Advanced Microeconomic Theory

Chapter 6: Partial Equilibrium

Outline

- Partial equilibrium analysis
- Comparative statics
- Welfare analysis
- Appendix: Marshall–Hicks four laws of derived demand

Partial Equilibrium Analysis

- In a competitive equilibrium (CE), all agents must select an optimal allocation given their resources:
  - Firms choose profit-maximizing production plans given their technology;
  - Consumers choose utility-maximizing bundles given their budget constraint.
- A competitive equilibrium allocation will emerge at a price that makes consumers' purchasing plans coincide with the firms' production decisions.

Partial Equilibrium Analysis

- ***Firm***:
  - Assume that firm j's cost function is strictly convex, $c_j''(q_j) > 0$.
  - Given the market price p^*, firm j's equilibrium output level q_j^* must solve

$$\max_{q_j \geq 0} \; p^* q_j - c_j(q_j)$$

Partial Equilibrium Analysis

- ***Firm*** (continued):
  - Since c_j is strictly convex, this problem yields the necessary and sufficient condition

$$p^* \leq c_j'(q_j^*), \qquad \text{with equality if } \; q_j^* > 0$$

  - That is, every firm j produces up to the point at which its marginal cost, $c_j'(q_j^*)$, coincides with the current market price.

Partial Equilibrium Analysis

- ***Consumers***:
  - Consider a quasilinear utility function

$$u_i(m_i, x_i) = m_i + \phi_i(x_i)$$

where m_i denotes the numeraire, and $\phi_i'(x_i) > 0$, $\phi_i''(x_i) < 0$ for all $x_i > 0$.

  - Normalizing, $\phi_i(0) = 0$. Recall that with quasilinear utility functions, the wealth effects for all non-numeraire commodities are zero.

Partial Equilibrium Analysis

– Consumer i's UMP is

$$\max_{m_i \in \mathbb{R}, x_i \in \mathbb{R}_+} \quad m_i + \phi_i(x_i)$$

$$\text{s.t.} \quad \underbrace{m_i + p^* x_i}_{\text{Total expend.}} \leq \underbrace{w_{m_i} + \sum_{j=1}^{J} \theta_{ij} \left(\underbrace{p^* q_j^* - c_j\left(q_j^*\right)}_{\text{Profits}} \right)}_{\text{Total resources (endowment + profits)}}$$

– where w_{m_i} is consumer i's endowment of the numeraire and θ_{ij} is his ownership share of firm j.

Partial Equilibrium Analysis

- The budget constraint binds because u_i is strictly increasing in m_i. Hence,

$$m_i = -p^* x_i + \left[w_{m_i} + \sum_{j=1}^{J} \theta_{ij} \left(p^* q_j^* - c_j\left(q_j^*\right) \right) \right]$$

- Since $m_i \in \mathbb{R}$, we do not need to impose $m_i \geq 0$ when we substitute this expression into the objective function.

Partial Equilibrium Analysis

– Substituting the budget constraint into the objective function,

$$\max_{x_i \in \mathbb{R}_+} \quad \phi_i(x_i) - p^* x_i + w_{m_i} + \sum_{j=1}^{J} \theta_{ij} \left(p^* q_j^* - c_j(q_j^*) \right)$$

– The FOC with respect to x_i yields

$$\phi_i'(x_i^*) \leq p^*, \qquad \text{with equality if } x_i^* > 0$$

– That is, the consumer increases the amount he buys of good x up to the point at which the marginal utility he obtains coincides with the market price he has to pay for it.

Partial Equilibrium Analysis

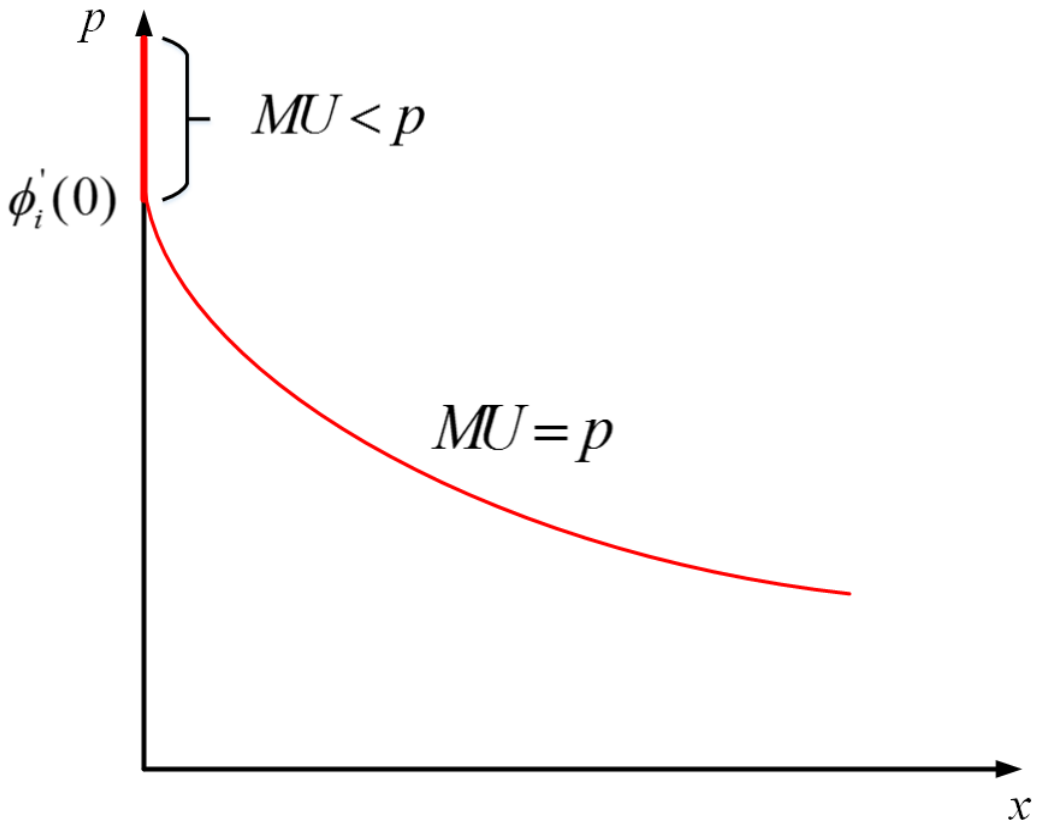
- Hence, an allocation $\left(x_1^*, x_2^*, \ldots, x_I^*, q_1^*, q_2^*, \ldots, q_J^*\right)$ and a price $p^* \in \mathbb{R}_+$ constitute a CE if:
  - $p^* \leq c_j'\left(q_j^*\right)$, with equality if $q_j^* > 0$, for every firm j;
  - $\phi_i'(x_i^*) \leq p^*$, with equality if $x_i^* > 0$, for every consumer i; and

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$$

- Note that these conditions do not depend on consumers' endowments w_{m_i} or ownership shares θ_{ij}.

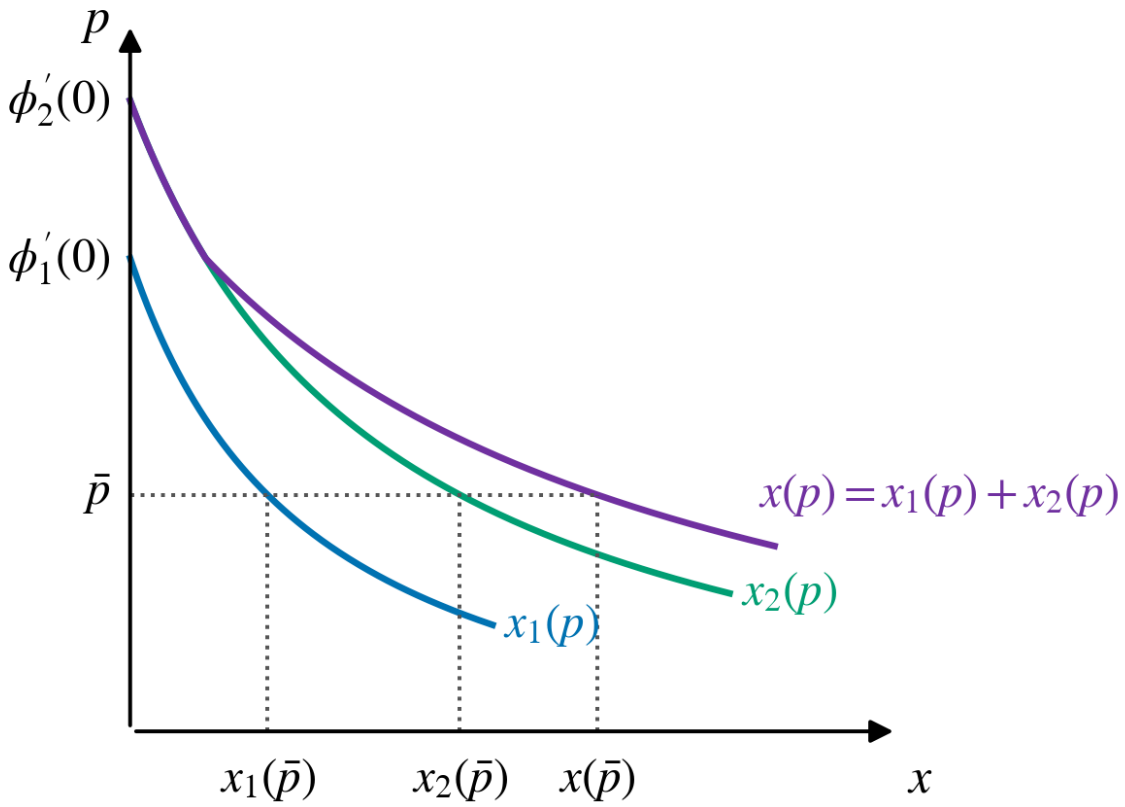
Partial Equilibrium Analysis

- The individual demand curve, where $\phi_i'(x_i^*) \leq p^*$



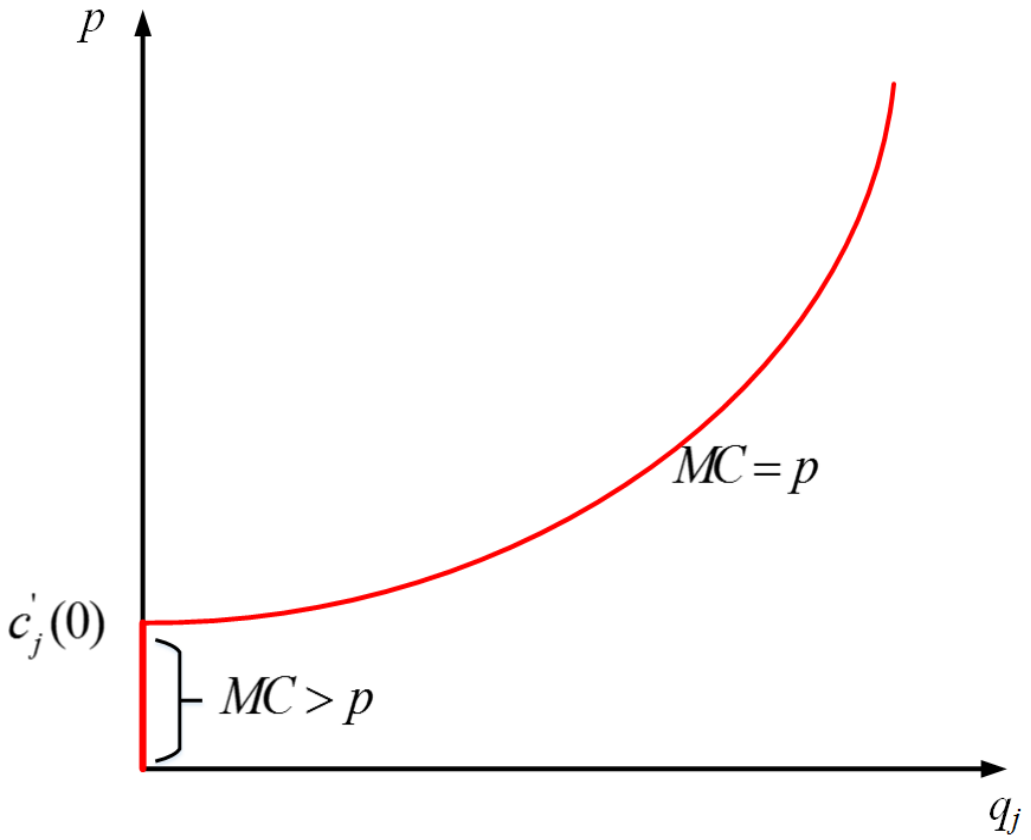
Partial Equilibrium Analysis

- Horizontally summing individual demand curves yields the aggregate demand curve.



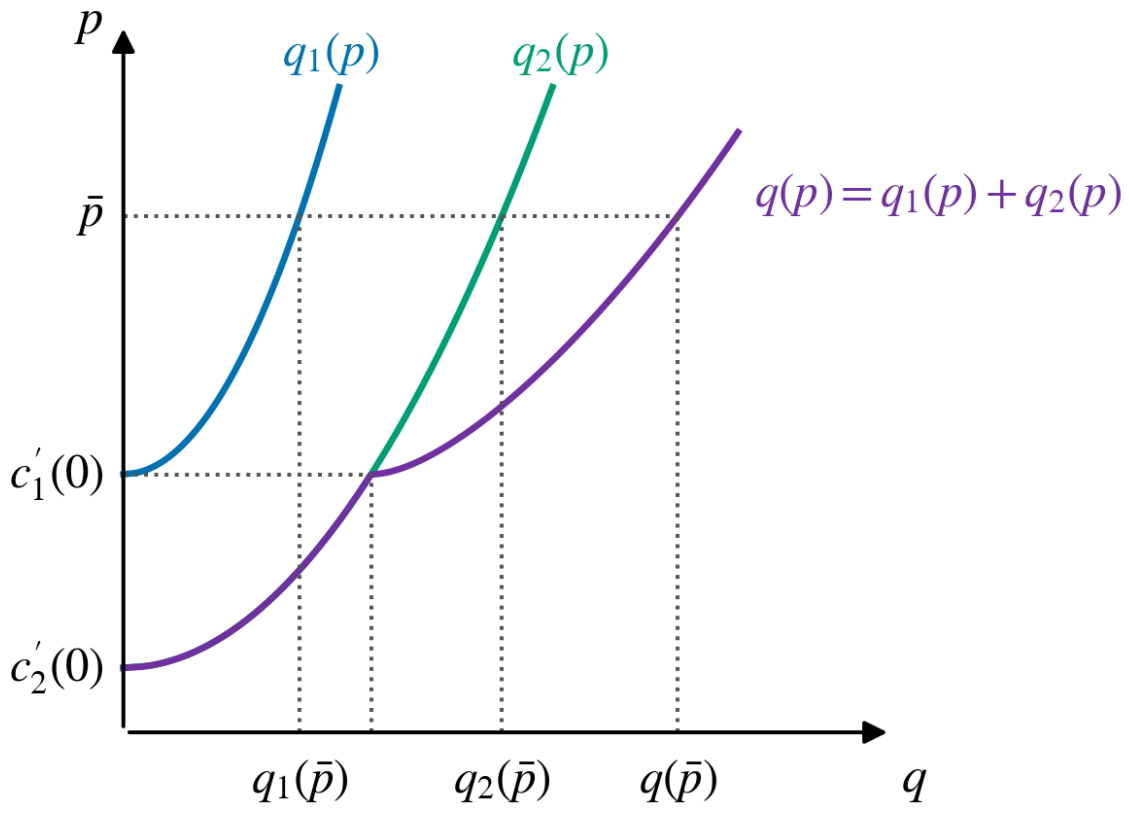
Partial Equilibrium Analysis

- The individual supply curve, where $p^* \leq c_j'(q_j^*)$



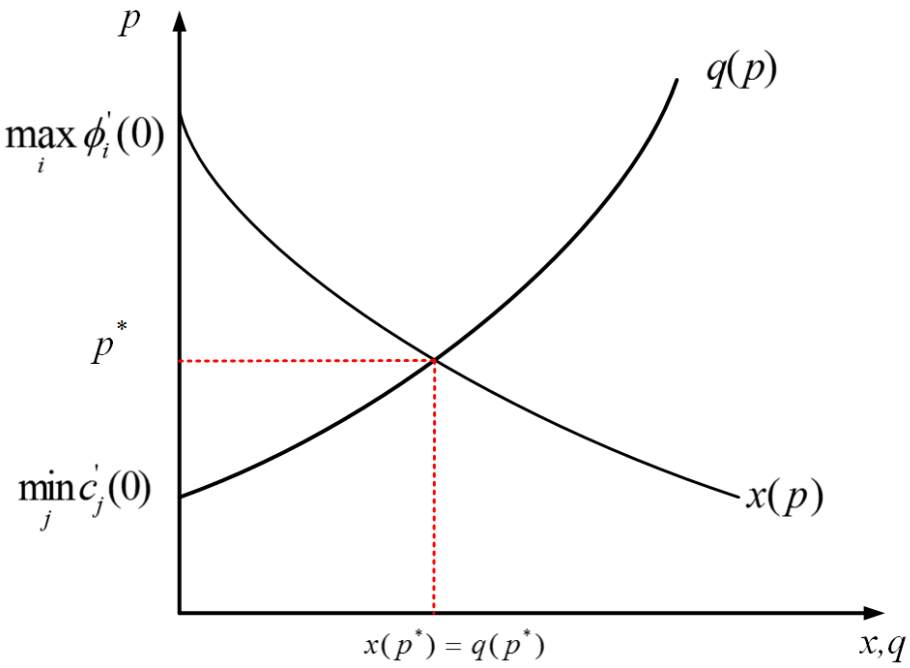
Partial Equilibrium Analysis

- Horizontally summing individual supply curves yields the aggregate supply curve.



Partial Equilibrium Analysis

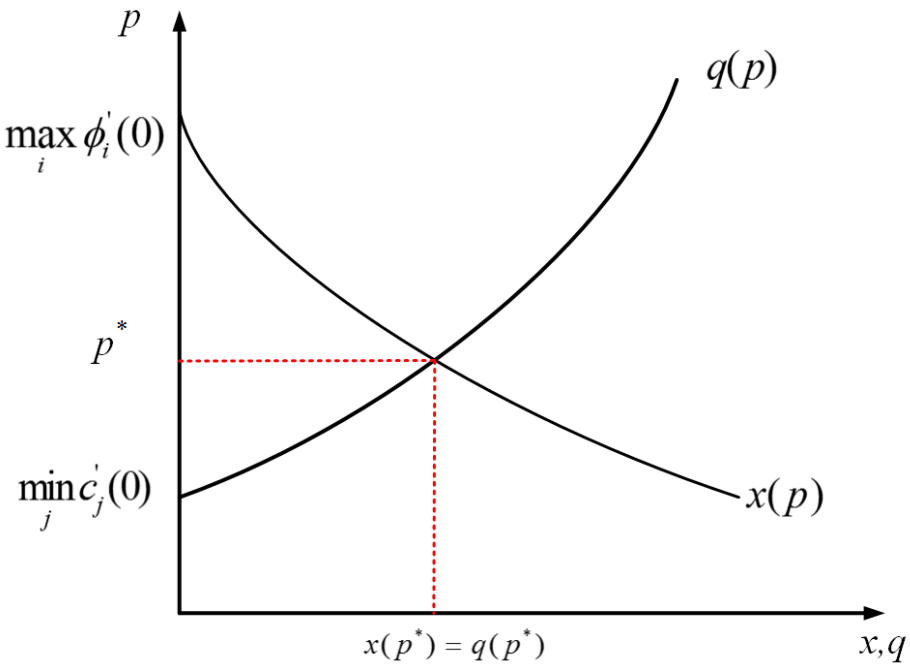
- Superimposing aggregate demand and aggregate supply curves, we obtain the CE allocation of good x.



Partial Equilibrium Analysis

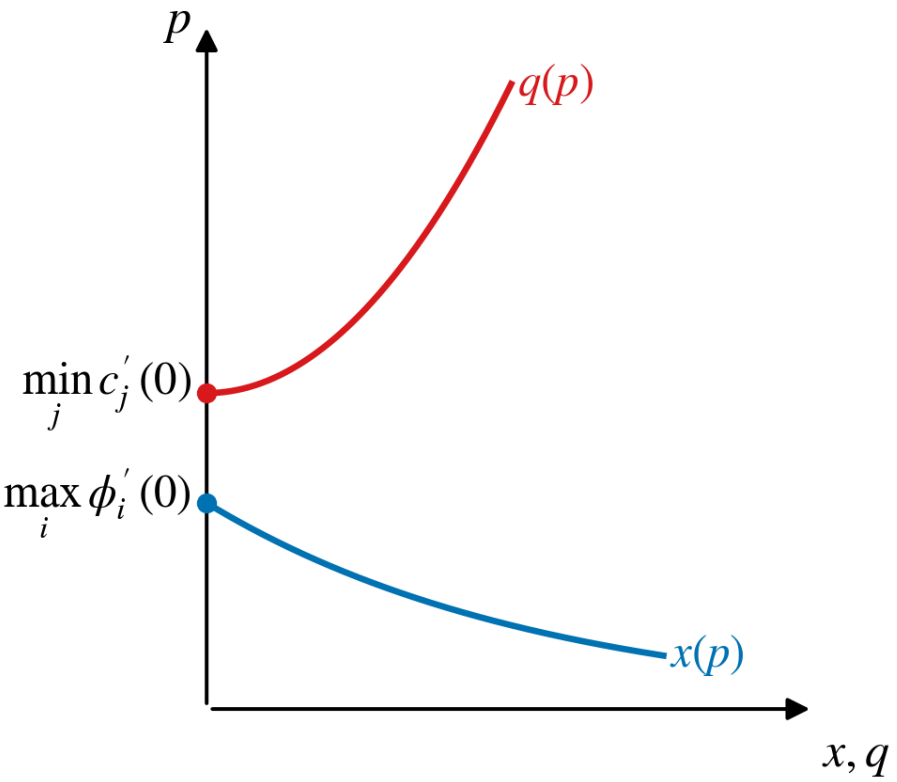
- For the CE to involve positive production and consumption, we need
$$\max_i \phi_i'(0) > \min_j c_j'(0)$$
- The equilibrium price then satisfies

$$\max_i \phi_i'(0) > p^* > \min_j c_j'(0)$$



Partial Equilibrium Analysis

- If we have
  $\max_i \phi_i'(0) < \min_j c_j'(0)$
  then there is no positive production or consumption of good x.
- The CE then has
  $x^* = q^* = 0$



Partial Equilibrium Analysis

- Also, since $\phi_i'(x_i)$ is downward sloping in x_i, and $c_j'(q_j)$ is upward sloping in q_j (since c_j is strictly convex), aggregate demand and supply cross at a unique point, given $\max_i \phi_i'(0) > \min_j c_j'(0)$.
  - Hence, the CE allocation is unique.

Partial Equilibrium Analysis

- ***Example 6.1***:
  - Assume a perfectly competitive industry consisting of two types of firms: 100 firms of type *A* and 30 firms of type *B.*
  - Short-run supply curve of type *A* firm is
$$s_A(p) = 2p$$
  - Short-run supply curve of type *B* firm is
$$s_B(p) = 10p$$
  - The Walrasian market demand curve is
$$x(p) = 5000 - 500p$$

Partial Equilibrium Analysis

- ***Example 6.1*** (continued):
  - Summing the individual supply curves of the 100 type-A firms and the 30 type-B firms,

$$S(p) = 100 \cdot 2p + 30 \cdot 10p = 500p$$

  - The short-run equilibrium occurs at the price at which quantity demanded equals quantity supplied,

$$5000 - 500p = 500p, \qquad \text{or} \quad p = 5$$

Partial Equilibrium Analysis

- ***Example 6.1*** (continued):
  - Each type-A firm supplies
$$s_A(5) = 2 \cdot 5 = 10$$
  - Each type-B firm supplies
$$s_B(5) = 10 \cdot 5 = 50$$
  - The equilibrium quantity is

$$x(5) = 5000 - 2500 = 2500 = 100 \cdot 10 + 30 \cdot 50$$

Comparative Statics

Comparative Statics

- Parameters:
  - Let us assume that the consumer's preferences are affected by a vector of parameters $\alpha \in \mathbb{R}^M$.
    - Then, consumer i's utility from good x is $\phi_i(x_i, \alpha)$.
  - Similarly, firms' technology is affected by a vector of parameters $\beta \in \mathbb{R}^S$.
    - Then, firm j's cost function is $c_j(q_j, \beta)$.

Comparative Statics

- Tax notation:
  - $\hat{p}_C(p,t)$ is the effective price paid by consumers.
  - $\hat{p}_F(p,t) = p$ is the effective price received by firms.
  - Per-unit tax: $\hat{p}_C(p,t) = p + t$.
    - Example: $t = \$2$, regardless of the price p.
  - Ad valorem tax: $\hat{p}_C(p,t) = p + pt = p(1+t)$.
    - Example: $t = 0.1$ (10%).

Comparative Statics

- If consumption and production are strictly positive in the CE, then
  - $\phi_i'(x_i^*, \alpha) = \hat{p}_C(p^*, t)$ for every consumer i;
  - $c_j'(q_j^*, \beta) = \hat{p}_F(p^*, t)$ for every firm j; and

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$$

- Then we have $I + J + 1$ equations, which depend on parameter values α, β and t, in the $I + J + 1$ unknowns (x_i^*, q_j^*, p^*).

Comparative Statics

- In order to understand how x_i^* or q_j^* depends on parameters α, β and t, we can use the Implicit Function Theorem.
  - The above functions have to be differentiable.

Comparative Statics

- Implicit Function Theorem:
  - Let $u(x, y)$ be a utility function, where x and y are amounts of two goods.
  - If $\frac{\partial u(\overline{x},\overline{y})}{\partial y} \neq 0$, then the equation $u(x, y) = u(\overline{x}, \overline{y})$ implicitly defines $y(x)$ near $(\overline{x}, \overline{y})$, and along it

$$\frac{\partial u(x, y)}{\partial x} dx + \frac{\partial u(x, y)}{\partial y} dy = 0$$

Comparative Statics

– Rearranging, we obtain

$$\frac{dy(x)}{dx} = -\frac{\frac{\partial u(x,y)}{\partial x}}{\frac{\partial u(x,y)}{\partial y}}$$

– Similarly, at every $(\overline{x}, \overline{y})$ where $\frac{\partial u(\overline{x},\overline{y})}{\partial x} \neq 0$, the equation $u(x,y) = u(\overline{x}, \overline{y})$ implicitly defines $x(y)$, with

$$\frac{dx(y)}{dy} = -\frac{\frac{\partial u(x,y)}{\partial y}}{\frac{\partial u(x,y)}{\partial x}}$$

Comparative Statics

- In comparative statics, we apply the IFT to the equilibrium condition rather than to the utility function.
- If $F(x, \alpha) = 0$ (e.g., the FOC $\phi'(x, \alpha) - p = 0$) and $\frac{\partial F}{\partial x} \neq 0$, then

$$\frac{dx(\alpha)}{d\alpha} = -\frac{\partial F/\partial \alpha}{\partial F/\partial x} = -\frac{\phi''_{x\alpha}(x, \alpha)}{\phi''_{xx}(x, \alpha)}$$

- The left-hand side is unknown.
- The right-hand side is, however, easier to find.

Comparative Statics

- ***Per-unit tax: general case*** (Example 6.2):
  - The aggregate demand is now $x(p + t)$, because the effective price that the consumer pays is $p + t$.
  - In equilibrium, the market price after imposing the tax, $p^*(t)$, must hence satisfy

$$x(p^*(t) + t) = q(p^*(t))$$

Comparative Statics

- If the per-unit tax is marginally increased, and functions are differentiable at $p = p^*(t)$,
  $x'(p^*(t) + t) \cdot (p^{*\prime}(t) + 1) = q'(p^*(t)) \cdot p^{*\prime}(t)$
- Rearranging, we obtain

$$p^{*\prime}(t) \cdot [x'(p^*(t) + t) - q'(p^*(t))] = -x'(p^*(t) + t)$$

Comparative Statics

– Hence,

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t) - q'(p^*(t))}$$

– Since $x(p)$ is decreasing in prices, $x'(p^*(t)+t) < 0$, and $q(p)$ is increasing in prices, $q'(p^*(t)) > 0$,

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t) - q'(p^*(t))} = -\frac{-}{-} = -$$

$$\qquad\qquad\qquad\qquad\qquad\quad -\qquad\qquad\quad +$$

Comparative Statics

– Let

$$|x'| = -x'(p^*(t) + t) > 0 \quad \text{and} \quad q' = q'(p^*(t)) > 0$$

– Then

$$p^{*\prime}(t) = -\frac{|x'|}{|x'| + q'} \in (-1,0)$$

– and

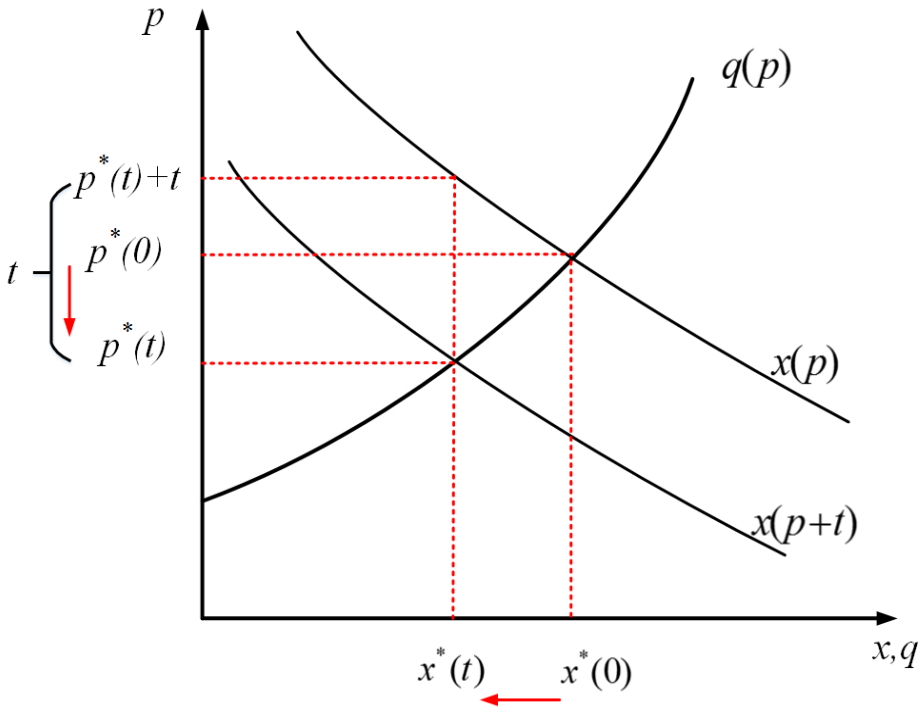
$$p^{*\prime}(t) + 1 = \frac{q'}{q' + |x'|} \in (0,1)$$

Comparative Statics

- Hence, $p^{*\prime}(t) < 0$.
- Moreover, $p^{*\prime}(t) \in (-1,0)$.
  - That is, the price received by producers falls with the tax, but by less than the tax increase.
- Additionally, since $p^*(t) + t$ is the price paid by consumers, $p^{*\prime}(t) + 1$ is the marginal increase in the price paid by consumers when the tax marginally increases.
  - Since $p^{*\prime}(t) > -1$, $p^{*\prime}(t) + 1 < 1$, and the price paid by consumers also rises by less than the tax increase.

Comparative Statics

- *No tax:*
  - CE occurs at $p^*(0)$ and $x^*(0)$
- *Tax:*
  - x^* decreases from $x^*(0)$ to $x^*(t)$
  - Consumers now pay $p^*(t) + t$
  - Producers now receive $p^*(t)$ for the $x^*(t)$ units they sell.



Comparative Statics

- ***Per-unit tax*** (extreme cases):
  - The supply is very responsive to price changes, i.e., $q'(p^*(t))$ is large.

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t) - q'(p^*(t))} \to 0$$

  - Therefore, $p^{*\prime}(t) \to 0$, and the price received by producers barely falls (in the limit, does not fall).

Comparative Statics

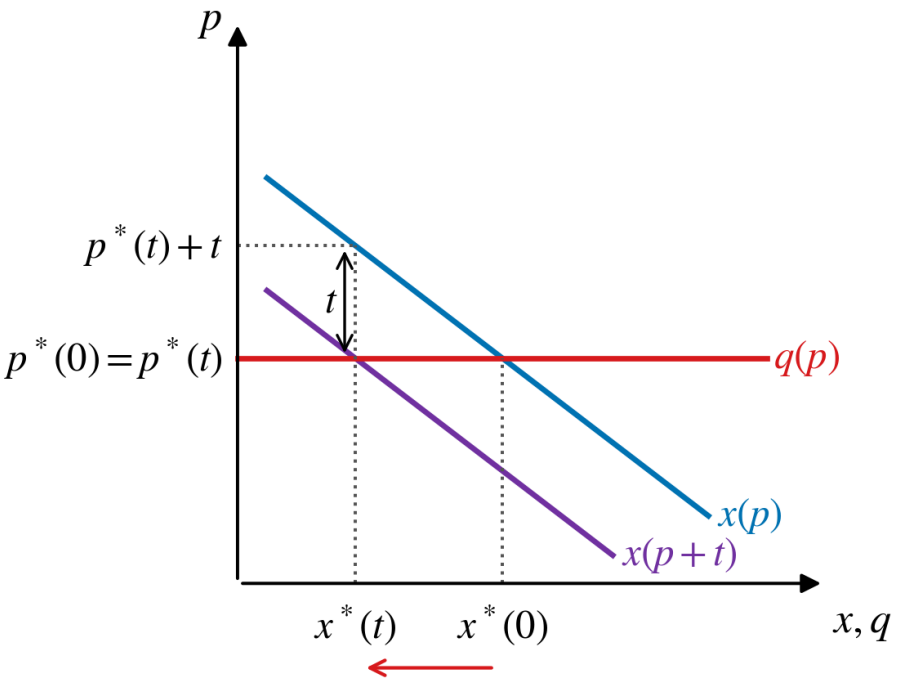
- ***Per-unit tax*** (extreme cases, continued):
  - However, consumers still have to pay $p^*(t) + t$.
    - A marginal increase in taxes therefore provides an increase in the consumer's price of

$$p^{*\prime}(t) + 1 \to 0 + 1 = 1$$

    - In the limit, the tax is borne entirely by consumers.

Comparative Statics

- Perfectly elastic supply (limit case):
  - The price received by producers does not fall.
  - But the price paid by consumers increases by the amount of the tax.



Comparative Statics

- ***Per-unit tax*** (extreme cases, continued):
  - The supply is not responsive to price changes, i.e., $q'(p^*(t))$ is close to zero.

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t) - q'(p^*(t))} \to -1$$

  - Therefore, $p^{*\prime}(t) \to -1$, and the price received by producers falls by \$1 for every extra dollar in taxes.
    - Producers bear the entire tax burden (in the limit).

Comparative Statics

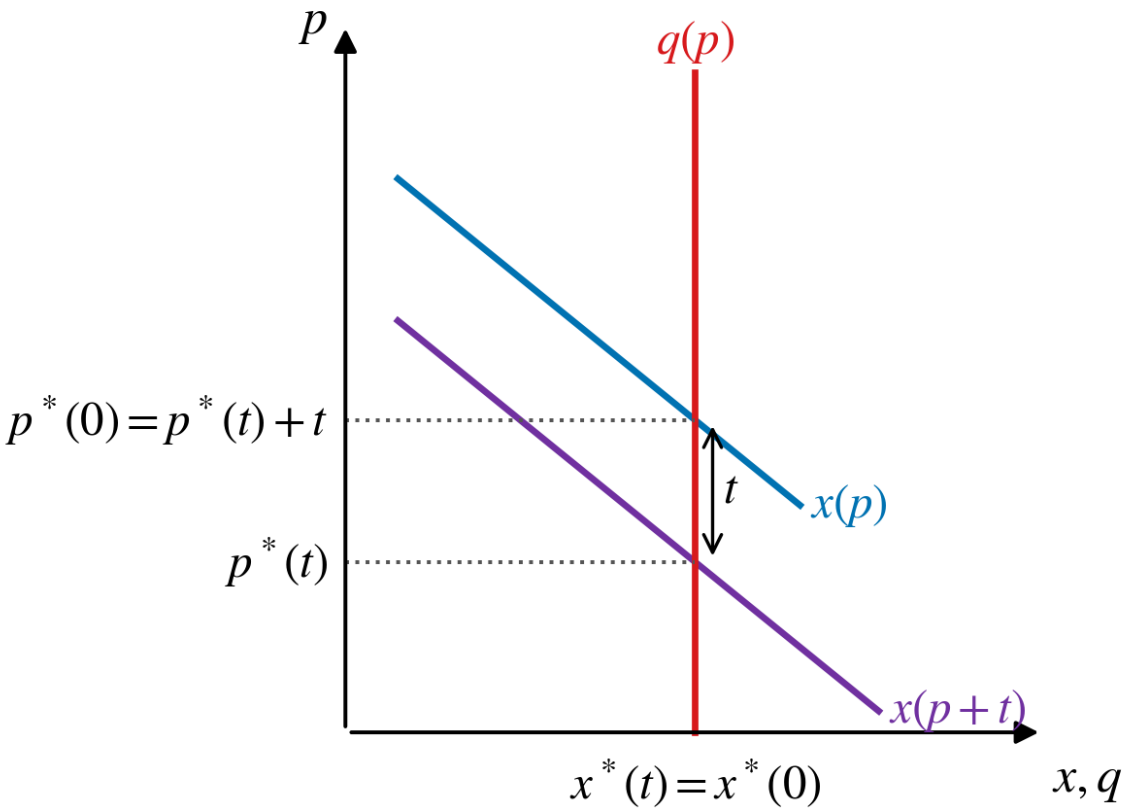
- ***Per-unit tax*** (extreme cases, continued):
  - In contrast, consumers pay $p^*(t) + t$.
    - A marginal increase in taxes produces an increase in the consumer's price of

$$p^{*\prime}(t) + 1 \rightarrow -1 + 1 = 0$$

    - Consumers do not bear the tax burden at all.

Comparative Statics

- Perfectly inelastic supply curve (limit case)



Comparative Statics

- ***Example 6.3***:
  - Consider a competitive market in which the government will be imposing a per-unit tax t.
  - Aggregate demand curve is $x(p) = Ap^{\varepsilon}$, where $A > 0$ and $\varepsilon < 0$, and aggregate supply curve is $q(p) = ap^{\gamma}$, where $a > 0$ and $\gamma > 0$.
  - Let us evaluate how the equilibrium price is affected by a marginal increase in the tax.

Comparative Statics

- ***Example 6.3*** (continued):
  - At $t = 0$, the change in the price received by producers is

$$p^{*\prime}(0) = -\frac{x'(p^*)}{x'(p^*) - q'(p^*)}$$

$$= -\frac{A\varepsilon(p^*)^{\varepsilon-1}}{A\varepsilon(p^*)^{\varepsilon-1} - a\gamma(p^*)^{\gamma-1}} = -\frac{A\varepsilon(p^*)^{\varepsilon}}{A\varepsilon(p^*)^{\varepsilon} - a\gamma(p^*)^{\gamma}}$$

$$= -\frac{\varepsilon x(p^*)}{\varepsilon x(p^*) - \gamma q(p^*)} = -\frac{\varepsilon}{\varepsilon - \gamma}$$

  - since $x(p^*) = q(p^*)$ at $t = 0$.

Comparative Statics

- ***Example 6.3*** (continued):
  - At $t = 0$, the change in the price paid by consumers is

$$p^{*\prime}(0) + 1 = -\frac{\varepsilon}{\varepsilon - \gamma} + 1 = -\frac{\gamma}{\varepsilon - \gamma}$$

Comparative Statics

- ***Example 6.3*** (continued):
  - In the limit $\gamma \to 0$ (i.e., supply is perfectly inelastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax since

$$p^{*\prime}(0) \to -\frac{\varepsilon}{\varepsilon - 0} = -1$$

    - That is, producers bear the full burden of the tax.

Comparative Statics

- ***Example 6.3*** (continued):
  - In the limit $\varepsilon \to 0$ (i.e., demand is perfectly inelastic), the price received by producers is unchanged, and the price paid by consumers increases by the amount of the tax because

$$p^{*\prime}(0) + 1 \to -\frac{\gamma}{0 - \gamma} = 1$$

    - That is, consumers bear the full burden of the tax.

Comparative Statics

- ***Example 6.3*** (continued):
  - When $\varepsilon \to -\infty$ (i.e., demand is perfectly elastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax since

$$\lim_{\varepsilon \to -\infty} p^{*\prime}(0) = -\frac{1}{1 - \lim\limits_{\varepsilon \to -\infty} \frac{\gamma}{\varepsilon}} = -1$$

Comparative Statics

- ***Example 6.3*** (continued):
  - When $\gamma \to +\infty$ (i.e., supply is perfectly elastic), the price received by producers is unchanged, and the price paid by consumers increases by the amount of the tax because

$$\lim_{\gamma \to +\infty} p^{*\prime}(0) + 1 = -\frac{1}{\lim_{\gamma \to +\infty} \frac{\varepsilon}{\gamma} - 1} = 1$$

Welfare Analysis

Welfare Analysis

- Let us now measure the changes in the aggregate social welfare due to a change in the allocation.
- Consider the aggregate surplus

$$S = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$$

Welfare Analysis

- Take a differential change in the quantity of good x that individuals consume and that firms produce such that $\sum_{i=1}^{I} dx_i = \sum_{j=1}^{J} dq_j$.
- The change in the aggregate surplus is

$$dS = \sum_{i=1}^{I} \phi_i'(x_i)dx_i - \sum_{j=1}^{J} c_j'(q_j)dq_j$$

Welfare Analysis

- Let $P(x)$ denote the inverse aggregate demand and $C(q)$ the aggregate (industry) cost function.
- Since
  - $\phi_i'(x_i) = P(x)$ for all consumers with $x_i > 0$ and
    - That is, every individual consumes until $\mathrm{MB} = P(x)$.
  - $c_j'(q_j) = C'(q)$ for all firms,
    - That is, every firm's MC coincides with aggregate MC,

Welfare Analysis

- then the change in surplus can be rewritten as

$$dS = \sum_{i=1}^{I} P(x)dx_i - \sum_{j=1}^{J} C'(q)dq_j$$

$$= P(x)\sum_{i=1}^{I} dx_i - C'(q)\sum_{j=1}^{J} dq_j$$

Welfare Analysis

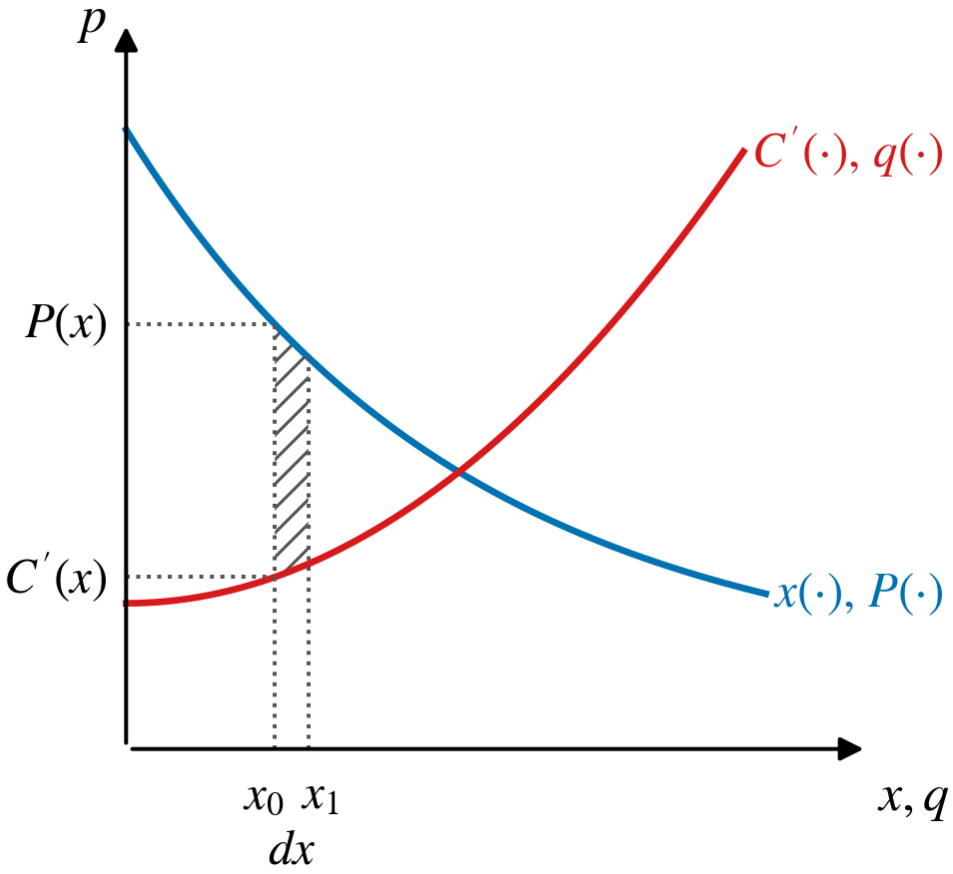
- But since $\sum_{i=1}^{I} dx_i = \sum_{j=1}^{J} dq_j = dx$, and $x = q$ by market feasibility, then

$$dS = [P(x) - C'(q)]dx$$

- *Intuition*:
  - The change in surplus of a marginal increase in consumption (and production) reflects the difference between the consumers' additional utility and the firms' additional cost of production.

Welfare Analysis

- Differential change in surplus



Welfare Analysis

- We can also integrate the above expression, eliminating the differentials, in order to obtain the total surplus for an aggregate consumption level of x:

$$S(x) = S_0 + \int_0^x [P(s) - C'(s)]ds$$

- where $S_0 = S(0)$ is the constant of integration, and represents the aggregate surplus when aggregate consumption is zero, $x = 0$.

Welfare Analysis

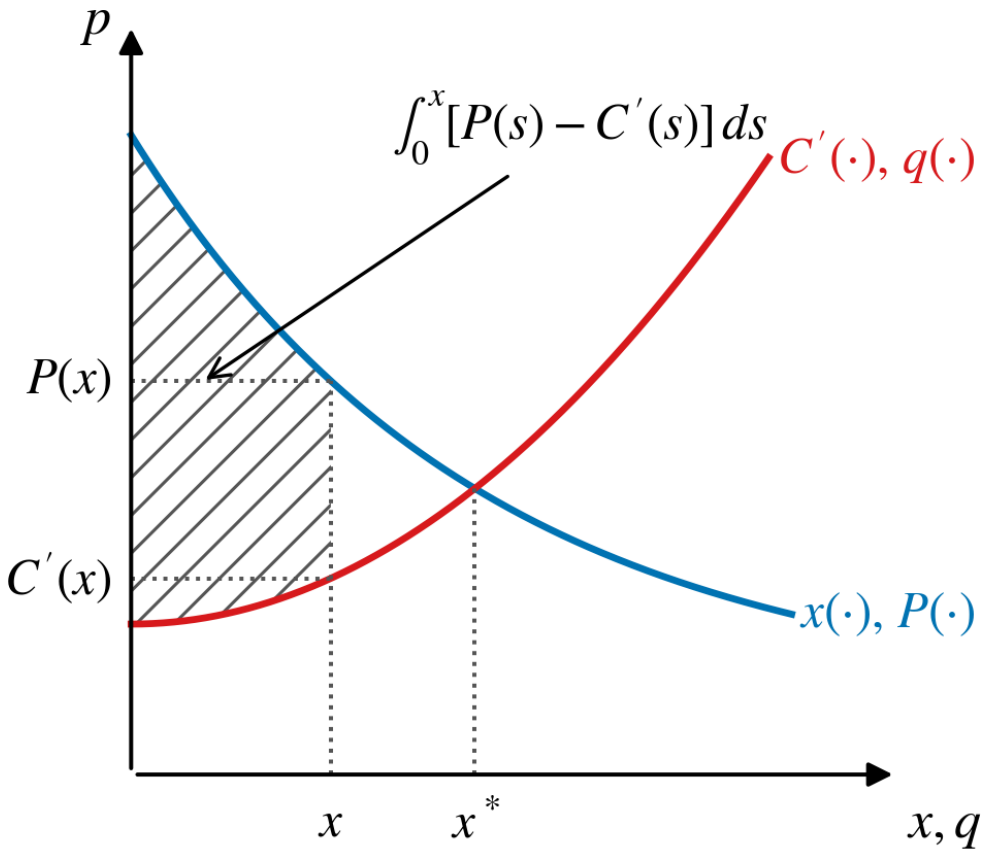
- Since $\phi_i(0) = 0$ for every consumer,

$$S_0 = -\sum_{j=1}^{J} c_j(0)$$

  - $S_0 = 0$ if there are no fixed costs, $c_j(0) = 0$ for all J firms.

Welfare Analysis

- Surplus at aggregate consumption x



Welfare Analysis

- For which consumption level is aggregate surplus $S(x)$ maximized?
  - Differentiating $S(x)$ with respect to x,
    $$S'(x^*) = P(x^*) - C'(x^*) \leq 0$$
  - with equality if $x^* > 0$; or, $P(x^*) \leq C'(x^*)$.
  - The second-order (sufficient) condition is
    $$S''(x) = \underset{-}{P'(x)} - \underset{+}{C''(x)} < 0, \qquad \text{for all} \quad x$$
    - Hence, $S(x)$ is concave.
    - Then, when $x^* > 0$, aggregate surplus $S(x)$ is maximized at $P(x^*) = C'(x^*)$.

Welfare Analysis

- In the CE, $P(x^*) = p^* = C'(x^*)$, so the CE satisfies the condition on the previous slide.
- Therefore, the CE allocation maximizes aggregate surplus.
- With quasilinear utility, maximizing aggregate surplus is equivalent to Pareto optimality.
- This is the First Welfare Theorem:
  - Every CE is Pareto optimal (PO).

Welfare Analysis

- ***Example 6.4***:
  - Consider an aggregate demand $x(p) = a - bp$ and aggregate supply $q(p) = J \cdot \frac{p}{2}$, where J is the number of firms in the industry.
  - The CE price solves

$$a - bp = J \cdot \frac{p}{2}, \qquad \text{or} \quad p^* = \frac{2a}{2b + J}$$

Welfare Analysis

- ***Example 6.4*** (continued):
  - Intuitively, as demand increases (as the number of firms increases), the equilibrium price increases (decreases):

$$\frac{\partial p^*}{\partial a} = \frac{2}{2b + J} > 0$$

$$\frac{\partial p^*}{\partial J} = -\frac{2a}{(2b + J)^2} < 0$$

Welfare Analysis

- ***Example 6.4*** (continued):
  - Therefore, equilibrium output is

$$x^* = a - b\frac{2a}{2b+J} = \frac{aJ}{2b+J}$$

  - Surplus is

$$S(x^*) = \int_0^{x^*} \big(P(x) - C'(x)\big)dx$$

Welfare Analysis

- ***Example 6.4*** (continued):
  - where $P(x) = \frac{a-x}{b}$ and, since $q = \frac{pJ}{2}$, solving for p and rearranging yields the marginal cost function

$$p = C'(x) = \frac{2x}{J}$$

  - so that

$$S(x^*) = \int_0^{x^*} \left(\frac{a-x}{b} - \frac{2x}{J} \right) dx$$

Welfare Analysis

- ***Example 6.4*** (continued):
  - Integrating,

$$S(x^*) = \left[\frac{ax}{b} - \frac{x^2}{2b} - \frac{x^2}{J}\right]_0^{x^*}$$

$$= \frac{a^2 J}{4b^2 + 2bJ} = \frac{a^2}{2b\left(1 + \frac{2b}{J}\right)}$$

  - which is increasing in the number of firms J,

$$\frac{\partial S(x^*)}{\partial J} = \frac{a^2}{(2b + J)^2} > 0$$

Appendix:
Marshall–Hicks Four Laws of Derived Demand

Marshall–Hicks Four Laws

- Consider a production function $q = f(K, L)$, with positive marginal products, $f_L, f_K > 0$.
- Assume that the supply of capital is positively sloped, $r'(K) > 0$, while the supply of labor, $w(L)$, can be upward sloping, $w'(L) > 0$, or backward bending, $w'(L) < 0$.
- Demand for output is given by $q = g(p)$, which satisfies $g'(p) < 0$.

Marshall–Hicks Four Laws

- The total cost is $w(L)L + r(K)K$.
- Assume that the capital market is perfectly competitive, but the labor and output markets are not necessarily competitive.

Marshall–Hicks Four Laws

- Define:
  - $\varepsilon_{q,p} = -\left(\frac{\partial q}{\partial p}\right)\left(\frac{p}{q}\right) > 0$ as the absolute price elasticity of output demand
  - $s_{K,r} = \left(\frac{\partial K}{\partial r}\right)\left(\frac{r}{K}\right)$ as the elasticity of the (derived) demand for capital with respect to its price r
  - $s_{L,w} = \left(\frac{\partial L}{\partial w}\right)\left(\frac{w}{L}\right)$ as the elasticity of labor supply to a change in its price
  - σ as the elasticity of substitution between inputs

Marshall–Hicks Four Laws

- We use superscript i to refer to the elasticity that an individual firm faces ($\varepsilon_{q,p}^{i}$).
- The industry elasticities do not include superscripts ($\varepsilon_{q,p}$).

Marshall–Hicks Four Laws

- Let $\theta_L \equiv \frac{wL}{pq}$ and $\theta_K \equiv \frac{rK}{pq}$ be the cost of labor and capital, respectively, relative to total sales.
- Under constant returns and zero profits,

$$pq = wL + rK, \qquad \text{so} \quad \theta_L = 1 - \theta_K$$

- For compactness, let us define

$$A \equiv 1 - \frac{1}{\varepsilon^i_{q,p}}$$

$$B \equiv 1 + \frac{1}{s^i_{L,w}}$$

Marshall–Hicks Four Laws

- Marshall, Hicks, and Allen analyze how the demand for an input bought in a competitive market, such as capital, is affected by a marginal change in the price of capital:

$$s_{K,r} = \frac{-A\left[\theta_K \varepsilon_{q,p} + \sigma\left(\varepsilon_{q,p}/s_{L,w}\right)A + \sigma\theta_L B\right]}{\left(\theta_K + \theta_L B\right)^2 + \theta_K\left(\sigma/s_{L,w}\right)A + \theta_L\left(\varepsilon_{q,p}/s_{L,w}\right)AB}$$

Sources: Marshall (1920), *Principles of Economics*, 8th ed.; Hicks (1932), *The Theory of Wages*; Allen (1938), *Mathematical Analysis for Economists*.

Marshall–Hicks Four Laws

- Marshall–Hicks's four laws of input demand ("derived demand") state that an input demand becomes more elastic ($s_{K,r}$ decreases) as each of the following increases:
  - the elasticity of substitution between inputs, σ
  - the price elasticity of output demand, $\varepsilon_{q,p}$
  - the cost of the input relative to total sales, θ_K (if $\varepsilon_{q,p} > \sigma$ and $s_{L,w} > 0$)
  - the elasticity of the other input's supply to a change in its price, $s_{L,w}$

Marshall–Hicks Four Laws

- We analyze these four comparative statics in two cases:
  - Marshall's presentation: $\varepsilon_{q,p}^{i} = s_{L,w}^{i} = \infty, \sigma = 0$
  - Hicks's presentation: $\varepsilon_{q,p}^{i} = s_{L,w}^{i} = \infty$ (no assumptions on σ)

Marshall's Presentation

- Assumptions:
  - Output and input markets are perfectly competitive, $\varepsilon_{q,p}^{i} = s_{L,w}^{i} = \infty$, for every firm i, so $A = B = 1$.
  - Inputs cannot be substituted in the production process, $\sigma = 0$.
- The expression for $s_{K,r}$ can be simplified to

$$s_{K,r} = -\frac{\theta_K \varepsilon_{q,p} s_{L,w}}{s_{L,w} + \theta_L \varepsilon_{q,p}}$$

Marshall's Presentation

- The derivatives testing the laws are:

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{\theta_K \left(s_{L,w}\right)^2}{\left(s_{L,w} + \theta_L \varepsilon_{q,p}\right)^2}$$

$$\frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{s_{L,w} \cdot \varepsilon_{q,p}\left(s_{L,w} + \varepsilon_{q,p}\right)}{\left(s_{L,w} + \theta_L \varepsilon_{q,p}\right)^2}$$

$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{\theta_K \theta_L \left(\varepsilon_{q,p}\right)^2}{\left(s_{L,w} + \theta_L \varepsilon_{q,p}\right)^2}$$

Marshall's Presentation

- If labor supply is upward sloping ($s_{L,w} > 0$), the three derivatives are all negative (the three laws hold).
- If labor supply is backward bending ($s_{L,w} < 0$), $s_{K,r}$ is still decreasing in $\varepsilon_{q,p}$ and in $s_{L,w}$, but not necessarily in θ_K.

Hicks's Presentation

- Assumptions:
  - Output and input markets are perfectly competitive, $\varepsilon_{q,p}^{i} = s_{L,w}^{i} = \infty$, for every firm i, so $A = B = 1$.
  - No condition imposed on the substitution of inputs (σ).
- The expression for $s_{K,r}$ can be simplified to

$$s_{K,r} = -\frac{\theta_K \varepsilon_{q,p} s_{L,w} + \sigma\left(\varepsilon_{q,p} + \theta_L s_{L,w}\right)}{s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}}$$

Hicks's Presentation

- The derivatives testing the laws are

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{\theta_K \left(s_{L,w} + \sigma\right)^2}{\left(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}\right)^2}$$

$$\frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{\left(\varepsilon_{q,p} + s_{L,w}\right)\left(\varepsilon_{q,p} - \sigma\right)\left(s_{L,w} + \sigma\right)}{\left(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}\right)^2}$$

$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{\theta_K \theta_L \left(\varepsilon_{q,p} - \sigma\right)^2}{\left(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}\right)^2}$$

$$\frac{\partial s_{K,r}}{\partial \sigma} = -\frac{\theta_L \left(\varepsilon_{q,p} + s_{L,w}\right)^2}{\left(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}\right)^2}$$

Hicks's Presentation

- Hence, $s_{K,r}$ decreases in $\varepsilon_{q,p}$, $s_{L,w}$, and σ (the three laws hold).
- If labor supply is upward sloping ($s_{L,w} > 0$), $s_{K,r}$ also decreases in θ_K when the elasticity of substitution is below the output-demand elasticity, $\sigma < \varepsilon_{q,p}$.
- If labor supply is backward bending ($s_{L,w} < 0$), $s_{K,r}$ does not necessarily decrease in θ_K.