \quad d \ln q \left(1 + \frac{\theta_L \varepsilon_{q,p}}{s_{L,w}} \right) = -\theta_K \varepsilon_{q,p}\, d \ln r.$$

Since $d \ln K = d \ln q$, multiplying both sides by $s_{L,w}$ and solving,

$$s_{K,r} = \frac{d \ln K}{d \ln r} = -\frac{\theta_K \varepsilon_{q,p} s_{L,w}}{s_{L,w} + \theta_L \varepsilon_{q,p}}, \tag{13}$$

which is the expression in slide 75. A higher r raises the output price, which reduces output and, with it, the use of both inputs. The fall in labor demand lowers the wage when $s_{L,w} > 0$, which partly offsets the increase in the output price. With $\theta_L = 1 - \theta_K$, the derivatives of (13) are (slide 76)

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{\theta_K s_{L,w}^2}{\left(s_{L,w} + \theta_L \varepsilon_{q,p} \right)^2}, \qquad \frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{s_{L,w} \varepsilon_{q,p} \left(s_{L,w} + \varepsilon_{q,p} \right)}{\left(s_{L,w} + \theta_L \varepsilon_{q,p} \right)^2},$$
$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{\theta_K \theta_L \varepsilon_{q,p}^2}{\left(s_{L,w} + \theta_L \varepsilon_{q,p} \right)^2}. \tag{14}$$

The first and third derivatives are negative for any sign of $s_{L,w}$. The second is negative when $s_{L,w} > 0$, but it can be positive when labor supply is backward bending (slide 77).

Example 17 (Marshall's laws with numbers). Let $\theta_K = 0.4$ (so $\theta_L = 0.6$), $\varepsilon_{q,p} = 2$, and $s_{L,w} = 1.5$. The denominator in (13) is $1.5 + 0.6 \cdot 2 = 2.7$, so

$$s_{K,r} = -\frac{0.4 \cdot 2 \cdot 1.5}{2.7} = -\frac{1.2}{2.7} = -\frac{4}{9} \approx -0.444.$$

A 1 percent increase in the price of capital reduces the use of capital by about 0.44 percent. From (14), with $2.7^2 = 7.29$,

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{0.4 \cdot 2.25}{7.29} = -\frac{10}{81} \approx -0.123, \qquad \frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{1.5 \cdot 2 \cdot 3.5}{7.29} = -\frac{350}{243} \approx -1.440,$$
$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{0.24 \cdot 4}{7.29} = -\frac{32}{243} \approx -0.132.$$

All three laws hold. Now let labor supply be backward bending, with $s_{L,w} = -1.5$. The denominator becomes $-1.5 + 1.2 = -0.3$, so $s_{K,r} = -\frac{0.4 \cdot 2 \cdot (-1.5)}{-0.3} = -\frac{1.2}{0.3} = -4$. The derivative with respect to θ_K is now

$$\frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{(-1.5) \cdot 2 \cdot (-1.5 + 2)}{(-0.3)^2} = \frac{1.5}{0.09} \approx 16.7 > 0,$$

so a larger cost share of capital makes its demand less elastic. Indeed, with $\theta_K = 0.5$, the denominator is $-1.5 + 0.5 \cdot 2 = -0.5$, and $s_{K,r} = -\frac{0.5 \cdot 2 \cdot (-1.5)}{-0.5} = -3$, which is above -4. Here $s_{L,w} + \theta_L \varepsilon_{q,p} < 0$: labor supply, with elasticity -1.5, responds more to the wage than the derived demand for labor, whose elasticity is $-\theta_L \varepsilon_{q,p} = -1.2$, so the labor-market equilibrium is unstable. In the stable range $-1.2 < s_{L,w} < 0$, the denominator is positive and $s_{K,r} > 0$, so the demand for capital slopes upward. In general, the sign of $\frac{\partial s_{K,r}}{\partial \theta_K}$ in (14) is the sign of $-s_{L,w}(s_{L,w} + \varepsilon_{q,p})$, which is positive when $-\varepsilon_{q,p} < s_{L,w} < 0$.

5.3 Hicks's presentation: substitution between inputs

When inputs can be substituted ($\sigma \geq 0$), the elasticity becomes (slide 78)

$$s_{K,r} = -\frac{\theta_K \varepsilon_{q,p} s_{L,w} + \sigma \left(\varepsilon_{q,p} + \theta_L s_{L,w}\right)}{s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}}, \tag{15}$$

which reduces to (13) when $\sigma = 0$. Its derivatives are (slide 79)

$$\begin{aligned}
\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} &= -\frac{\theta_K \left(s_{L,w} + \sigma\right)^2}{D^2}, & \frac{\partial s_{K,r}}{\partial \theta_K} &= -\frac{\left(\varepsilon_{q,p} + s_{L,w}\right)\left(\varepsilon_{q,p} - \sigma\right)\left(s_{L,w} + \sigma\right)}{D^2}, \\
\frac{\partial s_{K,r}}{\partial s_{L,w}} &= -\frac{\theta_K \theta_L \left(\varepsilon_{q,p} - \sigma\right)^2}{D^2}, & \frac{\partial s_{K,r}}{\partial \sigma} &= -\frac{\theta_L \left(\varepsilon_{q,p} + s_{L,w}\right)^2}{D^2},
\end{aligned} \tag{16}$$

where $D = s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}$. Three derivatives are always weakly negative. The derivative with respect to θ_K is negative when $s_{L,w} > 0$ and $\varepsilon_{q,p} > \sigma$, but positive when $s_{L,w} > 0$ and $\varepsilon_{q,p} < \sigma$ (slide 80).

A useful special case is a perfectly elastic labor supply. Dividing the numerator and the denominator of (15) by $s_{L,w}$ and letting $s_{L,w} \to \infty$, the terms without $s_{L,w}$ vanish, and

$$s_{K,r} \to -\left(\theta_K \varepsilon_{q,p} + \theta_L \sigma\right). \tag{17}$$

The elasticity of derived demand is a weighted average of two effects: a scale effect, $\varepsilon_{q,p}$ (a higher r raises the output price and reduces output), with weight θ_K; and a substitution effect, σ (the firm replaces capital with labor), with weight $\theta_L = 1 - \theta_K$. A higher θ_K moves weight from σ to $\varepsilon_{q,p}$, so it makes demand more elastic if and only if $\varepsilon_{q,p} > \sigma$. This is the intuition behind the condition in the third law.

Example 18 (Hicks's laws with numbers). Keep $\theta_K = 0.4$, $\varepsilon_{q,p} = 2$, and $s_{L,w} = 1.5$, and let $\sigma = 1$. The numerator of (15) is $0.4 \cdot 2 \cdot 1.5 + 1 \cdot (2 + 0.6 \cdot 1.5) = 1.2 + 2.9 = 4.1$ and the denominator is $D = 1.5 + 0.4 + 1.2 = 3.1$, so

$$s_{K,r} = -\frac{4.1}{3.1} = -\frac{41}{31} \approx -1.323.$$

Compared with Example 17, the possibility of substitution makes the demand for capital about three times more elastic. From (16), with $D^2 = 9.61$,

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{0.4 \cdot 6.25}{9.61} \approx -0.260, \qquad \frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{3.5 \cdot 1 \cdot 2.5}{9.61} \approx -0.911,$$
$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{0.24 \cdot 1}{9.61} \approx -0.025, \qquad \frac{\partial s_{K,r}}{\partial \sigma} = -\frac{0.6 \cdot 12.25}{9.61} \approx -0.765.$$

All four laws hold, since $\varepsilon_{q,p} = 2 > 1 = \sigma$. With $\sigma = 3 > \varepsilon_{q,p}$, instead, the derivative with respect to θ_K becomes

$$\frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{3.5 \cdot (2-3) \cdot 4.5}{(1.5+1.2+1.2)^2} = \frac{15.75}{15.21} \approx 1.036 > 0.$$

Figure 9 plots the absolute elasticity $|s_{K,r}|$ against θ_K for $\sigma \in \{0, 1, 3\}$. When $\theta_K = 1$, (15) gives $s_{K,r} = -\frac{\varepsilon_{q,p}(s_{L,w}+\sigma)}{s_{L,w}+\sigma} = -\varepsilon_{q,p} = -2$ for every σ: if capital is the only cost, the demand for capital inherits the elasticity of output demand. When $\theta_K = 0$, (15) gives $|s_{K,r}| = \frac{\sigma(2+1.5)}{1.5+2} = \sigma$, so the curves start at 0, 1, and 3.

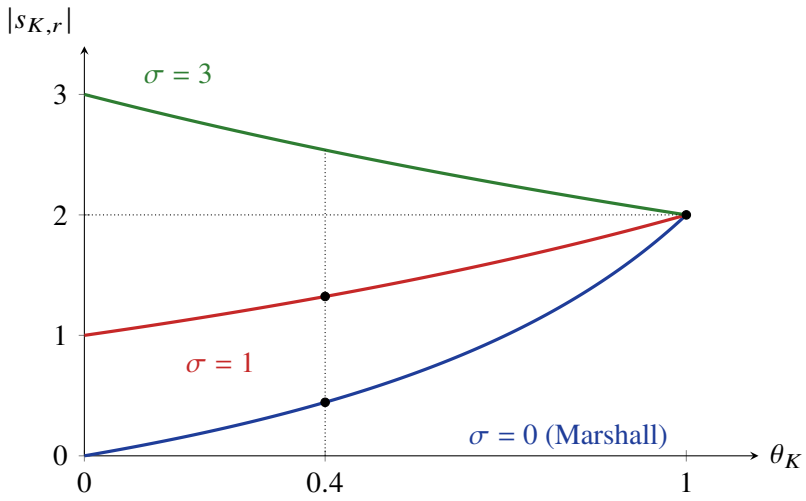


Figure 9: Absolute elasticity of the demand for capital, $|s_{K,r}|$, as a function of the cost share θ_K, with $\varepsilon_{q,p} = 2$ and $s_{L,w} = 1.5$. When $\sigma < \varepsilon_{q,p}$, the elasticity increases in θ_K; when $\sigma > \varepsilon_{q,p}$, it decreases. All curves reach $\varepsilon_{q,p} = 2$ at $\theta_K = 1$. The dots at $\theta_K = 0.4$ are Examples 17 and 18.

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