# General Equilibrium: Worked Examples

EconS 501, a handout on general equilibrium in Chapter 6

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## 1 Introduction

This handout extends the Chapter 6 slides. Up to this point, we studied one consumer or one firm at a time, and we took prices as given. General equilibrium studies several consumers and firms who interact in several markets at the same time, so prices become an outcome of everyone's choices rather than a given.

The handout follows the order of the slides, but every example is new, and we show the intermediate steps that the slides skip. When a definition or a proof already appears in the slides, we refer to the slide numbers instead of repeating it. Most examples have two consumers and two goods, so we can depict every allocation in an Edgeworth box. We return to the same few economies in several sections. This allows you to see how the contract curve, the core, and the competitive equilibrium of one economy relate to each other, and to check that different approaches give the same answer.

To find a Walrasian equilibrium, we always follow the same recipe. First, we solve each consumer's utility maximization problem to obtain her Walrasian demands, which depend on prices and on the market value of her endowment. Second, we impose market clearing for one good, since Walras' law guarantees that the other market then clears too. We never use the welfare theorems as a shortcut to find an equilibrium; instead, we use them to describe the equilibrium once we have found it.

We first describe exchange economies and the Edgeworth box, and then study Pareto efficiency, the core, and Walrasian equilibrium. Next, we present the two welfare theorems. We then add firms and examine comparative statics and taxes. The Appendix shows that the core shrinks to the set of Walrasian equilibria as the economy becomes large.

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# 2 Pure exchange economies and the Edgeworth box

Consider an economy with two goods, 1 and 2, and two consumers, $A$ and $B$, where consumer $i$ owns an endowment $e^i = (e^i_1, e^i_2)$ and consumes a bundle $x^i = (x^i_1, x^i_2)$. An allocation is feasible if $x^A_k + x^B_k \leq e^A_k + e^B_k \equiv \bar{e}_k$ for each good $k$, and every allocation that uses up the aggregate endowment is a point in the Edgeworth box (Edgeworth, 1881), with consumer $A$'s bundle measured from the bottom-left corner, $0^A$, and consumer $B$'s from the top-right corner, $0^B$ (see slides 5–13). Throughout the handout, utility functions are continuous, strictly increasing, and strictly quasiconcave. (In our Cobb–Douglas, quasilinear, and Example 9 utilities, these properties hold for strictly positive bundles.) We refer to these properties as "the assumptions on preferences above."

Table 1 lists the economies that we use in the examples.

| Economy | $u^A$ | $u^B$ | $e^A$ | $e^B$ |
|---|---|---|---|---|
| Cobb–Douglas | $(x^A_1)^{3/4}(x^A_2)^{1/4}$ | $(x^B_1)^{1/4}(x^B_2)^{3/4}$ | $(4, 6)$ | $(4, 2)$ |
| Quasilinear | $\ln x^A_1 + x^A_2$ | $2 \ln x^B_1 + x^B_2$ | $(1, 6)$ | $(8, 2)$ |
| Three equilibria | $x^A_1 - \frac{1}{2}(x^A_2)^{-2}$ | $-\frac{1}{2}(x^B_1)^{-2} + x^B_2$ | $(2, \frac{2}{7})$ | $(\frac{2}{7}, 2)$ |

Table 1. Exchange economies used in the handout. The Cobb–Douglas economy appears in Examples 1–2, 4, 6–7, and 10–12; the quasilinear economy in Examples 3, 5, 8, 10, and 21; and the last economy in Example 9.

**Example 1 (An Edgeworth box).** In the Cobb–Douglas economy, the aggregate endowment is $\bar{e} = (4 + 4, 6 + 2) = (8, 8)$, so the Edgeworth box is a square with sides of length 8.

- The endowment point $e$ is at $(4, 6)$ when measured from $0^A$. Measured from $0^B$, the same point is at $(8 - 4, 8 - 6) = (4, 2)$, which is $e^B$.
- Consider the allocation $x^A = (6, 3)$ and $x^B = (2, 5)$. It is feasible, since $6 + 2 = 8$ and $3 + 5 = 8$, so it is also a point in the box.
- Relative to the endowment, consumer $A$ obtains two more units of good 1 and gives up three units of good 2. Her utility increases from

$$u^A(4, 6) = 4^{3/4}\, 6^{1/4} \approx 4.43 \qquad \text{to} \qquad u^A(6, 3) = 6^{3/4}\, 3^{1/4} \approx 5.05.$$

  Consumer $B$'s utility also increases, from $u^B(4, 2) = 4^{1/4}\, 2^{3/4} \approx 2.38$ to $u^B(2, 5) = 2^{1/4}\, 5^{3/4} \approx 3.98$.

Both consumers prefer $x$ to their endowments, because consumer $A$ cares mostly about good 1 (the exponent $\frac{3}{4}$), while consumer $B$ cares mostly about good 2. Trading good 2 for good 1 benefits both. As we show next, however, $x$ still leaves room for further gains from trade.

# 3 Pareto efficiency and the contract curve

Slides 14–23 define Pareto efficiency and derive its tangency condition by maximizing one consumer's utility while holding the other at a given utility level.

*Result 1 (Pareto efficiency in the interior of the box).* An allocation in the interior of the box is Pareto efficient if and only if the consumers' indifference curves are tangent,

$$MRS^A_{1,2}(x^A) = MRS^B_{1,2}(x^B), \qquad \text{where} \qquad MRS^i_{1,2} = \frac{MU^i_1}{MU^i_2} \tag{1}$$

and $MU^i_k = \frac{\partial u^i}{\partial x^i_k}$ denotes consumer $i$'s marginal utility of good $k$.

Intuitively, if consumer $A$ were willing to give up more units of good 2 for an extra unit of good 1 than consumer $B$ requires, $MRS^A_{1,2} > MRS^B_{1,2}$, then $B$ could give a small amount $\varepsilon$ of good 1 to $A$ in exchange for an amount of good 2 between $MRS^B_{1,2}\varepsilon$ and $MRS^A_{1,2}\varepsilon$, making both better off. Conversely, with tangency, quasiconcavity puts $A$'s better-than set on one side of the common tangent line and $B$'s on the other, so no feasible reallocation makes one consumer better off without making the other worse off. The set of Pareto-efficient allocations is the contract curve. On the edges of the box, tangency is no longer necessary, as Example 3 shows.

**Example 2 (Contract curve with Cobb–Douglas preferences).** In the Cobb–Douglas economy,

$$MRS^A_{1,2} = \frac{\frac{3}{4}(x^A_1)^{-1/4}(x^A_2)^{1/4}}{\frac{1}{4}(x^A_1)^{3/4}(x^A_2)^{-3/4}} = \frac{3x^A_2}{x^A_1} \qquad \text{and} \qquad MRS^B_{1,2} = \frac{\frac{1}{4}(x^B_1)^{-3/4}(x^B_2)^{3/4}}{\frac{3}{4}(x^B_1)^{1/4}(x^B_2)^{-1/4}} = \frac{x^B_2}{3x^B_1}.$$

- *First step: feasibility.* Since utility is strictly increasing, both feasibility constraints bind, so $x^B_1 = 8 - x^A_1$ and $x^B_2 = 8 - x^A_2$.
- *Second step: tangency.* Substituting into (1),

$$\frac{3x^A_2}{x^A_1} = \frac{8 - x^A_2}{3(8 - x^A_1)}.$$

- *Third step: solve for $x^A_2$.* Cross-multiplying yields $9x^A_2(8 - x^A_1) = x^A_1(8 - x^A_2)$, that is,

$$72x^A_2 - 9x^A_1x^A_2 = 8x^A_1 - x^A_1x^A_2, \qquad \text{or} \qquad x^A_2\left(72 - 8x^A_1\right) = 8x^A_1.$$

  Dividing both sides by $8(9 - x^A_1)$, the contract curve is

$$x^A_2 = \frac{x^A_1}{9 - x^A_1} \qquad \text{for } x^A_1 \in [0, 8]. \tag{2}$$

The curve connects the two origins: at $x^A_1 = 0$ we obtain $x^A_2 = 0$, and at $x^A_1 = 8$ we obtain $x^A_2 = \frac{8}{1} = 8$. On the edges of the box, only the two origins are efficient: on the bottom edge, for

instance, $MRS^A_{1,2} = 0 < MRS^B_{1,2}$, so both consumers gain if $A$ sells some good 1 for good 2. Since $\frac{x_1^A}{9-x_1^A} < x_1^A$ whenever $0 < x_1^A < 8$, the contract curve lies below the diagonal of the box: at every interior Pareto-efficient allocation, consumer $A$, who values good 1 relatively more, holds more units of good 1 than of good 2. Substituting (2) into $MRS^A_{1,2}$, the common MRS along the contract curve is

$$MRS^A_{1,2} = \frac{3}{x_1^A} \cdot \frac{x_1^A}{9 - x_1^A} = \frac{3}{9 - x_1^A}, \tag{3}$$

which increases from $\frac{1}{3}$ at $0^A$ to 3 at $0^B$. Table 2 reports some points on the curve, and Figure 1 depicts the indifference curves at two of them. At $x^A = (4.5, 1)$, for instance, $MRS^A_{1,2} = \frac{3(1)}{4.5} = \frac{2}{3}$ and, with $x^B = (3.5, 7)$, $MRS^B_{1,2} = \frac{7}{3(3.5)} = \frac{2}{3}$. Returning to the allocation of Example 1, $x^A = (6, 3)$ and $x^B = (2, 5)$, we have $MRS^A_{1,2} = \frac{3(3)}{6} = \frac{3}{2}$ while $MRS^B_{1,2} = \frac{5}{3(2)} = \frac{5}{6}$, so the indifference curves cross at $x$; the contract curve at $x_1^A = 6$ has $x_2^A = 2$, not 3.

| $x_1^A$ | $x_2^A$ | $x_1^B$ | $x_2^B$ | $MRS^A_{1,2} = MRS^B_{1,2}$ |
|---|---|---|---|---|
| 3 | $\frac{3}{6} = 0.5$ | 5 | 7.5 | $\frac{1}{2}$ |
| 4.5 | $\frac{4.5}{4.5} = 1$ | 3.5 | 7 | $\frac{2}{3}$ |
| 6 | $\frac{6}{3} = 2$ | 2 | 6 | 1 |
| 7.2 | $\frac{7.2}{1.8} = 4$ | 0.8 | 4 | $\frac{5}{3}$ |

Table 2. Points on the contract curve of the Cobb–Douglas economy.

**Example 3 (Contract curve with quasilinear preferences).** In the quasilinear economy, $u^A = \ln x_1^A + x_2^A$ and $u^B = 2\ln x_1^B + x_2^B$, and the aggregate endowment is $\bar{e} = (1 + 8, 6 + 2) = (9, 8)$. The marginal utility of good 2 is one for both consumers, so

$$MRS^A_{1,2} = \frac{1}{x_1^A} \qquad \text{and} \qquad MRS^B_{1,2} = \frac{2}{x_1^B}.$$

- *Interior allocations.* Tangency requires $\frac{1}{x_1^A} = \frac{2}{x_1^B}$, that is, $x_1^B = 2x_1^A$. Using feasibility, $x_1^A + 2x_1^A = 9$, so $x_1^A = 3$ and $x_1^B = 6$. Every allocation with $x_1^A = 3$ and $x_2^A \in (0, 8)$ is Pareto efficient: the contract curve is a vertical line. With quasilinear utility, the efficient amount of good 1 does not depend on how good 2 is distributed.
- *Bottom edge, $x_2^A = 0$.* Consumer $A$ has no good 2 to give, so the only possible trade moves good 1 from $A$ to $B$, paid for with good 2. Consumer $A$ requires $MRS^A_{1,2} = \frac{1}{x_1^A}$ units of good 2 per unit of good 1 she gives up, and $B$ is willing to pay up to $MRS^B_{1,2} = \frac{2}{9-x_1^A}$. Gains from trade exist if and only if

  $$\frac{2}{9 - x_1^A} > \frac{1}{x_1^A}, \qquad \text{that is,} \qquad 2x_1^A > 9 - x_1^A, \qquad \text{or} \qquad x_1^A > 3.$$

  Hence, the allocations on the bottom edge with $x_1^A \le 3$ are Pareto efficient.

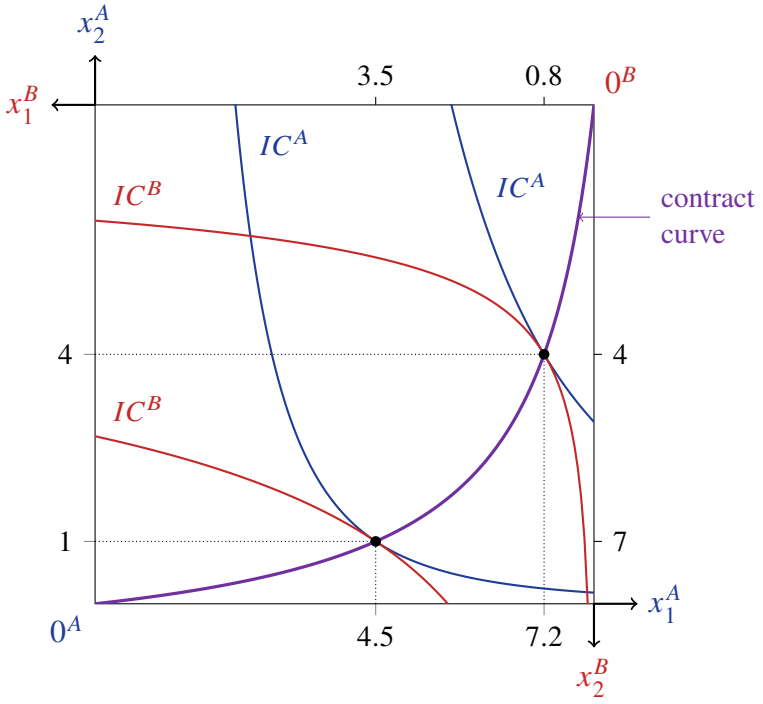



Figure 1. Contract curve $x_2^A = \frac{x_1^A}{9-x_1^A}$ of the Cobb–Douglas economy, with the tangent indifference curves at $x^A = (4.5, 1)$, where both MRSs equal $\frac{2}{3}$, and at $x^A = (7.2, 4)$, where both equal $\frac{5}{3}$.

- *Top edge,* $x_2^A = 8$. Now consumer $B$ has no good 2, and the same argument (with the roles reversed) shows that the allocations on the top edge with $x_1^A \geq 3$ are Pareto efficient.

The contract curve thus runs along the bottom edge from $0^A$ to $(3, 0)$, up the vertical line $x_1^A = 3$, and along the top edge from $(3, 8)$ to $0^B$. Figure 3 below depicts it.

# 4 Barter equilibrium and the core

Slides 30–32 define blocking coalitions and the core, $C(e)$, the set of feasible allocations that no coalition blocks. With two consumers, the core is the part of the contract curve inside the lens between the two indifference curves through $e$: neither consumer alone blocks (individual rationality), and the grand coalition does not block (efficiency). To find it, we solve for the two points where the contract curve crosses the indifference curves through $e$: the first gives the lowest utility that consumer $A$ accepts, and the second gives the lowest utility that consumer $B$ accepts.

**Example 4 (The core of the Cobb–Douglas economy).** The endowment yields $u^A(e^A) = 4^{3/4}6^{1/4} = 384^{1/4}$ and $u^B(e^B) = 4^{1/4}2^{3/4} = 2^{5/4}$.

- *First step: consumer A's bound.* Along the contract curve (2), consumer $A$'s utility is

$$u^A = (x_1^A)^{3/4}\left(\frac{x_1^A}{9-x_1^A}\right)^{1/4} = \frac{x_1^A}{(9-x_1^A)^{1/4}},$$

which increases in $x_1^A$. Setting it equal to $384^{1/4}$ and raising both sides to the fourth power, $(x_1^A)^4 = 384(9 - x_1^A)$, or

$$(x_1^A)^4 + 384x_1^A - 3456 = 0,$$

whose positive root is $x_1^A \approx 5.882$, with $x_2^A = \frac{x_1^A}{9-x_1^A} \approx 1.887$.

- *Second step: consumer B's bound.* Along the contract curve, $x_1^B = 8 - x_1^A$ and $x_2^B = 8 - \frac{x_1^A}{9-x_1^A} = \frac{9(8-x_1^A)}{9-x_1^A}$. Hence,

$$u^B = (8 - x_1^A)^{1/4} \left( \frac{9(8 - x_1^A)}{9 - x_1^A} \right)^{3/4} = \frac{9^{3/4}(8 - x_1^A)}{(9 - x_1^A)^{3/4}},$$

which decreases in $x_1^A$. Setting it equal to $2^{5/4}$ and raising both sides to the fourth power yields

$$729(8 - x_1^A)^4 = 32(9 - x_1^A)^3,$$

whose root in $[0, 8]$ is $x_1^A \approx 7.327$, with $x_2^A = \frac{x_1^A}{9-x_1^A} \approx 4.378$.

The core is therefore the part of the contract curve with $5.882 \le x_1^A \le 7.327$. Figure 2 depicts it. Consumer $A$ obtains her lowest core utility at the lower-left end of the core and her highest at the upper-right end, while the opposite holds for consumer $B$.

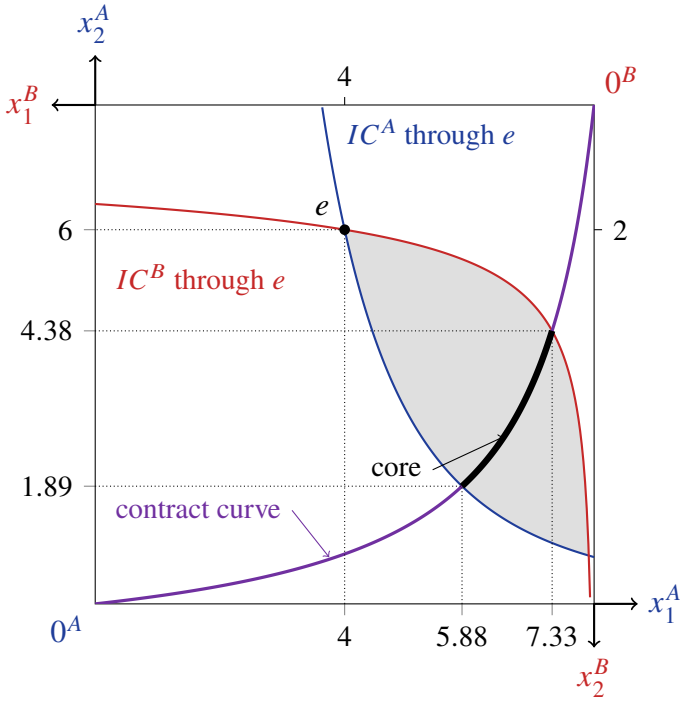



Figure 2. The lens between the indifference curves through $e$ (shaded) and the core of the Cobb–Douglas economy (thick segment), $5.882 \le x_1^A \le 7.327$.

**Example 5 (The core of the quasilinear economy).** With $e^A = (1, 6)$ and $e^B = (8, 2)$, the endowment utilities are $u^A(e^A) = \ln 1 + 6 = 6$ and $u^B(e^B) = 2 \ln 8 + 2$. In the interior of the box, the

contract curve is $x_1^A = 3$. On the bottom edge, $A$ would need $\ln x_1^A \geq 6$ (her utility at $e$), impossible with $x_1^A \leq 9$. On the top edge, $B$ would need $2\ln(9 - x_1^A) \geq 2\ln 8 + 2$, that is, $\ln \frac{9-x_1^A}{8} \geq 1$, also impossible, since $9 - x_1^A \leq 9$. So the core lies on $x_1^A = 3$, and we only need to find the range of $x_2^A$.

- *Consumer A.* She accepts $(3, x_2^A)$ only if $\ln 3 + x_2^A \geq 6$, that is,

$$x_2^A \geq 6 - \ln 3 \approx 6 - 1.099 = 4.901.$$

- *Consumer B.* He receives $(6, 8 - x_2^A)$, which he accepts only if $2\ln 6 + 8 - x_2^A \geq 2\ln 8 + 2$, that is,

$$x_2^A \leq 6 - 2(\ln 8 - \ln 6) = 6 - 2\ln\frac{4}{3} \approx 6 - 0.575 = 5.425.$$

The core is the vertical segment $x_1^A = 3$, $4.901 \leq x_2^A \leq 5.425$. In this economy, the core bounds have closed forms because, along the contract curve, utility changes one-for-one with good 2. The lens is bounded by $A$'s indifference curve, $x_2^A = 6 - \ln x_1^A$, and by $B$'s indifference curve, which in $A$'s coordinates satisfies $2\ln(9 - x_1^A) + 8 - x_2^A = 2\ln 8 + 2$, that is, $x_2^A = 6 + 2\ln\frac{9-x_1^A}{8}$. The two curves cross at $e$ and where $-\ln x_1^A = 2\ln\frac{9-x_1^A}{8}$, that is, where $x_1^A(9 - x_1^A)^2 = 64$. This cubic factors as

$$(x_1^A)^3 - 18(x_1^A)^2 + 81x_1^A - 64 = (x_1^A - 1)\left[(x_1^A)^2 - 17x_1^A + 64\right] = 0,$$

so, besides $x_1^A = 1$ (the endowment), the lens ends at $x_1^A = \frac{17-\sqrt{33}}{2} \approx 5.628$. Figure 3 depicts the contract curve, the lens, and the core.

# 5 Walrasian equilibrium

## 5.1 Walrasian demand and excess demand

Consumer $i$ takes prices $p = (p_1, p_2) \gg 0$ as given and maximizes utility subject to $p \cdot x^i \leq p \cdot e^i$, which yields her Walrasian demand $x^i(p, p \cdot e^i)$. The excess demand for good $k$, $z_k(p)$, is aggregate demand minus aggregate endowment (slides 35–40). Unlike in the UMP of Chapter 2, income changes with prices, since it is the value of the endowment.

*Result 2 (Properties of excess demand).* Under the assumptions on preferences above, the excess demand function $z(p) = (z_1(p), z_2(p))$ satisfies Walras' law, $p \cdot z(p) = p_1 z_1(p) + p_2 z_2(p) = 0$, it is continuous, and it is homogeneous of degree zero, $z(\lambda p) = z(p)$ for every $\lambda > 0$ (slides 41–47). Hence, if market 1 clears, so does market 2, and we can normalize one price, for instance $p_2 = 1$.

**Example 6 (Excess demand in the Cobb–Douglas economy).** A Cobb–Douglas consumer with exponents $\alpha$ and $1 - \alpha$ spends a share $\alpha$ of her income on good 1 (we derive this in Example 7). Hence, with incomes $p \cdot e^A = 4p_1 + 6p_2$ and $p \cdot e^B = 4p_1 + 2p_2$,

$$x_1^A = \frac{3}{4} \cdot \frac{4p_1 + 6p_2}{p_1} = 3 + \frac{9}{2} \cdot \frac{p_2}{p_1} \qquad \text{and} \qquad x_1^B = \frac{1}{4} \cdot \frac{4p_1 + 2p_2}{p_1} = 1 + \frac{1}{2} \cdot \frac{p_2}{p_1}.$$

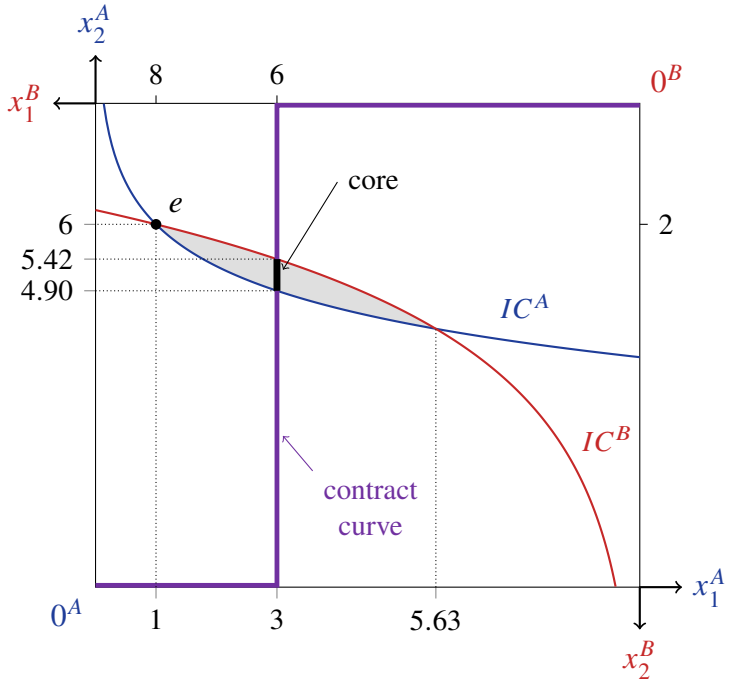



Figure 3. The quasilinear economy. The contract curve (purple) runs along the bottom edge to $(3, 0)$, up the line $x_1^A = 3$, and along the top edge to $0^B$. The core is the segment of $x_1^A = 3$ inside the lens, $4.901 \le x_2^A \le 5.425$.

Subtracting the aggregate endowment of good 1, which is 8,

$$z_1(p) = 3 + \frac{9}{2} \cdot \frac{p_2}{p_1} + 1 + \frac{1}{2} \cdot \frac{p_2}{p_1} - 8 = \frac{5p_2}{p_1} - 4. \tag{4}$$

Similarly, $x_2^A = \frac{1}{4} \cdot \frac{4p_1+6p_2}{p_2} = \frac{p_1}{p_2} + \frac{3}{2}$ and $x_2^B = \frac{3}{4} \cdot \frac{4p_1+2p_2}{p_2} = \frac{3p_1}{p_2} + \frac{3}{2}$, so

$$z_2(p) = \frac{p_1}{p_2} + \frac{3}{2} + \frac{3p_1}{p_2} + \frac{3}{2} - 8 = \frac{4p_1}{p_2} - 5. \tag{5}$$

We can now check each property in Result 2:

- Walras' law: $p_1 z_1(p) + p_2 z_2(p) = (5p_2 - 4p_1) + (4p_1 - 5p_2) = 0$.
- Continuity: both excess demands are continuous at every $p \gg 0$.
- Homogeneity: both excess demands depend only on $\frac{p_1}{p_2}$, so doubling both prices leaves them unchanged.

We also note the boundary behavior used in Result 3: as $p_1 \to 0$, $z_1(p) \to +\infty$. When a good becomes almost free, consumers want unboundedly many units of it. At $p_1 = p_2 = 1$, for instance, $z_1 = 5 - 4 = 1 > 0$ and $z_2 = 4 - 5 = -1 < 0$: good 1 is in excess demand, so its relative price is too low.

## 5.2 Walrasian equilibrium

A price vector $p^* \gg 0$ is a Walrasian equilibrium if $z(p^*) = 0$, and the resulting allocation $x(p^*)$ is a Walrasian equilibrium allocation (WEA); $W(e)$ denotes the set of WEAs (slides 48–49). We find it in four steps: solve consumer $A$'s UMP, solve consumer $B$'s UMP, impose market clearing for one good, and recover the allocation.

**Example 7 (WEA in the Cobb–Douglas economy).**

- *First step: consumer A's UMP.* She solves $\max (x_1^A)^{3/4}(x_2^A)^{1/4}$ subject to $p_1x_1^A + p_2x_2^A = 4p_1 + 6p_2$. At an interior optimum, her MRS equals the price ratio,

$$\frac{3x_2^A}{x_1^A} = \frac{p_1}{p_2}, \qquad \text{so} \qquad p_2x_2^A = \frac{1}{3}\,p_1x_1^A.$$

  Substituting into her budget constraint, $p_1x_1^A + \frac{1}{3}p_1x_1^A = 4p_1 + 6p_2$, so $\frac{4}{3}p_1x_1^A = 4p_1 + 6p_2$, and

$$x_1^A(p) = \frac{3(4p_1 + 6p_2)}{4p_1} \qquad \text{and} \qquad x_2^A(p) = \frac{4p_1 + 6p_2}{4p_2}.$$

  She spends three fourths of the value of her endowment on good 1 and one fourth on good 2.

- *Second step: consumer B's UMP.* His tangency condition is $\frac{x_2^B}{3x_1^B} = \frac{p_1}{p_2}$, so $p_2x_2^B = 3p_1x_1^B$. Substituting into $p_1x_1^B + p_2x_2^B = 4p_1 + 2p_2$ gives $4p_1x_1^B = 4p_1 + 2p_2$, so

$$x_1^B(p) = \frac{4p_1 + 2p_2}{4p_1} \qquad \text{and} \qquad x_2^B(p) = \frac{3(4p_1 + 2p_2)}{4p_2}.$$

- *Third step: market clearing for good 1.* Normalizing $p_2 = 1$, the condition $x_1^A(p) + x_1^B(p) = 8$ reads

$$\frac{3(4p_1 + 6)}{4p_1} + \frac{4p_1 + 2}{4p_1} = 8, \qquad \text{that is,} \qquad 12p_1 + 18 + 4p_1 + 2 = 32p_1,$$

  so $16p_1 = 20$ and $p_1^* = \frac{5}{4}$. This is the price at which $z_1 = \frac{5}{p_1} - 4$ in (4) equals zero.

- *Fourth step: the allocation.* At $p^* = (\frac{5}{4}, 1)$, consumer $A$'s income is $\frac{5}{4}(4)+6 = 11$ and consumer $B$'s is $\frac{5}{4}(4) + 2 = 7$. Hence,

$$x_1^A = \frac{3}{4}\cdot 11 \cdot \frac{4}{5} = \frac{33}{5} = 6.6, \quad x_2^A = \frac{11}{4} = 2.75,$$
$$x_1^B = \frac{1}{4}\cdot 7 \cdot \frac{4}{5} = \frac{7}{5} = 1.4, \qquad x_2^B = \frac{3(7)}{4} = 5.25.$$

The market for good 2 clears as well, $2.75 + 5.25 = 8$, as Walras' law predicts. Consumer $A$ buys $6.6 - 4 = 2.6$ units of good 1 and pays for them with $6 - 2.75 = 3.25$ units of good 2, which is the value of her purchase, $\frac{5}{4}(2.6) = 3.25$. Both consumers gain from trade:

$$u^A(6.6, 2.75) = 6.6^{3/4}\,2.75^{1/4} \approx 5.30 > 4.43 \approx u^A(4, 6),$$
$$u^B(1.4, 5.25) = 1.4^{1/4}\,5.25^{3/4} \approx 3.77 > 2.38 \approx u^B(4, 2).$$

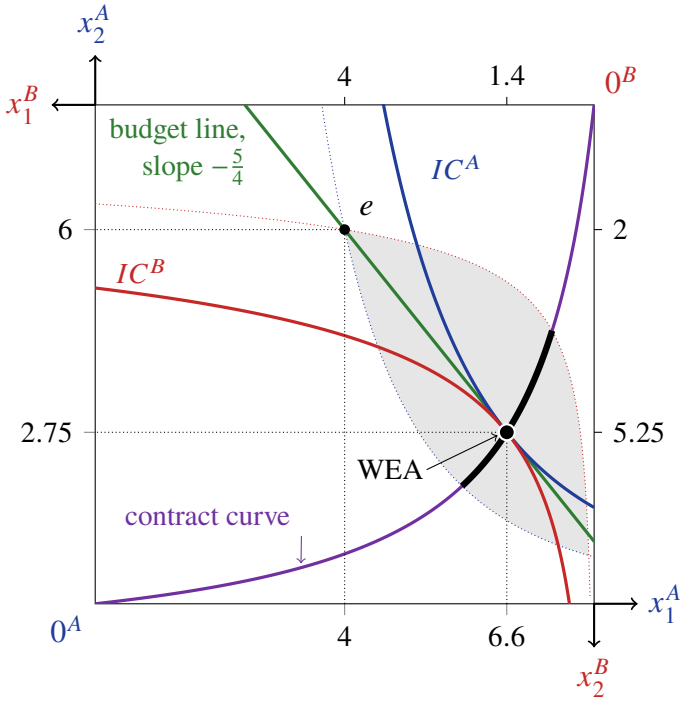



Figure 4. WEA of the Cobb–Douglas economy. The budget line through $e$ has slope $-\frac{5}{4}$, and the indifference curves are tangent to it at $x^A = (6.6, 2.75)$, which lies on the contract curve and inside the core (thick segment). Dotted curves are the indifference curves through $e$.

Figure 4 depicts the equilibrium: the budget line goes through $e$ with slope $-\frac{p_1^*}{p_2^*} = -\frac{5}{4}$, and both indifference curves are tangent to it at the WEA.

**Example 8 (WEA in the quasilinear economy).** Normalize $p_2 = 1$.

- *First step: consumer A's UMP.* She solves max $\ln x_1^A + x_2^A$ subject to $p_1 x_1^A + x_2^A = p_1 + 6$. Her tangency condition, $MRS_{1,2}^A = \frac{1}{x_1^A} = p_1$, gives $x_1^A = \frac{1}{p_1}$, and her budget gives $x_2^A = p_1 + 6 - p_1 \cdot \frac{1}{p_1} = p_1 + 5$, which is positive at every price, so the solution is interior.
- *Second step: consumer B's UMP.* From $\frac{2}{x_1^B} = p_1$, $x_1^B = \frac{2}{p_1}$, and $x_2^B = 8p_1 + 2 - 2 = 8p_1$, which is positive as well.
- *Third step: market clearing for good 1.* $\frac{1}{p_1} + \frac{2}{p_1} = 9$, so $p_1^* = \frac{1}{3}$.
- *Fourth step: the allocation.* $x^A = (3, \frac{1}{3} + 5) = (3, \frac{16}{3})$ and $x^B = (6, \frac{8}{3})$.

Good 2 clears, $\frac{16}{3} + \frac{8}{3} = 8$. Both consumers set their MRS equal to $p_1^*$, so the equilibrium amounts of good 1 satisfy the tangency condition of Example 3, $x_1^A = 3$ and $x_1^B = 6$. With quasilinear utility, these amounts do not depend on income, so the endowments only determine how good 2 is split. Consumer $A$ buys two units of good 1 for $\frac{2}{3}$ units of good 2. Figure 5 depicts the equilibrium.

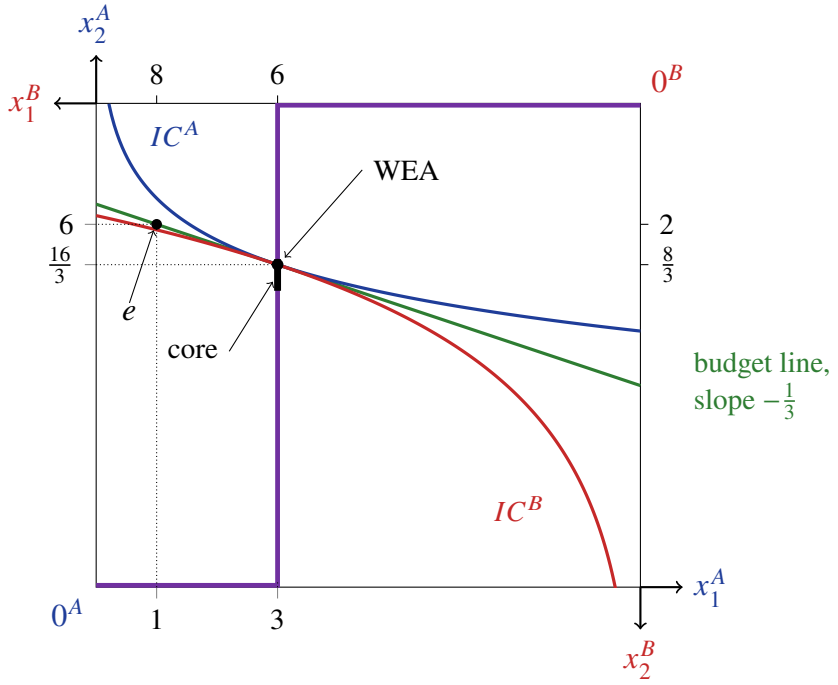



Figure 5. WEA of the quasilinear economy, $x^A = (3, \frac{16}{3})$, at $p_1^* = \frac{1}{3}$. The WEA lies on the contract curve, inside the core (thick segment) of Example 5.

## 5.3 Existence and uniqueness

*Result 3 (Existence).* Under the assumptions on preferences above, if the aggregate endowment of every good is strictly positive, there is a price vector $p^* \gg 0$ such that $z(p^*) = 0$.

Slides 50–51 list the conditions that the proof uses; see Arrow and Debreu (1954) and Mas-Colell, Whinston, and Green (1995, Propositions 17.B.2 and 17.C.1). In a two-good economy, the idea is simple. Normalize $p_2 = 1$. When $p_1$ is close to zero, $z_1 > 0$; when $p_1$ is very large, good 2 is almost free relative to good 1, so $z_2 > 0$ and, by Walras' law, $z_1 < 0$. Since $z_1$ is continuous in $p_1$, it must cross zero in between. Existence does not guarantee uniqueness, however, since $z_1$ may cross zero several times.

*Result 4 (Uniqueness under gross substitutes).* If all goods are gross substitutes, that is, $z_k(p)$ is increasing in $p_l$ for every $l \neq k$, then the equilibrium price vector is unique, up to the normalization of one price (slides 52–53).

With two goods, homogeneity gives $z_1(p_1, p_2) = z_1(\frac{p_1}{p_2}, 1)$, so if $z_1$ increases in $p_2$, it decreases in the relative price $\frac{p_1}{p_2}$, and it can cross zero only once. In the Cobb–Douglas economy, $\frac{\partial z_1}{\partial p_2} = \frac{5}{p_1} > 0$ and $\frac{\partial z_2}{\partial p_1} = \frac{4}{p_2} > 0$ from (4) and (5), so $p_1^* = \frac{5}{4}$ (with $p_2 = 1$) is the only equilibrium, as the left panel of Figure 6 shows. The next example shows what can happen without gross substitutes.

**Example 9 (Three equilibria).** Consider the economy in the last row of Table 1, where $u^A = x_1^A - \frac{1}{2}(x_2^A)^{-2}$ and $u^B = -\frac{1}{2}(x_1^B)^{-2} + x_2^B$, with $e^A = (2, \frac{2}{7})$ and $e^B = (\frac{2}{7}, 2)$. Normalize $p_2 = 1$.

- *First step: consumer A's UMP.* Her marginal utilities are 1 and $(x_2^A)^{-3}$, so $MRS_{1,2}^A = (x_2^A)^3$.

Setting it equal to $p_1$ yields $x_2^A = p_1^{1/3}$, and her budget, $p_1 x_1^A + x_2^A = 2p_1 + \frac{2}{7}$, yields

$$x_1^A = 2 + \frac{2}{7p_1} - p_1^{-2/3}.$$

- *Second step: consumer B's UMP.* His MRS is $(x_1^B)^{-3}$, so $x_1^B = p_1^{-1/3}$, and his budget, $p_1 x_1^B + x_2^B = \frac{2}{7}p_1 + 2$, yields $x_2^B = 2 + \frac{2}{7}p_1 - p_1^{2/3}$.
- *Third step: market clearing for good 1.* Subtracting the aggregate endowment, $2 + \frac{2}{7}$, the excess demand for good 1 is

$$z_1(p_1) = \frac{2}{7p_1} - p_1^{-2/3} + p_1^{-1/3} - \frac{2}{7}. \tag{6}$$

  Let $q \equiv p_1^{-1/3}$, so that $\frac{1}{p_1} = q^3$ and $p_1^{-2/3} = q^2$. Multiplying $z_1 = 0$ by 7,

$$2q^3 - 7q^2 + 7q - 2 = (q-1)(2q-1)(q-2) = 0,$$

  so $q \in \{\frac{1}{2}, 1, 2\}$, and since $p_1 = q^{-3}$, the equilibrium prices are $p_1^* \in \{8, 1, \frac{1}{8}\}$.

The demands are interior at all three prices. In fact, $x_1^A(p_1)$ and $x_2^B(p_1)$ are positive at every $p_1 > 0$, with minimum $\frac{5}{27}$ (reached at $p_1 = \frac{27}{343}$ and $p_1 = \frac{343}{27}$, respectively), so (6) is valid at every price, including the whole range plotted in Figure 6. For instance, at $p_1^* = 8$, $x^A = (2 + \frac{2}{56} - \frac{1}{4}, 2) = (\frac{25}{14}, 2)$ and $x^B = (\frac{1}{2}, 2 + \frac{16}{7} - 4) = (\frac{1}{2}, \frac{2}{7})$; at $p_1^* = 1$, $x^A = (\frac{9}{7}, 1)$ and $x^B = (1, \frac{9}{7})$; and at $p_1^* = \frac{1}{8}$, $x^A = (\frac{2}{7}, \frac{1}{2})$ and $x^B = (2, \frac{25}{14})$. The economy is symmetric, which is why $p_1^* = 8$ and $p_1^* = \frac{1}{8}$ come in pairs, with the roles of the consumers reversed. Consumer $A$, who owns most of good 1, prefers the equilibrium with $p_1^* = 8$, since her endowment is then worth the most: her utility is $\frac{25}{14} - \frac{1}{8} \approx 1.66$, compared with $\frac{9}{7} - \frac{1}{2} \approx 0.79$ at $p_1^* = 1$ and $\frac{2}{7} - 2 \approx -1.71$ at $p_1^* = \frac{1}{8}$.

Good 1 is not a gross substitute here. Between $p_1 = 1$ and $p_1 = 8$, we have $q \in (\frac{1}{2}, 1)$, so $(q-1) < 0$, $(2q-1) > 0$, and $(q-2) < 0$, which makes $z_1 > 0$; since $z_1 = 0$ at $p_1 = 1$, excess demand for good 1 increases with its own price right above $p_1 = 1$. By homogeneity, $z_1(1, p_2) = z_1(\frac{1}{p_2}, 1)$ then decreases in $p_2$ near $p_2 = 1$, which violates gross substitutes. The reason is a strong income effect: consumer $A$ sells good 1, and a higher $p_1$ raises her income so much that her own demand for good 1 increases. The right panel of Figure 6 depicts (6).

# 6 Walrasian equilibria, the core, and the first welfare theorem

*Result 5 (Every WEA is in the core).* Under the assumptions on preferences above, $W(e) \subseteq C(e)$.

Slides 70–75 prove this result. Since a WEA exists (Result 3), the core is nonempty, and since every core allocation is Pareto efficient, we obtain the first welfare theorem (slides 76–79).

*Result 6 (First welfare theorem).* Under the assumptions on preferences above, every WEA is Pareto efficient.

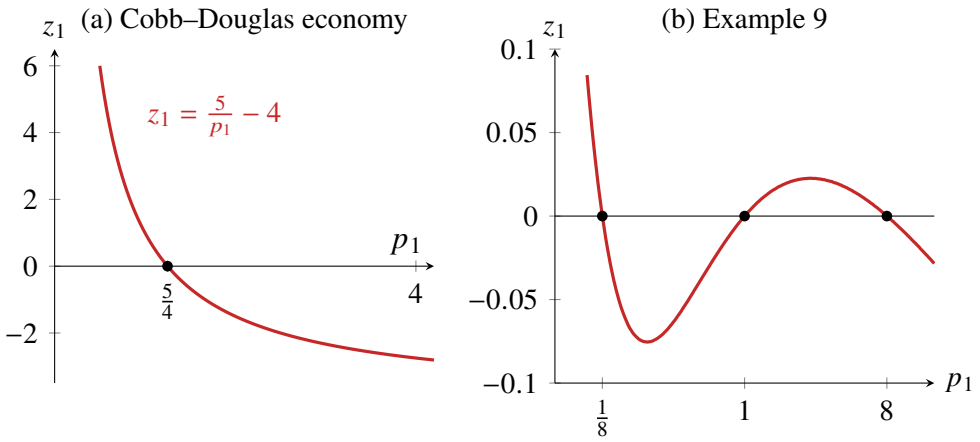



Figure 6. Excess demand for good 1, with $p_2 = 1$. (a) In the Cobb–Douglas economy, $z_1$ decreases in $p_1$ and crosses zero once, at $p_1^* = \frac{5}{4}$. (b) In Example 9, $z_1$ crosses zero three times (the horizontal axis uses a logarithmic scale).

Graphically, at a WEA, each consumer's indifference curve is tangent to the same budget line, so both MRSs equal $\frac{p_1^*}{p_2^*}$, and the indifference curves are tangent to each other, which is the efficiency condition (1).

**Example 10 (Checking Results 5 and 6 in our economies).** We found each WEA from the UMPs and market clearing alone. We can now check the welfare properties that the two results predict.

- *Cobb–Douglas economy.* At the WEA $x^A = (6.6, 2.75)$ and $x^B = (1.4, 5.25)$,

$$MRS_{1,2}^A = \frac{3(2.75)}{6.6} = \frac{8.25}{6.6} = \frac{5}{4} \qquad \text{and} \qquad MRS_{1,2}^B = \frac{5.25}{3(1.4)} = \frac{5.25}{4.2} = \frac{5}{4},$$

  so the WEA is Pareto efficient. It also lies on the contract curve (2), since $\frac{6.6}{9-6.6} = \frac{6.6}{2.4} = 2.75$, and inside the core of Example 4, since $5.882 \le 6.6 \le 7.327$. From (3), the common MRS on the contract curve at $x_1^A = 6.6$ is $\frac{3}{2.4} = \frac{5}{4}$, the equilibrium price.

- *Quasilinear economy.* At the WEA, $x_1^A = 3$, on the contract curve, and $x_2^A = \frac{16}{3} \approx 5.333$ lies between the core bounds 4.901 and 5.425 of Example 5. Both MRSs equal $\frac{1}{3} = p_1^*$.

- *Three equilibria.* In Example 9, all three WEAs are Pareto efficient: at $p_1^* = 8$, for instance, $MRS_{1,2}^A = (x_2^A)^3 = 8$ and $MRS_{1,2}^B = (x_1^B)^{-3} = (\frac{1}{2})^{-3} = 8$. The first welfare theorem holds for every WEA, but it does not tell us which one the economy reaches.

The core is in general much larger than the set of WEAs. In the Cobb–Douglas economy, the core contains all the contract-curve allocations with $x_1^A$ between 5.882 and 7.327, but only one of them is a WEA. The Appendix shows that this gap disappears as the economy grows.

# 7 The second welfare theorem

*Result 7 (Second welfare theorem).* Under the assumptions on preferences above, let $\bar{x}$ be an interior Pareto-efficient allocation, and let $\bar{p}$ be the price vector with $\frac{\bar{p}_1}{\bar{p}_2} = MRS^i_{1,2}(\bar{x}^i)$ for every consumer $i$. Then the planner can implement $\bar{x}$ as a competitive equilibrium at prices $\bar{p}$ with lump-sum transfers $T^i = \bar{p} \cdot \bar{x}^i - \bar{p} \cdot e^i$, which add up to zero, or by redistributing endowments to any $\omega$ with $\sum_i \omega^i = \sum_i e^i$ and $\bar{p} \cdot \omega^i = \bar{p} \cdot \bar{x}^i$ for every $i$, for example, $\omega = \bar{x}$ (slides 80–82).

To apply the theorem, we follow three steps: find the supporting price, find the transfers (of income or of goods), and check that each consumer demands her target bundle.

**Example 11 (Lump-sum transfers in the Cobb–Douglas economy).** The planner wants to implement the Pareto-efficient allocation $\bar{x}^A = (4.5, 1)$ and $\bar{x}^B = (3.5, 7)$ from Table 2, which favors consumer $B$ relative to the WEA of Example 7.

- *First step: supporting price.* From Table 2, both MRSs at $\bar{x}$ equal $\frac{2}{3}$, so we normalize $\bar{p}_2 = 1$ and set $\bar{p}_1 = \frac{2}{3}$.
- *Second step: transfers.* At $\bar{p}$, the endowments are worth

  $$\bar{p} \cdot e^A = \frac{2}{3}(4) + 6 = \frac{26}{3} \qquad \text{and} \qquad \bar{p} \cdot e^B = \frac{2}{3}(4) + 2 = \frac{14}{3},$$

  while the target bundles cost

  $$\bar{p} \cdot \bar{x}^A = \frac{2}{3}(4.5) + 1 = 4 \qquad \text{and} \qquad \bar{p} \cdot \bar{x}^B = \frac{2}{3}(3.5) + 7 = \frac{28}{3}.$$

  Hence, $T^A = 4 - \frac{26}{3} = -\frac{14}{3}$ and $T^B = \frac{28}{3} - \frac{14}{3} = \frac{14}{3}$: consumer $A$ pays $\frac{14}{3} \approx 4.67$ units of good 2 (in value) to consumer $B$, and the transfers add up to zero.
- *Third step: check the demands.* Using the Walrasian demands of Example 7, consumer $A$ spends $\frac{3}{4}$ of her income, 4, on good 1, and consumer $B$ spends $\frac{1}{4}$ of his income, $\frac{28}{3}$, on good 1, so

  $$x_1^A = \frac{3}{4} \cdot 4 \cdot \frac{3}{2} = \frac{9}{2}, \quad x_2^A = \frac{1}{4}(4) = 1, \quad x_1^B = \frac{1}{4} \cdot \frac{28}{3} \cdot \frac{3}{2} = \frac{7}{2}, \quad x_2^B = \frac{3}{4} \cdot \frac{28}{3} = 7.$$

  Both markets clear, since $4.5 + 3.5 = 8$ and $1 + 7 = 8$.

Consumer $A$ ends up worse off than at her endowment, since $u^A(4.5, 1) = 4.5^{3/4} \approx 3.09 < 4.43$. The second welfare theorem can thus implement allocations outside the core: with transfers, voluntary trade is no longer the only way to change the allocation.

**Example 12 (Redistributing goods in the Cobb–Douglas economy).** Consider now that the planner cannot transfer income, but can move units of the goods themselves before markets open. Consumer $A$ gives $t_1$ units of good 1 and $t_2$ units of good 2 to consumer $B$ (a negative $t_k$ means that $B$ gives units of good $k$ to $A$), so the new endowments are

$$\omega^A = (4 - t_1,\ 6 - t_2) \qquad \text{and} \qquad \omega^B = (4 + t_1,\ 2 + t_2).$$

We look for the transfers $(t_1, t_2)$ whose WEA is the target $\bar{x}$ of Example 11.

- *First step: the equilibrium price for any endowment.* Repeating the third step of Example 7 with $A$'s endowment $\omega^A = (\omega_1, \omega_2)$ and $B$'s endowment $(8-\omega_1, 8-\omega_2)$, market clearing for good 1 (with $p_2 = 1$) requires

$$\frac{3}{4} \cdot \frac{p_1\omega_1 + \omega_2}{p_1} + \frac{1}{4} \cdot \frac{p_1(8-\omega_1) + 8 - \omega_2}{p_1} = 8.$$

  Multiplying by $p_1$ and collecting terms, $p_1\left(\frac{\omega_1}{2} + 2\right) + \frac{\omega_2}{2} + 2 = 8p_1$, so

$$p_1^* = \frac{4 + \omega_2}{12 - \omega_1}. \tag{7}$$

  With the original endowment, $\omega = (4, 6)$, this gives $\frac{10}{8} = \frac{5}{4}$, as in Example 7.

- *Second step: impose the target price.* We need $p_1^* = \frac{2}{3}$, that is, $3(4 + \omega_2) = 2(12 - \omega_1)$, or $2\omega_1 + 3\omega_2 = 12$. Substituting $\omega_1 = 4 - t_1$ and $\omega_2 = 6 - t_2$,

$$2(4 - t_1) + 3(6 - t_2) = 12, \qquad \text{that is,} \qquad 2t_1 + 3t_2 = 14. \tag{8}$$

- *Third step: check the demands.* When (8) holds, consumer $A$'s income is $\frac{2}{3}\omega_1 + \omega_2 = \frac{1}{3}(2\omega_1 + 3\omega_2) = 4$, the cost of $\bar{x}^A$, so by Example 11 she demands $(4.5, 1)$; consumer $B$'s income is the value of the aggregate endowment minus $A$'s, $\frac{2}{3}(8) + 8 - 4 = \frac{28}{3}$, so he demands $(3.5, 7)$.

Condition (8) says that the endowment must be moved to the line through $\bar{x}$ with slope $-\frac{2}{3}$, which in $A$'s coordinates is $x_2^A = 4 - \frac{2}{3}x_1^A$. Since nobody can give more than she owns, $\omega^A \geq 0$ and $\omega^B \geq 0$, which requires $-2 \leq t_1 \leq 4$ along (8). Table 3 reports three transfers that work.

| $t_1$ | $t_2$ | $\omega^A$ | $\omega^B$ | Trade at $\bar{p}_1 = \frac{2}{3}$ |
|---|---|---|---|---|
| 1 | 4 | $(3, 2)$ | $(5, 6)$ | $A$ buys 1.5 units of good 1 with 1 unit of good 2 |
| 4 | 2 | $(0, 4)$ | $(8, 4)$ | $A$ buys 4.5 units of good 1 with 3 units of good 2 |
| $-\frac{1}{2}$ | 5 | $(4.5, 1)$ | $(3.5, 7)$ | no trade: $\omega = \bar{x}$ |

Table 3. Transfers of goods that implement $\bar{x}^A = (4.5, 1)$, $\bar{x}^B = (3.5, 7)$.

In every row, $A$ moves along the line $x_2^A = 4 - \frac{2}{3}x_1^A$ to $\bar{x}$. In the first row, for instance, she buys $4.5 - 3 = 1.5$ units of good 1, worth $\frac{2}{3}(1.5) = 1$, by selling $2 - 1 = 1$ unit of good 2. The third row is the transfer $\omega = \bar{x}$ in Result 7: the planner hands each consumer her target bundle, and markets then confirm it, since nobody wants to trade at $\bar{p}$. Figure 7 depicts the target, the supporting budget line, and the endowments in Table 3.

# 8 Economies with production

## 8.1 Firms, profits, and equilibrium

Slides 108–122 add $J$ firms, where firm $j$ chooses a production plan $y^j$ from its production set $Y^j$ to maximize profit $p \cdot y^j$, and consumer $i$ owns a share $\theta^{ij}$ of firm $j$, so her income becomes

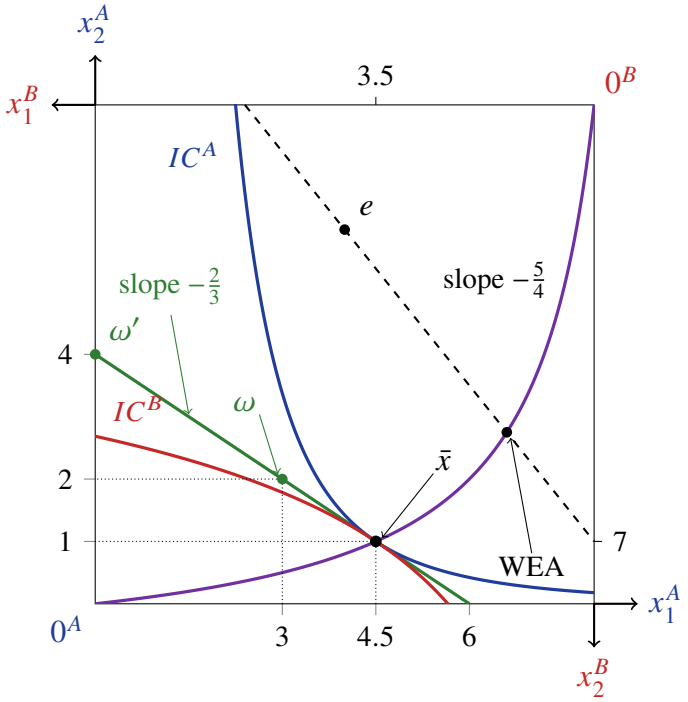



Figure 7. Second welfare theorem. The target $\bar{x}^A = (4.5, 1)$ is supported by the green budget line with slope $-\frac{2}{3}$. Lump-sum transfers move consumer $A$ from the dashed budget line through $e$ (slope $-\frac{5}{4}$, which leads to the WEA of Example 7) to the green line. Redistributing goods moves the endowment to any point on the green line, such as $\omega = (3, 2)$ or $\omega' = (0, 4)$.

$m^i(p) = p \cdot e^i + \sum_j \theta^{ij} \pi^j(p)$. A WEA requires that consumers solve their UMPs, firms solve their PMPs, and every market clears. Walras' law still holds, so with $n$ markets we only need to impose $n - 1$ market-clearing conditions. In Examples 14–20, technologies have constant returns, so at equilibrium prices each firm earns zero profit and is indifferent about its scale; market clearing then determines its output.

**Example 13 (One firm owned by two consumers).** There are two goods: a consumption good, with price normalized to 1, and labor, with wage $w$. Consumers $A$ and $B$ each own 6 hours of time and no consumption good, and both have utility $u^i = \ln c^i + \ln \ell^i$, where $c^i$ is consumption and $\ell^i$ is leisure. A single firm produces the consumption good with $q = 4L^{1/2}$, and consumer $A$ owns $\theta^A = \frac{1}{4}$ of the firm while $B$ owns $\theta^B = \frac{3}{4}$.

- *First step: the firm's PMP.* The firm solves $\max_L \; 4L^{1/2} - wL$. The first-order condition $2L^{-1/2} = w$ gives

$$L(w) = \frac{4}{w^2}, \qquad q(w) = 4 \cdot \frac{2}{w} = \frac{8}{w}, \qquad \text{and} \qquad \pi(w) = \frac{8}{w} - w \cdot \frac{4}{w^2} = \frac{4}{w}.$$

  With decreasing returns, the firm earns positive profits, which its owners receive.

- *Second step: the UMPs.* Consumer $i$'s income is the value of her time plus her share of profits, $m^i = 6w + \theta^i \frac{4}{w}$, and she solves $\max \; \ln c^i + \ln \ell^i$ subject to $c^i + w\ell^i = m^i$. By the Cobb–Douglas

share rule of Example 7, she spends half of her income on each good,

$$c^i = \frac{m^i}{2} \qquad \text{and} \qquad \ell^i = \frac{m^i}{2w}.$$

- *Third step: market clearing for the consumption good.* Aggregate income is $m^A + m^B = 12w + \frac{4}{w}$, since the shares add up to one. Setting aggregate consumption equal to output,

$$\frac{1}{2}\left(12w + \frac{4}{w}\right) = \frac{8}{w}, \qquad \text{that is,} \qquad 6w = \frac{6}{w},$$

  so $w^* = 1$. The firm hires $L = 4$ hours, produces $q = 8$, and earns $\pi = 4$.

- *Fourth step: the allocation.* Consumer $A$'s income is $6 + \frac{1}{4}(4) = 7$, so $c^A = \ell^A = 3.5$ and she works $6 - 3.5 = 2.5$ hours. Consumer $B$'s income is $6 + \frac{3}{4}(4) = 9$, so $c^B = \ell^B = 4.5$ and he works 1.5 hours.

The labor market clears, $2.5 + 1.5 = 4 = L(1)$, as Walras' law predicts, and consumption adds up to output, $3.5 + 4.5 = 8$. Consumer $B$ works less than $A$ although they own the same time, because his larger share of profits allows him to buy more leisure.

Since both consumers have the same homothetic preferences, aggregate demand depends only on aggregate income, so the aggregate outcome coincides with that of a single consumer who owns all 12 hours and the whole firm (a "Robinson Crusoe" economy). Figure 8 depicts it. The production frontier is $c = 4(12 - \ell)^{1/2}$, where $\ell$ is aggregate leisure, and the budget line is $c = \pi + w(12 - \ell) = 16 - \ell$. At $\ell = 8$, the frontier has slope $-\frac{2}{(12-8)^{1/2}} = -1$, the budget line has slope $-w = -1$, and the indifference curve $c\,\ell = 64$ has slope $-\frac{c}{\ell} = -1$: the three are tangent at $(8, 8)$. The budget line meets the vertical line $\ell = 12$ at $c = \pi = 4$: if consumers took all their time as leisure, they could still consume the firm's profits.

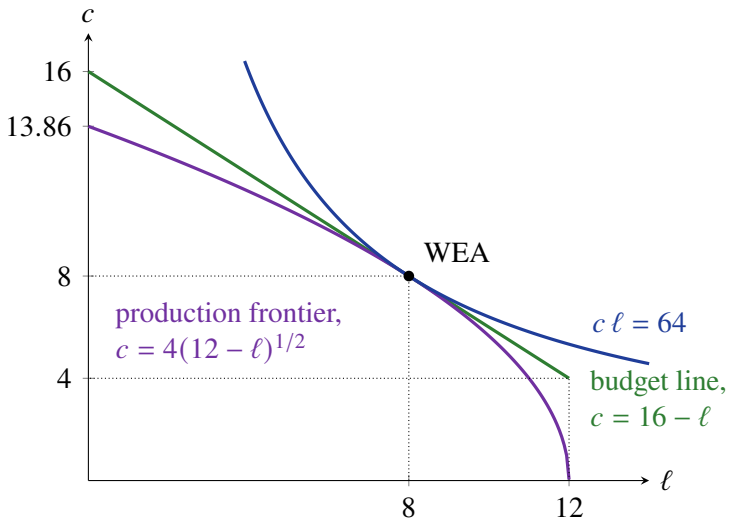



Figure 8. Aggregate outcome of Example 13. The production frontier (purple), the budget line (green, intercept $\pi = 4$ at $\ell = 12$), and the indifference curve (blue) are tangent at $(\ell, c) = (8, 8)$.

## 8.2 Efficiency conditions with production

Consider now the economy in the slides: two consumers, two goods, and two firms, where firm $k$ produces good $k$ with capital $K_k$ and labor $L_k$, using the technology $q_k = f_k(K_k, L_k)$, and consumers own the aggregate endowments $\bar{K}$ and $\bar{L}$. Slides 155–164 derive the following conditions.

*Result 8 (Pareto efficiency with production).* An interior allocation of this economy is Pareto efficient if and only if it is feasible and satisfies efficiency in consumption, $MRS^A_{1,2} = MRS^B_{1,2}$; efficiency in production, $MRTS^1_{L,K} = MRTS^2_{L,K}$; and efficiency in the product mix, $MRS^i_{1,2} = MRT_{1,2}$. Here, $MRTS^k_{L,K} = \frac{MP^k_L}{MP^k_K}$ and $MRT_{1,2} = \frac{MP^2_L}{MP^1_L}$, where $MP^k_L$ and $MP^k_K$ denote firm $k$'s marginal products of labor and capital.

As in Result 1, the three conditions are also sufficient when utilities are quasiconcave and production functions are concave, as in our examples, since the set of feasible outputs is then convex. Graphically, $MRT_{1,2}$ is the absolute value of the slope of the production possibility frontier (PPF). At a competitive equilibrium, the three conditions hold because every agent faces the same prices, so that $MRS^i_{1,2} = MRT_{1,2} = \frac{p_1}{p_2}$ and $MRTS^k_{L,K} = \frac{w}{r}$ for both firms (slides 123–131).

**Example 14 (WEA with two firms and two inputs).** Firm 1 produces good 1 with $q_1 = K_1^{2/3} L_1^{1/3}$ and firm 2 produces good 2 with $q_2 = K_2^{1/3} L_2^{2/3}$. Both technologies exhibit constant returns to scale, and firm 1 uses capital more intensively. Both consumers have utility $u^i = (x^i_1)^{1/2}(x^i_2)^{1/2}$ and own no goods; consumer $A$ owns $(K^A, L^A) = (5, 1)$ and consumer $B$ owns $(K^B, L^B) = (1, 2)$, so $\bar{K} = 6$ and $\bar{L} = 3$. Normalize $p_2 = 1$.

- *First step: the firms' PMPs.* Firm 1 solves $\max\ p_1 K_1^{2/3} L_1^{1/3} - rK_1 - wL_1$, with first-order conditions

$$r = \frac{2}{3} p_1 \left(\frac{L_1}{K_1}\right)^{1/3} = \frac{2}{3} \cdot \frac{p_1 q_1}{K_1} \qquad \text{and} \qquad w = \frac{1}{3} p_1 \left(\frac{K_1}{L_1}\right)^{2/3} = \frac{1}{3} \cdot \frac{p_1 q_1}{L_1}.$$

  Hence, firm 1 pays $\frac{2}{3}$ of its revenue to capital and $\frac{1}{3}$ to labor, and dividing the two conditions, $\frac{w}{r} = \frac{K_1}{2L_1}$. Similarly, firm 2 pays $rK_2 = \frac{1}{3} p_2 q_2$ and $wL_2 = \frac{2}{3} p_2 q_2$, so $\frac{w}{r} = \frac{2K_2}{L_2}$. With constant returns, both firms earn zero profits.

- *Second step: the UMPs.* Consumer $i$'s income is $m^i = rK^i + wL^i$, and she spends half of it on each good, $x^i_1 = \frac{m^i}{2p_1}$ and $x^i_2 = \frac{m^i}{2p_2}$. Aggregate income is $Y = 6r + 3w$.

- *Third step: market clearing.* With four markets (two goods and two inputs), we impose three. Good 1 clears when $p_1 q_1 = p_1(x^A_1 + x^B_1) = \frac{Y}{2}$. Since profits are zero, the value of output equals aggregate income, $p_1 q_1 + p_2 q_2 = Y$, so $p_2 q_2 = \frac{Y}{2}$ as well. The labor market clears when $w(L_1 + L_2) = 3w$, that is, using the labor payments of the first step,

$$\frac{1}{3} \cdot \frac{Y}{2} + \frac{2}{3} \cdot \frac{Y}{2} = 3w, \qquad \text{so} \qquad Y = 6w.$$

  Since $Y = 6r + 3w$, we obtain $6r = 3w$, that is, $\frac{w}{r} = 2$. The firms' conditions then give $K_1 = 2 \cdot \frac{w}{r} L_1 = 4L_1$ and $K_2 = \frac{1}{2} \cdot \frac{w}{r} L_2 = L_2$. Combining these with the capital market,

$4L_1 + L_2 = 6$, and the labor market, $L_1 + L_2 = 3$, we obtain $3L_1 = 3$, so

$$(K_1, L_1) = (4, 1) \qquad \text{and} \qquad (K_2, L_2) = (2, 2).$$

- *Fourth step: outputs and prices.* Outputs are $q_1 = 4^{2/3} \cdot 1^{1/3} = 2^{4/3} \approx 2.520$ and $q_2 = 2^{1/3}2^{2/3} = 2$. Firm 2's conditions give $r = \frac{1}{3}\left(\frac{2}{2}\right)^{2/3} = \frac{1}{3}$ and $w = \frac{2}{3}\left(\frac{2}{2}\right)^{1/3} = \frac{2}{3}$, so $Y = 6 \cdot \frac{1}{3} + 3 \cdot \frac{2}{3} = 4$, and $p_1 q_1 = \frac{Y}{2} = 2$ gives
$$p_1^* = \frac{2}{2^{4/3}} = 2^{-1/3} \approx 0.794.$$

- *Fifth step: consumption.* Consumer $A$'s income is $5 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{7}{3}$ and consumer $B$'s is $1 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = \frac{5}{3}$. Hence, $x^A = \left(\frac{7}{6p_1^*}, \frac{7}{6}\right) = \frac{7}{6}\left(2^{1/3}, 1\right) \approx (1.470, 1.167)$ and $x^B = \frac{5}{6}\left(2^{1/3}, 1\right) \approx (1.050, 0.833)$.

As a check, good 1 clears, $\frac{7}{6}2^{1/3} + \frac{5}{6}2^{1/3} = 2 \cdot 2^{1/3} = 2^{4/3} = q_1$, and so does good 2, $\frac{7}{6} + \frac{5}{6} = 2 = q_2$. Firm 1's capital condition holds as well: $\frac{2}{3}p_1^*\left(\frac{1}{4}\right)^{1/3} = \frac{2}{3} \cdot 2^{-1/3} \cdot 2^{-2/3} = \frac{1}{3} = r$.

To draw the PPF, we maximize $q_2$ for each level of $q_1$. Efficiency in production requires $MRTS^1_{L,K} = MRTS^2_{L,K}$, that is, $\frac{K_1}{2L_1} = \frac{2K_2}{L_2}$, or $K_1(3 - L_1) = 4(6 - K_1)L_1$. Solving for $K_1$,

$$K_1 = \frac{24L_1}{3 + 3L_1} = \frac{8L_1}{1 + L_1}, \tag{9}$$

and each $L_1 \in [0, 3]$ yields a point $\left(K_1^{2/3}L_1^{1/3},\ (6 - K_1)^{1/3}(3 - L_1)^{2/3}\right)$ on the PPF. At $L_1 = 1$, (9) gives $K_1 = 4$, the equilibrium point. The PPF meets the axes at $q_2 = (6 \cdot 9)^{1/3} = 54^{1/3} \approx 3.780$ (all inputs in firm 2) and $q_1 = (36 \cdot 3)^{1/3} = 108^{1/3} \approx 4.762$ (all inputs in firm 1). Figure 9 depicts it.

## 8.3 Welfare theorems with production

*Result 9 (First welfare theorem with production).* Under the assumptions on preferences above, every WEA of an economy with production is Pareto efficient (see slides 165–172 for the proof).

**Example 15 (Checking efficiency in Example 14).** We check the three conditions of Result 8 at the WEA.

- *Consumption.* With $u^i = (x_1^i)^{1/2}(x_2^i)^{1/2}$, $MRS^i_{1,2} = \frac{x_2^i}{x_1^i}$. Both bundles are proportional to $(2^{1/3}, 1)$, so both ratios equal $\frac{1}{2^{1/3}} = 2^{-1/3} = \frac{p_1^*}{p_2^*}$.

- *Production.* For firm 1,
$$MRTS^1_{L,K} = \frac{MP^1_L}{MP^1_K} = \frac{\frac{1}{3}K_1^{2/3}L_1^{-2/3}}{\frac{2}{3}K_1^{-1/3}L_1^{1/3}} = \frac{K_1}{2L_1} = \frac{4}{2} = 2,$$
and, for firm 2, $MRTS^2_{L,K} = \frac{2K_2}{L_2} = \frac{4}{2} = 2$. Both equal $\frac{w}{r} = 2$.

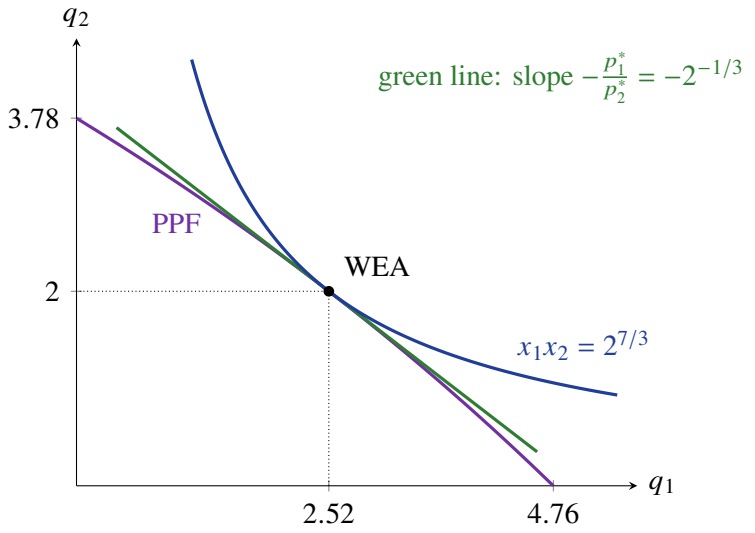



Figure 9. The PPF of Example 14 (purple) and the WEA $(q_1, q_2) = (2^{4/3}, 2)$. The price line (green) and the indifference curve $x_1x_2 = q_1q_2 = 2^{7/3}$ of the consumers' common preferences (blue) are tangent to the PPF at the WEA.

- *Product mix.* The marginal products of labor are $MP_L^1 = \frac{1}{3}\left(\frac{K_1}{L_1}\right)^{2/3} = \frac{4^{2/3}}{3}$ and $MP_L^2 = \frac{2}{3}\left(\frac{K_2}{L_2}\right)^{1/3} = \frac{2}{3}$, so

$$MRT_{1,2} = \frac{MP_L^2}{MP_L^1} = \frac{2}{3} \cdot \frac{3}{4^{2/3}} = \frac{2}{2^{4/3}} = 2^{-1/3},$$

  which equals the MRS of both consumers.

In Figure 9, the slope of the PPF at the WEA, $-2^{-1/3} \approx -0.794$, coincides with the slope of the price line and with the slope of the indifference curve, $-\frac{x_2}{x_1} = -\frac{2}{2^{4/3}} = -2^{-1/3}$.

*Result 10 (Second welfare theorem with production).* Under the assumptions on consumers and firms above, every Pareto-efficient allocation $(\bar{x}, \bar{y})$ is a WEA at some prices $\bar{p}$ after lump-sum transfers that add up to zero (slides 178–180). As in the exchange economy, the planner can alternatively redistribute the consumers' endowments, here of capital and labor.

**Example 16 (Second welfare theorem in Example 14).** Since both consumers have the same Cobb–Douglas preferences, efficiency in consumption, $\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B}$, requires both consumers to hold goods in the same proportion as aggregate output. Then $MRS_{1,2} = \frac{q_2}{q_1}$ for both, and efficiency in the product mix, $MRT_{1,2} = \frac{q_2}{q_1}$, is the first-order condition for maximizing $q_1q_2$ along the PPF. Since the PPF is concave, there is a single such point, and Example 15 shows that the WEA of Example 14 satisfies it. Hence, every Pareto-efficient allocation gives consumer $A$ a fraction $s \in [0, 1]$ of each output, $\bar{x}^A = s\,(2^{4/3}, 2)$, and consumer $B$ the rest. The WEA corresponds to $s = \frac{7}{12}$, since $A$'s income is $\frac{7}{3}$ out of $Y = 4$. Consider the target $s = \frac{1}{2}$, where both consumers receive $(2^{1/3}, 1)$.

- *First step: supporting prices.* The production plan does not change, so the prices of Example 14 support it: $(p_1, p_2, r, w) = (2^{-1/3}, 1, \frac{1}{3}, \frac{2}{3})$. Aggregate demand depends only on aggregate

income, $Y = 4$, which does not change with transfers, so markets clear at these prices.

- *Second step: lump-sum transfers.* Each target bundle costs $2^{-1/3} \cdot 2^{1/3} + 1 = 2$, so $T^A = 2 - \frac{7}{3} = -\frac{1}{3}$ and $T^B = 2 - \frac{5}{3} = \frac{1}{3}$.
- *Third step: redistributing inputs.* Alternatively, consumer $A$ gives $t_K$ units of capital and $t_L$ units of labor to $B$. Consumer $A$'s income becomes $\frac{1}{3}(5 - t_K) + \frac{2}{3}(1 - t_L)$, which must equal 2:

$$\frac{7}{3} - \frac{t_K}{3} - \frac{2t_L}{3} = 2, \qquad \text{that is,} \qquad t_K + 2t_L = 1.$$

  For instance, $(t_K, t_L) = (1, 0)$ leaves $A$ with $(4, 1)$ and $B$ with $(2, 2)$; $(t_K, t_L) = (0, \frac{1}{2})$ leaves $A$ with $(5, \frac{1}{2})$ and $B$ with $(1, \frac{5}{2})$; and $(t_K, t_L) = (3, -1)$, where $A$ gives three units of capital and receives one unit of labor, leaves $A$ with $(2, 2)$ and $B$ with $(4, 1)$. In each case, both incomes equal 2 at the supporting prices, so each consumer demands $(\frac{2}{2p_1}, \frac{2}{2}) = (2^{1/3}, 1)$.

# 9 Comparative statics

As in slides 189–192, consider the two-good, two-factor economy with constant returns, and let $c_k(r, w)$ denote firm $k$'s unit cost function, so that, if both goods are produced, zero profits require $c_1(r, w) = p_1$ and $c_2(r, w) = p_2$. Let $a_{Kk}$ and $a_{Lk}$ denote the units of capital and labor that firm $k$ uses per unit of output (its unit factor demands). Good 1 is more capital intensive than good 2 if $\frac{a_{K1}}{a_{L1}} > \frac{a_{K2}}{a_{L2}}$, that is, if $D \equiv a_{K1}a_{L2} - a_{K2}a_{L1} > 0$.

*Result 11 (Stolper–Samuelson).* With constant returns and both goods produced, if good 1 is more capital intensive and its price increases, the return to capital increases and the wage decreases: $\frac{dr}{dp_1} = \frac{a_{L2}}{D} > 0$ and $\frac{dw}{dp_1} = -\frac{a_{K2}}{D} < 0$.

Slides 193–198 derive this result, due to Stolper and Samuelson (1941), by differentiating the zero-profit conditions.

**Example 17 (Stolper–Samuelson in Example 14).** For a Cobb–Douglas technology $K^a L^{1-a}$, the unit cost function is $\frac{r^a w^{1-a}}{a^a (1-a)^{1-a}}$ (from cost minimization, $\frac{K}{L} = \frac{a}{1-a} \cdot \frac{w}{r}$, substituted into $K^a L^{1-a} = 1$). With $a = \frac{2}{3}$ for firm 1 and $a = \frac{1}{3}$ for firm 2, both denominators equal $\kappa \equiv \left(\frac{2}{3}\right)^{2/3} \left(\frac{1}{3}\right)^{1/3} = \frac{2^{2/3}}{3}$, so

$$c_1(r, w) = \frac{r^{2/3} w^{1/3}}{\kappa} \qquad \text{and} \qquad c_2(r, w) = \frac{r^{1/3} w^{2/3}}{\kappa}.$$

- *First step: factor prices as functions of $p_1$.* Dividing $c_1 = p_1$ by $c_2 = 1$ gives $\left(\frac{r}{w}\right)^{1/3} = p_1$, so $r = p_1^3 w$. Substituting into $c_2 = 1$, $(p_1^3 w)^{1/3} w^{2/3} = \kappa$, that is, $p_1 w = \kappa$. Hence,

$$w(p_1) = \frac{\kappa}{p_1} \qquad \text{and} \qquad r(p_1) = \kappa\, p_1^2.$$

  At $p_1 = 2^{-1/3}$, these give $w = \frac{2^{2/3}}{3} \cdot 2^{1/3} = \frac{2}{3}$ and $r = \frac{2^{2/3}}{3} \cdot 2^{-2/3} = \frac{1}{3}$, as in Example 14.

- *Second step: the formulas of Result 11.* The unit factor demands at the equilibrium of Example 14 are the inputs per unit of output: $a_{K1} = \frac{4}{2^{4/3}} = 2^{2/3}$, $a_{L1} = \frac{1}{2^{4/3}} = 2^{-4/3}$, $a_{K2} = \frac{2}{2} = 1$, and $a_{L2} = \frac{2}{2} = 1$. Hence,

$$D = 2^{2/3} \cdot 1 - 1 \cdot 2^{-4/3} = 2^{-4/3}(4-1) = 3 \cdot 2^{-4/3} \approx 1.19 > 0,$$

  confirming that good 1 is capital intensive, and

$$\frac{dr}{dp_1} = \frac{a_{L2}}{D} = \frac{2^{4/3}}{3} \approx 0.840 \qquad \text{and} \qquad \frac{dw}{dp_1} = -\frac{a_{K2}}{D} = -\frac{2^{4/3}}{3} \approx -0.840.$$

  The first step confirms them: $\frac{dr}{dp_1} = 2\kappa p_1 = 2 \cdot \frac{2^{2/3}}{3} \cdot 2^{-1/3} = \frac{2^{4/3}}{3}$ and $\frac{dw}{dp_1} = -\frac{\kappa}{p_1^2} = -\frac{2^{2/3}}{3} \cdot 2^{2/3} = -\frac{2^{4/3}}{3}$.

- *Third step: a discrete change.* If $p_1$ increases from $2^{-1/3} \approx 0.794$ to 1, that is, by $2^{1/3} - 1 \approx 26.0$ percent, then $r$ increases from $\frac{1}{3}$ to $\kappa = \frac{2^{2/3}}{3} \approx 0.529$, by $2^{2/3} - 1 \approx 58.7$ percent, while $w$ decreases from $\frac{2}{3}$ to $\kappa \approx 0.529$, by $1 - 2^{-1/3} \approx 20.6$ percent.

The return to capital increases by more than the price of good 1, a magnification effect: in logs, $\ln r = \ln \kappa + 2 \ln p_1$ and $\ln w = \ln \kappa - \ln p_1$, so the elasticities are 2 and $-1$. Capital owners can then buy more of both goods, and workers less of both.

*Result 12 (Rybczynski).* Holding goods prices (and thus factor prices) fixed, if the endowment of one factor increases while the other stays fixed, the output of the good that uses that factor more intensively increases more than proportionally, and the output of the other good decreases.

Slides 204–214 prove this result, due to Rybczynski (1955), by differentiating the full-employment conditions at constant unit factor demands. Writing $\lambda_{Kk}$ and $\lambda_{Lk}$ for the shares of capital and labor employed by firm $k$, and $\hat{v}$ for the percentage change in $v$, this yields

$$\hat{K} = \lambda_{K1}\hat{q}_1 + \lambda_{K2}\hat{q}_2 \qquad \text{and} \qquad \hat{L} = \lambda_{L1}\hat{q}_1 + \lambda_{L2}\hat{q}_2, \tag{10}$$

so each factor's growth is a weighted average of output growth.

**Example 18 (Rybczynski in Example 14).** Consider that the labor endowment increases from $\bar{L} = 3$ to $\bar{L} = 4.5$, a 50 percent increase, while $\bar{K} = 6$ and the goods prices $(p_1, p_2) = (2^{-1/3}, 1)$ stay fixed.

- *First step: factor prices and input ratios do not change.* By the first step of Example 17, $r = \frac{1}{3}$ and $w = \frac{2}{3}$, so the firms keep their capital-labor ratios, $K_1 = 4L_1$ and $K_2 = L_2$.
- *Second step: full employment.* $4L_1 + L_2 = 6$ and $L_1 + L_2 = 4.5$. Subtracting, $3L_1 = 1.5$, so $L_1 = 0.5$, $K_1 = 2$, $L_2 = 4$, and $K_2 = 4$.
- *Third step: outputs.* $q_1 = 2^{2/3}\left(\frac{1}{2}\right)^{1/3} = \frac{2^{2/3}}{2^{1/3}} = 2^{1/3} \approx 1.260$ and $q_2 = 4^{1/3}4^{2/3} = 4$.

Relative to Example 14, the output of good 2, which is labor intensive, doubles (from 2 to 4, a 100 percent increase), while the output of good 1 falls by half (from $2^{4/3}$ to $2^{1/3}$). Using (10) with the shares of Example 14, $\lambda_{K1} = \frac{4}{6} = \frac{2}{3}$, $\lambda_{K2} = \frac{1}{3}$, $\lambda_{L1} = \frac{1}{3}$, and $\lambda_{L2} = \frac{2}{3}$, the system

$$0 = \frac{2}{3}\hat{q}_1 + \frac{1}{3}\hat{q}_2 \qquad \text{and} \qquad 0.5 = \frac{1}{3}\hat{q}_1 + \frac{2}{3}\hat{q}_2$$

gives $\hat{q}_2 = -2\hat{q}_1$ from the first equation, and then $0.5 = \frac{1}{3}\hat{q}_1 - \frac{4}{3}\hat{q}_1 = -\hat{q}_1$, so $\hat{q}_1 = -0.5$ and $\hat{q}_2 = 1$. Hence,

$$\hat{q}_2 = 100\% > \hat{L} = 50\% > \hat{K} = 0 > \hat{q}_1 = -50\%,$$

the magnification effect for quantities. Since the unit factor demands are constant, (10) holds for discrete changes too. Figure 10 depicts the outward shift of the PPF. With $\bar{L} = 4.5$, efficiency in production requires $K_1(4.5 - L_1) = 4(6 - K_1)L_1$, that is, $K_1 = \frac{8L_1}{1.5+L_1}$, which at $L_1 = 0.5$ gives $K_1 = 2$, and the new PPF is tangent to a price line with the same slope, $-2^{-1/3}$.

If, instead, goods prices adjusted to a new WEA, the steps of Example 14 with $\bar{L} = 4.5$ give $Y = 6r + 4.5w$ and, from the labor market, $\frac{Y}{2} = 4.5w$, so $\frac{w}{r} = \frac{4}{3}$, $(K_1, L_1) = (4, 1.5)$, and $(K_2, L_2) = (2, 3)$. Both outputs then increase, to $q_1 = 24^{1/3} \approx 2.884$ and $q_2 = 18^{1/3} \approx 2.621$, and good 1 becomes relatively more expensive, $\frac{p_1}{p_2} = \frac{q_2}{q_1} = \left(\frac{3}{4}\right)^{1/3} \approx 0.909$.

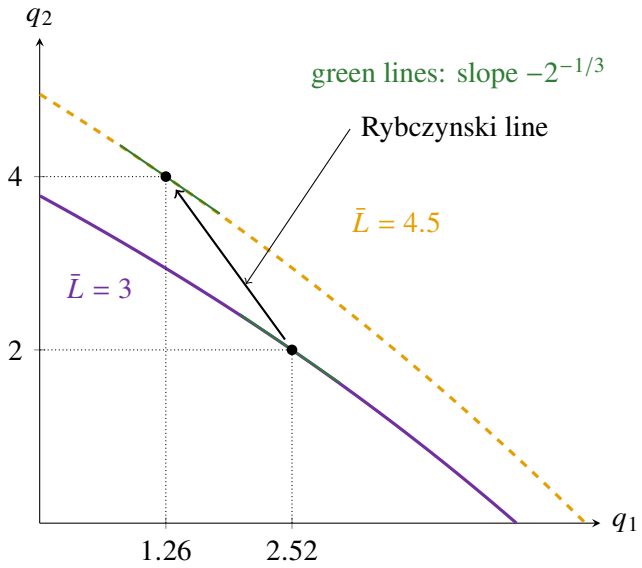



Figure 10. Rybczynski theorem. When $\bar{L}$ increases from 3 to 4.5 at fixed goods prices, production moves along the Rybczynski line (arrow) from $(2^{4/3}, 2)$ on the original PPF (purple) to $(2^{1/3}, 4)$ on the new PPF (dashed orange), where the price line has the same slope.

# 10 Taxes in general equilibrium

## 10.1 A tax on one good

Consider a sales tax $\tau_k$ on good $k$, so that consumers pay $(1 + \tau_k)p_k$ while producers receive $p_k$, and the government returns the revenue to consumers as a lump sum. Slides 224–229 show that a tax on good 1 alone keeps production on the PPF but yields $MRS_{1,2} = \frac{(1+\tau_1)p_1}{p_2} > \frac{p_1}{p_2} = MRT_{1,2}$, so the economy produces too little good 1.

**Example 19 (A 50 percent tax on good 1 in Example 14).** Let $\tau_1 = \frac{1}{2}$, with the revenue returned lump-sum, and normalize $p_2 = 1$.

- *First step: spending.* Consumers' total income $Y$ now includes the tax revenue. They spend half of it on each good, so $(1+\tau_1)p_1q_1 = \frac{Y}{2}$ and $p_2q_2 = \frac{Y}{2}$. Firm 1's revenue is therefore $p_1q_1 = \frac{Y}{2(1+\tau_1)}$.
- *Second step: input allocation.* The firms' first-order conditions of Example 14 still hold at the producer prices. Firm 1 pays $\frac{1}{3}p_1q_1$ to labor and firm 2 pays $\frac{2}{3}p_2q_2$, so the share of labor employed by firm 1 is

$$\frac{L_1}{\bar{L}} = \frac{\frac{1}{3}\cdot\frac{Y}{2(1+\tau_1)}}{\frac{1}{3}\cdot\frac{Y}{2(1+\tau_1)} + \frac{2}{3}\cdot\frac{Y}{2}} = \frac{1}{1+2(1+\tau_1)} = \frac{1}{3+2\tau_1}.$$

  Similarly, firm 1 pays $\frac{2}{3}p_1q_1$ to capital and firm 2 pays $\frac{1}{3}p_2q_2$, so

$$\frac{K_1}{\bar{K}} = \frac{\frac{2}{3}\cdot\frac{1}{1+\tau_1}}{\frac{2}{3}\cdot\frac{1}{1+\tau_1} + \frac{1}{3}} = \frac{2}{3+\tau_1}.$$

  Without the tax, these shares are $\frac{1}{3}$ and $\frac{2}{3}$, as in Example 14. With $\tau_1 = \frac{1}{2}$, they become $\frac{1}{4}$ and $\frac{4}{7}$, so $L_1 = \frac{3}{4}$, $K_1 = \frac{24}{7}$, $L_2 = \frac{9}{4}$, and $K_2 = \frac{18}{7}$.
- *Third step: outputs and prices.* Outputs are

$$q_1 = \left(\frac{24}{7}\right)^{2/3}\left(\frac{3}{4}\right)^{1/3} \approx 2.066 \qquad \text{and} \qquad q_2 = \left(\frac{18}{7}\right)^{1/3}\left(\frac{9}{4}\right)^{2/3} \approx 2.352.$$

  Firm 2's conditions give $w = \frac{2}{3}\left(\frac{8}{7}\right)^{1/3} \approx 0.697$ and $r = \frac{1}{3}\left(\frac{7}{8}\right)^{2/3} \approx 0.305$, and firm 1's capital condition then gives the producer price $p_1 \approx 0.759$, so consumers pay $(1.5)(0.759) \approx 1.139$.

Production stays on the PPF: (9) at $L_1 = \frac{3}{4}$ gives $K_1 = \frac{8\cdot3}{4+3} = \frac{24}{7}$, after multiplying the numerator and denominator by 4. But the product mix is wrong: consumers' MRS is $\frac{q_2}{q_1} \approx \frac{2.352}{2.066} \approx 1.139$, the consumer price, while $MRT_{1,2} = p_1 \approx 0.759$. Relative to Example 14, the economy produces less good 1 (2.066 instead of 2.520) and more good 2 (2.352 instead of 2). Each consumer's utility equals her income share times $(q_1q_2)^{1/2}$, so the sum of utilities equals this index. Its fall, from $(2^{4/3}\cdot 2)^{1/2} \approx 2.245$ to $(2.066\cdot 2.352)^{1/2} \approx 2.204$, means that without the tax, lump-sum transfers could make both consumers better off than with it.

## 10.2 A tax on inputs

Consider now that firm 2 pays $(1+t_K)r$ for capital and $(1+t_L)w$ for labor, while firm 1 pays $r$ and $w$. Slides 230–234 show that, if $t_L > t_K$, then $MRTS^2_{L,K} > MRTS^1_{L,K}$, so moving labor from firm 1 to firm 2 and capital from firm 2 to firm 1 would raise the output of both goods: production is inside the PPF.

**Example 20 (A tax on the labor used by firm 2).** In Example 14, let $t_L = 1$ and $t_K = 0$, so firm 2 pays $2w$ per unit of labor, and return the revenue lump-sum. Goods are untaxed, so consumers spend half of their income on each good, $p_1q_1 = p_2q_2 = \frac{Y}{2}$.

- *First step: capital.* Firm 1 pays $rK_1 = \frac{2}{3} \cdot \frac{Y}{2}$ and firm 2 pays $rK_2 = \frac{1}{3} \cdot \frac{Y}{2}$, so $K_1 = 2K_2$, that is, $K_1 = 4$ and $K_2 = 2$, as without the tax.
- *Second step: labor.* Firm 1 pays $wL_1 = \frac{1}{3} \cdot \frac{Y}{2}$ and firm 2 pays $2wL_2 = \frac{2}{3} \cdot \frac{Y}{2}$, so $wL_2 = \frac{1}{3} \cdot \frac{Y}{2} = wL_1$. Hence $L_1 = L_2 = \frac{3}{2}$.
- *Third step: outputs.* $q_1 = 4^{2/3} \left(\frac{3}{2}\right)^{1/3} = 24^{1/3} \approx 2.884$ and $q_2 = 2^{1/3} \left(\frac{3}{2}\right)^{2/3} = \left(\frac{9}{2}\right)^{1/3} \approx 1.651$.

The MRTS differ across firms: $MRTS^1_{L,K} = \frac{K_1}{2L_1} = \frac{4}{3}$, while $MRTS^2_{L,K} = \frac{2K_2}{L_2} = \frac{8}{3}$. To see that production is inefficient, find the efficient allocation that yields the same $q_1 = 24^{1/3}$: solving $\left(\frac{8L_1}{1+L_1}\right)^{2/3} L_1^{1/3} = 24^{1/3}$ gives $L_1 \approx 1.231$ and $K_1 \approx 4.415$, which leave enough inputs to firm 2 to produce $q_2 \approx 1.705 > 1.651$. The tax thus wastes about 0.054 units of good 2. Each consumer's utility again equals her income share times $(q_1 q_2)^{1/2}$, and this sum of utilities falls from 2.245 to $(2.884 \cdot 1.651)^{1/2} \approx 2.182$, so without the tax, lump-sum transfers could make both consumers better off than with it. Figure 11 compares the two taxes.

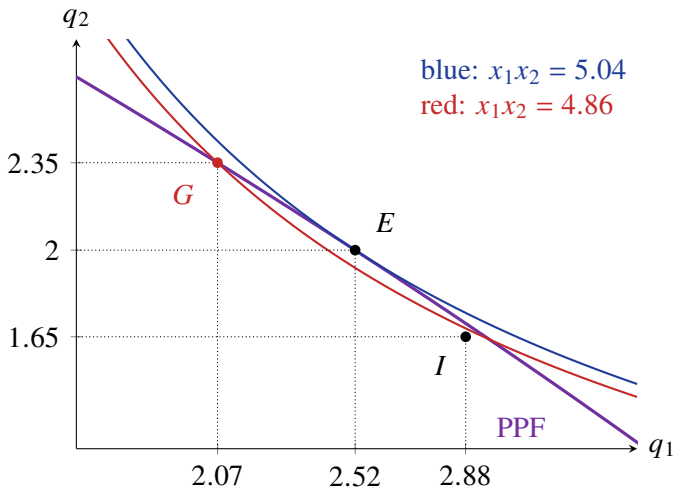



Figure 11. Taxes in Example 14 (detail of the PPF, in purple, around the equilibria). Without taxes, the economy produces at $E$, where the indifference curve (blue) is tangent to the PPF. A 50 percent tax on good 1 moves production along the PPF to $G$, where the indifference curve (red) crosses the PPF. A tax on the labor used by firm 2 moves production inside the PPF, to $I$.

# Appendix: Large economies and the core

We showed that every WEA is in the core, but Example 10 showed that the core can be much larger. This appendix shows that, as the economy is replicated, the core shrinks to the set of WEAs, a result due to Debreu and Scarf (1963). Slides 236–239 define the $r$-fold replica economy, $\mathcal{E}_r$, which has $r$ consumers of each type, all with the same preferences and endowment as their type. We denote by $C_r$ the core of $\mathcal{E}_r$. Slides 240–280 prove the following three results.

*Result 13 (Equal treatment).* Under the assumptions on preferences above, every allocation in $C_r$ gives the same bundle to all consumers of the same type.

By Result 13, a core allocation of $\mathcal{E}_r$ is described by one bundle per type, which we can view as an allocation of the original economy $\mathcal{E}_1$.

*Result 14 (The core shrinks).* $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$.

*Result 15 (Limit theorem on the core).* If the $r$-fold copy of an allocation $x$ belongs to $C_r$ for every $r \geq 1$, then $x$ is a WEA of $\mathcal{E}_1$.

Since the $r$-fold copy of every WEA is a WEA of $\mathcal{E}_r$, and thus belongs to $C_r$, the allocations that remain in the core for every $r$ are the WEAs. To show that an allocation $x$ that is not a WEA eventually leaves the core, the proof of Result 15 (slides 269–280) uses, after relabeling types if necessary, the coalition with all $r$ type-$A$ consumers and $r-1$ type-$B$ consumers: each type-$A$ member receives $\frac{1}{r}e^A + \frac{r-1}{r}x^A$, which she strictly prefers to $x^A$ when $r$ is large enough, and each type-$B$ member keeps $x^B$.

**Example 21 (The core of the replicated quasilinear economy).** By Result 13, a core allocation of $\mathcal{E}_r$ gives every type-$A$ consumer the same bundle $(3, x_2^A)$ on the contract curve and every type-$B$ consumer $(6, 8-x_2^A)$, with utilities $v^A = \ln 3 + x_2^A$ and $v^B = 2\ln 6 + 8 - x_2^A$. Quasilinear utility makes it simple to check which coalitions block. Since good 2 enters each utility linearly, a coalition can move utility among its members one-for-one by moving good 2. Hence, a coalition with $a$ type-$A$ consumers and $b$ type-$B$ consumers blocks if and only if the largest total utility it can achieve with its own endowment, $S(a,b)$, exceeds the total utility of its members at the allocation, $a\,v^A + b\,v^B$.

- *First step: the coalition's largest total utility.* The coalition owns $a+8b$ units of good 1 and $6a+2b$ units of good 2. Maximizing $a\ln y_1^A + 2b\ln y_1^B$ subject to $a\,y_1^A + b\,y_1^B = a+8b$ gives $\frac{1}{y_1^A} = \frac{2}{y_1^B}$, so $y_1^B = 2y_1^A$ and

$$y_1^A = \frac{a+8b}{a+2b}, \qquad \text{so} \qquad S(a,b) = a\ln y_1^A + 2b\ln\left(2y_1^A\right) + 6a + 2b.$$

- *Second step: the strongest coalitions.* Consider the coalition of the proof of Result 15, with $a = r$ type-$A$ consumers and $b = r-1$ type-$B$ consumers. Then $y_1^A = \frac{9r-8}{3r-2}$. Using $\ln y_1^A - \ln 3 = \ln\frac{9r-8}{9r-6}$ and $2\ln(2y_1^A) - 2\ln 6 = 2\ln\frac{9r-8}{9r-6}$, the coalition's gain is

$$S(r,r-1) - r\,v^A - (r-1)v^B = (3r-2)\ln\frac{9r-8}{9r-6} + 6 - x_2^A,$$

  which is positive, so that the coalition blocks, if and only if

$$x_2^A < \underline{x}_r \equiv 6 - (3r-2)\ln\frac{9r-6}{9r-8}. \tag{11}$$

  The coalition can split this gain while giving every member a nonnegative amount of good 2, since, near the core, type-$A$ consumers hold about 5 units of good 2 and type-$B$ consumers about 3. Symmetrically, the coalition with $r-1$ type-$A$ and $r$ type-$B$ consumers blocks if and only if

$$x_2^A > \overline{x}_r \equiv 6 - (3r-1)\ln\frac{9r-1}{9r-3}. \tag{12}$$

A numerical check of every coalition with $a, b \le r$, for $r$ up to 100, shows that these two coalitions give the tightest bounds, so $C_r$ is the set of allocations with $\underline{x}_r \le x_2^A \le \overline{x}_r$.

- *Third step: the limit.* For $r = 1$, (11) and (12) give $6 - \ln 3$ and $6 - 2\ln\frac{4}{3}$, the bounds of Example 5. As $r$ grows, $\ln\frac{9r-6}{9r-8} = \ln\left(1 + \frac{2}{9r-8}\right) \approx \frac{2}{9r-8}$, so

$$(3r-2)\ln\frac{9r-6}{9r-8} \approx \frac{2(3r-2)}{9r-8} \to \frac{2}{3},$$

and similarly $(3r-1)\ln\frac{9r-1}{9r-3} \to \frac{2}{3}$. Both bounds converge to $6 - \frac{2}{3} = \frac{16}{3}$, the WEA of Example 8.

Table 4 and Figure 12 report the bounds. The core of $\mathcal{E}_2$ already excludes the core allocation of Example 5 that leaves consumer $A$ indifferent to her endowment, $x_2^A = 6 - \ln 3 \approx 4.901$, since $4.901 < 5.271 = \underline{x}_2$, and by $r = 50$ the core is more than 170 times narrower than in the original economy.

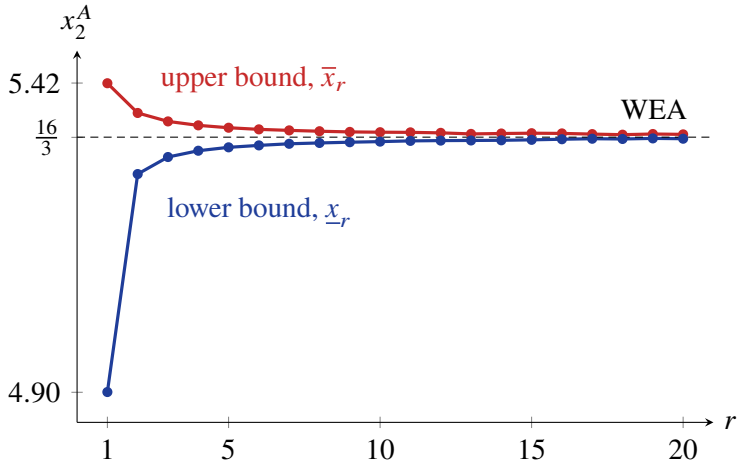



Figure 12. Core bounds of the $r$-fold replica of the quasilinear economy. Both bounds converge to the WEA, $x_2^A = \frac{16}{3}$.

| $r$ | $\underline{x}_r$ | $\overline{x}_r$ | Width of $C_r$ |
|---|---|---|---|
| 1 | 4.901 | 5.425 | 0.523 |
| 2 | 5.271 | 5.374 | 0.103 |
| 3 | 5.299 | 5.360 | 0.060 |
| 5 | 5.316 | 5.349 | 0.033 |
| 10 | 5.325 | 5.341 | 0.016 |
| 50 | 5.332 | 5.335 | 0.003 |
| Limit | $\frac{16}{3} \approx 5.333$ | $\frac{16}{3} \approx 5.333$ | 0 |

Table 4. Core of the $r$-fold replica of the quasilinear economy, in terms of each type-$A$ consumer's amount of good 2 (widths computed from unrounded bounds).

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