Advanced Microeconomic Theory

Chapter 6: General Equilibrium

Outline

- General equilibrium without production
- Competitive markets and the welfare theorems
- General equilibrium with production
- Comparative statics
- Taxes in general equilibrium
- Appendix: Large economies and the core

General Equilibrium

General Equilibrium

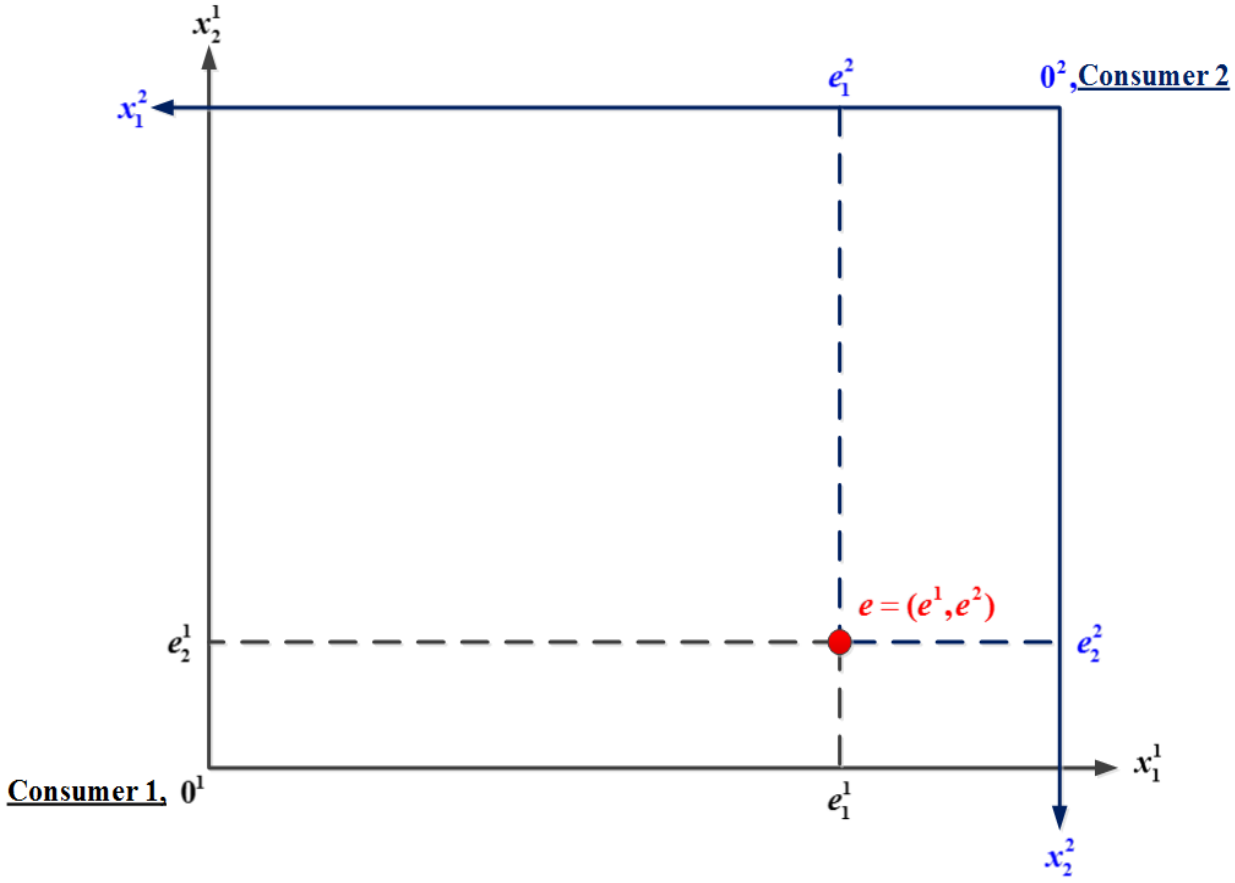
- So far, we explored equilibrium conditions in:
  - a single market with
  - a single type of consumer.
- Now we examine settings with markets for:
  - different goods and
  - multiple consumers.

General Equilibrium: No Production

- Consider an economy with two goods and two consumers, $i \in \{1,2\}$.
- Each consumer i is initially endowed with $\mathbf{e}^i \equiv (e_1^i, e_2^i)$ units of goods 1 and 2.
- Any other bundle for consumer i is denoted by $\mathbf{x}^i \equiv (x_1^i, x_2^i)$.

General Equilibrium: No Production

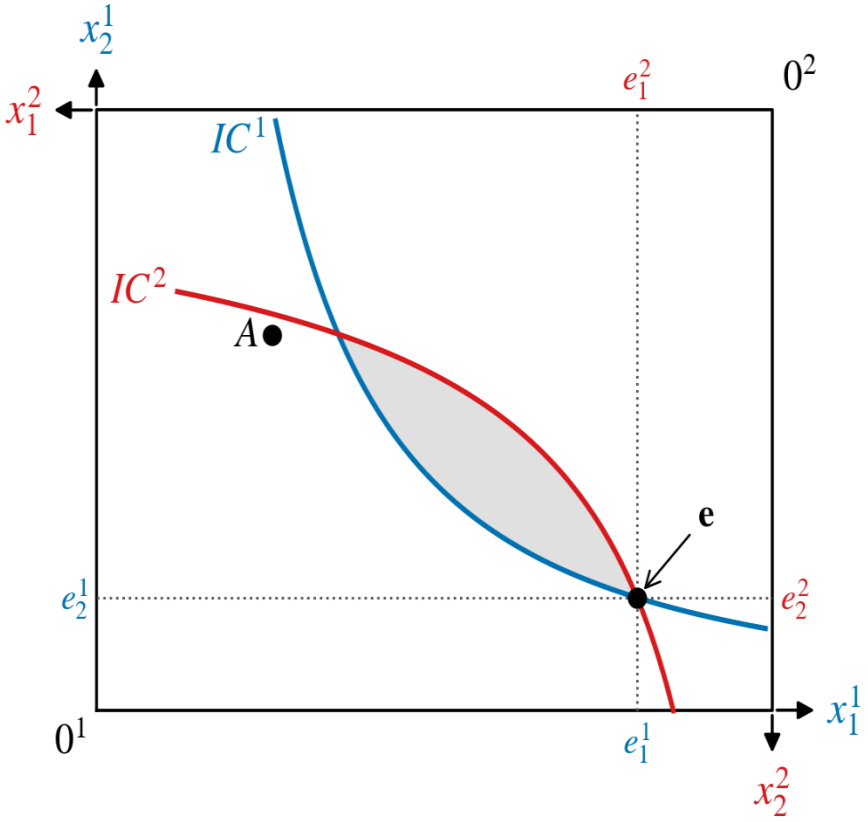
- ***Edgeworth box***:



General Equilibrium: No Production

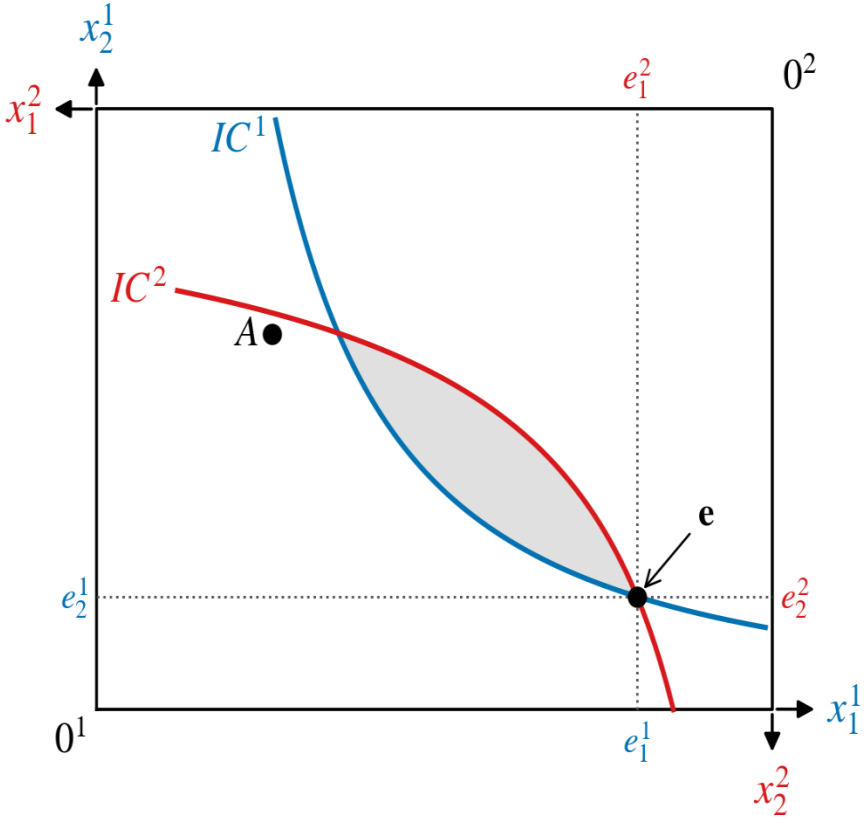
- IC^i is the indifference curve of consumer i, which passes through his endowment point $\mathbf{e}^i$.
- The shaded area represents the set of allocations $(\mathbf{x}^1, \mathbf{x}^2)$ satisfying

$$u^1(x_1^1, x_2^1) \geq u^1(e_1^1, e_2^1)$$
$$u^2(x_1^2, x_2^2) \geq u^2(e_1^2, e_2^2)$$



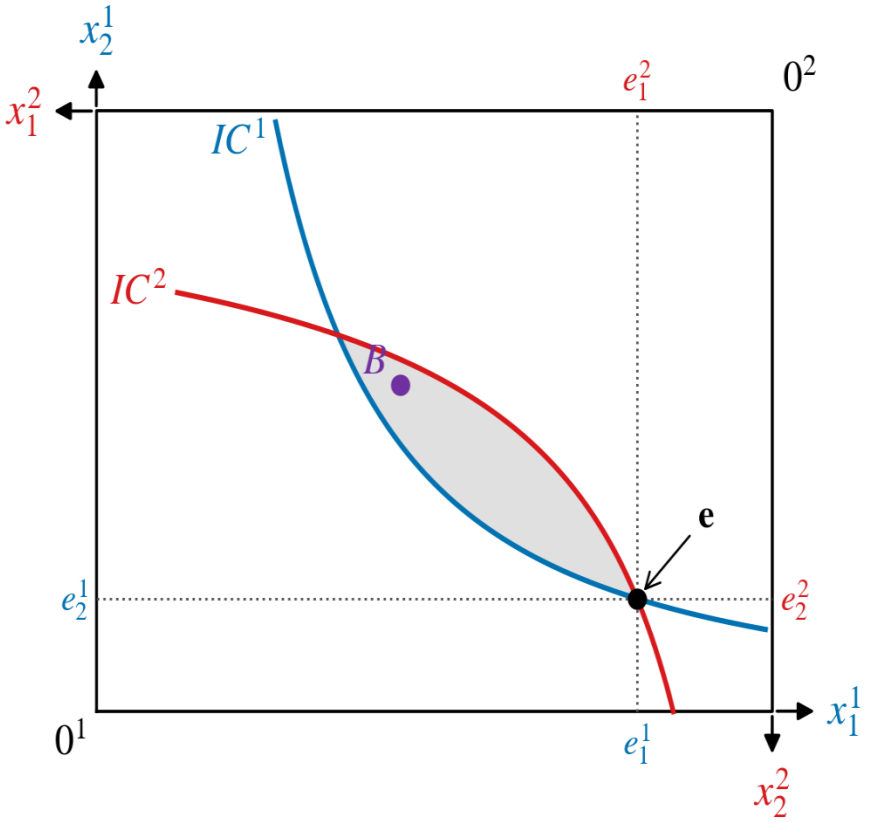
General Equilibrium: No Production

- Allocation A cannot be a barter equilibrium (defined formally below):
  - Consumer 1 prefers his endowment $\mathbf{e}^1$ to his bundle at A (A lies below IC^1), so he does not trade to A.



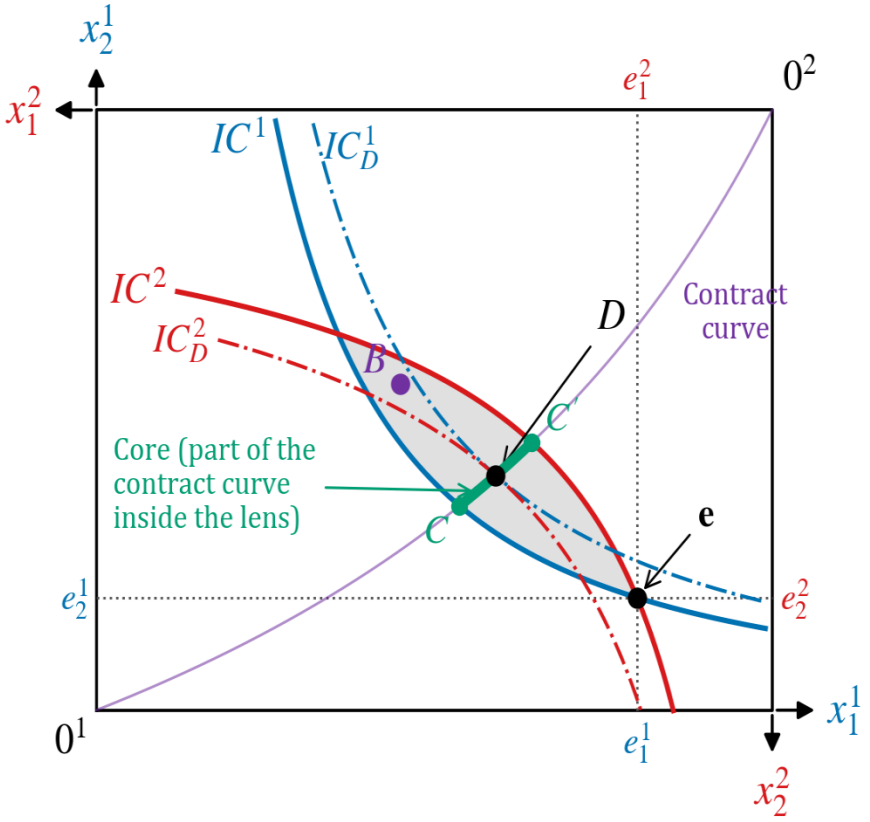
General Equilibrium: No Production

- Not all points in the lens-shaped area are barter equilibria!
- Allocation B lies inside the lens-shaped area
  - Thus, it yields a higher utility level than the initial endowment **e** for both consumers.



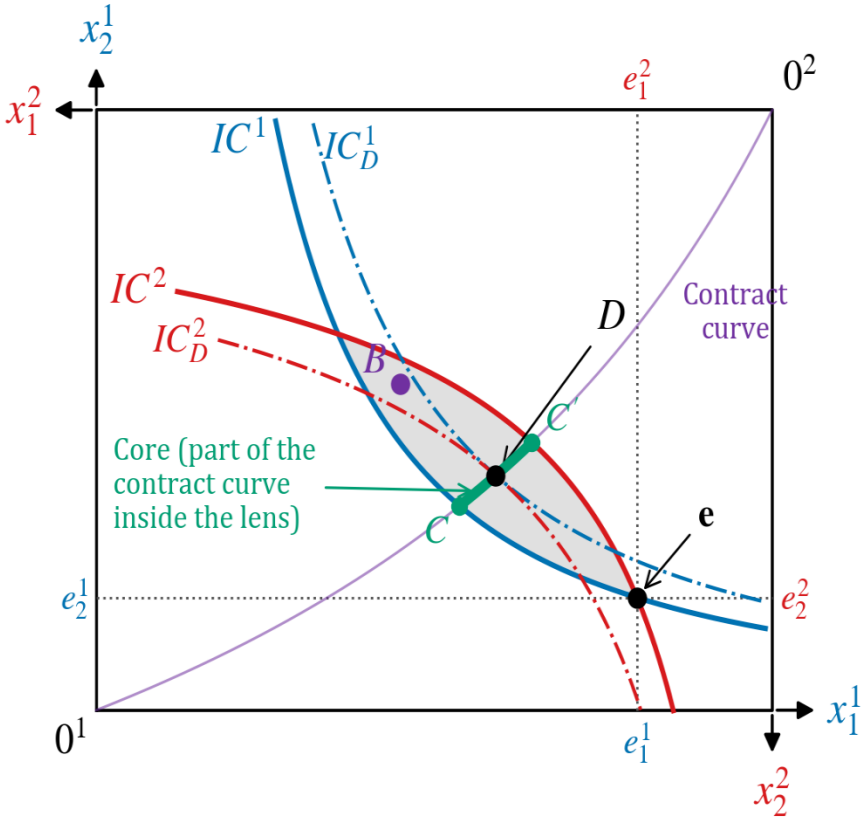
General Equilibrium: No Production

- Allocation D, however, makes both consumers better off than allocation B.
  - It lies on the ***contract curve***, in which the indifference curves are tangent to one another.



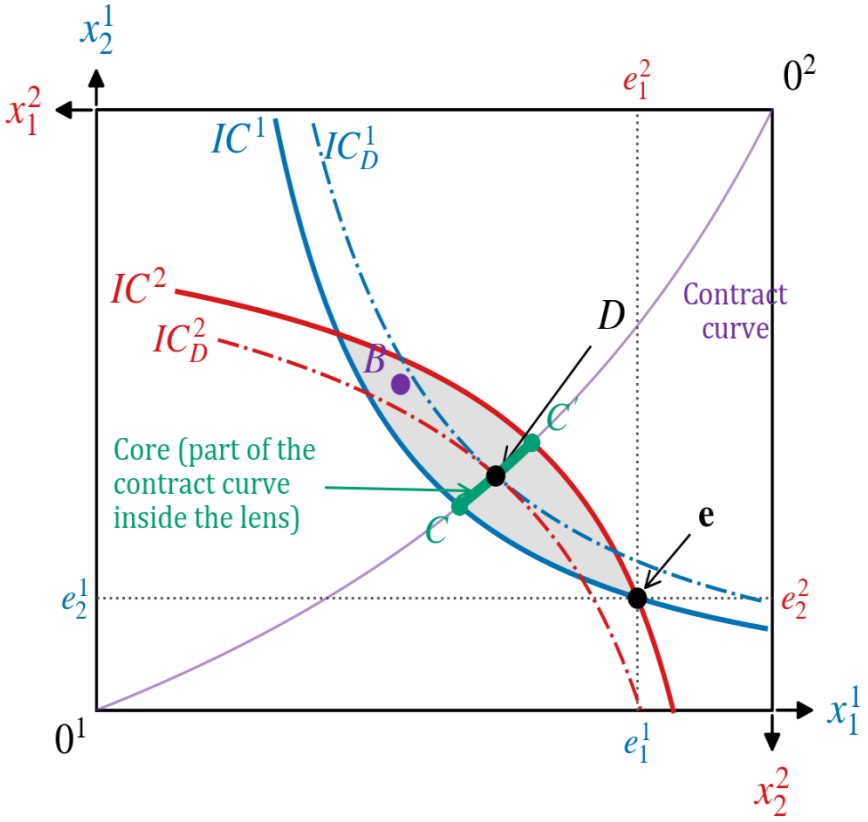
General Equilibrium: No Production

- Allocation D is an equilibrium: it lies in the lens (no consumer prefers his endowment) and no Pareto improvement is possible.



General Equilibrium: No Production

- Being in the lens-shaped area is then a necessary, but not sufficient, condition for an allocation to be a barter equilibrium.



General Equilibrium: No Production

- ***Feasible allocation***:
  - An allocation $\mathbf{x} \equiv (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$ is *feasible* if it satisfies

$$\sum_{i=1}^{I} \mathbf{x}^i \leq \sum_{i=1}^{I} \mathbf{e}^i$$

  - In words, the aggregate amount of goods in allocation $\mathbf{x}$ does not exceed the aggregate initial endowment $\mathbf{e} \equiv \sum_{i=1}^{I} \mathbf{e}^i$.

General Equilibrium: No Production

- ***Pareto-efficient allocations***: A feasible allocation $\mathbf{x}$ is Pareto efficient if there is no other feasible allocation $\mathbf{y}$ which is:
  - weakly preferred by all consumers, $\mathbf{y}^i \succsim^i \mathbf{x}^i$ for all $i \in \{1, \ldots, I\}$, and
  - strictly preferred by at least one consumer, $\mathbf{y}^i \succ^i \mathbf{x}^i$ for some i.

General Equilibrium: No Production

- That is, allocation $\mathbf{x}$ is Pareto efficient if there is no other feasible allocation $\mathbf{y}$ making:
  - all individuals at least as well off as under $\mathbf{x}$, and
  - at least one individual strictly better off.

General Equilibrium: No Production

- ***Pareto-efficient allocations***:
  - Consider the program

$$\max_{\mathbf{x}^1,\ldots,\mathbf{x}^I \geq \mathbf{0}} u^i(\mathbf{x}^i)$$

$$\text{s.t. } u^j(\mathbf{x}^j) \geq \overline{u}^j \text{ for every } j \neq i, \quad \text{and}$$

$$\sum_{h=1}^{I} \mathbf{x}^h \leq \sum_{h=1}^{I} \mathbf{e}^h \quad \text{(feasibility)}$$

  - where $\mathbf{x}^i \equiv (x_1^i, x_2^i)$.

General Equilibrium: No Production

- With strictly increasing utilities, an allocation $(\mathbf{x}^1, \ldots, \mathbf{x}^I)$ is Pareto efficient if and only if it solves this program for some levels $\overline{u}^j$, $j \neq i$; varying $\overline{u}^j$ traces out the set.
- That is, a Pareto-efficient allocation maximizes individual i's utility without reducing the utility of any other individual j below a given level $\overline{u}^j$, and satisfies feasibility.

General Equilibrium: No Production

- The Lagrangian is

$$\mathcal{L} = u^i(\mathbf{x}^i) + \sum_{j \neq i} \lambda^j [u^j(\mathbf{x}^j) - \overline{u}^j]$$
$$+ \sum_{k=1}^{2} \mu_k [\sum_{h=1}^{I} e_k^h - \sum_{h=1}^{I} x_k^h]$$

  - where μ_k is the multiplier on the feasibility constraint of good k, so we have one multiplier for each good.

General Equilibrium: No Production

- FOC with respect to $\mathbf{x}^i = (x_1^i, x_2^i)$:

$$\frac{\partial \mathcal{L}}{\partial x_k^i} = \frac{\partial u^i(\mathbf{x}^i)}{\partial x_k^i} - \mu_k \leq 0$$

  - for every good k of consumer i.

- For any individual $j \neq i$, the FOCs become

$$\frac{\partial \mathcal{L}}{\partial x_k^j} = \frac{\lambda^j \partial u^j(\mathbf{x}^j)}{\partial x_k^j} - \mu_k \leq 0$$

General Equilibrium: No Production

- FOCs with respect to λ^j and μ_k yield $u^j(\mathbf{x}^j) \geq \overline{u}^j$ and $\sum_{h=1}^{I} x_k^h \leq \sum_{h=1}^{I} e_k^h$, respectively.
- Interior solutions: the FOCs of consumer i hold with equality,

$$\frac{\partial u^i(\mathbf{x}^i)}{\partial x_1^i} = \mu_1 \quad \text{and} \quad \frac{\partial u^i(\mathbf{x}^i)}{\partial x_2^i} = \mu_2$$

General Equilibrium: No Production

- Dividing the first condition by the second,

$$\frac{\frac{\partial u^i(\mathbf{x}^i)}{\partial x_1^i}}{\frac{\partial u^i(\mathbf{x}^i)}{\partial x_2^i}} = \frac{\mu_1}{\mu_2}$$

- Similarly, for every consumer $j \neq i$,

$$\frac{\lambda^j \partial u^j(\mathbf{x}^j)}{\partial x_1^j} = \mu_1 \quad \text{and} \quad \frac{\lambda^j \partial u^j(\mathbf{x}^j)}{\partial x_2^j} = \mu_2$$

General Equilibrium: No Production

- Dividing the first condition by the second, λ^j cancels out,

$$\frac{\frac{\partial u^j(\mathbf{x}^j)}{\partial x_1^j}}{\frac{\partial u^j(\mathbf{x}^j)}{\partial x_2^j}} = \frac{\mu_1}{\mu_2}$$

- Both ratios are equal to μ_1/μ_2.

General Equilibrium: No Production

- Hence, a compact condition for Pareto efficiency is

$$MRS_{1,2}^{i} = \frac{\mu_1}{\mu_2} = MRS_{1,2}^{j}$$

  - for every consumer $j \neq i$.
- Graphically, consumers' indifference curves are tangent to one another at the Pareto-efficient allocations.

General Equilibrium: No Production

- ***Example 6.5*** (Pareto efficiency):
  - We now call the two consumers A and B.
  - Consider a barter economy with two goods, 1 and 2, and two consumers, A and B, with initial endowments
    $$\mathbf{e}^A = (100,350) \quad \text{and} \quad \mathbf{e}^B = (100,50)$$
  - Each consumer's utility function is Cobb-Douglas,
    $$u^i(x_1^i, x_2^i) = x_1^i x_2^i \quad \text{for every} \quad i \in \{A, B\}$$
  - Let us find the set of Pareto efficient allocations.

General Equilibrium: No Production

- ***Example 6.5*** (continued):
  - Pareto efficient allocations are reached at points where $MRS_{1,2}^{A} = MRS_{1,2}^{B}$. Hence,

$$MRS_{1,2}^{A} = MRS_{1,2}^{B}, \qquad \text{that is,} \qquad \frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B}$$

  - or, cross-multiplying,

$$x_2^A x_1^B = x_2^B x_1^A$$

General Equilibrium: No Production

- ***Example 6.5*** (continued):
  - Using the feasibility constraints for goods 1 and 2 (feasibility binds because utility is strictly increasing),

$$e_1^A + e_1^B = x_1^A + x_1^B$$
$$e_2^A + e_2^B = x_2^A + x_2^B$$

  - we obtain

$$x_1^B = e_1^A + e_1^B - x_1^A$$
$$x_2^B = e_2^A + e_2^B - x_2^A$$

General Equilibrium: No Production

- ***Example 6.5*** (continued):
  - Combining the tangency condition and the feasibility constraints yields

$$x_2^A(e_1^A + e_1^B - x_1^A) = (e_2^A + e_2^B - x_2^A)x_1^A$$

  - Expanding both sides,

$$x_2^A(e_1^A + e_1^B) - x_1^A x_2^A = x_1^A(e_2^A + e_2^B) - x_1^A x_2^A$$

General Equilibrium: No Production

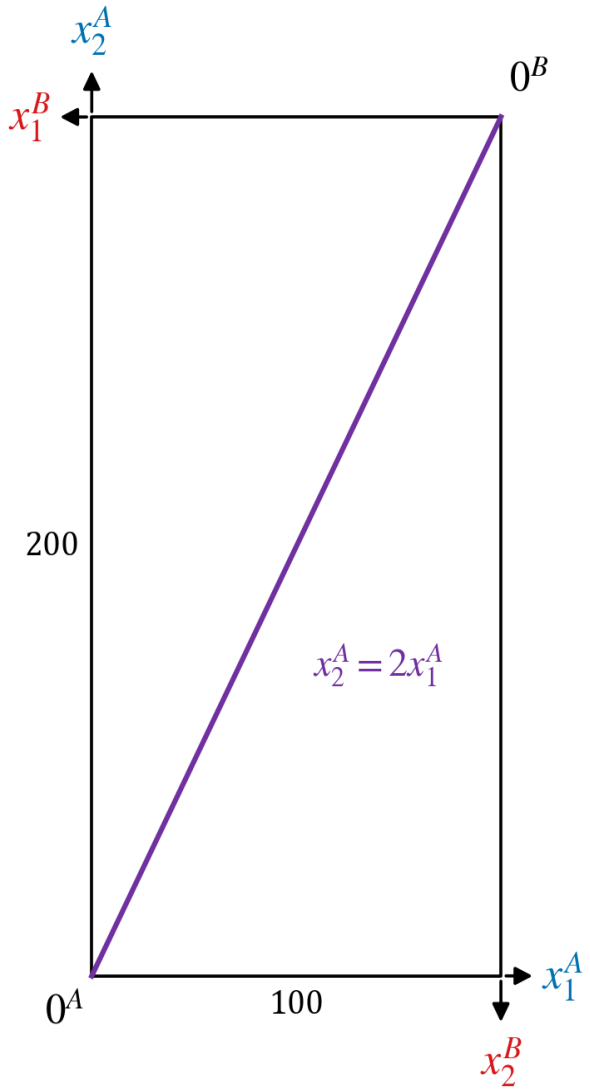
- ***Example 6.5*** (continued):
  - The $x_1^A x_2^A$ terms cancel, so

$$x_2^A = \frac{e_2^A + e_2^B}{e_1^A + e_1^B} x_1^A = \frac{350 + 50}{100 + 100} x_1^A = 2x_1^A$$

  - for all $x_1^A \in [0,200]$.

General Equilibrium: No Production

- ***Example 6.5*** (continued):
  - The set of Pareto-efficient allocations is the diagonal of the Edgeworth box, $x_2^A = 2x_1^A$.
  - It connects the origins of consumers A and B, since the aggregate endowment is (200, 400).



General Equilibrium: No Production

- ***Blocking coalitions***: Consider a nonempty coalition $S \subseteq \{1, \ldots, I\}$. We say that S blocks the feasible allocation $\mathbf{x}$ if there are bundles $(\mathbf{y}^i)_{i \in S}$ such that:
  - (1) they are feasible for S:

$$\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$$

  - (2) $\mathbf{y}^i \gtrsim^i \mathbf{x}^i$ for all $i \in S$, and $\mathbf{y}^i \succ^i \mathbf{x}^i$ for some $i \in S$.

General Equilibrium: No Production

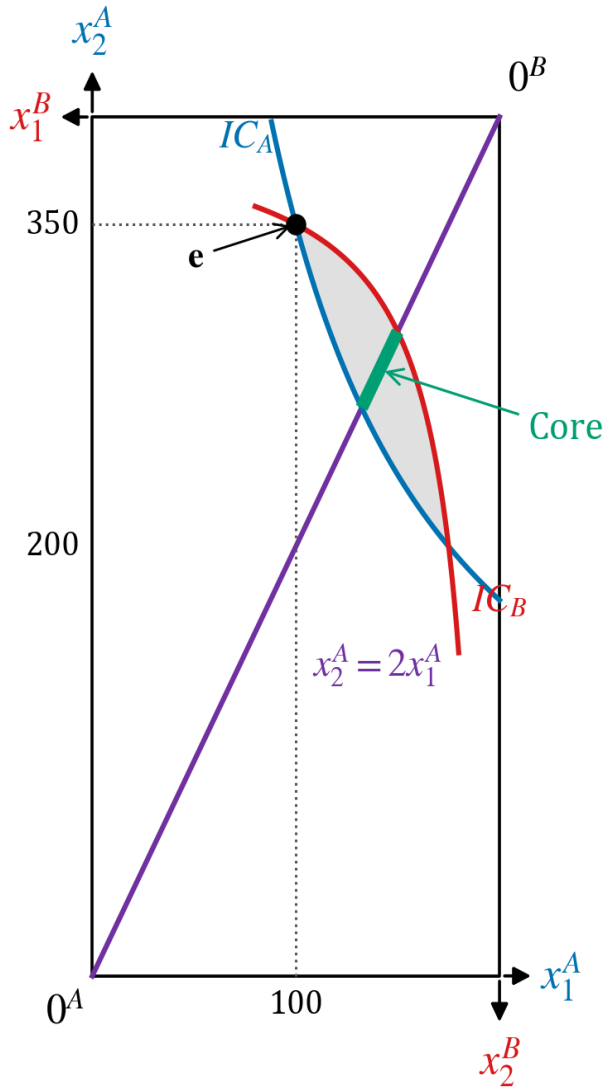
- ***Core***: The core of an exchange economy with endowment $\mathbf{e}$, denoted $C(\mathbf{e})$, is the set of all unblocked feasible allocations.
- ***Equilibrium in a barter economy***: An allocation is an equilibrium of the barter economy if it is in the core $C(\mathbf{e})$, the set of feasible allocations that no coalition blocks.

General Equilibrium: No Production

- Core allocations:
  - a) leave no consumer worse off than at his endowment, since a single consumer would otherwise block (in the box: the lens, including its boundary);
  - b) do not allow for further Pareto improvements (the grand coalition cannot block), so they lie on the contract curve.

General Equilibrium: No Production

- ***Example 6.5*** (continued):
  - The indifference curves through the endowment point form the lens-shaped (shaded) area.
  - The core is the segment of the contract curve inside this area.



General Equilibrium: Competitive Markets

- The barter economy did not require prices for an equilibrium to arise.
- Now we explore the equilibrium in economies where we allow prices to emerge.
- Order of analysis:
  - consumers' preferences
  - the excess demand function
  - the equilibrium allocations in competitive markets (i.e., *Walrasian equilibrium allocations*)

General Equilibrium: Competitive Markets

- ***Consumers***:
  - Consider each consumer's utility function to be continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}_+^L$.
  - Hence, for every price vector $\mathbf{p} \gg \mathbf{0}$, the UMP of every consumer i, with budget constraint
$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i$$
  - yields a unique solution, the Walrasian demand $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$.

General Equilibrium: Competitive Markets

- A solution exists since the budget set is compact for $\mathbf{p} \gg \mathbf{0}$, and it is unique by strict quasiconcavity.
- $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ is continuous in $\mathbf{p}$ (theorem of the maximum).

General Equilibrium: Competitive Markets

– Intuitively, individual i's income comes from selling his endowment $\mathbf{e}^i$ at market prices $\mathbf{p}$, producing $\mathbf{p} \cdot \mathbf{e}^i = p_1 e_1^i + \cdots + p_L e_L^i$ dollars to be used in the purchase of bundle $\mathbf{x}^i$.

General Equilibrium: Competitive Markets

- ***Excess demand***:
  - Summing the Walrasian demands $x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ for good k of every individual in the economy, we obtain the aggregate demand for good k.
  - The difference between the aggregate demand and the aggregate endowment of good k yields the excess demand of good k:

$$z_k(\mathbf{p}) = \sum_{i=1}^{I} x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - \sum_{i=1}^{I} e_k^i$$

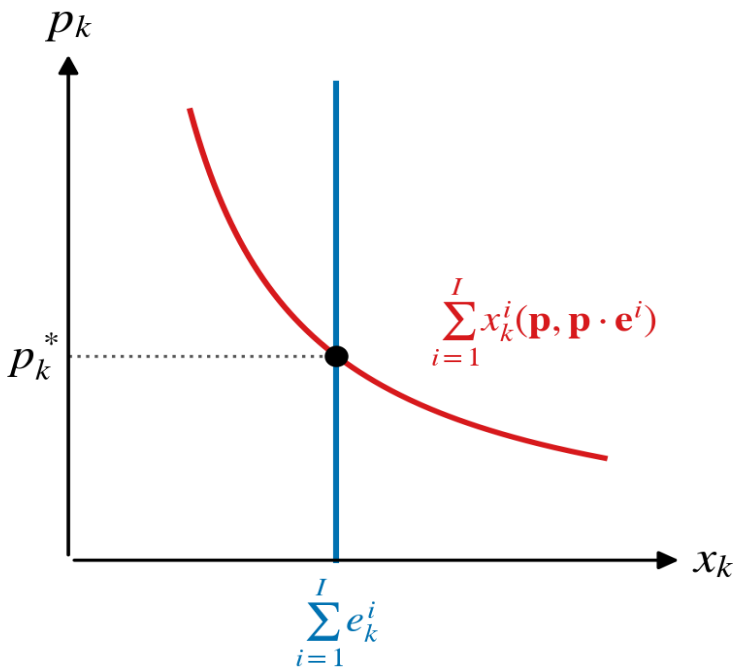
  - where $z_k(\mathbf{p}) \in \mathbb{R}$.

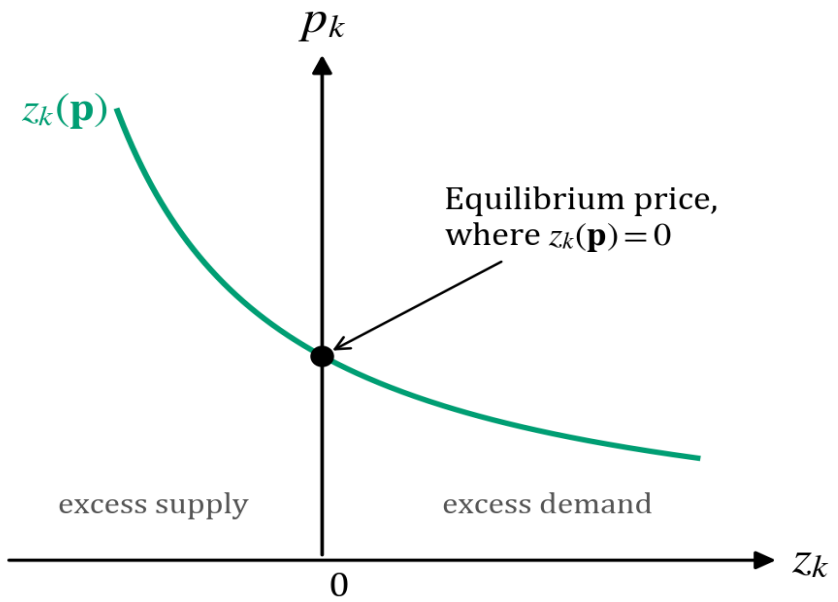
General Equilibrium: Competitive Markets

- When $z_k(\mathbf{p}) > 0$, the aggregate demand for good k exceeds its aggregate endowment.
  - Excess demand of good k
- When $z_k(\mathbf{p}) < 0$, the aggregate demand for good k falls short of its aggregate endowment.
  - Excess supply of good k

General Equilibrium: Competitive Markets

- Aggregate demand, aggregate endowment, and excess demand of good k (other prices held fixed)





General Equilibrium: Competitive Markets

- The excess demand function $\mathbf{z}(\mathbf{p}) \equiv (z_1(\mathbf{p}), \ldots, z_L(\mathbf{p}))$ satisfies the following properties:
- ***Walras' law***: $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$.
  - Since utility is strictly increasing, every consumer exhausts his income at every $\mathbf{p} \gg \mathbf{0}$,

$$\sum_{k=1}^{L} p_k x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) = \sum_{k=1}^{L} p_k e_k^i$$

General Equilibrium: Competitive Markets

- ***Walras' law*** (continued):
  - Equivalently, for every consumer i,

$$\sum_{k=1}^{L} p_k [x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - e_k^i] = 0$$

General Equilibrium: Competitive Markets

– Summing over all individuals,

$$\sum_{i=1}^{I}\sum_{k=1}^{L} p_k\left[x_k^i\left(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i\right) - e_k^i\right] = 0$$

– We can re-write the above expression as

$$\sum_{k=1}^{L}\sum_{i=1}^{I} p_k\left[x_k^i\left(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i\right) - e_k^i\right] = 0$$

which is equivalent to

$$\sum_{k=1}^{L} p_k \underbrace{\left(\sum_{i=1}^{I}\left[x_k^i\left(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i\right)\right] - \sum_{i=1}^{I} e_k^i\right)}_{z_k(\mathbf{p})} = 0$$

– Hence,

$$\sum_{k=1}^{L} p_k \cdot z_k(\mathbf{p}) = \mathbf{p}\cdot\mathbf{z}(\mathbf{p}) = 0$$

General Equilibrium: Competitive Markets

- In a two-good economy, Walras' law implies
$$p_1 z_1(\mathbf{p}) = -p_2 z_2(\mathbf{p})$$
- Intuition: if there is excess demand in market 1, $z_1(\mathbf{p}) > 0$, there must be excess supply in market 2, $z_2(\mathbf{p}) < 0$.

General Equilibrium: Competitive Markets

- Hence, if market 1 is in equilibrium, $z_1(\mathbf{p}) = 0$, then so is market 2, $z_2(\mathbf{p}) = 0$.
- More generally, if the markets of $L - 1$ goods are in equilibrium, then so is the Lth market (since $p_L > 0$, Walras' law gives $p_L z_L(\mathbf{p}) = 0$, so $z_L(\mathbf{p}) = 0$).

General Equilibrium: Competitive Markets

- ***Continuity***: $\mathbf{z}(\mathbf{p})$ is continuous at every $\mathbf{p} \gg \mathbf{0}$.
  - This follows from individual Walrasian demands being continuous in prices.
- ***Homogeneity***: $\mathbf{z}(\lambda\mathbf{p}) = \mathbf{z}(\mathbf{p})$ for all $\lambda > 0$.
  - This follows from Walrasian demands being homogeneous of degree zero in prices.
  - In other words, Walrasian demand is unaffected by a common change in prices, λ, so the excess demand function is not affected either.

General Equilibrium: Competitive Markets

- In summary, the excess demand function satisfies three properties: Walras' law, continuity, and homogeneity of degree zero.
- We now use excess demand to define a Walrasian equilibrium allocation.

General Equilibrium: Competitive Markets

- ***Walrasian equilibrium***:
  - A price vector $\mathbf{p}^* \gg \mathbf{0}$ is a Walrasian equilibrium if aggregate excess demand is zero at that price vector, $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.
  - In words, price vector $\mathbf{p}^*$ clears all markets.
  - Alternatively, $\mathbf{p}^* \gg \mathbf{0}$ is a Walrasian equilibrium if each consumer solves his UMP and aggregate demand equals aggregate supply,

$$\sum_{i=1}^{I} \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i) = \sum_{i=1}^{I} \mathbf{e}^i$$

General Equilibrium: Competitive Markets

- The allocation

$$\mathbf{x}(\mathbf{p}^*) = (\mathbf{x}^1(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^1), \ldots, \mathbf{x}^I(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^I))$$

- is a ***Walrasian equilibrium allocation*** (WEA).
- $W(\mathbf{e})$ denotes the set of WEAs.

General Equilibrium: Competitive Markets

- ***Existence of a Walrasian equilibrium***:
  - Under the assumptions on preferences above, and if the aggregate endowment is strictly positive, $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, the excess demand function $\mathbf{z}(\mathbf{p})$:
    - is continuous and homogeneous of degree zero, and satisfies Walras' law;
    - is bounded below; and
    - satisfies a boundary condition: if some price approaches zero, excess demand for some good grows without bound.

General Equilibrium: Competitive Markets

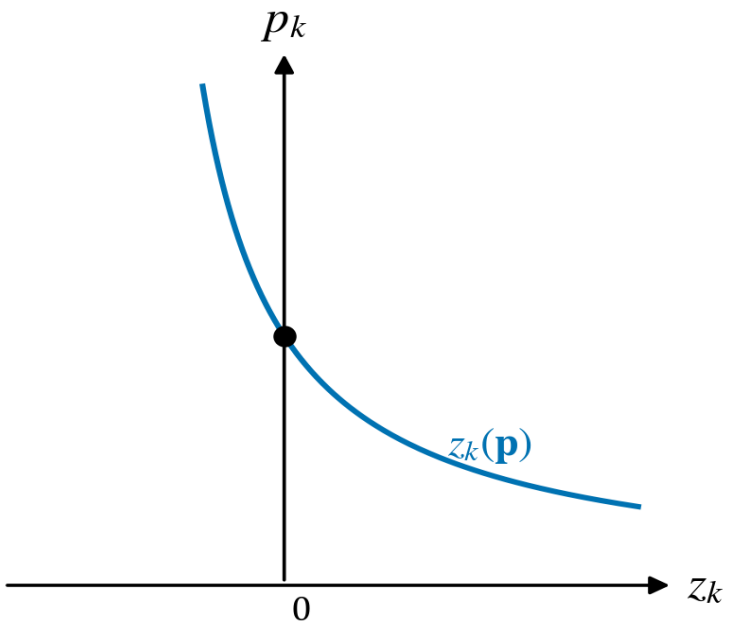
- ***Existence of a Walrasian equilibrium*** (continued):
  - Then, there is a price vector $\mathbf{p}^* \gg \mathbf{0}$ with $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.
  - See Mas-Colell, Whinston and Green (1995, Propositions 17.B.2 and 17.C.1); see also Varian (1992).

General Equilibrium: Competitive Markets

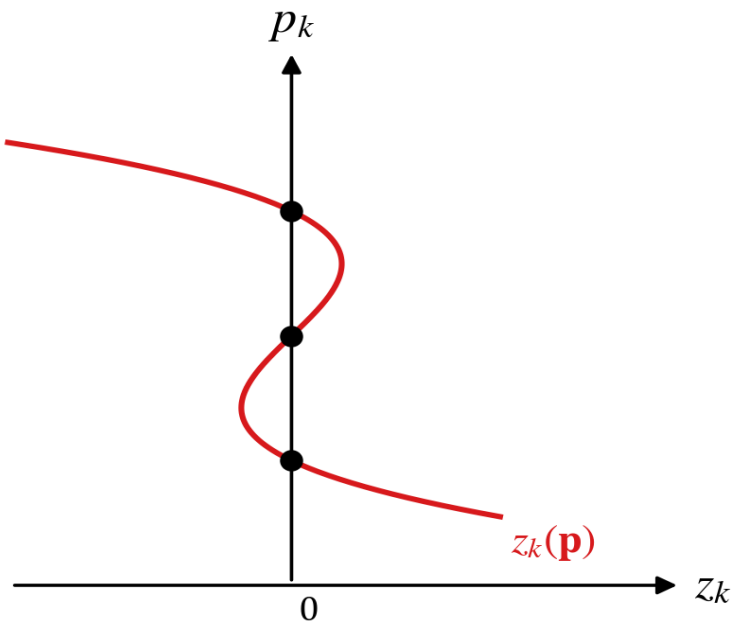
- ***Uniqueness of equilibrium prices***:
  - The equilibrium price is unique when excess demand crosses zero only once.
  - For instance, when all goods are gross substitutes: $z_k(\mathbf{p})$ is increasing in p_j for every $j \neq k$.

General Equilibrium: Competitive Markets

- ***Uniqueness of equilibrium prices***:



Crosses zero once:
unique equilibrium price



Crosses zero three times:
three equilibrium prices

General Equilibrium: Competitive Markets

- ***Example 6.6*** (Walrasian equilibrium allocation):
  - Consider the economy in Example 6.5. Consumer A solves

$$\max_{x_1^A, x_2^A} x_1^A x_2^A$$

  - subject to

$$p_1 x_1^A + p_2 x_2^A = 100p_1 + 350p_2$$

  - where the right side is the market value of his endowment, $\mathbf{e}^A$ = (100, 350).

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - The Lagrangian is
    $$\mathcal{L} = x_1^A x_2^A + \lambda_A[100p_1 + 350p_2 - p_1 x_1^A - p_2 x_2^A]$$
  - FOCs:
    $$x_2^A = \lambda_A p_1 \quad \text{and} \quad x_1^A = \lambda_A p_2$$
  - Dividing the first FOC by the second,
    $$MRS_{1,2}^A = \frac{x_2^A}{x_1^A} = \frac{p_1}{p_2}$$
  - or, rearranging,
    $$p_1 x_1^A = p_2 x_2^A$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Substituting this into consumer A's budget constraint yields

$$p_1 x_1^A + p_1 x_1^A = p_1 \cdot 100 + p_2 \cdot 350$$

$$\text{or } x_1^A = 50 + 175\frac{p_2}{p_1}$$

    which is consumer A's Walrasian demand for good 1.
  - Plugging this value back into $p_1 x_1^A = p_2 x_2^A$ yields

$$p_1\left(50 + 175\frac{p_2}{p_1}\right) = p_2 x_2^A$$

$$\text{or } x_2^A = 175 + 50\frac{p_1}{p_2}$$

    which is consumer A's Walrasian demand for good 2.

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Consumer B solves the same problem, but with endowment $\mathbf{e}^B$ = (100, 50):

$$\max_{x_1^B, x_2^B} x_1^B x_2^B$$

  - subject to

$$p_1 x_1^B + p_2 x_2^B = 100p_1 + 50p_2$$

  - The Lagrangian is

$$\mathcal{L} = x_1^B x_2^B + \lambda_B [100p_1 + 50p_2 - p_1 x_1^B - p_2 x_2^B]$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - FOCs:

$$x_2^B = \lambda_B p_1 \quad \text{and} \quad x_1^B = \lambda_B p_2$$

  - Dividing the first FOC by the second,

$$MRS_{1,2}^B = \frac{x_2^B}{x_1^B} = \frac{p_1}{p_2}$$

  - or, rearranging,

$$p_1 x_1^B = p_2 x_2^B$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Substituting $p_1 x_1^B = p_2 x_2^B$ into consumer B's budget constraint yields

$$p_1 x_1^B + p_1 x_1^B = 100p_1 + 50p_2$$

  - or

$$x_1^B = 50 + 25\frac{p_2}{p_1}$$

  - which is consumer B's Walrasian demand for good 1.

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Plugging this value back into $p_1 x_1^B = p_2 x_2^B$ yields

$$p_1(50 + 25\frac{p_2}{p_1}) = p_2 x_2^B$$

  - or

$$x_2^B = 25 + 50\frac{p_1}{p_2}$$

  - which is consumer B's Walrasian demand for good 2.

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - For good 1, the feasibility constraint is
$$x_1^A + x_1^B = 100 + 100$$
  - Substituting the Walrasian demands,

$$(50 + 175\frac{p_2}{p_1}) + (50 + 25\frac{p_2}{p_1}) = 200, \qquad \text{so} \quad \frac{p_2}{p_1} = \frac{1}{2}$$

  - Plugging the relative prices into the Walrasian demands yields the WEA:

$$(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2}) = (137.5{,}275; 62.5{,}125; 2)$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Where do these numbers come from? The feasibility constraint for good 1 simplifies to

$$100 + 200\frac{p_2}{p_1} = 200, \qquad \text{so} \quad \frac{p_2}{p_1} = \frac{1}{2} \quad \text{and} \quad \frac{p_1}{p_2} = 2$$

  - Plugging the price ratio into consumer A's Walrasian demands:

$$x_1^A = 50 + 175(\frac{1}{2}) = 137.5$$

$$x_2^A = 175 + 50(2) = 275$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Similarly, for consumer B:

$$x_1^B = 50 + 25(\frac{1}{2}) = 62.5$$

$$x_2^B = 25 + 50(2) = 125$$

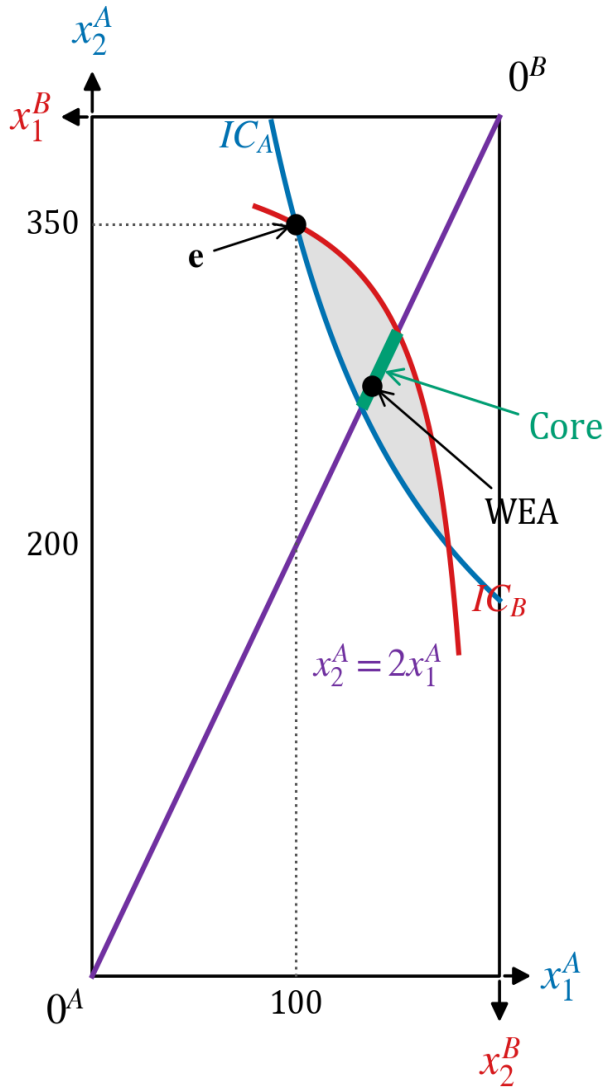
  - The market for good 2 clears as well,

$$x_2^A + x_2^B = 275 + 125 = 400 = 350 + 50$$

  - as Walras' law predicts.

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Initial allocation,
  - core allocation, and
  - Walrasian equilibrium allocations (WEA).



General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Core allocations lie on the contract curve, $x_2^A = 2x_1^A$, where both consumers are at least as well off as at their endowments.
  - ***Lower bound***: consumer A must reach his utility at $\mathbf{e}^A$, $u^A = x_1^A(2x_1^A) = 2(x_1^A)^2 \geq 100 \cdot 350 = 35{,}000$
  - which yields

$$x_1^A \geq \sqrt{17{,}500} \approx 132.3, \qquad \text{so} \quad x_2^A \geq 264.6$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - On the contract curve, consumer B's bundle also lies on the ray from his origin,
    $$x_2^B = 400 - 2x_1^A = 2(200 - x_1^A) = 2x_1^B$$
  - ***Upper bound***: B must reach his utility at $\mathbf{e}^B$,
    $$u^B = x_1^B(2x_1^B) = 2(x_1^B)^2 \geq 100 \cdot 50 = 5{,}000$$
  - which yields
    $$x_1^B \geq \sqrt{2{,}500} = 50, \qquad \text{so} \quad x_2^B \geq 100$$

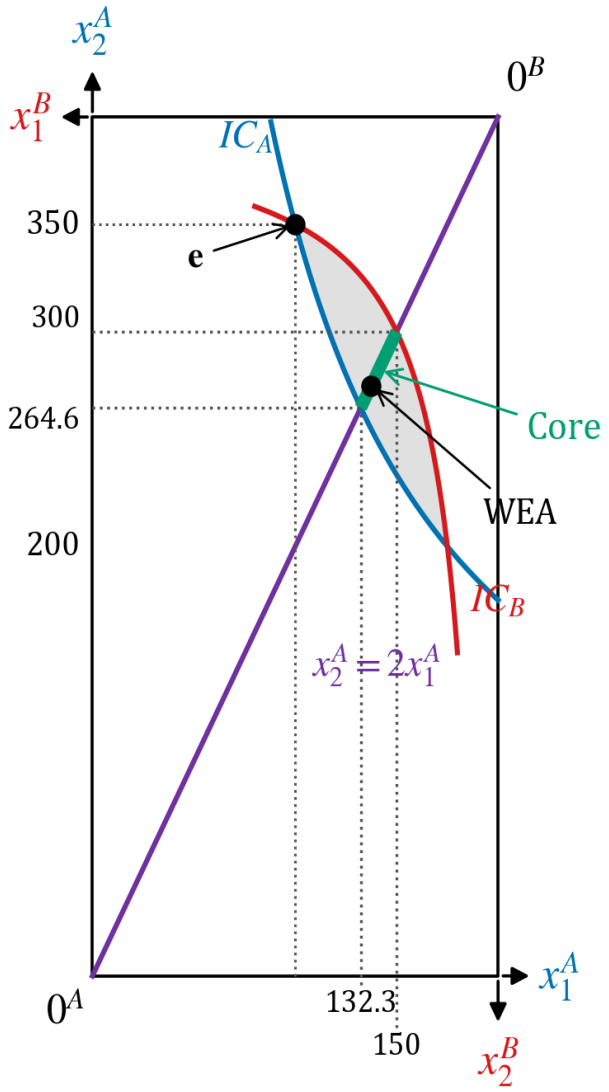
General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Since consumer A receives the rest of the aggregate endowment, the upper bound for consumer A is

$$x_1^A = 200 - x_1^B \leq 200 - 50 = 150$$
$$x_2^A = 400 - x_2^B \leq 400 - 100 = 300$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Hence, the core is
    $$132.3 \leq x_1^A \leq 150,$$
    $$\text{with} \quad x_2^A = 2x_1^A$$
  - where consumer B receives the rest of the endowment.
  - The WEA, $x_1^A = 137.5$, lies in the core.



General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Consumer A: utility $137.5 \times 275 = 37{,}812.5$ at the WEA, and $100 \times 350 = 35{,}000$ at $\mathbf{e}^A$,
    $$u^A(WEA) = 37{,}812.5 > 35{,}000 = u^A(\mathbf{e}^A)$$
  - Consumer B: utility $62.5 \times 125 = 7{,}812.5$ at the WEA, and $100 \times 50 = 5{,}000$ at $\mathbf{e}^B$,
    $$u^B(WEA) = 7{,}812.5 > 5{,}000 = u^B(\mathbf{e}^B)$$
  - Both consumers gain from trade, as every core allocation requires.

General Equilibrium: Competitive Markets

- ***Equilibrium allocations must be in the core***:
  - Under the assumptions on preferences above, every WEA is in the core, $W(\mathbf{e}) \subseteq C(\mathbf{e})$.
- *Proof* (by contradiction):
  - Take a WEA $\mathbf{x}(\mathbf{p}^*)$ with equilibrium price $\mathbf{p}^*$, but $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$.
  - Since $\mathbf{x}(\mathbf{p}^*)$ is a WEA, it must be feasible.

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - If $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$, we can find a coalition S and bundles $(\mathbf{y}^i)_{i \in S}$ such that
    $$u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)) \quad \text{for all} \quad i \in S$$
  - with strict inequality for at least one $i \in S$,
    $$u^i(\mathbf{y}^i) > u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i))$$

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - Allocation $\mathbf{y}$ is feasible for the coalition,

$$\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$$

  - Multiplying both sides of the feasibility condition by $\mathbf{p}^*$ yields

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$$

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - Since $\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)$ maximizes u^i on consumer i's budget set, any bundle he strictly prefers costs more: $\mathbf{p}^* \cdot \mathbf{y}^i > \mathbf{p}^* \cdot \mathbf{e}^i$ if $u^i(\mathbf{y}^i) > u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i))$

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - If $u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i))$ but $\mathbf{p}^* \cdot \mathbf{y}^i < \mathbf{p}^* \cdot \mathbf{e}^i$, strict monotonicity gives an affordable bundle strictly better than $\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)$, a contradiction.
  - Hence,

$$\mathbf{p}^* \cdot \mathbf{y}^i \geq \mathbf{p}^* \cdot \mathbf{e}^i \quad \text{for all} \quad i \in S$$

  - with strict inequality for at least one $i \in S$.

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - Hence, summing over all consumers in the coalition S, we obtain

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i > \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{x}^i\left(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i\right) = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$$

    which contradicts $\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$.

  - Therefore, all WEAs must be part of the core, i.e.,

$$\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$$

General Equilibrium: Competitive Markets

- ***Remarks***:
  - Since a WEA exists (existence result above) and every WEA is in the core, the core is nonempty.
  - All core allocations are Pareto efficient (the grand coalition cannot block a core allocation): we cannot increase the welfare of one consumer without decreasing that of at least one other consumer.
  - Hence, all WEAs (which are part of the core) are also Pareto efficient.

General Equilibrium: Competitive Markets

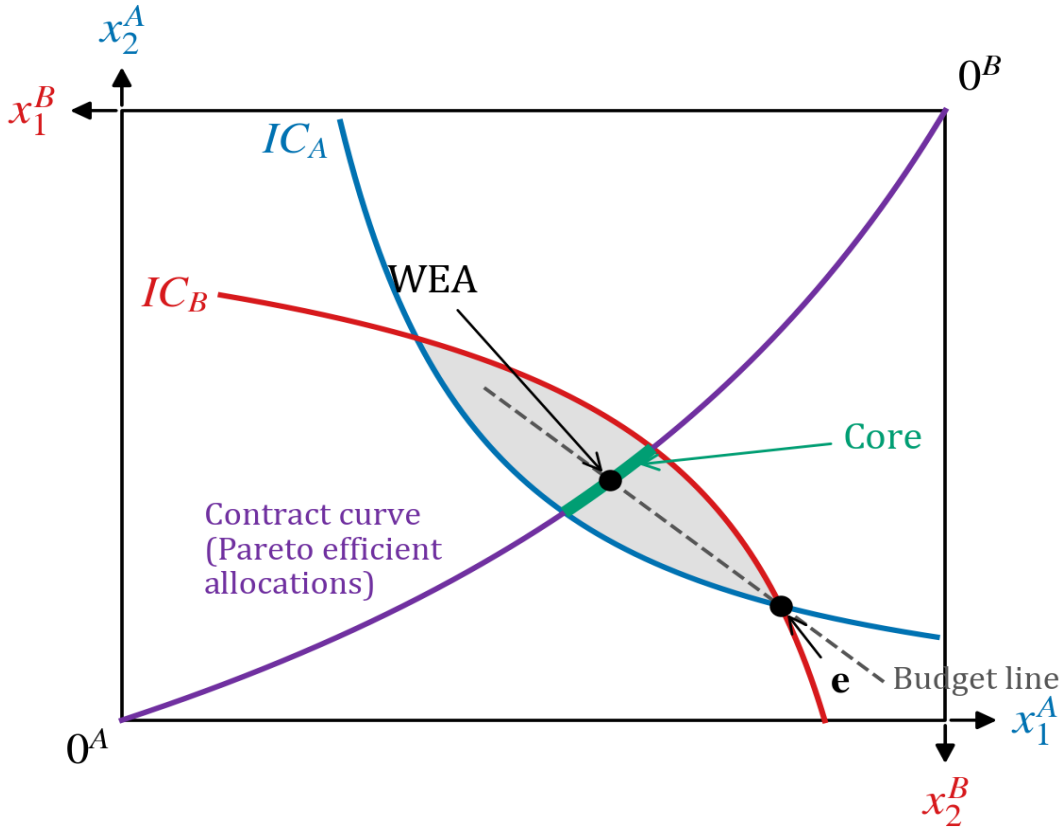
- ***First welfare theorem***: Under the assumptions on preferences above (strictly increasing utility is the key one; local nonsatiation is enough), every WEA is Pareto efficient.
  - Every WEA is in the core (previous result), and no core allocation can be Pareto dominated, since the grand coalition would block it.

General Equilibrium: Competitive Markets

- ***First welfare theorem*** (continued):
  - The WEA lies in the core (the segment of the contract curve within the lens-shaped area).
  - The core is a subset of the set of Pareto efficient allocations.

General Equilibrium: Competitive Markets

- First welfare theorem



General Equilibrium: Competitive Markets

- ***Second welfare theorem***:
  - Under the assumptions on preferences above, let $\overline{\mathbf{x}} \gg \mathbf{0}$ be a Pareto-efficient allocation.
  - Let $\mathbf{p}^* \gg \mathbf{0}$ be the price vector with

$$\frac{{p_1}^*}{{p_2}^*} = MRS^i_{1,2}(\overline{\mathbf{x}}^i)$$

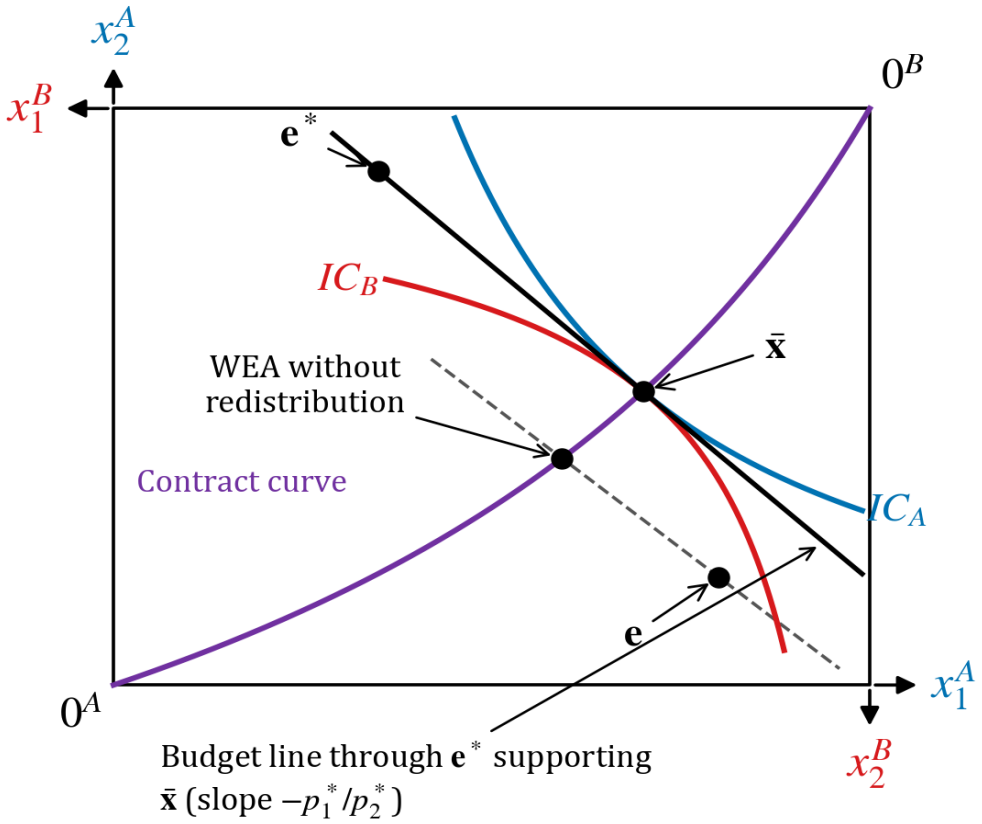
  - for every consumer i.

General Equilibrium: Competitive Markets

- ***Second welfare theorem*** (continued):
  - If endowments are redistributed to any $\mathbf{e}^{*i}$ with $\sum_{i=1}^{I} \mathbf{e}^{*i} = \sum_{i=1}^{I} \mathbf{e}^{i}$ and $\mathbf{p}^{*} \cdot \mathbf{e}^{*i} = \mathbf{p}^{*} \cdot \bar{\mathbf{x}}^{i}$ for every i (for example, $\mathbf{e}^{*i} = \bar{\mathbf{x}}^{i}$),
  - then $(\bar{\mathbf{x}}, \mathbf{p}^{*})$ is a Walrasian equilibrium of the economy with endowments $\mathbf{e}^{*}$.

General Equilibrium: Competitive Markets

- Second welfare theorem



General Equilibrium: Competitive Markets

- ***Example 6.7*** (WEA and second welfare theorem):
  - Consider an economy with utility functions
    $$u^A = x_1^A x_2^A \quad \text{and} \quad u^B = \min\{x_1^B, x_2^B\}$$
  - and initial endowments
    $$\mathbf{e}^A = (3,1) \quad \text{and} \quad \mathbf{e}^B = (1,3)$$
  - Good 2 is the numeraire, $p_2 = 1$.
  - Consumer B's utility is not strictly increasing, but it is locally nonsatiated, which is all the first welfare theorem needs.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  1) Pareto efficient allocations (PEAs):

  – At a Pareto efficient allocation, B cannot hold more of one good than the other: the excess could go to A, raising u^A without lowering u^B. Hence,

$$x_1^B = x_2^B$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Given the feasibility constraints

$$x_1^A + x_1^B = 4$$
$$x_2^A + x_2^B = 4$$

  - substitute x_2^B for x_1^B in the first constraint to get

$$x_2^B = 4 - x_1^A$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  1) Pareto efficient allocations:

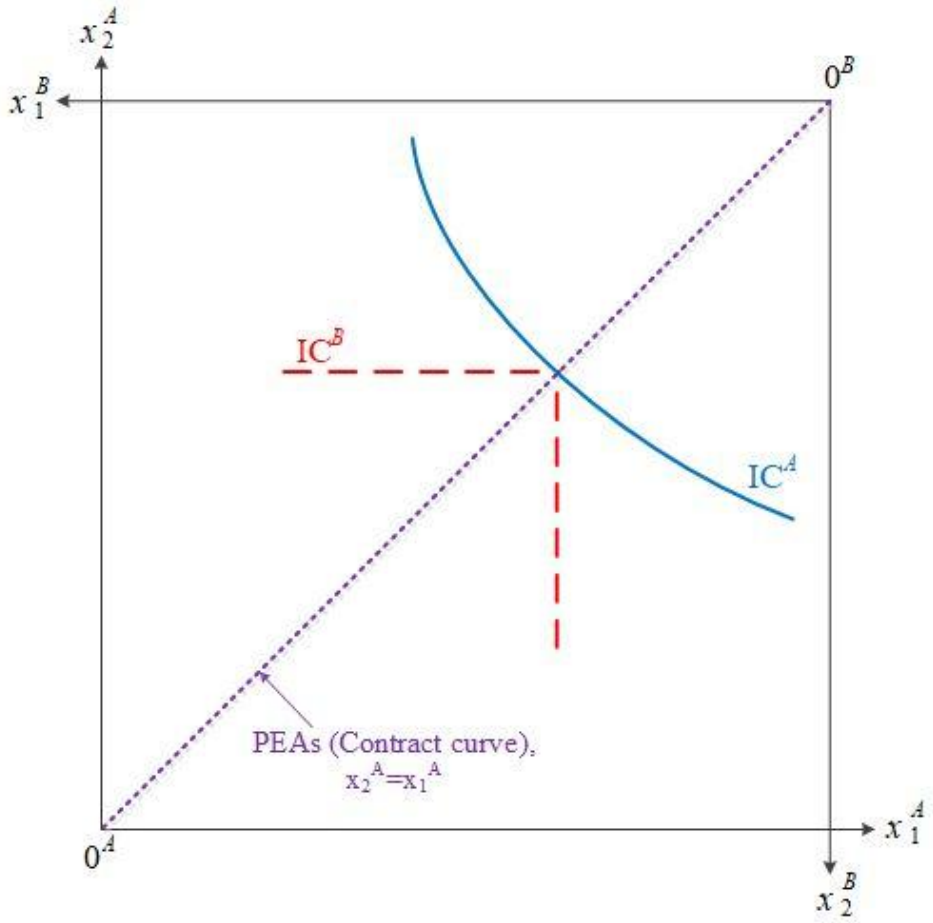
  – Substituting the above expression in the second constraint yields

$$x_2^A + \underbrace{(4 - x_1^A)}_{x_2^B} = 4 \iff x_2^A = x_1^A$$

  – This defines the contract curve, i.e., the set of Pareto efficient allocations.

General Equilibrium: Competitive Markets

- Contract curve (PEAs)



General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  2) WEA:

  – Consumer A's maximization problem is

$$\max_{x_1^A,\, x_2^A} \; x_1^A x_2^A$$

$$\text{s.t. } p_1 x_1^A + x_2^A \leq p_1 \cdot 3 + 1$$

  – FOCs:

$$x_2^A - \lambda p_1 = 0$$

$$x_1^A - \lambda = 0$$

$$p_1 x_1^A + x_2^A = 3p_1 + 1$$

  where λ is the Lagrange multiplier.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Combining the first two FOCs,

$$\lambda = \frac{x_2^A}{p_1} = x_1^A, \qquad \text{so} \quad x_2^A = p_1 x_1^A$$

  - Hence, consumer A spends the same amount on each good. Substituting into the budget constraint,

$$p_1 x_1^A + p_1 x_1^A = 3p_1 + 1$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Hence, with income $m^A = 3p_1 + 1$ (the value of his endowment), consumer A's Walrasian demands are

$$x_1^A(p_1) = \frac{m^A}{2p_1} = \frac{3p_1 + 1}{2p_1}$$

$$x_2^A(p_1) = \frac{m^A}{2} = \frac{3p_1 + 1}{2}$$

  - Next, we find consumer B's Walrasian demands.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Consumer B solves

$$\max_{x_1^B, x_2^B} \min\{x_1^B, x_2^B\} \quad \text{s.t.} \quad p_1 x_1^B + x_2^B \leq p_1 \cdot 1 + 3$$

  - He consumes at the kink, $x_1^B = x_2^B$, so

$$x_1^B = x_2^B = \frac{p_1 + 3}{p_1 + 1}$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Market clearing for good 1 requires
$$\frac{3p_1 + 1}{2p_1} + \frac{p_1 + 3}{p_1 + 1} = 4$$
  - which simplifies to
$$3{p_1}^2 - 2p_1 - 1 = (3p_1 + 1)(p_1 - 1) = 0$$
  - whose unique positive root is $p_1 = 1$.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Substituting $p_1 = 1$ into the Walrasian demands,

$$x_1^A = \frac{3+1}{2} = 2 \quad \text{and} \quad x_2^A = \frac{3+1}{2} = 2$$

$$x_1^B = x_2^B = \frac{1+3}{1+1} = 2$$

  - The market for good 2 clears as well, as Walras' law predicts,

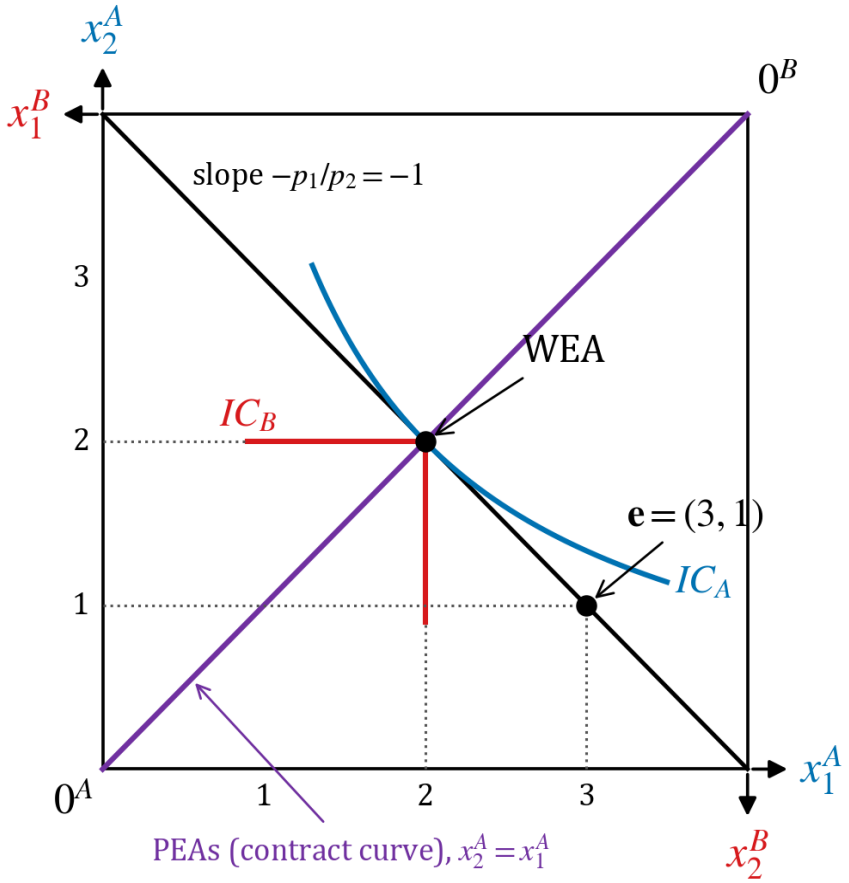
$$x_2^A + x_2^B = 2 + 2 = 4 = 1 + 3$$

  - Thus, the WEA is

$$(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2}) = (2,2; 2,2; 1)$$

General Equilibrium: Competitive Markets

- Second welfare theorem: WEA and PEAs



General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued): Second welfare theorem.
  - Suppose that we want to reach the Pareto-efficient allocation

$$\hat{\mathbf{x}}^A = (1,1) \quad \text{and} \quad \hat{\mathbf{x}}^B = (3,3)$$

  - which lies on the contract curve and favors consumer B.
  - Without transfers, the market yields the WEA

$$(x_1^A, x_2^A; x_1^B, x_2^B) = (2,2; 2,2)$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - ***Step 1***: find the price ratio that supports $\hat{\mathbf{x}}^A$. Consumer A's MRS at (1,1) is

$$MRS_{1,2}^A = \frac{x_2^A}{x_1^A} = \frac{1}{1} = 1, \qquad \text{so} \quad \frac{p_1}{p_2} = 1$$

  - Consumer B's indifference curve has a kink at $\hat{\mathbf{x}}^B$, so he demands $x_1^B = x_2^B$ at any prices, including $p_1 = p_2 = 1$.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - ***Step 2***: find the lump-sum transfers T_A and T_B, with $T_A = -T_B$, so that each consumer can afford his bundle in $\hat{\mathbf{x}}^A$ and $\hat{\mathbf{x}}^B$.
  - At $p_1 = p_2 = 1$, the endowments are worth

$$\mathbf{p} \cdot \mathbf{e}^A = 1(3) + 1(1) = 4 \quad \text{and} \quad \mathbf{p} \cdot \mathbf{e}^B = 1(1) + 1(3) = 4$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Consumer A's income after the transfer must cover the cost of $\hat{\mathbf{x}}^A$:

$$4 + T_A = \mathbf{p} \cdot \hat{\mathbf{x}}^A = 1(1) + 1(1) = 2, \qquad \text{so} \quad T_A = -2$$

  - Similarly, for consumer B:

$$4 + T_B = \mathbf{p} \cdot \hat{\mathbf{x}}^B = 1(3) + 1(3) = 6, \qquad \text{so} \quad T_B = 2$$

  - Hence, $T_A = -T_B$: consumer A transfers 2 to consumer B, so the transfers add up to zero.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - ***Step 3***: check that $\hat{\mathbf{x}}^A$ and $\hat{\mathbf{x}}^B$ form a WEA given the transfers. Using consumer A's Walrasian demands with income $m^A = 4 + T_A = 2$,

$$x_k^A = \frac{m^A}{2p_k} = \frac{2}{2(1)} = 1 \quad \text{for} \quad k = 1,2$$

  - Using consumer B's Walrasian demand with income $m^B = 4 + T_B = 6$,

$$x_1^B = x_2^B = \frac{m^B}{p_1 + p_2} = \frac{6}{1 + 1} = 3$$

General Equilibrium: Competitive Markets

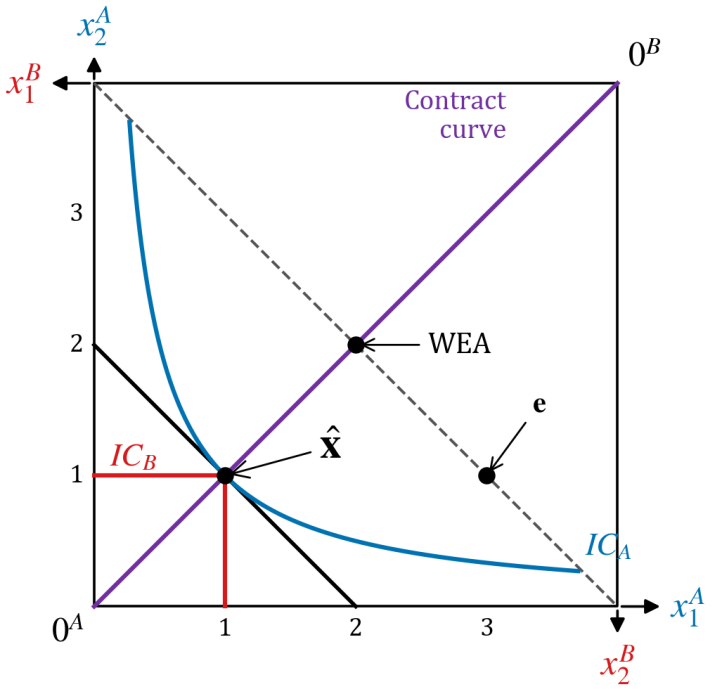
- ***Example 6.7*** (continued):
  - Both markets clear,

$$x_1^A + x_1^B = 1 + 3 = 4 = 3 + 1$$
$$x_2^A + x_2^B = 1 + 3 = 4 = 1 + 3$$

  - Hence, the Pareto-efficient allocation is a WEA with transfers $T_A = -2$ and $T_B = 2$: consumer A sells two units of good 1 to finance his transfer.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Dashed line: no transfers, WEA at (2,2).
  - Solid line: budget line after the transfer.



General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - ***Redistributing goods***: suppose instead that the planner reaches the same allocation by moving units of the goods, not money.
  - Consumer A gives t_1 units of good 1 and t_2 units of good 2 to consumer B (a negative t_k means that B gives to A), so the new endowments are

$$\hat{\mathbf{e}}^A = (3 - t_1, 1 - t_2)$$
$$\hat{\mathbf{e}}^B = (1 + t_1, 3 + t_2)$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - We look for t_1 and t_2 such that the WEA from these new endowments is

$$\hat{\mathbf{x}}^A = (1,1) \quad \text{and} \quad \hat{\mathbf{x}}^B = (3,3)$$

  - As in Step 1, consumer A's MRS at (1,1) equals 1, so the supporting prices are $p_1 = p_2 = 1$.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Consumer A's income is now the value of his new endowment,
    $$m^A = p_1(3 - t_1) + p_2(1 - t_2) = 4 - t_1 - t_2$$
  - His Walrasian demands give $\hat{\mathbf{x}}^A = (1,1)$ only if
    $$x_k^A = \frac{m^A}{2p_k} = \frac{4 - t_1 - t_2}{2} = 1$$
  - that is,
    $$t_1 + t_2 = 2$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Consumer B's income is then
    $$m^B = (1 + t_1) + (3 + t_2) = 4 + t_1 + t_2 = 6$$
  - so his Walrasian demand is
    $$x_1^B = x_2^B = \frac{m^B}{p_1 + p_2} = \frac{6}{2} = 3$$
  - as desired, and both markets clear.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Hence, any transfer with $t_1 + t_2 = 2$ works, as long as consumer A owns enough of each good,

$$t_1 \leq 3 \quad \text{and} \quad t_2 \leq 1$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - For instance,

$$(t_1, t_2) = (2,0): \quad \hat{\mathbf{e}}^A = (1,1)$$
$$(t_1, t_2) = (1,1): \quad \hat{\mathbf{e}}^A = (2,0)$$
$$(t_1, t_2) = (3,-1): \quad \hat{\mathbf{e}}^A = (0,2)$$

  - In the first case no trade is needed; in the other two, A trades one unit of a good for one unit of the other.

General Equilibrium: Production

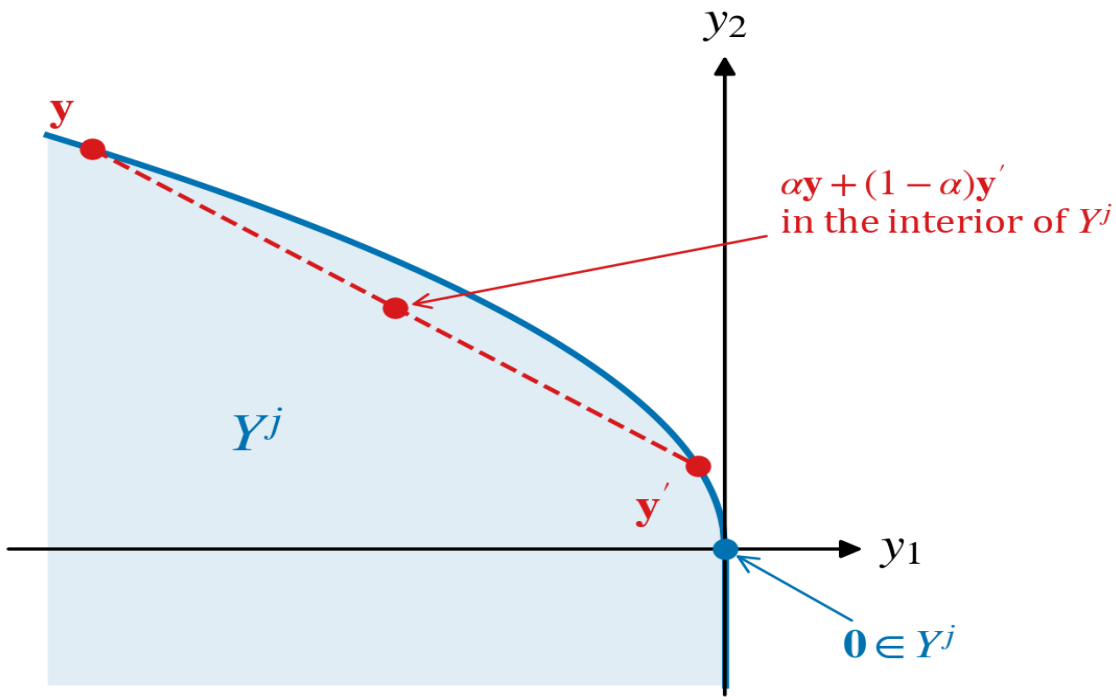
- Let us now extend our previous results to a setting where firms are also active.
- Assume J firms in the economy, each with production set Y^j, which satisfies:
  - Inaction is possible, $\mathbf{0} \in Y^j$.

General Equilibrium: Production

- Production set Y^j (continued):
  - Y^j is closed and bounded (compact), so a profit-maximizing plan exists at every $\mathbf{p}$; closedness also puts the frontier in Y^j.
  - Y^j is strictly convex, so convex combinations of two production plans lie in the interior of the production set.

General Equilibrium: Production

- Production set Y^j for firm j



General Equilibrium: Production

- Every firm j facing a fixed price vector $\mathbf{p} \gg \mathbf{0}$ independently and simultaneously solves

$$\max_{\mathbf{y}^j \in Y^j} \mathbf{p} \cdot \mathbf{y}^j$$

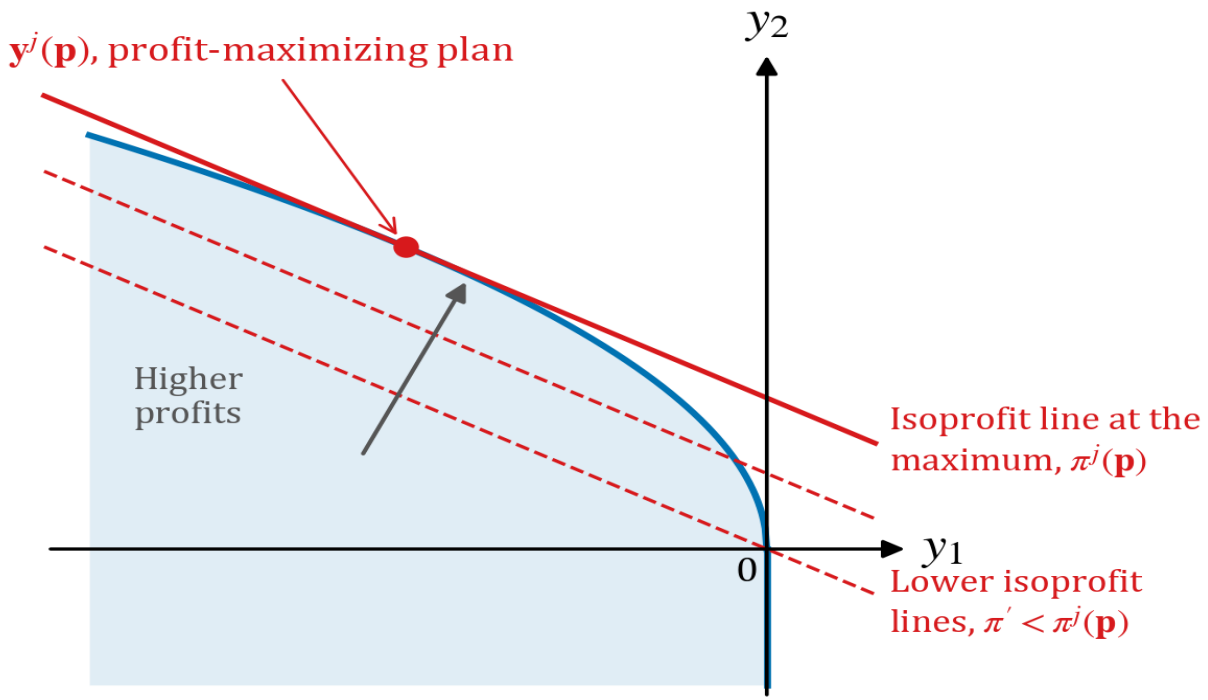
- A profit-maximizing production plan $\mathbf{y}^j(\mathbf{p})$ exists (Y^j is compact) and is unique (Y^j is strictly convex).

General Equilibrium: Production

- By the theorem of the maximum, both the argmax, $\mathbf{y}^j(\mathbf{p})$, and the value function, $\pi^j(\mathbf{p}) \equiv \mathbf{p} \cdot \mathbf{y}^j(\mathbf{p})$, are continuous in $\mathbf{p}$.

General Equilibrium: Production

- $\mathbf{y}^j(\mathbf{p})$ exists and is unique



General Equilibrium: Production

- ***Aggregate production set***:
  - The aggregate production set is the set of all sums of feasible plans,

$$Y = \{\mathbf{y}: \mathbf{y} = \sum_{j=1}^{J} \mathbf{y}^j, \qquad \text{where } \mathbf{y}^j \in Y^j \text{ for every } j\}$$

General Equilibrium: Production

- Define

$$\mathbf{y}(\mathbf{p}) = \sum_{j=1}^{J} \mathbf{y}^j(\mathbf{p})$$

- Then $\mathbf{y}(\mathbf{p})$ maximizes $\mathbf{p} \cdot \mathbf{y}$ over Y: aggregate profit maximization equals firm-by-firm maximization.

General Equilibrium: Production

- In an economy with J firms, each of them earning $\pi^j(\mathbf{p})$ profits in equilibrium, how are profits distributed?
- Assume that each individual i owns a share θ^{ij} of firm j's profits, where

$$0 \leq \theta^{ij} \leq 1 \text{ and } \sum_{i=1}^{I} \theta^{ij} = 1$$

General Equilibrium: Production

- This allows for multiple sharing profiles:
  - $\theta^{ij} = 1$: individual i owns all shares of firm j;
  - $\theta^{ij} = \frac{1}{I}$: all individuals hold equal shares of firm j.

General Equilibrium: Production

- Consumer i's budget constraint becomes

$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + \sum_{j=1}^{J} \theta^{ij} \pi^j(\mathbf{p})$$

- where $\sum_{j=1}^{J} \theta^{ij} \pi^j(\mathbf{p})$ is new relative to the standard budget constraint.

General Equilibrium: Production

- Let us express the budget constraint using

$$m^i(\mathbf{p}) \equiv \mathbf{p} \cdot \mathbf{e}^i + \sum_{j=1}^{J} \theta^{ij} \pi^j(\mathbf{p})$$

- that is, $\mathbf{p} \cdot \mathbf{x}^i \leq m^i(\mathbf{p})$, where $m^i(\mathbf{p}) > 0$, since $\pi^j(\mathbf{p}) \geq 0$ (inaction is possible) and $\mathbf{p} \cdot \mathbf{e}^i > 0$.

General Equilibrium: Production

- ***Equilibrium with production***:
  - We start by defining excess demand functions and use such a definition to identify the set of equilibrium allocations.
  - *Excess demand*: The excess demand function for good k is

$$z_k(\mathbf{p}) \equiv \sum_{i=1}^{I} x_k^i(\mathbf{p}, m^i(\mathbf{p})) - \sum_{i=1}^{I} e_k^i - \sum_{j=1}^{J} y_k^j(\mathbf{p})$$

    where $\sum_{j=1}^{J} y_k^j(\mathbf{p})$ is a new term relative to the analysis of general equilibrium without production.

General Equilibrium: Production

- Hence, the aggregate excess demand vector is

$$\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), z_2(\mathbf{p}), \dots, z_L(\mathbf{p}))$$

General Equilibrium: Production

- ***WEA with production***: $\mathbf{p}^* \gg \mathbf{0}$ is a Walrasian equilibrium price vector, and $(\mathbf{x}(\mathbf{p}^*), \mathbf{y}(\mathbf{p}^*))$ the WEA, if:
  - 1) $\mathbf{x}^i(\mathbf{p}^*, m^i(\mathbf{p}^*))$ solves consumer i's UMP;
  - 2) $\mathbf{y}^j(\mathbf{p}^*)$ solves firm j's PMP;
  - 3) markets clear,

$$\sum_{i=1}^{I} \mathbf{x}^i(\mathbf{p}^*, m^i(\mathbf{p}^*)) = \sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j(\mathbf{p}^*)$$

  - that is, $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

General Equilibrium: Production

- Setup: two firms; firm k produces good k with $F_k(K_k, L_k)$; input prices r and w; $F_{kK} = \frac{\partial F_k}{\partial K_k}$, and similarly for labor (here L denotes labor).
- From every consumer solving his UMP, the marginal rate of substitution between goods 1 and 2 satisfies, for any consumers i and h,

$$MRS_{1,2}^{i} = MRS_{1,2}^{h} = \frac{p_1}{p_2}$$

General Equilibrium: Production

- Similarly, from every firm solving its PMP, we obtain

$$p_1 F_{1K} = r \quad \text{and} \quad p_1 F_{1L} = w$$

  - for firm 1, and

$$p_2 F_{2K} = r \quad \text{and} \quad p_2 F_{2L} = w$$

  - for firm 2.

General Equilibrium: Production

- Dividing these expressions yields

$$MRTS^1_{L,K} \equiv \frac{F_{1L}}{F_{1K}} = \frac{w}{r}$$

  - for firm 1, and

$$MRTS^2_{L,K} \equiv \frac{F_{2L}}{F_{2K}} = \frac{w}{r}$$

  - for firm 2.

General Equilibrium: Production

- Therefore, $MRTS^1_{L,K} = MRTS^2_{L,K}$.
- Moving dK units of capital from firm 2 to firm 1 changes outputs by

$$dy_1 = F_{1K}dK \quad \text{and} \quad dy_2 = -F_{2K}dK$$

  - so the marginal rate of transformation is

$$MRT^K_{1,2} = -\frac{dy_2}{dy_1} = \frac{F_{2K}}{F_{1K}}$$

General Equilibrium: Production

- Dividing $p_1 F_{1K} = r$ by $p_2 F_{2K} = r$ yields

$$\frac{p_1 F_{1K}}{p_2 F_{2K}} = \frac{r}{r} = 1, \qquad \text{so} \quad \frac{p_1}{p_2} = \frac{F_{2K}}{F_{1K}} = MRT_{1,2}^{K}$$

- Similarly, dividing $p_1 F_{1L} = w$ by $p_2 F_{2L} = w$ yields

$$\frac{p_1}{p_2} = \frac{F_{2L}}{F_{1L}} = MRT_{1,2}^{L}$$

General Equilibrium: Production

- Combining $MRS_{1,2}^{i} = \frac{p_1}{p_2}$ for every consumer i with $\frac{p_1}{p_2} = MRT_{1,2}^{m}$ for every input $m \in \{K, L\}$ yields

$$MRS_{1,2}^{i} = MRT_{1,2}^{m} = \frac{p_1}{p_2}$$

General Equilibrium: Production

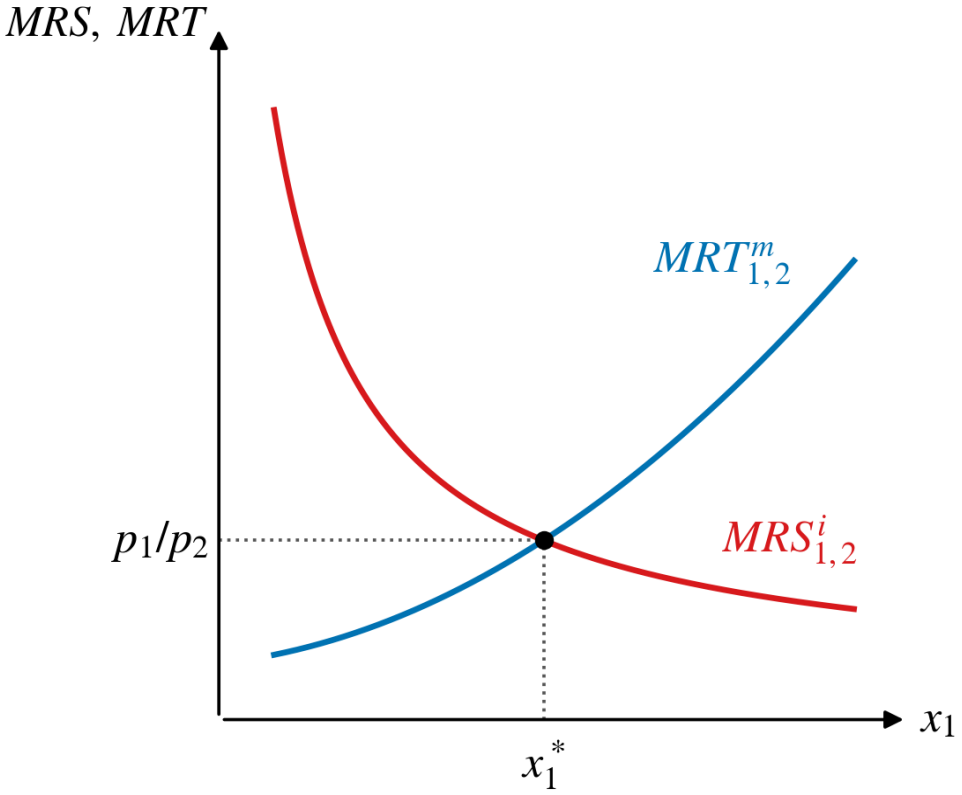
- Inspecting $MRS^i_{1,2} = MRT^m_{1,2} = \frac{p_1}{p_2}$, we see that
  - $MRS^i_{1,2} = \frac{MU^i_1}{MU^i_2}$ decreases in x_1 along an indifference curve, by strict quasiconcavity of u^i.
  - In contrast, $MRT^m_{1,2} = \frac{F_{2m}}{F_{1m}}$ increases in x_1 (by diminishing marginal products).

General Equilibrium: Production

- Intuitively, to increase x_1, we move units of input m from firm 2 to firm 1, which raises the marginal product of this input in firm 2 and lowers it in firm 1.
- Plotting $MRS^i_{1,2}$ and $MRT^m_{1,2}$ as functions of x_1, they cross at the equilibrium level of good 1, x_1^*, at a height of $\frac{p_1}{p_2}$.

General Equilibrium: Production

- Equilibrium with production



General Equilibrium: Production

- ***Existence***: Under the assumptions on preferences and production sets above, assume that the aggregate endowment is strictly positive in every good,

$$\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$$

- Then, there is a price vector $\mathbf{p}^* \gg \mathbf{0}$ such that a WEA exists, that is, $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

General Equilibrium: Production

- ***Example 6.8*** (Equilibrium with production):
  - Consider a two-consumer, two-good economy where consumer $i \in \{A, B\}$ has utility function $u^i = x_1^i x_2^i$.
  - There are two firms in this economy; each uses capital (K) and labor (L) to produce one of the two goods.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Firm 1 produces good 1 according to $y_1 = K_1^{0.75} L_1^{0.25}$.
  - Firm 2 produces good 2 according to $y_2 = K_2^{0.25} L_2^{0.75}$.
  - Consumer A is endowed with $(K^A, L^A) = (1,1)$, while consumer B is endowed with $(K^B, L^B) = (2,1)$.
  - Let us find a WEA in this economy with production.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - UMPs: Consumer i's maximization problem is

$$\max_{x_1^i, x_2^i} x_1^i x_2^i$$

  - subject to

$$p_1 x_1^i + p_2 x_2^i = rK^i + wL^i$$

  - where r and w are the prices of capital and labor, respectively.
  - Consumers are not endowed with goods 1 and 2. Each consumer i is only endowed with capital and labor, K^i and L^i, which he sells to the firms, earning rK^i from capital and wL^i from labor.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Both technologies exhibit constant returns to scale, so firms earn zero profits and income is $rK^i + wL^i$.
  - FOC:

$$\frac{p_1}{p_2} = MRS^i_{1,2} = \frac{x^i_2}{x^i_1}, \qquad \text{so} \quad p_1 x^i_1 = p_2 x^i_2$$

  - for $i \in \{A, B\}$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Taking the above equation for consumers A and B, and adding them together yields

$$p_1(x_1^A + x_1^B) = p_2(x_2^A + x_2^B)$$

    where $x_1^A + x_1^B$ is the left side of the feasibility condition $x_1^A + x_1^B = y_1 = K_1^{0.75} L_1^{0.25}$.
  - Substituting both feasibility conditions into the above expression, and re-arranging, yields

$$\frac{p_1}{p_2} = \frac{K_2^{0.25} L_2^{0.75}}{K_1^{0.75} L_1^{0.25}}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - PMPs: Firm 1's maximization problem is

$$\max_{K_1, L_1} p_1 K_1^{0.75} L_1^{0.25} - rK_1 - wL_1$$

  - FOCs:

$$r = 0.75 p_1 K_1^{-0.25} L_1^{0.25}$$
$$w = 0.25 p_1 K_1^{0.75} L_1^{-0.75}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Dividing the second FOC by the first gives the tangency condition for profit maximization

$$MRTS_{L,K}^{1} = \frac{w}{r}, \qquad \text{so} \quad \frac{w}{r} = \frac{1}{3}\frac{K_1}{L_1}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Likewise, firm 2's PMP gives the following FOCs:

$$r = 0.25 p_2 K_2^{-0.75} L_2^{0.75}$$
$$w = 0.75 p_2 K_2^{0.25} L_2^{-0.25}$$

  - Dividing the second FOC by the first gives the tangency condition for profit maximization

$$MRTS_{L,K}^2 = \frac{w}{r}, \qquad \text{so} \quad \frac{w}{r} = 3\frac{K_2}{L_2}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Combining both $MRTS_{L,K}$ yields

$$\frac{1}{3}\frac{K_1}{L_1} = 3\frac{K_2}{L_2}, \qquad \text{so} \quad \frac{K_1}{L_1} = 9\frac{K_2}{L_2}$$

  - Hence, firm 1 is more capital intensive than firm 2, that is, its capital-to-labor ratio is higher.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Setting both firms' price of capital, r, equal to each other yields

$$0.75p_1K_1^{-0.25}L_1^{0.25} = 0.25p_2K_2^{-0.75}L_2^{0.75}$$

  - so

$$\frac{p_1}{p_2} = \frac{1}{3}\left(\frac{K_1}{L_1}\right)^{0.25}\left(\frac{K_2}{L_2}\right)^{-0.75}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Setting both firms' price of labor, w, equal to each other yields

$$0.25p_1K_1^{0.75}L_1^{-0.75} = 0.75p_2K_2^{0.25}L_2^{-0.25}$$

  - so

$$\frac{p_1}{p_2} = 3\left(\frac{K_1}{L_1}\right)^{-0.75}\left(\frac{K_2}{L_2}\right)^{0.25}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Setting the price ratio from consumers' UMP equal to the first price ratio from firms' PMP yields

$$\frac{K_2^{0.25} L_2^{0.75}}{K_1^{0.75} L_1^{0.25}} = \frac{1}{3}\left(\frac{K_1}{L_1}\right)^{0.25}\left(\frac{K_2}{L_2}\right)^{-0.75}$$

  - Since $L_1^{-0.25}$ and $L_2^{0.75}$ appear on both sides, they cancel:

$$\frac{1}{3} K_1^{0.25} K_2^{-0.75} = K_2^{0.25} K_1^{-0.75}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Multiplying both sides by $3K_1^{0.75}K_2^{0.75}$ yields

$$K_1 = 3K_2, \qquad \text{that is,} \qquad K_2 = \frac{K_1}{3}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - By the feasibility conditions, we know that $K_1 + K_2 = K^A + K^B = 3$ or $K_2 = 3 - K_1$.
  - Substituting the above expression into $K_1 = 3K_2$, we find the equilibrium capital used by firms 1 and 2:

$$K_1 = 3(3 - K_1) \implies K_1^* = \frac{9}{4}$$

$$K_2^* = \frac{1}{3}K_1^* = \frac{3}{4}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Setting price ratio from consumers' UMP equal to the second price ratio from firms' PMP yields

$$\frac{K_2^{0.25}L_2^{0.75}}{K_1^{0.75}L_1^{0.25}} = 3\left(\frac{K_1}{L_1}\right)^{-0.75}\left(\frac{K_2}{L_2}\right)^{0.25} \Longrightarrow L_1 = \frac{1}{3}L_2$$

  - By the feasibility conditions, we know that $L_1 + L_2 = L^A + L^B = 2$ or $L_2 = 2 - L_1$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Substituting the above expression into $L_1 = \frac{1}{3}L_2$, we find the equilibrium labor used by firms 1 and 2:

$$L_1 = \frac{1}{3}(2 - L_1) \implies L_1^* = \frac{1}{2}$$

$$L_2^* = 3L_1^* = \frac{3}{2}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Substituting the capital and labor used by firms 1 and 2 into the price ratio from consumers' UMP yields

$$\frac{p_1}{p_2} = \frac{\left(\frac{3}{4}\right)^{0.25}\left(\frac{3}{2}\right)^{0.75}}{\left(\frac{9}{4}\right)^{0.75}\left(\frac{1}{2}\right)^{0.25}} = \sqrt{\frac{2}{3}} \approx 0.82$$

  - Normalizing $p_2 = 1$ gives $p_1 = 0.82$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Furthermore, substituting our calculated values into firm 1's FOCs for r and w yields

$$r^* = 0.75\sqrt{\frac{2}{3}}\left(\frac{9}{4}\right)^{-0.25}\left(\frac{1}{2}\right)^{0.25} = \frac{2^{3/4}}{4} \approx 0.42$$

$$w^* = 0.25\sqrt{\frac{2}{3}}\left(\frac{9}{4}\right)^{0.75}\left(\frac{1}{2}\right)^{-0.75} = \frac{3 \cdot 2^{3/4}}{8} \approx 0.63$$

  - so that $\frac{w^*}{r^*} = \frac{3}{2}$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Using consumer A's tangency condition, $p_1 x_1^A = p_2 x_2^A$.
  - Substituting this into consumer A's budget constraint yields

$$2p_1 x_1^A = rK^A + wL^A$$

  - so consumer A spends half of his income on each good.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Plugging in our calculated values,

$$x_1^{A*} = \frac{rK^A + wL^A}{2p_1} = \frac{0.42 + 0.63}{2(0.82)} = 0.64$$

$$x_2^{A*} = \frac{rK^A + wL^A}{2p_2} = \frac{0.42 + 0.63}{2} = 0.53$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Performing the same steps for consumer B yields

$$x_1^{B*} = \frac{rK^B + wL^B}{2p_1} = \frac{2(0.42) + 0.63}{2(0.82)} = 0.90$$

$$x_2^{B*} = \frac{rK^B + wL^B}{2p_2} = \frac{2(0.42) + 0.63}{2} = 0.74$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Thus, our WEA is
    $$(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}) = (0.64,0.53; 0.90,0.74)$$
  - with firms using
    $$(K_1^*, L_1^*; K_2^*, L_2^*) = \left(\frac{9}{4}, \frac{1}{2}; \frac{3}{4}, \frac{3}{2}\right)$$
  - and prices
    $$\frac{p_1}{p_2} = 0.82 \quad \text{and} \quad (r^*, w^*) = (0.42,0.63)$$

General Equilibrium: Production

- ***Equilibrium with production: welfare***:
  - We extend the First and Second Welfare Theorems to economies with production, connecting WEA and Pareto efficient allocations.
  - ***Pareto efficiency***: A feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto efficient if there is no other feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that

$$u^i(\hat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i) \quad \text{for every} \quad i = 1, \ldots, I$$

  - with $u^i(\hat{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$ for at least one consumer.

General Equilibrium: Production

- In an economy with two goods, two consumers (i and h), two firms and two inputs (labor and capital), every Pareto efficient allocation solves, for some $\overline{u}^h$,

$$\max_{x_1^i, x_2^i, x_1^h, x_2^h, L_1, K_1, L_2, K_2 \geq 0} u^i(x_1^i, x_2^i)$$
$$\text{s.t. } u^h(x_1^h, x_2^h) \geq \overline{u}^h$$

General Equilibrium: Production

- and the feasibility constraints:
  - technological feasibility,

$$x_1^i + x_1^h \leq F_1(L_1, K_1), \qquad x_2^i + x_2^h \leq F_2(L_2, K_2)$$

  - input feasibility,

$$L_1 + L_2 \leq \overline{L}, \qquad K_1 + K_2 \leq \overline{K}$$

General Equilibrium: Production

- The Lagrangian is

$$\mathcal{L} = u^i(x_1^i, x_2^i) + \lambda[u^h(x_1^h, x_2^h) - \overline{u}^h]$$
$$+\mu_1[F_1(L_1, K_1) - x_1^i - x_1^h] + \mu_2[F_2(L_2, K_2) - x_2^i - x_2^h]$$
$$+\delta_L[\overline{L} - L_1 - L_2] + \delta_K[\overline{K} - K_1 - K_2]$$

General Equilibrium: Production

- In the case of interior solutions, the FOCs with respect to x_k^i and x_k^h yield

$$\frac{\partial u^i}{\partial x_k^i} = \mu_k \quad \text{and} \quad \lambda \frac{\partial u^h}{\partial x_k^h} = \mu_k$$

- Hence, as in barter economies, efficiency in consumption requires

$$MRS_{1,2}^i = MRS_{1,2}^h = \frac{\mu_1}{\mu_2}$$

General Equilibrium: Production

- FOCs with respect to L_j and K_j, where $j \in \{1,2\}$, yield

$$\mu_1 \frac{\partial F_1}{\partial L} = \delta_L = \mu_2 \frac{\partial F_2}{\partial L}$$
$$\mu_1 \frac{\partial F_1}{\partial K} = \delta_K = \mu_2 \frac{\partial F_2}{\partial K}$$

- Dividing the first condition by the second,

$$\frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}} = \frac{\delta_L}{\delta_K} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

General Equilibrium: Production

- This is a condition for efficiency that we encountered in production theory: the $MRTS_{L,K}$ between labor and capital must coincide across firms.
- Otherwise, welfare could be increased by assigning more labor to the firm with the higher $MRTS_{L,K}$ (and more capital to the other firm), which raises the output of both goods.

General Equilibrium: Production

- Combining the above two conditions for efficiency in consumption and production, we obtain

$$\frac{\frac{\partial u^i}{\partial x_1^i}}{\frac{\partial u^i}{\partial x_2^i}} = \frac{\mu_1}{\mu_2} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_1}{\partial L}}$$

General Equilibrium: Production

- That is, $MRS^{i}_{1,2}$ must coincide with the rate at which units of good 2 can be transformed into units of good 1, that is, the marginal rate of transformation $MRT_{1,2}$.
- Moving one unit of labor from firm 2 to firm 1 raises the output of good 1 by $\frac{\partial F_1}{\partial L}$ and lowers the output of good 2 by $\frac{\partial F_2}{\partial L}$.

General Equilibrium: Production

- Hence, one more unit of good 1 costs $MRT_{1,2}$ units of good 2, where

$$MRT_{1,2} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_1}{\partial L}}$$

- *Intuition*: for an allocation to be efficient, the rate at which consumers are willing to substitute goods 1 and 2 must coincide with the rate at which good 2 can be transformed into good 1.

General Equilibrium: Production

- ***First welfare theorem with production***: Under the assumptions on preferences above (strictly increasing utility is the key one), every WEA is Pareto efficient.
- *Proof* (by contradiction):
  - Suppose that $(\mathbf{x}, \mathbf{y})$ is a WEA at prices $\mathbf{p}^*$, but is not Pareto efficient.

General Equilibrium: Production

- Since $(\mathbf{x}, \mathbf{y})$ is a WEA, it is feasible:

$$\sum_{i=1}^{I} \mathbf{x}^i = \sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j$$

General Equilibrium: Production

- Because $(\mathbf{x}, \mathbf{y})$ is not Pareto efficient, there exists some other feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that

$$u^i(\hat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i) \quad \text{for every} \quad i = 1, \ldots, I$$

- with $u^i(\hat{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$ for at least one consumer.
- That is, $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ makes at least one consumer strictly better off than the WEA, without making anyone worse off.

General Equilibrium: Production

- But this implies that bundle $\hat{\mathbf{x}}^i$ is at least as costly as $\mathbf{x}^i$,

$$\mathbf{p}^* \cdot \hat{\mathbf{x}}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i$$

- for every consumer i (with at least one strict inequality):
  - if $u^i(\hat{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$, then $\mathbf{p}^* \cdot \hat{\mathbf{x}}^i > \mathbf{p}^* \cdot \mathbf{x}^i$; otherwise, i would have chosen $\hat{\mathbf{x}}^i$;
  - if $u^i(\hat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$ and $\mathbf{p}^* \cdot \hat{\mathbf{x}}^i < \mathbf{p}^* \cdot \mathbf{x}^i$, strict monotonicity gives an affordable bundle strictly better than $\mathbf{x}^i$.

General Equilibrium: Production

- Summing over all consumers yields

$$\mathbf{p}^* \cdot \sum_{i=1}^{I} \hat{\mathbf{x}}^i > \mathbf{p}^* \cdot \sum_{i=1}^{I} \mathbf{x}^i$$

- Since $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is feasible,

$$\sum_{i=1}^{I} \hat{\mathbf{x}}^i \leq \sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \hat{\mathbf{y}}^j$$

General Equilibrium: Production

- Because $\mathbf{p}^* \gg \mathbf{0}$,

$$\mathbf{p}^* \cdot \left(\sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \hat{\mathbf{y}}^j \right) \geq \mathbf{p}^* \cdot \sum_{i=1}^{I} \hat{\mathbf{x}}^i$$

$$> \mathbf{p}^* \cdot \sum_{i=1}^{I} \mathbf{x}^i = \mathbf{p}^* \cdot \left(\sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j \right)$$

General Equilibrium: Production

- or, rearranging,

$$\mathbf{p}^* \cdot \sum_{j=1}^{J} \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \sum_{j=1}^{J} \mathbf{y}^j$$

- However, this result implies that $\mathbf{p}^* \cdot \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \mathbf{y}^j$ for some firm j.

General Equilibrium: Production

- This indicates that production plan $\mathbf{y}^j$ was not profit-maximizing and, as a consequence, it cannot be part of a WEA.
- We therefore reached a contradiction.
- This implies that the original statement was true: if an allocation $(\mathbf{x}, \mathbf{y})$ is a WEA, it must also be Pareto efficient.

General Equilibrium: Production

- ***Example 6.9*** (WEA and PEA with production):
  - Consider the setting described in Example 6.8.
  - The set of Pareto efficient allocations must satisfy

$$MRS_{1,2}^{A} = MRS_{1,2}^{B}$$
$$MRTS_{L,K}^{1} = MRTS_{L,K}^{2}$$
$$MRS_{1,2}^{i} = MRT_{1,2}$$

General Equilibrium: Production

- ***Example 6.9*** (continued):
  - By consumer A's tangency condition,

$$MRS_{1,2}^{A} = \frac{x_2^A}{x_1^A} = \frac{p_1}{p_2} = 0.82$$

  - and, by consumer B's tangency condition,

$$MRS_{1,2}^{B} = \frac{x_2^B}{x_1^B} = \frac{p_1}{p_2} = 0.82$$

  - which implies that $MRS_{1,2}^{A} = MRS_{1,2}^{B}$.

General Equilibrium: Production

- ***Example 6.9*** (continued):
  - From Example 6.8, $\frac{K_1}{L_1} = \frac{9}{2}$ and $\frac{K_2}{L_2} = \frac{1}{2}$, so

$$MRTS_{L,K}^{1} = \frac{1}{3}\frac{K_1}{L_1} = \frac{1}{3} \cdot \frac{9}{2} = \frac{3}{2}$$

$$MRTS_{L,K}^{2} = 3\frac{K_2}{L_2} = 3 \cdot \frac{1}{2} = \frac{3}{2}$$

  - which implies that $MRTS_{L,K}^{1} = MRTS_{L,K}^{2}$.

General Equilibrium: Production

- ***Example 6.9*** (continued):
  - Finally, the marginal rate of transformation is

$$MRT_{1,2} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_1}{\partial L}} = \frac{0.75\left(\frac{K_2}{L_2}\right)^{0.25}}{0.25\left(\frac{K_1}{L_1}\right)^{0.75}}$$

$$= \frac{0.75\left(\frac{1}{2}\right)^{0.25}}{0.25\left(\frac{9}{2}\right)^{0.75}} \approx 0.82$$

General Equilibrium: Production

- ***Example 6.9*** (continued):
  - Summarizing,

$$MRS_{1,2}^{A} = MRS_{1,2}^{B} = 0.82$$

$$MRTS_{L,K}^{1} = MRTS_{L,K}^{2} = \frac{3}{2}$$

$$MRS_{1,2}^{i} = MRT_{1,2} = 0.82$$

  - Since all three conditions hold, our WEA from Example 6.8 is Pareto efficient.

General Equilibrium: Production

- ***Second welfare theorem with production***:
  - Consider the assumptions on consumers and producers described above.
  - Then, for every Pareto efficient allocation $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ we can find:
  - (a) a profile of income transfers $(T_1, T_2, \ldots, T_I)$ redistributing income among consumers, that is, satisfying $\sum_{i=1}^{I} T_i = 0$;
  - (b) a price vector $\overline{\mathbf{p}}$, such that:

General Equilibrium: Production

- (1) Bundle $\overline{\mathbf{x}}^i$ solves the UMP

$$\max_{\mathbf{x}^i} u^i(\mathbf{x}^i)$$

$$\text{s.t. } \overline{\mathbf{p}} \cdot \mathbf{x}^i \leq m^i(\overline{\mathbf{p}}) + T_i \quad \text{for every} \quad i = 1, \ldots, I$$

  - where individual i's original income $m^i(\overline{\mathbf{p}})$ is increased (decreased) if the transfer T_i is positive (negative).

General Equilibrium: Production

- (2) Production plan $\overline{\mathbf{y}}^j$ solves the PMP

$$\max_{\mathbf{y}^j} \overline{\mathbf{p}} \cdot \mathbf{y}^j$$

$$\text{s.t.} \quad \mathbf{y}^j \in Y^j \quad \text{for every} \quad j = 1, \ldots, J$$

General Equilibrium: Production

- ***Example 6.10*** (Second welfare theorem with production):
  - Consider Example 6.8. Both consumers have identical Cobb-Douglas preferences, so every Pareto efficient allocation uses the production plan of Example 6.8 and gives each consumer the same share of y_1 and y_2.

General Equilibrium: Production

- ***Example 6.10*** (continued):
  - Consider, for instance, the Pareto efficient allocation

$$(\overline{x}_1^A, \overline{x}_2^A; \overline{x}_1^B, \overline{x}_2^B) = (0.75, 0.61; 0.79, 0.65)$$

  - which is feasible since

$$0.75 + 0.79 = 1.54 = y_1$$
$$0.61 + 0.65 = 1.26 = y_2$$

General Equilibrium: Production

- ***Example 6.10*** (continued):
  - Prices remain unchanged: firms' input choices do not change, and aggregate demand depends only on aggregate income, since each consumer spends half of his income on each good.
  - Hence,

$$(p_1, p_2; r, w) = (0.82,1; 0.42,0.63)$$

General Equilibrium: Production

- ***Example 6.10*** (continued):
  - ***Transfers***: with a lump-sum transfer, consumer A's budget constraint becomes

$$p_1\overline{x}_1^A + p_2\overline{x}_2^A = rK^A + wL^A + T_A$$

  - Since $(K^A, L^A) = (1,1)$, at the prices above his income is

$$0.42(1) + 0.63(1) + T_A = 1.05 + T_A$$

General Equilibrium: Production

- ***Example 6.10*** (continued):
  - Consumer A can afford $(\overline{x}_1^A, \overline{x}_2^A) = (0.75, 0.61)$ only if his income equals its cost,
    $$0.82(0.75) + 1(0.61) \approx 1.22 = 1.05 + T_A$$
  - so $T_A \approx 0.17$.
  - This bundle is also his best choice: with Cobb-Douglas utility, he spends half of his income on each good, and indeed
    $$p_1\overline{x}_1^A = 0.82(0.75) \approx 0.61 = p_2\overline{x}_2^A$$

General Equilibrium: Production

- ***Example 6.10*** (continued):
  - Similarly, consumer B's income is
    $$0.42(2) + 0.63(1) + T_B = 1.47 + T_B$$
  - so that he can afford $(\overline{x}_1^B, \overline{x}_2^B) = (0.79, 0.65)$ only if
    $$0.82(0.79) + 1(0.65) \approx 1.30 = 1.47 + T_B$$
  - so $T_B \approx -0.17$.

General Equilibrium: Production

- ***Example 6.10*** (continued):
  - Consumer B also spends half of his income on each good,

$$p_1\overline{x}_1^B = 0.82(0.79) \approx 0.65 = p_2\overline{x}_2^B$$

  - Hence, $T_A + T_B = 0$ (up to rounding): consumer B transfers 0.17 to consumer A.
  - With these transfers, the Pareto efficient allocation is a WEA with transfers at the same prices.

Comparative Statics

Comparative Statics

- We analyze how equilibrium outcomes are affected by an increase in:
  - the price of one good;
  - the endowment of one input.
- Consider a setting with two goods, each produced with two factors under constant returns to scale (CRS): factor 1 is capital ($w_1 = r$) and factor 2 is labor ($w_2 = w$).

Comparative Statics

- If both goods are produced, a necessary condition for input prices (w_1^*, w_2^*) to be in equilibrium is

$$c_1(w_1, w_2) = p_1 \quad \text{and} \quad c_2(w_1, w_2) = p_2$$

  - where $c_j(w_1, w_2)$ is firm j's unit cost function.
- Under constant returns, marginal cost equals unit cost, so price equals unit cost and profits are zero.

Comparative Statics

- Let $z_{1j}(w)$ denote firm j's demand for factor 1, and $z_{2j}(w)$ its demand for factor 2, when producing one unit of output.
- These are the conditional factor demands evaluated at $q = 1$ (unit factor demands); under constant returns, $z(w, q) = q \cdot z(w, 1)$.

Comparative Statics

- The production of good 1 is more intensive in factor 1 than is the production of good 2 if

$$\frac{z_{11}(w)}{z_{21}(w)} > \frac{z_{12}(w)}{z_{22}(w)}$$

  - where $\frac{z_{1j}(w)}{z_{2j}(w)}$ represents firm j's demand for input 1 relative to that of input 2.

Comparative Statics: Price Change

1) ***Changes in the price of one good,*** p_j **(Stolper-Samuelson theorem)**:
   - Consider an economy with two consumers and two firms satisfying the above factor intensity assumption.
   - If the price of good j, p_j, increases, then:
     a) the equilibrium price of the factor more intensively used in the production of good j increases; while
     b) the equilibrium price of the other factor decreases.

Comparative Statics: Price Change

- *Proof*:
  - Let us first take the equilibrium conditions

$$c_1(w_1, w_2) = p_1 \text{ and } c_2(w_1, w_2) = p_2$$

  - Differentiating them yields

$$\frac{\partial c_1(w_1,w_2)}{\partial w_1} dw_1 + \frac{\partial c_1(w_1,w_2)}{\partial w_2} dw_2 = dp_1$$

$$\frac{\partial c_2(w_1,w_2)}{\partial w_1} dw_1 + \frac{\partial c_2(w_1,w_2)}{\partial w_2} dw_2 = dp_2$$

  - Applying Shephard's lemma, we obtain

$$z_{11}(w)dw_1 + z_{21}(w)dw_2 = dp_1$$

$$z_{12}(w)dw_1 + z_{22}(w)dw_2 = dp_2$$

Comparative Statics: Price Change

- If only price p_1 varies, then $dp_2 = 0$. Hence, we can rewrite the second expression as

$$z_{12}(w)dw_1 + z_{22}(w)dw_2 = 0, \qquad \text{so} \quad dw_2 = -\frac{z_{12}}{z_{22}}dw_1$$

- Substituting $dw_2 = -\frac{z_{12}}{z_{22}}dw_1$ into the first expression yields

$$\left[z_{11} - \frac{z_{21}z_{12}}{z_{22}}\right]dw_1 = dp_1$$

Comparative Statics: Price Change

- Solving for the ratio and multiplying its numerator and denominator by z_{22},

$$\frac{dw_1}{dp_1} = \frac{z_{22}}{z_{11}z_{22} - z_{12}z_{21}}$$

- Substituting this into $dw_2 = -\frac{z_{12}}{z_{22}} dw_1$ yields

$$\frac{dw_2}{dp_1} = -\frac{z_{12}}{z_{11}z_{22} - z_{12}z_{21}}$$

Comparative Statics: Price Change

- From the factor intensity condition,

$$\frac{z_{11}(w)}{z_{21}(w)} > \frac{z_{12}(w)}{z_{22}(w)}$$

- cross-multiplying (all factor demands are positive),

$$z_{11}z_{22} - z_{12}z_{21} > 0$$

Comparative Statics: Price Change

- Hence, the denominator in both $\frac{dw_1}{dp_1}$ and $\frac{dw_2}{dp_1}$ is positive.
- Since $z_{22} > 0$ and $z_{12} > 0$ (factor demands), $\frac{dw_1}{dp_1} > 0$ and $\frac{dw_2}{dp_1} < 0$.

Comparative Statics: Price Change

- ***Example 6.11***:
  - Let us find the unit factor demands in Example 6.8, where factor 1 is capital ($w_1 = r$) and factor 2 is labor ($w_2 = w$).
  - From firm 1's tangency condition,

$$MRTS_{L,K}^{1} = \frac{1}{3}\frac{K_1}{L_1} = \frac{w}{r}, \qquad \text{so} \quad L_1 = \frac{rK_1}{3w}$$

  - Substituting into $y_1 = K_1^{0.75} L_1^{0.25}$,

$$y_1 = K_1^{0.75}\left(\frac{rK_1}{3w}\right)^{0.25} = K_1\left(\frac{r}{3w}\right)^{0.25}$$

Comparative Statics: Price Change

- ***Example 6.11*** (continued):
  - Setting $y_1 = 1$ and using $L_1 = \frac{rK_1}{3w}$, firm 1's unit factor demands are

$$z_{11} = z_{K1} = \left(\frac{r}{3w}\right)^{-1/4} \quad \text{and} \quad z_{21} = z_{L1} = \left(\frac{r}{3w}\right)^{3/4}$$

  - From firm 2's tangency condition,

$$MRTS^2_{L,K} = 3\frac{K_2}{L_2} = \frac{w}{r}, \qquad \text{so} \quad K_2 = \frac{wL_2}{3r}$$

Comparative Statics: Price Change

- ***Example 6.11*** (continued):
  - Substituting into $y_2 = K_2^{0.25} L_2^{0.75}$,

$$y_2 = \left(\frac{wL_2}{3r}\right)^{0.25} L_2^{0.75} = L_2 \left(\frac{w}{3r}\right)^{0.25}$$

  - Setting $y_2 = 1$ and using $K_2 = \frac{wL_2}{3r}$, firm 2's unit factor demands are

$$z_{12} = z_{K2} = \left(\frac{w}{3r}\right)^{3/4} \quad \text{and} \quad z_{22} = z_{L2} = \left(\frac{w}{3r}\right)^{-1/4}$$

Comparative Statics: Price Change

- ***Example 6.11*** (continued):
  - Hence,

$$z_{11}z_{22} - z_{12}z_{21} = \left(\frac{r}{3w}\right)^{-1/4}\left(\frac{w}{3r}\right)^{-1/4} - \left(\frac{w}{3r}\right)^{3/4}\left(\frac{r}{3w}\right)^{3/4}$$

$$= \left(\frac{1}{9}\right)^{-1/4} - \left(\frac{1}{9}\right)^{3/4} = \sqrt{3} - \frac{1}{3\sqrt{3}} = \frac{8}{3\sqrt{3}} \approx 1.54 > 0$$

  - for every (r, w): firm 1 is more capital intensive than firm 2 at all factor prices.

Comparative Statics: Price Change

- ***Example 6.11*** (continued):
  - Using the expressions from the proof, with $w_1 = r$ and $w_2 = w$: at $(r, w) = (0.42, 0.63)$, $\frac{w}{3r} = \frac{1}{2}$, so

$$z_{22} = 2^{1/4} \approx 1.19 \quad \text{and} \quad z_{12} = 2^{-3/4} \approx 0.59$$

$$\frac{dr}{dp_1} = \frac{z_{22}}{z_{11}z_{22} - z_{12}z_{21}} = \frac{2^{1/4} \cdot 3\sqrt{3}}{8} \approx 0.77 > 0$$

$$\frac{dw}{dp_1} = -\frac{z_{12}}{z_{11}z_{22} - z_{12}z_{21}} = -\frac{2^{-3/4} \cdot 3\sqrt{3}}{8} \approx -0.39 < 0$$

Comparative Statics: Endowment Change

2) ***Changes in endowments*** **(Rybczynski theorem)**:
   - Consider an economy with two consumers and two firms satisfying the above factor intensity assumption.
   - Holding goods prices (p_1, p_2), and hence factor prices, constant, if the endowment of one factor increases while the other stays fixed, then
     a) production of the good that uses this factor more intensively increases; whereas
     b) the production of the other good decreases.

Comparative Statics: Endowment Change

- *Proof*:
  - Consider an economy with two factors, labor and capital, and two goods, 1 and 2.
  - Let $z_{Lj}(w)$ denote firm j's factor demand for labor (when producing one unit of output).
  - Similarly, let $z_{Kj}(w)$ denote firm j's factor demand for capital per unit of output.

Comparative Statics: Endowment Change

- Since goods prices, and hence factor prices, are fixed, the z's are constant.
- Then, factor feasibility requires

$$L = z_{L1}(w) \cdot y_1 + z_{L2}(w) \cdot y_2$$
$$K = z_{K1}(w) \cdot y_1 + z_{K2}(w) \cdot y_2$$

Comparative Statics: Endowment Change

- Differentiating the first condition (the z's are constant),

$$dL = z_{L1}dy_1 + z_{L2}dy_2$$

- Dividing both sides by L, and multiplying and dividing the term of each good j by y_j, we obtain

$$\frac{dL}{L} = \left(\frac{z_{L1}y_1}{L}\right)\left(\frac{dy_1}{y_1}\right) + \left(\frac{z_{L2}y_2}{L}\right)\left(\frac{dy_2}{y_2}\right)$$

Comparative Statics: Endowment Change

- We can express:
  - $\frac{z_{Lj}(w)y_j}{L} \equiv \gamma_{Lj}$, the share of labor used by firm j;
  - $\frac{dy_j}{y_j} \equiv \%\Delta y_j$, the percentage change in the output of good j;
  - $\frac{dL}{L} \equiv \%\Delta L$, the percentage increase in the endowment of labor in the economy.

Comparative Statics: Endowment Change

- Hence, the above expression becomes

$$\%\Delta L = \gamma_{L1} \cdot \%\Delta y_1 + \gamma_{L2} \cdot \%\Delta y_2$$

- A similar expression can be obtained for the endowment of capital:

$$\%\Delta K = \gamma_{K1} \cdot \%\Delta y_1 + \gamma_{K2} \cdot \%\Delta y_2$$

Comparative Statics: Endowment Change

- Note that $\gamma_{L1}, \gamma_{L2} \in (0,1)$ and $\gamma_{L1} + \gamma_{L2} = 1$ (full employment).
- Hence, $\%\Delta L$ is a weighted average of $\%\Delta y_1$ and $\%\Delta y_2$, so it lies between them.
- The same argument applies to $\%\Delta K$, where $\gamma_{K1}, \gamma_{K2} \in (0,1)$ and $\gamma_{K1} + \gamma_{K2} = 1$.

Comparative Statics: Endowment Change

- Capital is assumed to be more intensively used in firm 1,

$$\frac{K_1}{L_1} > \frac{K_2}{L_2}$$

- Since $K_2 = K - K_1$ and $L_2 = L - L_1$, this holds if and only if $K_1 L > L_1 K$, that is,

$$\gamma_{K1} = \frac{K_1}{K} > \frac{L_1}{L} = \gamma_{L1}$$

  - where $\gamma_{Kj} = \frac{z_{Kj}(w) y_j}{K} = \frac{K_j}{K}$; hence, $\gamma_{K2} < \gamma_{L2}$ for firm 2.

Comparative Statics: Endowment Change

- Subtracting the expression for $\%\Delta L$ from that for $\%\Delta K$, and using $\gamma_{K2} - \gamma_{L2} = -(\gamma_{K1} - \gamma_{L1})$,

$$\%\Delta K - \%\Delta L = (\gamma_{K1} - \gamma_{L1})(\%\Delta y_1 - \%\Delta y_2)$$

- Hence, if capital becomes relatively more abundant than labor, $\%\Delta K > \%\Delta L$, then $\%\Delta y_1 > \%\Delta y_2$.

Comparative Statics: Endowment Change

- If, in addition, the endowment of labor stays fixed, $\%\Delta L = 0$, then

$$0 = \gamma_{L1} \cdot \%\Delta y_1 + \gamma_{L2} \cdot \%\Delta y_2$$

  - so $\%\Delta y_1$ and $\%\Delta y_2$ have opposite signs, that is, $\%\Delta y_1 > 0 > \%\Delta y_2$.
- Since $\%\Delta K$ is a weighted average of $\%\Delta y_1$ and $\%\Delta y_2$,

$$\%\Delta y_1 > \%\Delta K > 0 > \%\Delta y_2$$

Comparative Statics: Endowment Change

- That is,

$$\%\Delta L = \gamma_{L1} \cdot \%\Delta y_1 + \gamma_{L2} \cdot \%\Delta y_2$$
$$\%\Delta K = \gamma_{K1} \cdot \%\Delta y_1 + \gamma_{K2} \cdot \%\Delta y_2$$

  - where $\gamma_{K1} > \gamma_{L1}$ and $\gamma_{K2} < \gamma_{L2}$: relative to labor, capital puts more weight on $\%\Delta y_1$.
- *Intuition*: the change in the input endowment produces a more-than-proportional increase in the output of the good whose production is intensive in that input, and the output of the other good falls.

Comparative Statics: Endowment Change

- ***Example 6.12*** (Rybczynski theorem):
  - Consider Example 6.8, where the aggregate endowments are $\overline{K} = 3$ and $\overline{L} = 2$.
  - The endowment of capital increases to $\overline{K} = 5$, while $\overline{L} = 2$ stays fixed.
  - Goods prices stay at $p_1 = 0.82$ and $p_2 = 1$, so factor prices stay at $(r, w) = (0.42, 0.63)$, with $\frac{r}{w} = \frac{2}{3}$.

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - From Example 6.11, with $\frac{r}{3w} = \frac{2}{9}$ and $\frac{w}{3r} = \frac{1}{2}$, the unit factor demands are

$$z_{K1} = \left(\frac{2}{9}\right)^{-1/4} \approx 1.46, \qquad z_{L1} = \left(\frac{2}{9}\right)^{3/4} \approx 0.32$$

$$z_{K2} = \left(\frac{1}{2}\right)^{3/4} \approx 0.59, \qquad z_{L2} = \left(\frac{1}{2}\right)^{-1/4} \approx 1.19$$

  - Since prices are fixed, these do not change.

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Factor feasibility with $\overline{K} = 5$ and $\overline{L} = 2$ requires

$$5 = 1.46y_1 + 0.59y_2$$
$$2 = 0.32y_1 + 1.19y_2$$

  - Each firm keeps its capital-labor ratio,

$$\frac{K_1}{L_1} = \frac{z_{K1}}{z_{L1}} = \frac{9}{2} \quad \text{and} \quad \frac{K_2}{L_2} = \frac{z_{K2}}{z_{L2}} = \frac{1}{2}$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Solving, the inputs used by each firm are

$$(K_1, L_1; K_2, L_2) = \left(\frac{9}{2}, 1; \frac{1}{2}, 1\right)$$

  - so outputs become

$$y_1 = \left(\frac{9}{2}\right)^{3/4} \approx 3.09 \quad \text{and} \quad y_2 = \left(\frac{1}{2}\right)^{1/4} \approx 0.84$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Relative to Example 6.8 ($y_1 \approx 1.54$ and $y_2 \approx 1.26$), the output of good 1 doubles (+100%), while that of good 2 falls by one third (−33.3%).
  - Firm 1 uses capital more intensively than firm 2: $\frac{K_1}{L_1} = \frac{9}{2} > \frac{1}{2} = \frac{K_2}{L_2}$.
  - If, instead, goods prices adjusted to a new WEA (Example 6.8 with $\overline{K} = 5$), then $\frac{p_1}{p_2} \approx 0.63$ and both outputs would rise, to $y_1 \approx 2.27$ and $y_2 \approx 1.43$.

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Let us now apply the proof. Since goods prices are fixed, the z's are those above.
  - Using $\gamma_{Kj} = \frac{K_j}{\overline{K}}$ and $\gamma_{Lj} = \frac{L_j}{\overline{L}}$ and the values from Example 6.8, we can assign the following values:

$$\gamma_{K1} = \frac{2.25}{3} = 0.75 \quad \text{and} \quad \gamma_{K2} = \frac{0.75}{3} = 0.25$$

$$\gamma_{L1} = \frac{0.5}{2} = 0.25 \quad \text{and} \quad \gamma_{L2} = \frac{1.5}{2} = 0.75$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Our two equations then become

$$0 = 0.25 \cdot \%\Delta y_1 + 0.75 \cdot \%\Delta y_2$$

$$\frac{2}{3} = 0.75 \cdot \%\Delta y_1 + 0.25 \cdot \%\Delta y_2$$

  - where $\%\Delta K = \frac{5-3}{3} = \frac{2}{3}$. Solving them simultaneously yields

$$\%\Delta y_1 = 1 = 100\% \quad \text{and} \quad \%\Delta y_2 = -\frac{1}{3} \approx -33.3\%$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Hence, an increase in the endowment of capital by 67% raises good 1's output by 100%, while that of good 2 decreases by 33.3%:

$$\%\Delta y_1 = 100\% > \%\Delta K \approx 67\% > 0 > \%\Delta y_2 \approx -33.3\%$$

  - These are the output changes found above (magnification effect).

Introducing Taxes

Introducing Taxes: Tax on Goods

- Assume that a sales tax t_k is imposed on good k. Then the price paid by consumers rises to $p_k^C = (1 + t_k)p_k^P$, where p_k^P is the price received by producers.
- If goods 1 and 2 face the same tax, $t_1 = t_2$, consumers and producers face the same price ratio:

$$\frac{p_1^C}{p_2^C} = \frac{(1 + t_1)p_1^P}{(1 + t_2)p_2^P} = \frac{p_1^P}{p_2^P}$$

Introducing Taxes: Tax on Goods

- Hence, with inputs in fixed supply and tax revenue returned lump-sum, the after-tax allocation is still Pareto efficient.
- However, if only good 1 is affected by the tax, i.e., $t_1 > 0$ while $t_2 = 0$, then the allocation will not be Pareto efficient.

Introducing Taxes: Tax on Goods

- In this setting, each firm's $MRTS_{L,K}$ still equals the common input price ratio $\frac{w_L}{w_K}$:

$$\frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}} = \frac{w_L}{w_K} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

- Therefore, the allocation of inputs still achieves productive efficiency.

Introducing Taxes: Tax on Goods

- Similarly, the $MRT_{1,2}$ still coincides with the price ratio of goods 1 and 2:

$$MRT_{1,2} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_1}{\partial L}} = \frac{p_1^P}{p_2} = \frac{\frac{\partial F_2}{\partial K}}{\frac{\partial F_1}{\partial K}}$$

  - where p_1^P is the price that producers of good 1 receive, net of the tax (since $t_2 = 0$, $p_2 = p_2^C = p_2^P$).

Introducing Taxes: Tax on Goods

- However, while $MRS_{1,2}$ equals the price ratio that consumers face, $\frac{p_1^C}{p_2} = \frac{(1+t_1)p_1^P}{p_2}$, it now becomes larger than the price ratio that producers face, $\frac{p_1^P}{p_2}$:

$$MRS_{1,2} = \frac{p_1^C}{p_2} = \frac{(1+t_1)p_1^P}{p_2} > \frac{p_1^P}{p_2} = MRT_{1,2}$$

Introducing Taxes: Tax on Goods

- *Intuition*: The rate at which consumers are willing to give up good 2 for an additional unit of good 1 is larger than the rate at which firms must give up good 2 to produce an additional unit of good 1.
- Thus, efficiency calls for producing more of good 1 and less of good 2: the tax leads to too little production of good 1.

Introducing Taxes: Tax on Inputs

- Similar arguments extend to the introduction of taxes on inputs.
- Assume that only inputs used by firm 2 are taxed: firm 2 pays $w_m^P = (1 + t_m)w_m^C$ for input $m \in \{L, K\}$, while firm 1 pays w_m^C.

Introducing Taxes: Tax on Inputs

- If both inputs are subject to the same tax, $t_L = t_K = t$, the input price ratios that firms 1 and 2 face coincide:

$$\frac{w_L^P}{w_K^P} = \frac{(1+t)w_L^C}{(1+t)w_K^C} = \frac{w_L^C}{w_K^C}$$

- Hence, the efficiency condition is unaffected.

Introducing Taxes: Tax on Inputs

- However, when taxes differ, $t_L \neq t_K$, productive efficiency no longer holds:
  - While firm 1 (untaxed) satisfies

$$\frac{w_L^C}{w_K^C} = \frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}}$$

  - and firm 2 (taxed) satisfies

$$\frac{w_L^P}{w_K^P} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

Introducing Taxes: Tax on Inputs

- Hence, the input price ratios that firms 1 and 2 face do not coincide:

$$MRTS_{L,K}^{1} = \frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}} = \frac{w_L^C}{w_K^C}$$

$$MRTS_{L,K}^{2} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}} = \frac{w_L^P}{w_K^P} = \frac{(1+t_L)w_L^C}{(1+t_K)w_K^C}$$

- That is, the two firms' $MRTS_{L,K}$ differ whenever $t_L \neq t_K$.

Introducing Taxes: Tax on Inputs

- For instance, if $t_L > t_K$, the $MRTS_{L,K}$ is larger for firm 2 than for firm 1:

$$MRTS^2_{L,K} - MRTS^1_{L,K} = \frac{w_L^C}{w_K^C}\frac{t_L - t_K}{1 + t_K} > 0$$

- Thus, the allocation of inputs is inefficient: labor is relatively more productive in firm 2 than in firm 1.
  - Moving labor from firm 1 to firm 2, and capital from firm 2 to firm 1, would raise the output of both goods.

Appendix:
Large Economies and the Core

Large Economies and the Core

- We know that Walrasian equilibrium allocations (WEAs) are part of the core.
- We now show that, as the economy becomes larger, the core shrinks and, in the limit, only WEAs remain in the core of every replica economy.

Large Economies and the Core

- Let us first consider an economy with I consumers, each with utility function u^i and endowment vector $\mathbf{e}^i$.
- Consider this economy's replica by doubling the number of consumers to $2I$, each of them still with utility function u^i and endowment vector $\mathbf{e}^i$.
  - There are now two consumers of each type, i.e., "twins," having identical preferences and endowments.

Large Economies and the Core

- Define an r-fold replica economy $\mathcal{E}_r$, having r consumers of each type, for a total of rI consumers.
- For any consumer type $i \in \{1, \ldots, I\}$, all r consumers of that type share the common utility function u^i and have identical endowments $\mathbf{e}^i \gg \mathbf{0}$.
- Preferences satisfy the assumptions above. We also assume that indifference curves are differentiable.

Large Economies and the Core

- When comparing two replica economies, $\mathcal{E}_r$ is larger than $\mathcal{E}_{r'}$ when $r > r'$, since it has more consumers of every type.
- Bundle $\mathbf{x}^{iq}$ indicates the vector of goods for the qth consumer of type i.
- The feasibility condition is

$$\sum_{i=1}^{I} \sum_{q=1}^{r} \mathbf{x}^{iq} = r \sum_{i=1}^{I} \mathbf{e}^{i}$$

Large Economies and the Core

- ***Equal treatment in the core***: Under the assumptions on preferences above (strict quasiconcavity is the key one), if $\mathbf{x}$ is in the core of $\mathcal{E}_r$, then every consumer of type i receives the same bundle,

$$\mathbf{x}^{iq} = \mathbf{x}^{iq'}$$

  - for every two consumers $q \neq q' \in \{1,2,\ldots,r\}$ of type i, and for every type $i \in \{1,\ldots,I\}$.

Large Economies and the Core

- That is, in the r-fold replica economy, consumers of the same type not only start with the same endowment vector $\mathbf{e}^i$, but they also end up with the same bundle in every core allocation.

Large Economies and the Core

- *Proof* (by contradiction):
  - Consider a two-fold replica economy $\mathcal{E}_2$. The argument extends to r-fold replicas (see below).
  - Suppose that allocation $\mathbf{x} \equiv (\mathbf{x}^{11}, \mathbf{x}^{12}, \mathbf{x}^{21}, \mathbf{x}^{22})$ is in the core of $\mathcal{E}_2$, $\mathbf{x} \in C_2$.
  - Since $\mathbf{x}$ is in the core, it must be feasible:

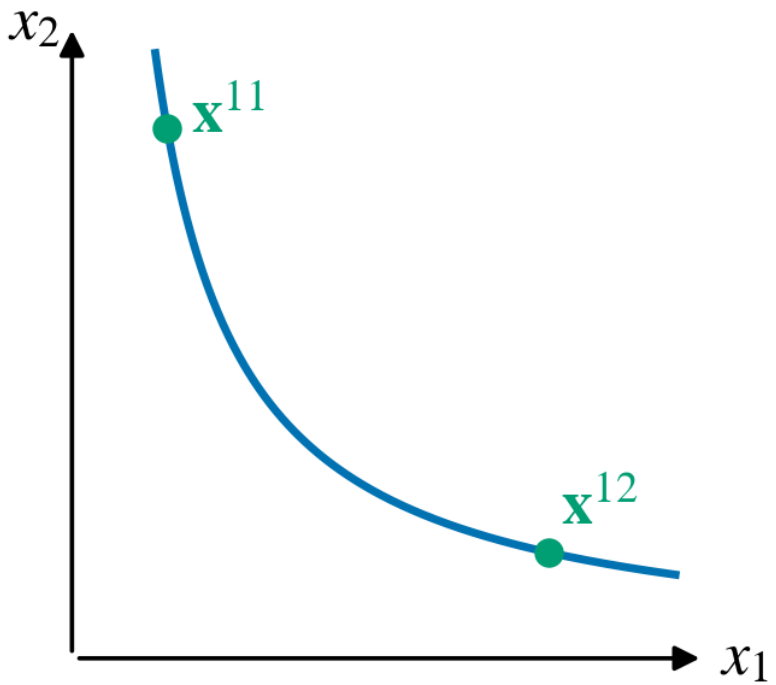
$$\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22} = 2\mathbf{e}^1 + 2\mathbf{e}^2$$

Large Economies and the Core

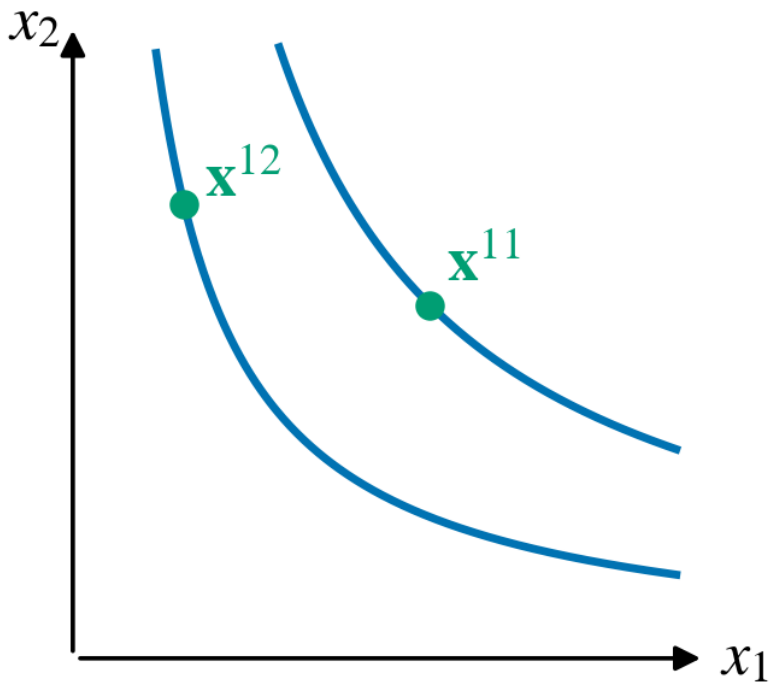
- Assume that allocation $\mathbf{x}$ does not assign the same consumption vectors to the two type-1 twins, $\mathbf{x}^{11} \neq \mathbf{x}^{12}$.
- Without loss of generality, label the type-1 twins so that $\mathbf{x}^{11} \gtrsim^{1} \mathbf{x}^{12}$.
  - This is true for both type-1 consumers, since they have the same preferences.
  - The other case follows by relabeling the twins.

Large Economies and the Core

- Type-1 twins treated unequally in the core



$\mathbf{x}^{11} \sim {}^1\mathbf{x}^{12}$
(same indifference curve)



$\mathbf{x}^{11} \succ {}^1\mathbf{x}^{12}$

Large Economies and the Core

- Similarly, label the type-2 twins so that $\mathbf{x}^{21} \succsim^{2} \mathbf{x}^{22}$.
- Hence, consumer 12 is the worst-off type-1 consumer and consumer 22 is the worst-off type-2 consumer.
- Let us take these two worst-off consumers, one of each type, and check if they can form a blocking coalition to oppose allocation $\mathbf{x}$.

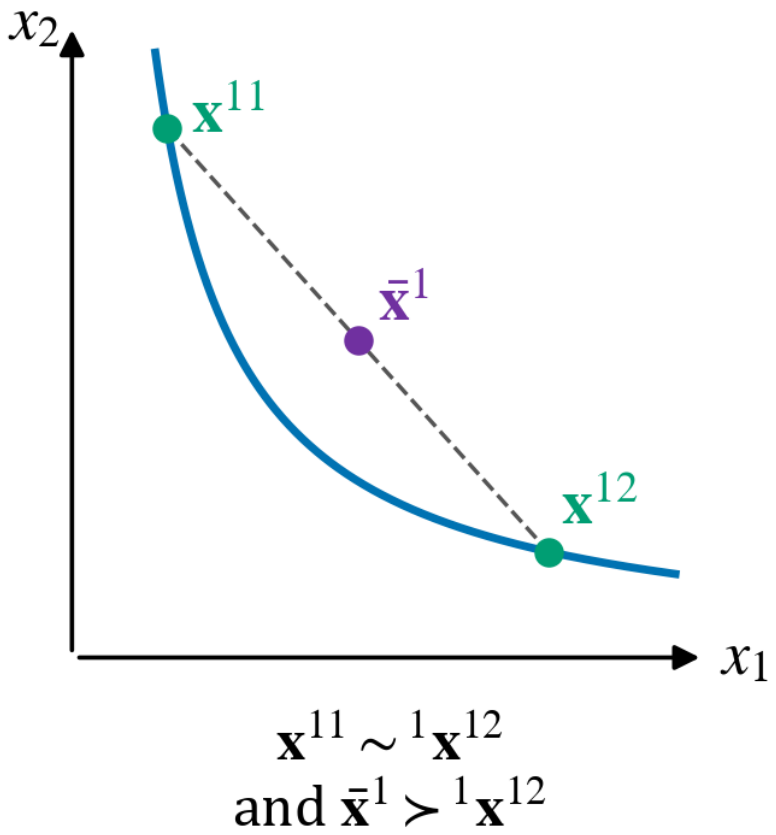
Large Economies and the Core

- The average bundles for type-1 and type-2 consumers are

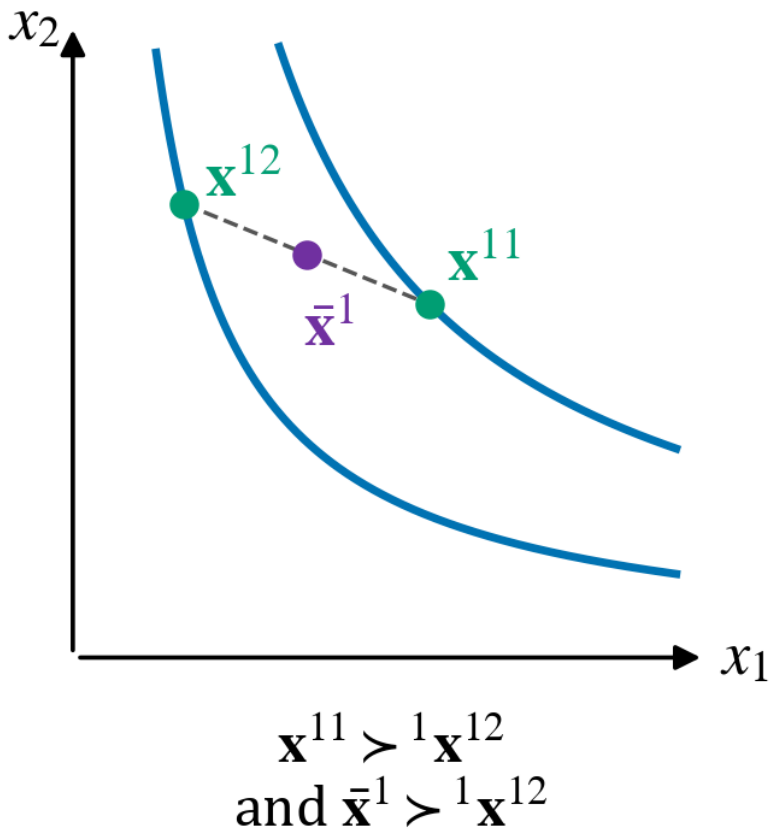
$$\bar{\mathbf{x}}^1 = \frac{\mathbf{x}^{11} + \mathbf{x}^{12}}{2} \quad \text{and} \quad \bar{\mathbf{x}}^2 = \frac{\mathbf{x}^{21} + \mathbf{x}^{22}}{2}$$

Large Economies and the Core

- Average bundles leading to a blocking coalition



$\mathbf{x}^{11} \sim {}^1\mathbf{x}^{12}$
and $\bar{\mathbf{x}}^1 \succ {}^1\mathbf{x}^{12}$



$\mathbf{x}^{11} \succ {}^1\mathbf{x}^{12}$
and $\bar{\mathbf{x}}^1 \succ {}^1\mathbf{x}^{12}$

Large Economies and the Core

- *Desirability*. By the strict quasiconcavity assumed above, and since $\mathbf{x}^{11} \neq \mathbf{x}^{12}$, the worst-off type-1 consumer strictly prefers $\overline{\mathbf{x}}^{1}$ to $\mathbf{x}^{12}$:

$$\overline{\mathbf{x}}^{1} \succ^{1} \mathbf{x}^{12}$$

  - That is, a convex combination (the midpoint) of $\mathbf{x}^{11}$ and $\mathbf{x}^{12}$ is strictly preferred to the original bundle $\mathbf{x}^{12}$.

Large Economies and the Core

- A similar argument applies to the worst-off type-2 consumer, $\bar{\mathbf{x}}^2 \gtrsim^2 \mathbf{x}^{22}$ (weakly, since $\mathbf{x}^{21} = \mathbf{x}^{22}$ is possible).
- Hence, $(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2)$ makes consumer 12 strictly better off and leaves consumer 22 at least as well off as at the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$.

Large Economies and the Core

- *Feasibility*. Can consumers 12 and 22 achieve $\left(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2\right)$?
- Summing the bundles that consumers 12 and 22 need, we obtain

$$\bar{\mathbf{x}}^1 + \bar{\mathbf{x}}^2 = \frac{\mathbf{x}^{11} + \mathbf{x}^{12}}{2} + \frac{\mathbf{x}^{21} + \mathbf{x}^{22}}{2}$$

$$= \frac{1}{2}(\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22}) = \frac{1}{2}(2\mathbf{e}^1 + 2\mathbf{e}^2) = \mathbf{e}^1 + \mathbf{e}^2$$

- Hence, the pair of bundles $\left(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2\right)$ is feasible.

Large Economies and the Core

- In summary, the pair of bundles $\left(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2\right)$:
  - makes consumer 12 strictly better off and consumer 22 no worse off than the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$;
  - is feasible for consumers 12 and 22.
- Thus, coalition {12, 22} blocks $\mathbf{x}$, contradicting $\mathbf{x} \in C_2$.
- Therefore, if an allocation is in the core of the replica economy, it must give consumers of the same type the same bundle.

Large Economies and the Core

- The argument extends to $\mathcal{E}_r$: give the worst-off consumer of each type i the average bundle

$$\bar{\mathbf{x}}^i = \frac{1}{r}\sum_{q=1}^{r} \mathbf{x}^{iq}$$

- These bundles are feasible for the coalition of worst-off consumers:

$$\sum_{i=1}^{I} \bar{\mathbf{x}}^i = \frac{1}{r}\sum_{i=1}^{I}\sum_{q=1}^{r} \mathbf{x}^{iq} = \sum_{i=1}^{I} \mathbf{e}^i$$

  - Each of them weakly prefers $\bar{\mathbf{x}}^i$, and strictly so for a type with unequal bundles.

Large Economies and the Core

- If $\mathbf{x}$ is in the core of an r-fold replica economy $\mathcal{E}_r$, $\mathbf{x} \in C_r$, then (by the equal treatment property) allocation $\mathbf{x}$ must be of the form

$$\mathbf{x} = \left(\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r \text{ times}}, \underbrace{\mathbf{x}^2, \ldots, \mathbf{x}^2}_{r \text{ times}}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r \text{ times}} \right)$$

  - All consumers of the same type must receive the same bundle.

Large Economies and the Core

- Core allocations in $\mathcal{E}_r$ are r-fold copies of feasible allocations in $\mathcal{E}_1$, $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$.
- Dividing the feasibility condition in $\mathcal{E}_r$ by r,

$$r\sum_{i=1}^{I}\mathbf{x}^i = r\sum_{i=1}^{I}\mathbf{e}^i, \qquad \text{so} \quad \sum_{i=1}^{I}\mathbf{x}^i = \sum_{i=1}^{I}\mathbf{e}^i$$

Large Economies and the Core

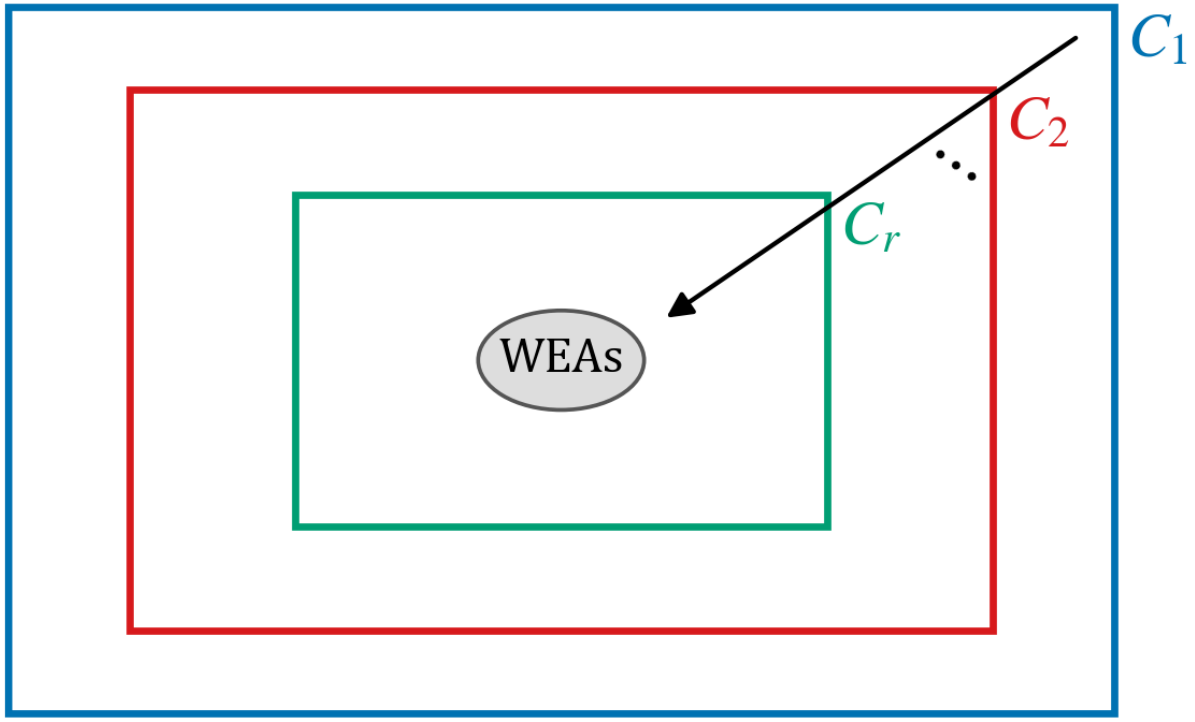
- The core shrinks as the economy enlarges: the sequence of core sets is nested,

$$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_r$$

  - where each core allocation is viewed as its type profile $(\mathbf{x}^1, \ldots, \mathbf{x}^I)$.
- That is, the core of the original (un-replicated) economy, C_1, is a superset of the core of the 2-fold replica economy, C_2; C_2 is a superset of the core of the 3-fold replica economy, C_3; etc.

Large Economies and the Core

- The core shrinks as r increases



$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_r$: as $r \to \infty$, C_r shrinks toward the WEAs

Large Economies and the Core

- *Proof*:
  - Since we seek to show that $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_{r-1} \supseteq C_r$, it suffices to show that, for every $r \geq 2$, $C_{r-1} \supseteq C_r$.
  - Suppose that allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I) \in C_r$.
  - There is no blocking coalition to $\mathbf{x}$ in the r-fold replica economy $\mathcal{E}_r$.
  - The $(r-1)$-fold copy of $\mathbf{x}$ is feasible in $\mathcal{E}_{r-1}$. We then need to show that $\mathbf{x}$ cannot be blocked by any coalition in $\mathcal{E}_{r-1}$ either.

Large Economies and the Core

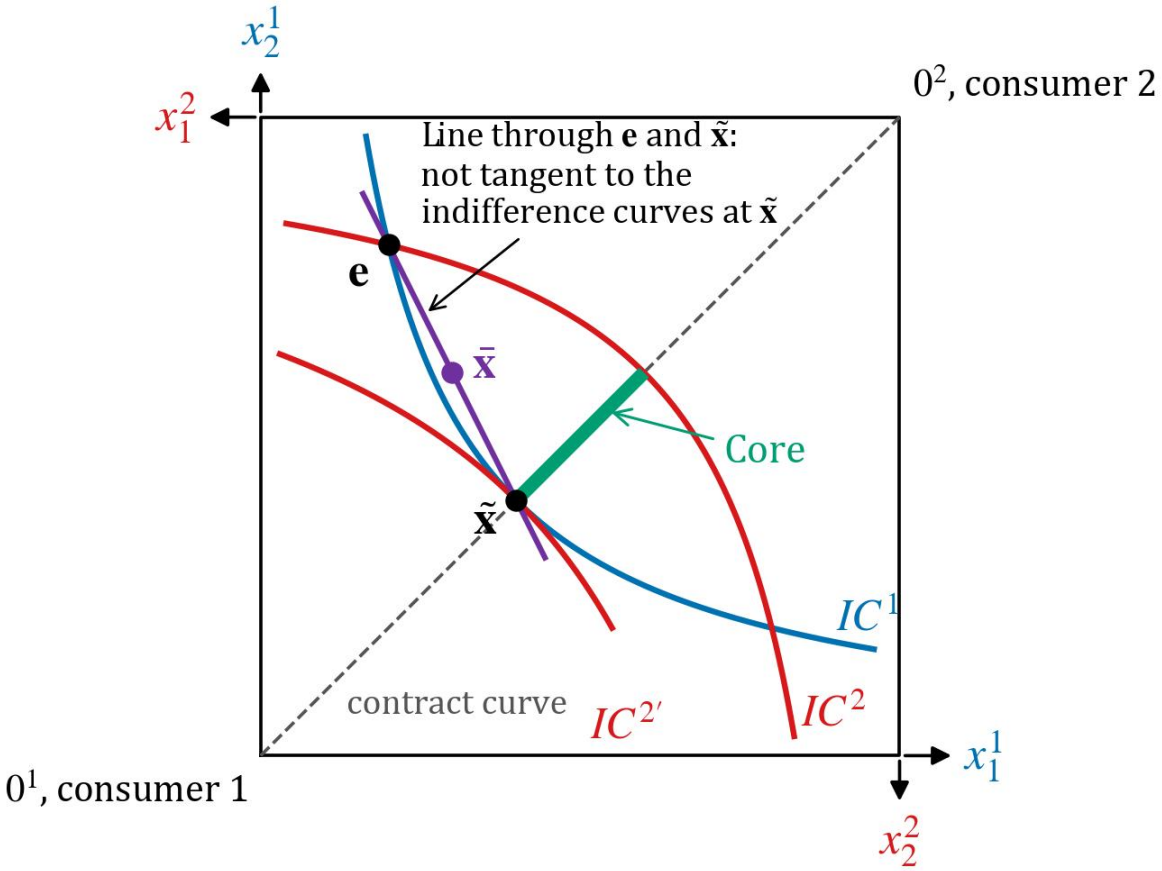
- If we could find a coalition S that blocks $\mathbf{x}$ in $\mathcal{E}_{r-1}$ with bundles $\mathbf{y}$, then we could also find a blocking coalition in $\mathcal{E}_r$:
  - all members of $\mathcal{E}_{r-1}$ are also present in $\mathcal{E}_r$, and their endowments have not changed.
- The same coalition S with the same bundles $\mathbf{y}$ is feasible in $\mathcal{E}_r$, which contradicts $\mathbf{x} \in C_r$. Hence, $\mathbf{x} \in C_{r-1}$.

Large Economies and the Core

- We now show that the core shrinks strictly: some allocations in C_1 are not in C_2.
- Consider an economy with two consumers and two goods. We show that the endpoint $\tilde{\mathbf{x}}$ of C_1, where consumer 1 is indifferent between $\tilde{\mathbf{x}}^1$ and $\mathbf{e}^1$, does not belong to the core of the 2-fold replica economy, C_2.

Large Economies and the Core

- Un-replicated economy $\mathcal{E}_1$



$\tilde{\mathbf{x}}$ gives consumer 1 the lowest utility among core allocations

Large Economies and the Core

- $\tilde{\mathbf{x}}$ is the endpoint of the core at which consumer 1 is indifferent to $\mathbf{e}$.
- $\tilde{\mathbf{x}}$ is not a WEA, since the line through $\tilde{\mathbf{x}}$ and $\mathbf{e}$ is not tangent to the consumers' indifference curves at $\tilde{\mathbf{x}}$.
- If the core shrinks as the economy enlarges, we should be able to show that $\tilde{\mathbf{x}} \notin C_2$.
- Let us build a blocking coalition against $\tilde{\mathbf{x}}$.

Large Economies and the Core

- *Desirability*. In $\mathcal{E}_2$, $\tilde{\mathbf{x}}$ gives $\tilde{\mathbf{x}}^{11} = \tilde{\mathbf{x}}^{12} = \tilde{\mathbf{x}}^{1}$ to the type-1 consumers and $\tilde{\mathbf{x}}^{21} = \tilde{\mathbf{x}}^{22} = \tilde{\mathbf{x}}^{2}$ to the type-2 consumers.
- Consider the coalition $S = \{11,12,21\}$ and the midpoint $\frac{1}{2}\mathbf{e}^1 + \frac{1}{2}\tilde{\mathbf{x}}^1$.
- Since $u^1(\mathbf{e}^1) = u^1(\tilde{\mathbf{x}}^1)$ and $\mathbf{e}^1 \neq \tilde{\mathbf{x}}^1$, strict quasiconcavity gives

$$u^1\left(\frac{1}{2}\mathbf{e}^1 + \frac{1}{2}\tilde{\mathbf{x}}^1\right) > u^1(\tilde{\mathbf{x}}^1)$$

Large Economies and the Core

- Hence, the midpoint is strictly preferred by both type-1 consumers.
- If the midpoint is offered to consumers 11 and 12, while consumer 21 keeps $\tilde{\mathbf{x}}^2$, they will all accept it:

$$\bar{\mathbf{x}}^{11} \equiv \frac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{11}) \succ^1 \tilde{\mathbf{x}}^{11}$$

$$\bar{\mathbf{x}}^{12} \equiv \frac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{12}) \succ^1 \tilde{\mathbf{x}}^{12}$$

- Consumer 21 receives $\bar{\mathbf{x}}^{21} = \tilde{\mathbf{x}}^2$, so he is indifferent.

Large Economies and the Core

- *Feasibility*. Since $\bar{\mathbf{x}}^{11} = \bar{\mathbf{x}}^{12}$, the sum of the proposed bundles is

$$\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \bar{\mathbf{x}}^{21} = 2 \times \frac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{11}) + \tilde{\mathbf{x}}^{21}$$
$$= \mathbf{e}^1 + \tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21}$$

- Recall that $\tilde{\mathbf{x}} \in C_1$, so it is feasible in $\mathcal{E}_1$:

$$\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2 = \mathbf{e}^1 + \mathbf{e}^2$$

- Therefore, since $\tilde{\mathbf{x}}^{11} = \tilde{\mathbf{x}}^1$ and $\tilde{\mathbf{x}}^{21} = \tilde{\mathbf{x}}^2$,

$$\tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21} = \mathbf{e}^1 + \mathbf{e}^2$$

Large Economies and the Core

- We can thus rewrite the above equality as

$$\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \bar{\mathbf{x}}^{21} = \mathbf{e}^1 + \underbrace{(\tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21})}_{=\mathbf{e}^1+\mathbf{e}^2} = 2\mathbf{e}^1 + \mathbf{e}^2$$

- This confirms feasibility.

Large Economies and the Core

- Hence, the endpoint $\tilde{\mathbf{x}}$ in the core of the un-replicated economy does not belong to the core of the two-fold replica economy, $\tilde{\mathbf{x}} \notin C_2$.
- There is a blocking coalition $S = \{11,12,21\}$ and an alternative allocation $\overline{\mathbf{x}} = \left(\overline{\mathbf{x}}^{11}, \overline{\mathbf{x}}^{12}, \overline{\mathbf{x}}^{21}\right)$ that makes consumers 11 and 12 strictly better off and leaves consumer 21 indifferent, and that is feasible for the coalition members.

Large Economies and the Core

- WEA in replicated economies:
  - Consider a WEA in the un-replicated economy $\mathcal{E}_1$, $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$, at prices $\mathbf{p}^*$.
  - $\mathbf{x}$ is a WEA of $\mathcal{E}_1$ at prices $\mathbf{p}^*$ if and only if its r-fold copy is a WEA of $\mathcal{E}_r$ at $\mathbf{p}^*$:

$$\mathbf{x} = \left(\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r \text{ times}}, \underbrace{\mathbf{x}^2, \ldots, \mathbf{x}^2}_{r \text{ times}}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r \text{ times}} \right)$$

Large Economies and the Core

- Each type-i consumer faces the same budget and has the same demand, so excess demand in $\mathcal{E}_r$ is r times that in $\mathcal{E}_1$.
- Hence, the r-fold copy of a WEA also belongs to the core of $\mathcal{E}_r$ (because every WEA is in the core, $W(\mathbf{e}) \subseteq C(\mathbf{e})$, shown earlier).

Large Economies and the Core

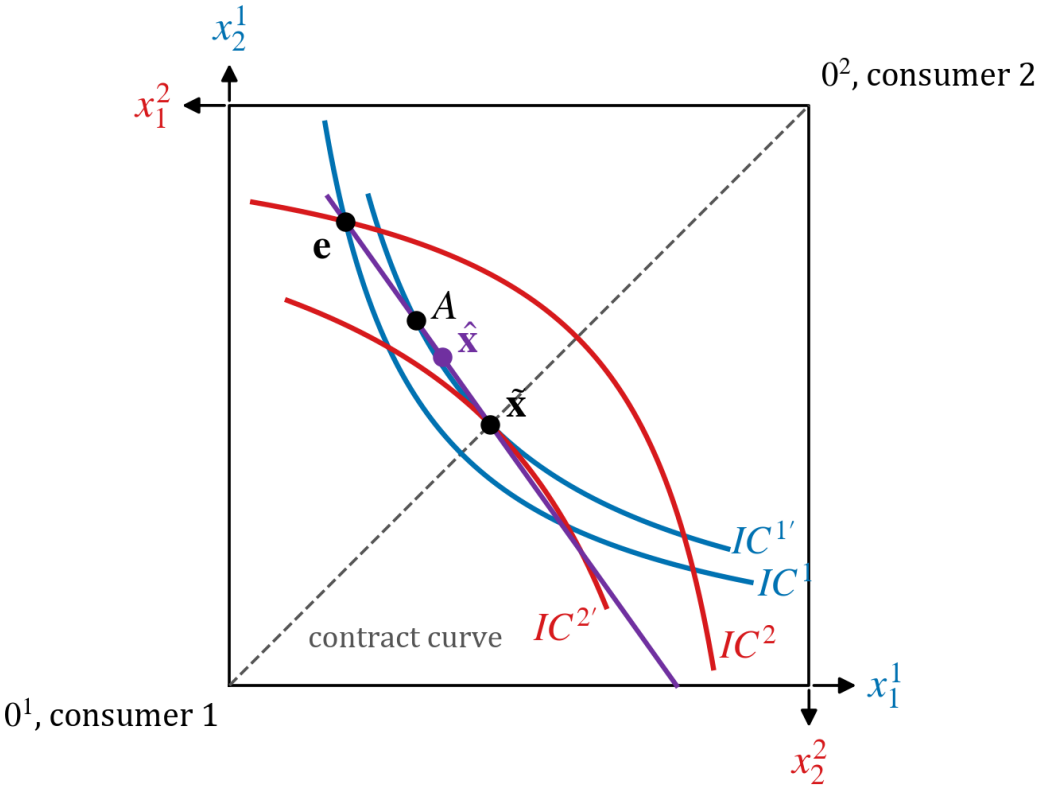
- ***A limit theorem on the core***: If $\mathbf{x}$ is an allocation of $\mathcal{E}_1$ whose r-fold copy belongs to C_r for every $r \geq 1$, then $\mathbf{x}$ must be a WEA of the un-replicated economy $\mathcal{E}_1$.
- We prove it for two types and two goods, with differentiable indifference curves (the general result, Debreu and Scarf 1963, uses a separating-hyperplane argument).

Large Economies and the Core

- *Proof* (by contradiction):
  - Consider that $\tilde{\mathbf{x}}$ belongs to C_r for every $r \geq 1$, but is not a WEA.
  - Since $\tilde{\mathbf{x}} \in C_r$ and $C_r \subseteq C_1$, $\tilde{\mathbf{x}} \in C_1$.
  - Allocation $\tilde{\mathbf{x}}$ must then be within the lens-shaped area and on the contract curve.

Large Economies and the Core

- A core allocation $\tilde{\mathbf{x}}$ that is not a WEA
  - A: the second intersection of the line through $\mathbf{e}$ and $\tilde{\mathbf{x}}$ with consumer 1's indifference curve through $\tilde{\mathbf{x}}$.



Large Economies and the Core

- Consider now the line connecting $\tilde{\mathbf{x}}$ and $\mathbf{e}$, and let $\frac{p_1}{p_2}$ be its slope (in absolute value).
- Since $\tilde{\mathbf{x}}$ is on the contract curve, $MRS^1_{1,2}(\tilde{\mathbf{x}}^1) = MRS^2_{1,2}(\tilde{\mathbf{x}}^2)$.
- Since $\tilde{\mathbf{x}}$ is not a WEA, $\frac{p_1}{p_2} \neq MRS^1_{1,2}(\tilde{\mathbf{x}}^1)$, so the line is tangent to neither indifference curve.
- Relabel types if needed so that the line cuts consumer 1's indifference curve at $\tilde{\mathbf{x}}$, as in the figure.

Large Economies and the Core

- Can allocation $\tilde{\mathbf{x}}$ be in C_r for every r and yet not be a WEA?
- Let us show that if $\tilde{\mathbf{x}}$ is not a WEA, it cannot be in C_r for every r.
- To demonstrate that $\tilde{\mathbf{x}} \notin C_r$ for r large enough, let us find a blocking coalition.

Large Economies and the Core

- By strict convexity of preferences, we can find a set of bundles (such as those between A and $\tilde{\mathbf{x}}^1$) that type-1 consumers strictly prefer to $\tilde{\mathbf{x}}^1$:

$$\hat{\mathbf{x}}^1 \equiv \frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\tilde{\mathbf{x}}^1$$

  - where r is large enough that $\hat{\mathbf{x}}^1$ lies between A and $\tilde{\mathbf{x}}^1$.

Large Economies and the Core

- Consider a coalition S with all r type-1 consumers and $r - 1$ type-2 consumers.
- Let us now show that allocation $\hat{\mathbf{x}}$ satisfies the properties of desirability and feasibility for the blocking coalition S.

Large Economies and the Core

- *Desirability*. If we give every type-1 consumer the bundle $\hat{\mathbf{x}}^1$, then $\hat{\mathbf{x}}^1 \succ^1 \tilde{\mathbf{x}}^1$.
- Each type-2 consumer in S keeps $\hat{\mathbf{x}}^2 = \tilde{\mathbf{x}}^2$ and is indifferent.

Large Economies and the Core

- *Feasibility*. The coalition's endowment is
$$r\mathbf{e}^1 + (r-1)\mathbf{e}^2$$
- Summing over the consumers in coalition S, their aggregate allocation is
$$r\hat{\mathbf{x}}^1 + (r-1)\hat{\mathbf{x}}^2 = r\left(\frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\tilde{\mathbf{x}}^1\right) + (r-1)\tilde{\mathbf{x}}^2$$
$$= \mathbf{e}^1 + (r-1)(\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2)$$

Large Economies and the Core

- Since $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2)$ is in the core of $\mathcal{E}_1$, it must be feasible:

$$\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2 = \mathbf{e}^1 + \mathbf{e}^2$$

- Combining the above two results, we find that

$$r\hat{\mathbf{x}}^1 + (r-1)\hat{\mathbf{x}}^2 = \mathbf{e}^1 + (r-1)\underbrace{(\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2)}_{=\mathbf{e}^1+\mathbf{e}^2}$$

$$= \mathbf{e}^1 + (r-1)(\mathbf{e}^1 + \mathbf{e}^2) = r\mathbf{e}^1 + (r-1)\mathbf{e}^2$$

Large Economies and the Core

- This confirms feasibility.
- Hence, r type-1 consumers and $r - 1$ type-2 consumers can form a coalition that blocks $\tilde{\mathbf{x}}$ in $\mathcal{E}_r$, so $\tilde{\mathbf{x}} \notin C_r$.

Large Economies and the Core

- Thus, if $\tilde{\mathbf{x}}$ is not a WEA, then $\tilde{\mathbf{x}} \notin C_r$ for every r large enough.
- As a consequence, if $\tilde{\mathbf{x}} \in C_r$ for all $r \geq 1$, then $\tilde{\mathbf{x}}$ must be a WEA.
- Together with the result on WEAs of replica economies, the allocations that remain in C_r for every r are the WEAs.