Advanced Microeconomic Theory

Chapter 1: Preferences and Utility

Outline

- Preference and Choice
- Preference-Based Approach
- Utility Function
- Indifference Sets, Convexity, and Quasiconcavity
- Special and Continuous Preference Relations
- Social and Reference-Dependent Preferences
- Hyperbolic and Quasi-Hyperbolic Discounting
- Choice-Based Approach
- Weak Axiom of Revealed Preference (WARP)
- Consumption Sets and Constraints

Preference and Choice

Preference and Choice

- We begin our analysis of individual decision-making in an abstract setting.
- Let $X \subseteq \mathbb{R}_+^N$ be a set of possible alternatives for a particular decision maker.
  - It might include the consumption bundles that an individual is considering to buy.
  - *Example*:

$$X = \{x, y, z, \dots\}$$

$$X = \{\text{Apple}, \text{Orange}, \text{Banana}, \dots\}$$

Preference and Choice

- Two ways to approach the decision making process:
  1) ***Preference-based approach***: analyzing how the individual uses his preferences to choose an element(s) from the set of alternatives X.
  2) ***Choice-based approach***: analyzing the actual choices the individual makes when he is called to choose element(s) from the set of possible alternatives.

Preference and Choice

- Advantages of the Choice-based approach:
  - It is based on observables (actual choices) rather than on unobservables (individual preferences)
- Advantages of Preference-based approach:
  - More tractable when the set of alternatives X has many elements.

Preference and Choice

- After describing both approaches, and the assumptions on each approach, we want to understand:

Rational Preferences $\Longrightarrow$ Consistent Choice behavior

Rational Preferences $\Longleftarrow$ Consistent Choice behavior

Preference-Based Approach

Preference-Based Approach

- ***Preferences***: "attitudes" of the decision-maker towards a set of possible alternatives X.
- For any $x, y \in X$, how do you compare x and y?
  - ❑ I prefer x to y $(x \succ y)$
  - ❑ I prefer y to x $(y \succ x)$
  - ❑ I am indifferent $(x \sim y)$

Preference-Based Approach

By asking:	We impose the assumption:
Check one box (i.e., not refrain from answering)	*Completeness*: individuals must compare any two alternatives, even the ones they don't know.
Check only one box	The individual is capable of comparing any pair of alternatives.
Don't add any new box in which the individual says, "I love x and hate y"	We don't allow the individual to specify the intensity of his preferences.

Preference-Based Approach

- ***Completeness***:
  - For any pair of alternatives $x, y \in X$, the individual decision maker:
    - $x \succ y$, or
    - $y \succ x$, or
    - both, i.e., $x \sim y$
  - (The decision maker is allowed to choose one, and only one, of the above boxes).

Preference-Based Approach

- ***Not all binary relations satisfy Completeness***.
- *Example*:
  - "Is the brother of": John $\not\succ$ Bob and Bob $\not\succ$ John if they are not brothers.
  - "Is the father of": John $\not\succ$ Bob and Bob $\not\succ$ John if the two individuals are not related.
- Not all pairs of alternatives are comparable according to these two relations.

Preference-Based Approach

- ***Weak preferences***:
  - Consider the following questionnaire:
  - For all $x, y \in X$, where x and y are not necessarily distinct, is x at least as preferred to y?
    - ❑ Yes $(x \succsim y)$
    - ❑ No $(y \succsim x)$
  - Respondents must answer yes, no, or both
    - Checking both boxes reveals that the individual is indifferent between x and y.
    - Note that the above statement relates to completeness, but in the context of weak preference $\succsim$ rather than strict preference $\succ$.

Preference-Based Approach

- ***Reflexivity***: every alternative x is weakly preferred to, at least, one alternative: itself.
- A preference relation satisfies reflexivity if for any alternative $x \in X$, we have that:
  1) $x \sim x$: any bundle is indifferent to itself.
  2) $x \gtrsim x$: any bundle is preferred or indifferent to itself.
  3) $x \succ x$: any bundle belongs to at least one indifference set, namely, the set containing itself if nothing else.

Preference-Based Approach

- The preference relation $\succsim$ is ***rational*** if it possesses the following two properties:

  a) *Completeness*: for all $x, y \in X$,
  either $x \succsim y$, or $y \succsim x$, or both.

  b) *Transitivity*: for all $x, y, z \in X$,
  if $x \succsim y$ and $y \succsim z$, then it must be that $x \succsim z$.

Preference-Based Approach

- ***Example 1.1*.**

Consider the preference relation

$$x \gtrsim y \quad \text{if and only if} \quad \sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} y_i$$

In words, the consumer prefers bundle x to y if the total number of goods in bundle x is larger than in bundle y.

Graphical interpretation in $\mathbb{R}^2$ (diagonal above another diagonal). Hyperplanes for $N > 2$.

Preference-Based Approach

- ***Example 1.1*** (continues)**.**
- *Completeness*:
  - either $\sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} y_i$ (which implies $x \succsim y$), or
  - $\sum_{i=1}^{N} y_i \geq \sum_{i=1}^{N} x_i$ (which implies $y \succsim x$), or
  - both, $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i$ (which implies $x \sim y$).

- *Transitivity:*
  - If $x \succsim y$, $\sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} y_i$, and
  - $y \succsim z$, $\sum_{i=1}^{N} y_i \geq \sum_{i=1}^{N} z_i$,
  - Then it must be that $\sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} z_i$ (which implies $x \succsim z$, as required).

Preference-Based Approach

- The assumption of transitivity is understood as that preferences should not cycle.
- Example violating transitivity:

$$\underbrace{apple \gtrsim banana \quad banana \gtrsim orange}_{apple \gtrsim orange \ (\text{by transitivity})}$$

  but $orange \succ apple$.

- Otherwise, we could start the cycle all over again, and extract infinite amount of money from individuals with intransitive preferences.

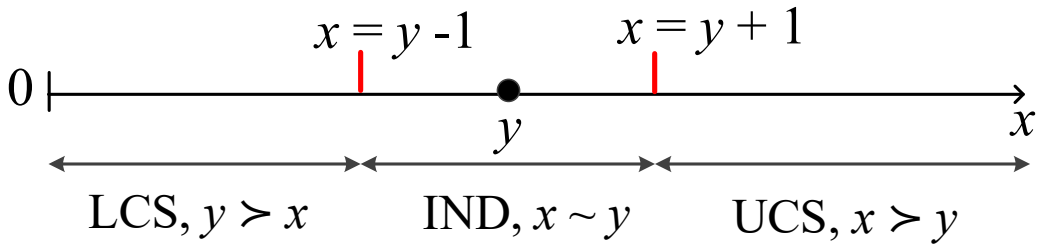
Preference-Based Approach

- Sources of intransitivity:
  a) Indistinguishable alternatives
  b) Framing effects
  c) Aggregation of criteria
  d) Change in preferences

Preference-Based Approach

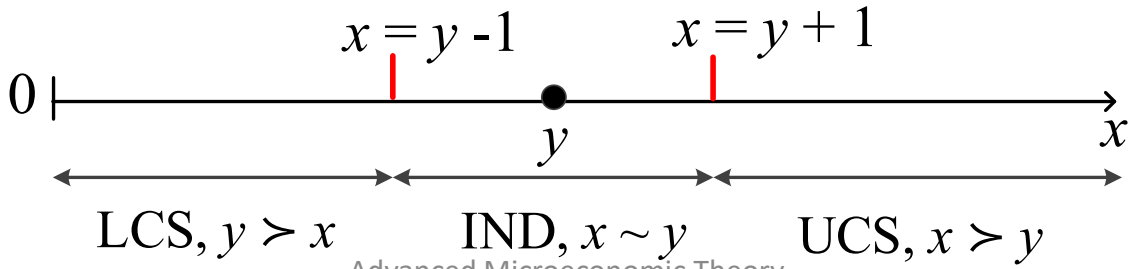
- ***Example 1.2*** (Indistinguishable alternatives):
  - Take $X = \mathbb{R}$, such as a piece of pie.
  - An individual is indifferent between x and y when $|x - y| < 1$.
    - This means that $-1 < x - y < 1$ or, after rearranging, $y - 1 < x < y + 1$.
    - In other words, when x satisfies both:
      - $x > y - 1$ and
      - $x < y + 1$,

      the individual is indifferent between x and y.
    - Intuitively, when alternatives are relatively similar (see figure), the individual cannot tell them apart.



Preference-Based Approach

- ***Example 1.2*** (Indistinguishable alternatives):
  - However, he strictly prefers x to y when $x \geq y + 1$, meaning that x is at least one unit larger than y.
    - See UCS at the right-hand of the figure.
  - In contrast, he strictly prefers y to x when $x \leq y - 1$, which means that y is at least one unit larger than x.
    - See LCS at the left-hand of the figure.



Preference-Based Approach

- ***Example 1.2*** (Indistinguishable alternatives):
  - **Completeness.** The above preference relation is complete:
    - For a given bundle x, another bundle y must lie in the UCS, IND, or LCS of x (see figure).
  - **Transitivity.** It does not hold:
    - Construct a counterexample, such as:

$$1.5 \sim 0.8 \text{ since } 1.5 - 0.8 = 0.7 < 1$$

$$0.8 \sim 0.3 \text{ since } 0.8 - 0.3 = 0.5 < 1$$

    - By transitivity, we would have $1.5 \sim 0.3$, but in fact $1.5 \succ 0.3$ (intransitive preference relation).

Preference-Based Approach

- *Other examples*:
  - similar shades of gray paint
  - milligrams of sugar in your coffee

Preference-Based Approach

- ***Example 1.3*** (Framing effects):
  - Transitivity might be violated because of the way in which alternatives are presented to the individual decision-maker.
  - What holiday package do you prefer?
    a) A weekend in Paris for \$574 at a four-star hotel.
    b) A weekend in Paris at the four-star hotel for \$574.
    c) A weekend in Rome at the five-star hotel for \$612.
  - By transitivity, we should expect that if $a \sim b$ and $b \succ c$, then $a \succ c$.

Preference-Based Approach

- ***Example 1.3*** (continued):
  - However, this did not happen!
  - More than 50% of the students responded $c \succ a$.
  - Such intransitive preference relation is induced by the framing of the options.

Preference-Based Approach

- ***Example 1.4*** (Aggregation of criteria):
  - Aggregation of several individual preferences might violate transitivity.
  - Consider $X = \{MIT, WSU, Home\ University\}$
  - When considering which university to attend, you might compare:
    a) Academic prestige (criterion #1)
    $$\succ_1 : \ MIT \succ_1 WSU \succ_1 Home\ Univ.$$
    b) City size/congestion (criterion #2)
    $$\succ_2 : \ WSU \succ_2 Home\ Univ. \succ_2 MIT$$
    c) Proximity to family and friends (criterion #3)
    $$\succ_3 : \ Home\ Univ. \succ_3 MIT \succ_3 WSU$$

Preference-Based Approach

- ***Example 1.4*** (continued):
  - By majority of these considerations:

$$MIT \underset{\text{criteria 1 \& 3}}{\succ} WSU \underset{\text{criteria 1 \& 2}}{\succ} Home\ Univ \underset{\text{criteria 2 \& 3}}{\succ} MIT$$

  - Transitivity is violated due to a cycle.
  - A similar argument can be used for the aggregation of individual preferences in *group decision-making*:
    - Every person in the group has a different (transitive) preference relation but the group preferences are not necessarily transitive ("***Condorcet paradox***").

Preference-Based Approach

- Intransitivity due to a *change in preferences*
  - When you start smoking

    $\text{One cigarette} \succsim \text{No smoking} \succsim \text{Smoking heavily}$

    By transitivity,

    $$\text{One cigarette} \succsim \text{Smoking heavily}$$
  - Once you started

    $\text{Smoking heavily} \succsim \text{One cigarette} \succsim \text{No smoking}$

    By transitivity,

    $$\text{Smoking heavily} \succsim \text{One cigarette}$$
  - But this contradicts the individual's past preferences when he started to smoke.

Utility Function

Utility Function

- A function $u\colon X \to \mathbb{R}$ is a ***utility function*** representing preference relations $\succsim$ if, for every pair of alternatives $x, y \in X$,

$$x \succsim y \iff u(x) \geq u(y)$$

Utility Function

- Two points:
  1) Only the ranking of alternatives matters.
     - That is, it does not matter if

$$u(x) = 14 \text{ or if } u(x) = 2000$$

$$u(y) = 10 \text{ or if } u(y) = 3$$

     - We do not care about *cardinality* (the number that the utility function associates with each alternative) but instead care about *ordinality* (ranking of utility values among alternatives).

Utility Function

2) If we apply any strictly increasing function $f(\cdot)$ on $u(x)$, i.e.,

$$f\colon \mathbb{R} \to \mathbb{R} \text{ such that } v(x) = f(u(x))$$

the new function keeps the ranking of alternatives intact and, therefore, the new function still represents the same preference relation.

– *Example*:

$$v(x) = 3u(x)$$
$$v(x) = 5u(x) + 8$$

Desirability

Desirability

- We can express desirability in different ways.
  - Monotonicity
  - Strong monotonicity
  - Non-satiation
  - Local non-satiation
- In all the above definitions, consider that x is an n-dimensional bundle

$$x \in \mathbb{R}^n, \text{ i.e., } x = (x_1, x_2, \dots, x_N)$$

where its k^{th} component represents the amount of good (or service) k, $x_k \in \mathbb{R}_+$.

Desirability

- ***Monotonicity***:
  - A preference relation satisfies monotonicity if, for all $x, y \in X$, where $x \neq y$,
    - *a)* $x_k \geq y_k$ for every good k implies $x \succsim y$
    - *b)* $x_k > y_k$ for every good k implies $x \succ y$
  - That is,
    - increasing the amounts of some commodities (without reducing the amount of any other commodity) cannot hurt, $x \succsim y$; and
    - increasing the amounts of all commodities is strictly preferred, $x \succ y$.

Desirability

- ***Strong Monotonicity***:
  - A preference relation satisfies strong monotonicity if, for all $x, y \in X$, where $x \neq y$,

$$x_k \geq y_k \text{ for every good } k \text{ implies } x \succ y$$

$$\text{and } x_l > y_l \text{ for at least one good } l$$

  - That is, even if we increase the amounts of only one of the commodities, we make the consumer strictly better off.

Desirability

- Relationship between **monotonicity** and utility function:
  - Monotonicity in preferences implies that the utility function is weakly monotonic (weakly increasing) in its arguments
    - That is, increasing some of its arguments weakly increases the value of the utility function, and increasing all its arguments strictly increases its value.
  - For any scalar $\alpha > 1$,

$$u(\alpha x_1, x_2) \geq u(x_1, x_2)$$
$$u(\alpha x_1, \alpha x_2) > u(x_1, x_2)$$

Desirability

- Relationship between **strong monotonicity** and utility function:
  - Strong monotonicity in preferences implies that the utility function is strictly monotonic (strictly increasing) in all its arguments.
    - That is, increasing some of its arguments strictly increases the value of the utility function.
  - For any scalar $\alpha > 1$,

$$u(\alpha x_1, x_2) > u(x_1, x_2)$$

Desirability

- ***Example 1.5***: $u(x_1, x_2) = \min\{x_1, x_2\}$
  - Monotone, since
    $$\min\{x_1 + \delta, x_2 + \delta\} > \min\{x_1, x_2\}$$
    for all $\delta > 0$.
  - Not strongly monotone, since
    $$\min\{x_1 + \delta, x_2\} \not\succ \min\{x_1, x_2\}$$
    if $\min\{x_1, x_2\} = x_2$.

Desirability

- ***Example 1.6***: $u(x_1, x_2) = x_1 + x_2$
  - Monotone, since
    $$(x_1 + \delta) + (x_2 + \delta) > x_1 + x_2$$
    for all $\delta > 0$.
  - Strongly monotone, since
    $$(x_1 + \delta) + x_2 > x_1 + x_2$$
- Hence, strong monotonicity implies monotonicity, but the converse is not necessarily true.

Desirability

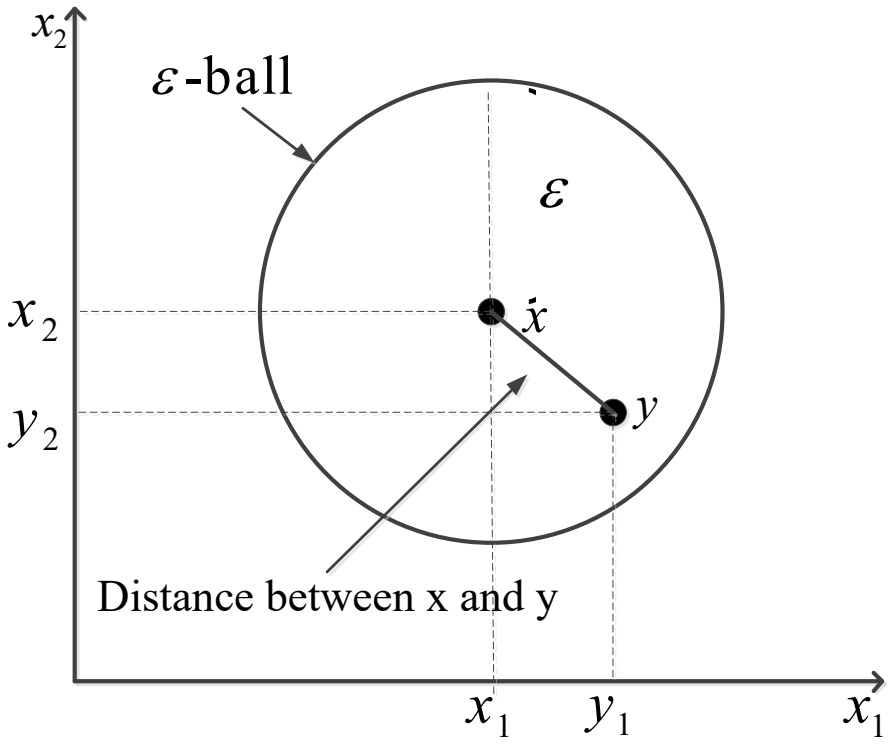
- ***Non-satiation*** (NS):
  - A preference relation satisfies NS if, for every $x \in X$, there is another bundle in set X, $y \in X$, which is strictly preferred to x, i.e., $y \succ x$.
    - NS is too general, since we could think about a bundle y containing extremely larger amounts of some goods than x.
    - How far away are y and x?

Desirability

- ***Local non-satiation*** (LNS):
  - A preference relation satisfies LNS if, for every bundle $x \in X$ and every $\varepsilon > 0$, there is another bundle $y \in X$ which is less than ε-away from x, $\|y - x\| < \varepsilon$, and for which $y \succ x$.
    - $\|y - x\| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$ is the Euclidean distance between x and y, where $x, y \in \mathbb{R}^2_+$.
    - In words, for every bundle x, and for ***every*** distance ε from x, we can find a more preferred bundle y.

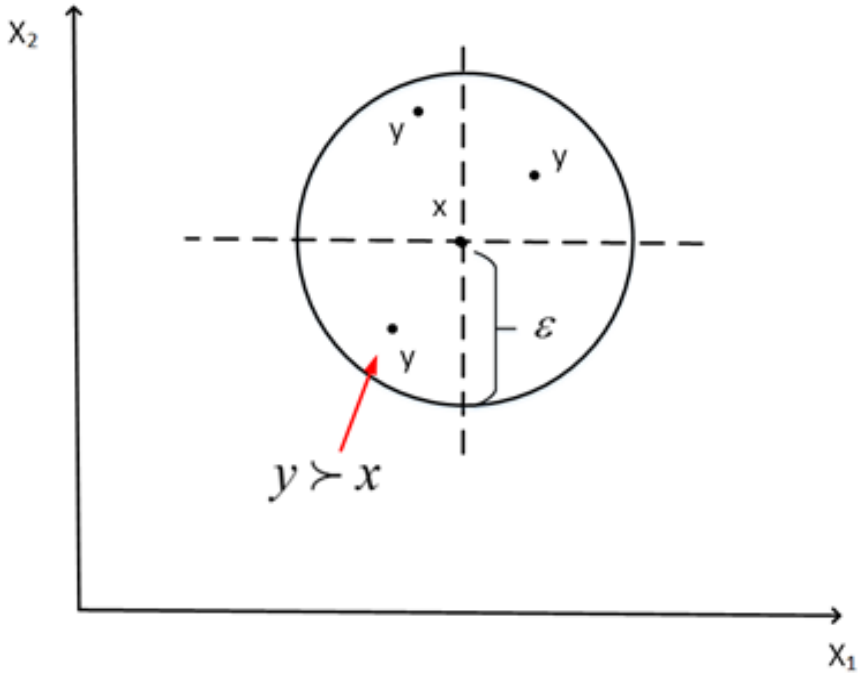
Desirability

– A preference relation satisfies $y \succ x$ even if bundle y contains less of good 2 (but more of good 1) than bundle x.



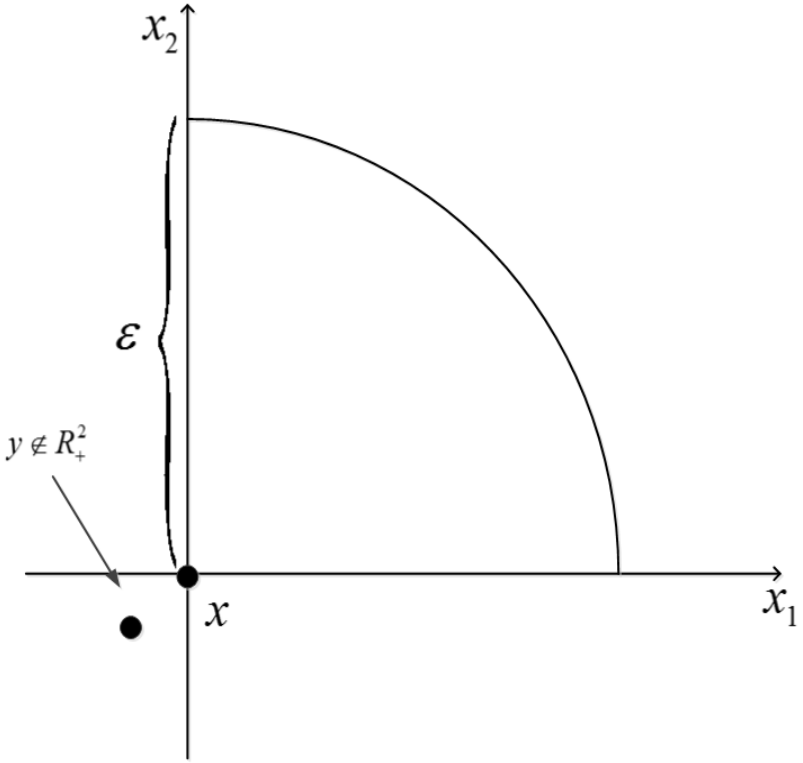
Desirability

– A preference relation satisfies $y \succ x$ even if bundle y contains less of *both* goods than bundle x.



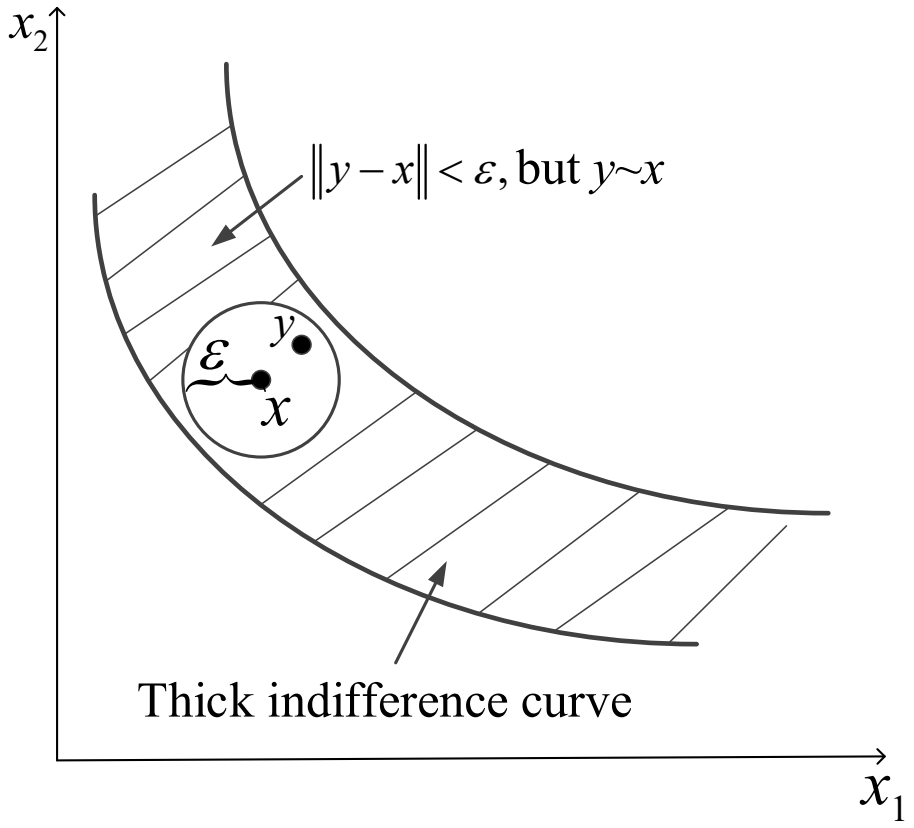
Desirability

- *Violation of LNS*
  - LNS rules out the case in which the decision-maker regards all goods as bads.
  - Although $y \succ x$, y is unfeasible given that it lies away from the consumption set, i.e., $y \notin \mathbb{R}^2_+$.



Desirability

- *Violation of LNS*
  - LNS also rules out "thick" indifference sets.
  - Bundles y and x lie on the same indifference curve.
  - Hence, decision maker is indifferent between x and y, i.e., $y \sim x$.



Desirability

- *Note*:
  - If a preference relation satisfies monotonicity, it must also satisfy LNS.
    - Given a bundle $x = (x_1, x_2)$, increasing all of its components yields a bundle $(x_1 + \delta, x_2 + \delta)$, which is strictly preferred to bundle (x_1, x_2) by monotonicity.
    - Hence, there is a bundle $y = (x_1 + \delta, x_2 + \delta)$ such that $y \succ x$ and $\|y - x\| < \varepsilon$.

Indifference sets

Indifference sets

- The indifference set of a bundle $x \in X$ is the set of all bundles $y \in X$, such that $y \sim x$.

$$IND(x) = \{y \in X : y \sim x\}$$

- The upper-contour set of bundle x is the set of all bundles $y \in X$, such that $y \succsim x$.

$$UCS(x) = \{y \in X : y \succsim x\}$$

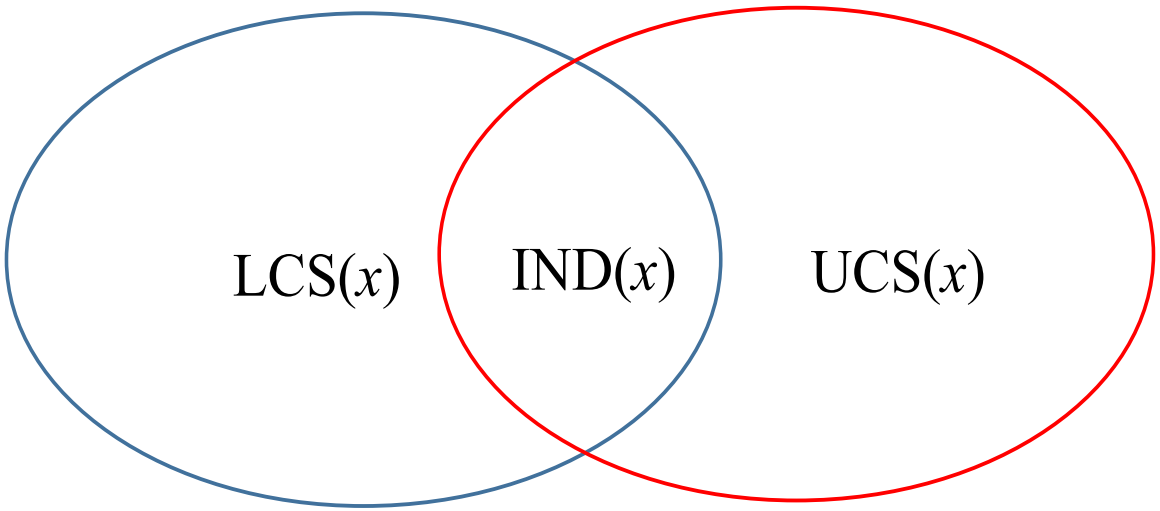
- The lower-contour set of bundle x is the set of all bundles $y \in X$, such that $x \succsim y$.

$$LCS(x) = \{y \in X : x \succsim y\}$$

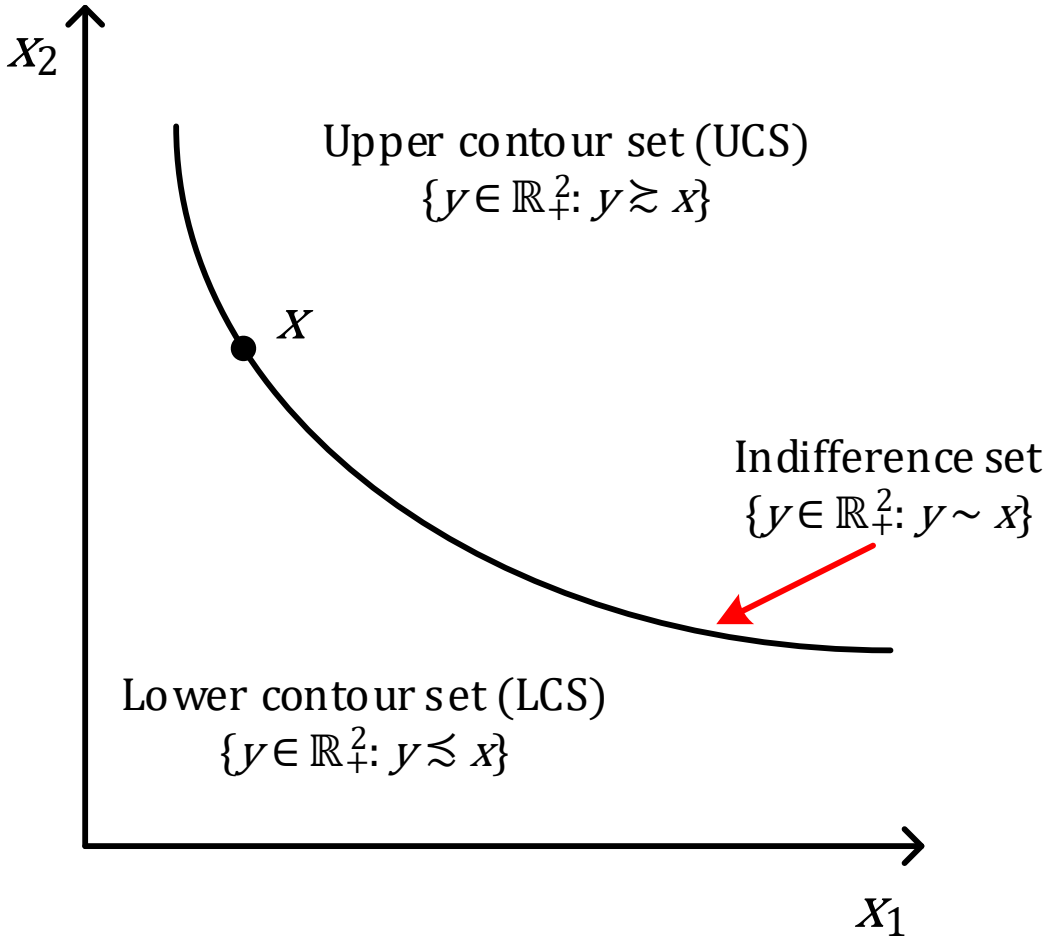
Indifference sets

- Therefore, $IND(x)$ is the intersection of $UCS(x)$ and $LCS(x)$, that is,

$$IND(x) = UCS(x) \cap LCS(x).$$

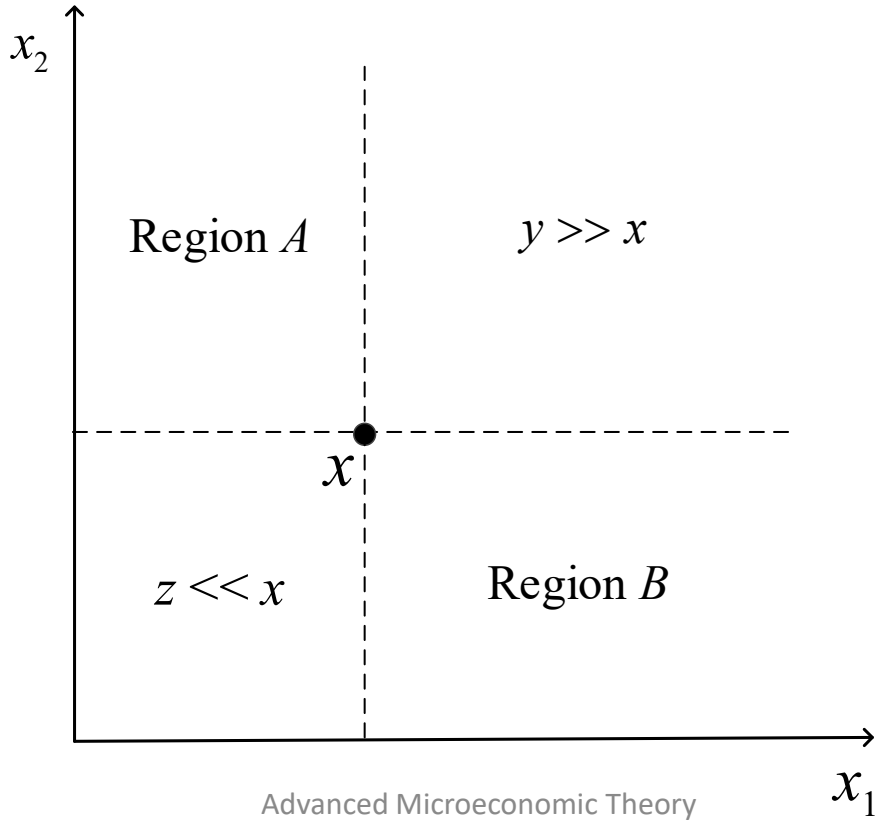


Indifference sets



Indifference sets

- Strong monotonicity implies that indifference curves must be negatively sloped.



Indifference sets

- *Note*:
  - Strong monotonicity implies that indifference curves must be negatively sloped.
  - In contrast, if an individual preference relation satisfies LNS, indifference curves can be upward sloping.
    - This can happen if, for instance, the individual regards good 2 as desirable but good 1 as a bad.

Convexity of Preferences

Convexity of Preferences

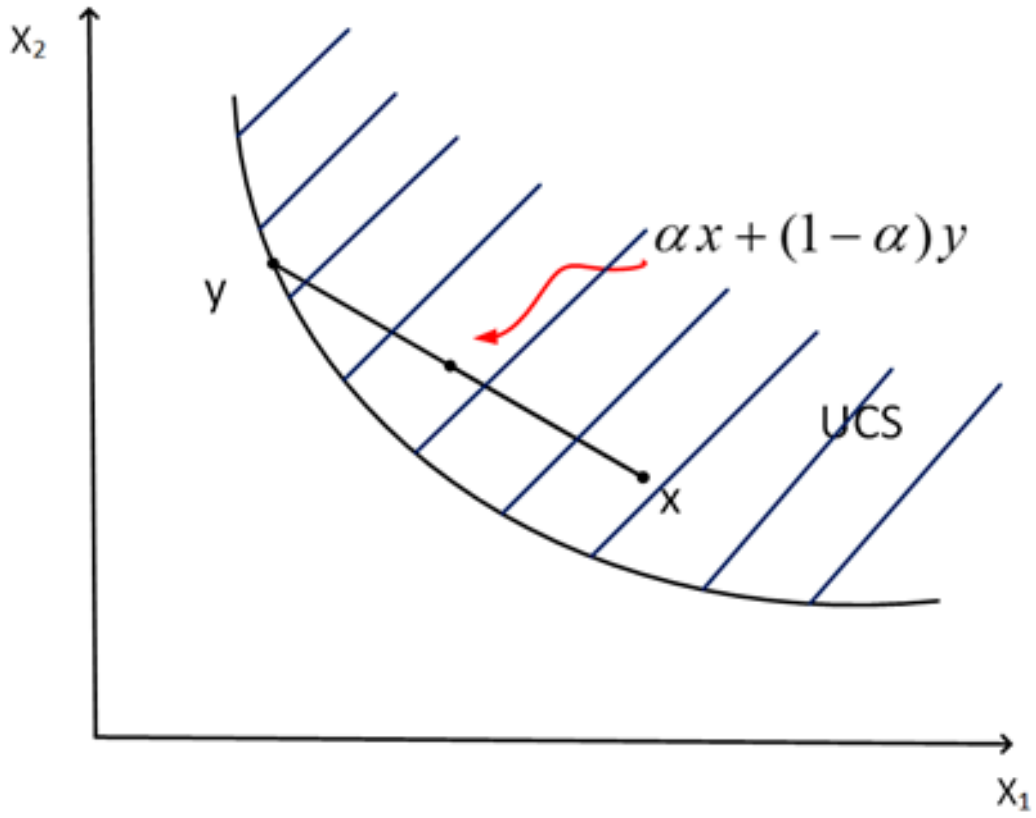
- ***Convexity 1***: A preference relation satisfies convexity if, for all $x, y \in X$,

$$x \succsim y \implies \alpha x + (1-\alpha)y \succsim y$$

for all $\alpha \in (0,1)$.

Convexity of Preferences

- Convexity 1



Convexity of Preferences

- ***Convexity 2***: A preference relation satisfies convexity if, for every bundle x, its upper contour set is convex.

$$UCS(x) = \{y \in X : y \succsim x\} \text{ is convex}$$

- That is, for every two bundles y and z,

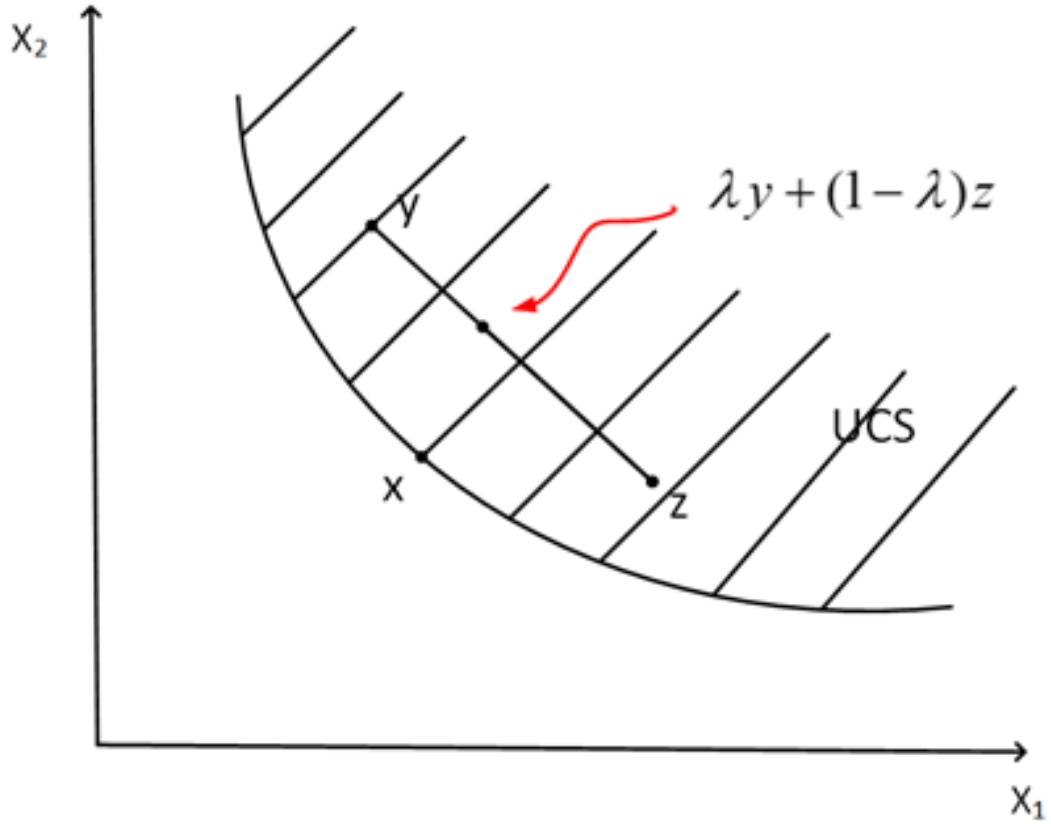
$$\begin{cases} y \succsim x \\ z \succsim x \end{cases} \Rightarrow \lambda y + (1-\lambda)z \succsim x$$

for any $\lambda \in (0,1)$.

- Hence, points y, z, and their convex combination belong to the UCS of x.

Convexity of Preferences

- Convexity 2



Convexity of Preferences

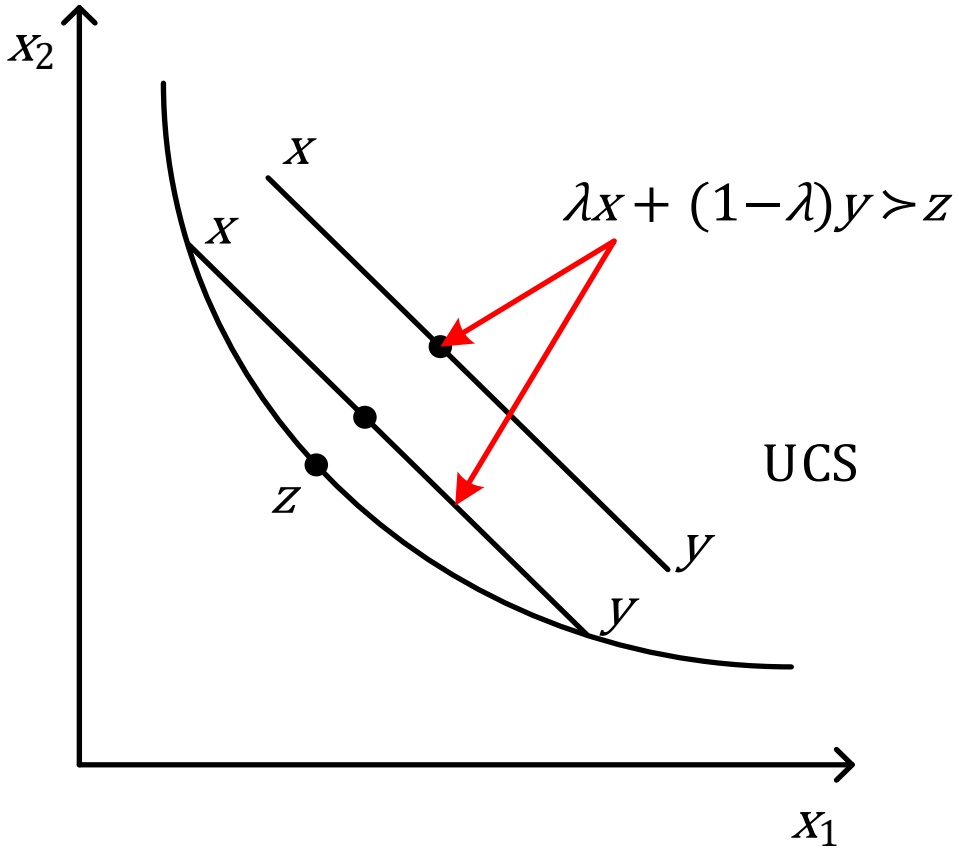
- ***Strict convexity***: A preference relation satisfies strict convexity if, for every $x, y \in X$ where $x \neq y$,

$$\begin{cases} x \gtrsim z \\ y \gtrsim z \end{cases} \Rightarrow \lambda x + (1-\lambda)y \succ z$$

for all $\lambda \in (0,1)$.

Convexity of Preferences

- Strictly convex preferences

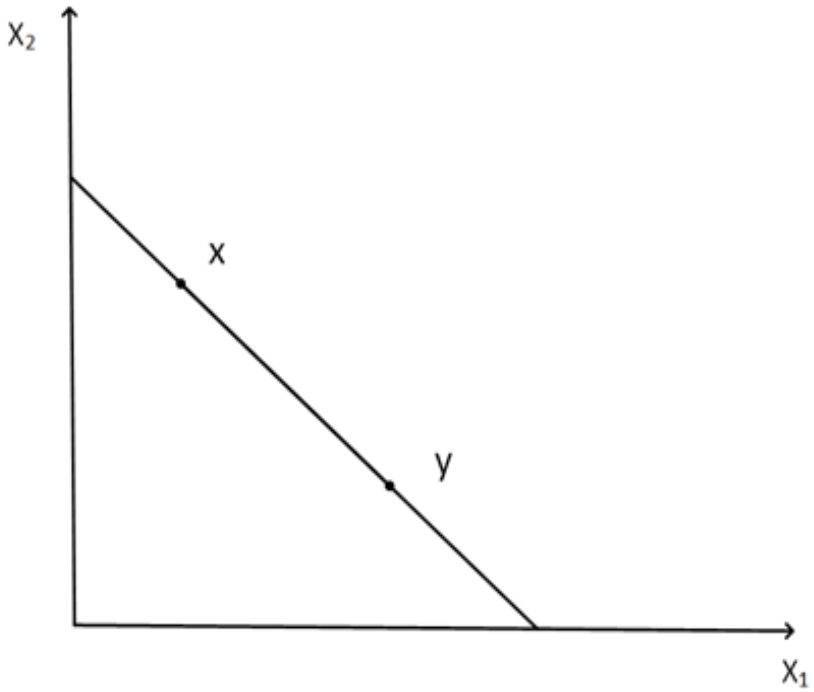


Convexity of Preferences

- **Convex but not strict convex preferences**
  - $\lambda x + (1-\lambda)y \sim z$
  - This type of preference relation is represented by linear utility functions such as

    $u(x_1, x_2) = ax_1 + bx_2$

    where x_1 and x_2 are regarded as substitutes.

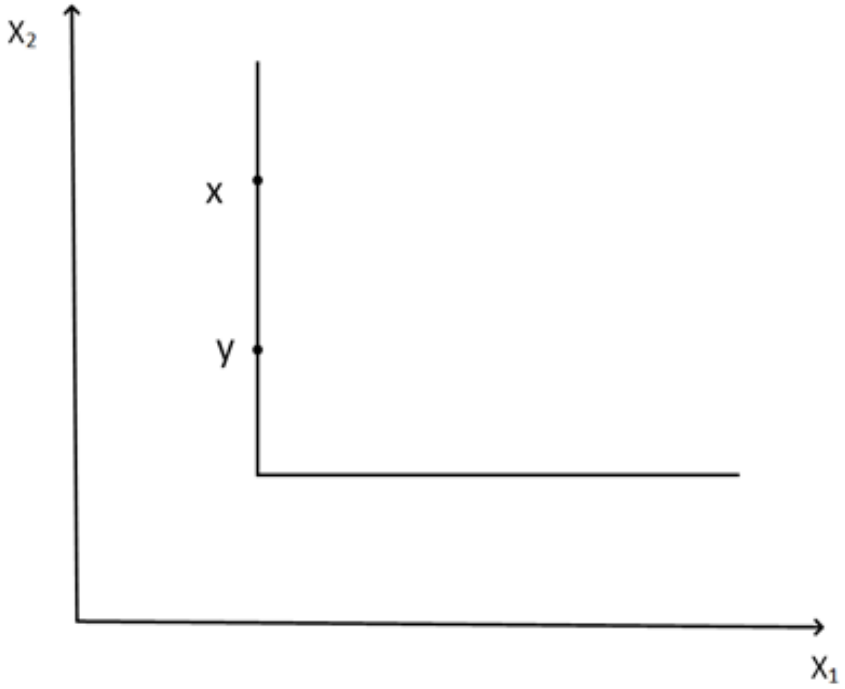


Convexity of Preferences

- **Convex but not strict convex preferences**
  - *Other example*: If a preference relation is represented by utility functions such as

$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$

where $a, b > 0$, then the pref. relation satisfies convexity, but not strict convexity.



Convexity of Preferences

- ***Example 1.7***

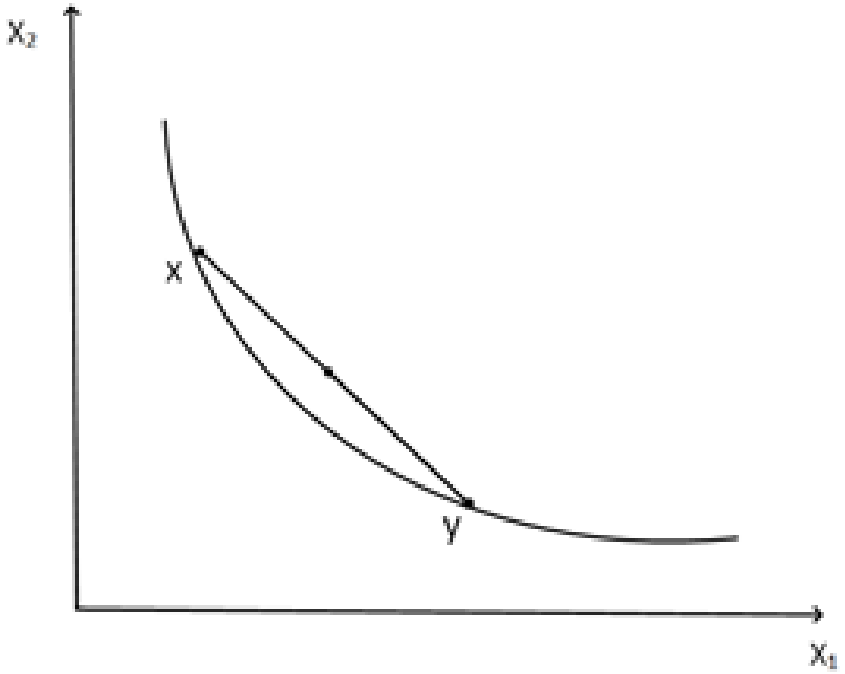
$u(x_1, x_2)$	Satisfies convexity	Satisfies strict convexity
$ax_1 + bx_2$	√	X
$\min\{ax_1, bx_2\}$	√	X
$ax_1^{\frac{1}{2}} \times bx_2^{\frac{1}{2}}$	√	√
$ax_1^2 \times bx_2^2$	√	√
$ax_1^{\frac{1}{2}} + bx_2^{\frac{1}{2}}$	√	√
$ax_1^2 + bx_2^2$	X	X

Convexity of Preferences

- ***Interpretation of convexity***

  1) Taste for diversification:

  – An individual with convex preferences prefers the convex combination of bundles x and y, than either of those bundles alone.



Convexity of Preferences

- ***Interpretation of convexity***

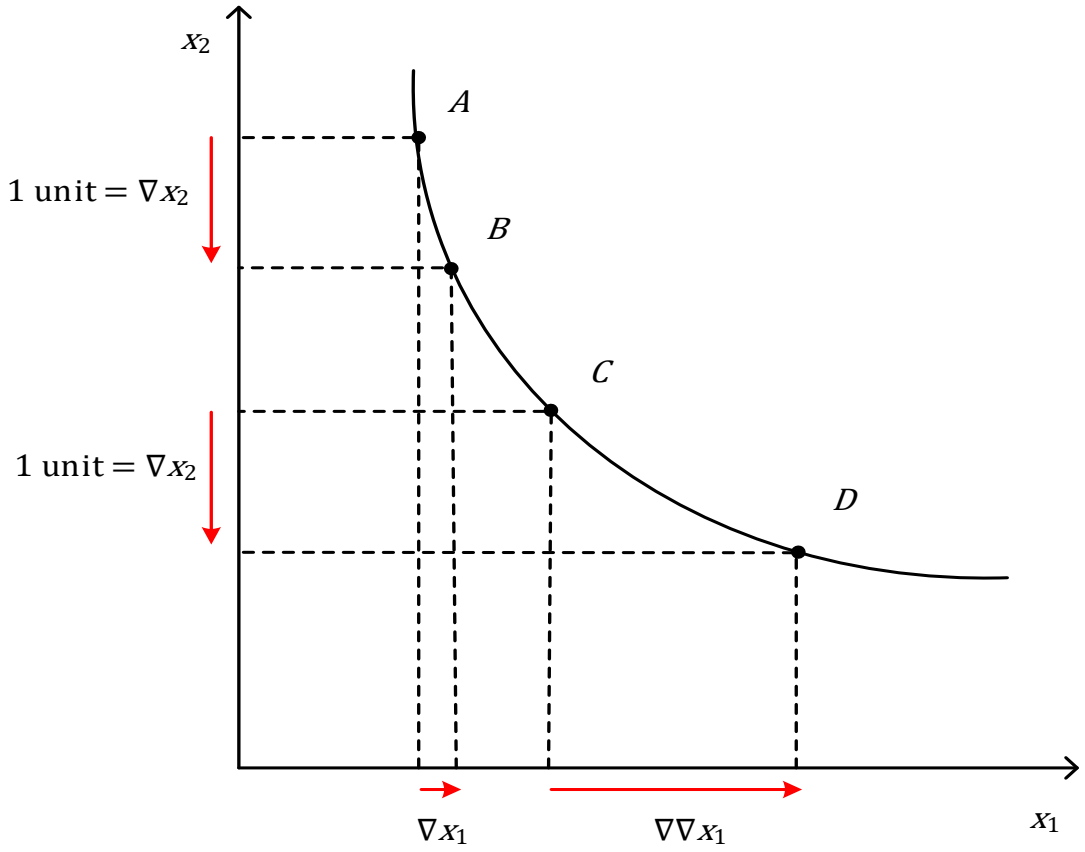
  2) Diminishing marginal rate of substitution:

$$MRS_{1,2} \equiv \frac{\partial u/\partial x_1}{\partial u/\partial x_2}$$

  - *MRS* describes the additional amount of good 1 that the consumer needs to receive in order to keep her utility level unaffected, when the amount of good 2 is reduced by one unit.
  - Hence, a *diminishing MRS* implies that the consumer needs to receive increasingly larger amounts of good 1 in order to accept further reductions of good 2.

Convexity of Preferences

- Diminishing marginal rate of substitution



Convexity of Preferences

- *Remark*:
  - Let us show that the slope of the indifference curve is given by the MRS.
  - Consider a continuous and differentiable utility function $u(x_1, x_2, \dots, x_n)$.
  - Totally differentiating, we obtain

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \cdots + \frac{\partial u}{\partial x_n} dx_n$$

  - But since we move along the same indifference curve, $du = 0$.

Convexity of Preferences

– Inserting $du = 0$,

$$0 = \frac{\partial u}{\partial x_i} dx_i + \frac{\partial u}{\partial x_j} dx_j$$

$$\text{or} \quad -\frac{\partial u}{\partial x_i} dx_i = \frac{\partial u}{\partial x_j} dx_j$$

– If we want to analyze the rate at which the consumer substitutes units of good i for good j, we must solve for $\frac{dx_j}{dx_i}$, to obtain

$$-\frac{dx_j}{dx_i} = \frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}} \equiv MRS_{i,j}$$

Quasiconcavity

Quasiconcavity

- A utility function $u(\cdot)$ is ***quasiconcave*** if, for every bundle $x \in X$, the set of all bundles for which the consumer experiences a higher utility, i.e., the $UCS(x) = \{y \in X \mid u(y) \geq u(x)\}$ is convex.
- The following three properties are equivalent:

Convexity of preferences $\Leftrightarrow$ $UCS(x)$ is convex $\Leftrightarrow$ $u(\cdot)$ is quasiconcave

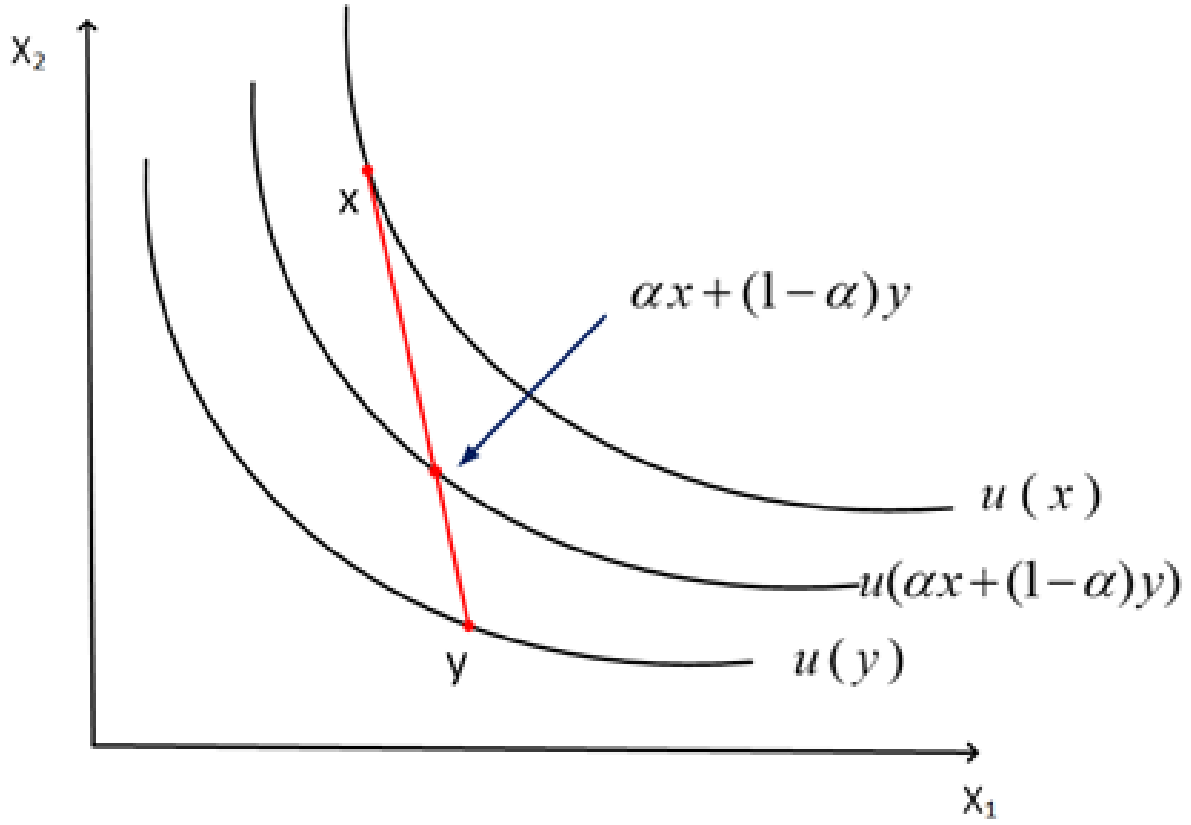
Quasiconcavity

- ***Alternative definition of quasiconcavity***:
  - A utility function $u(\cdot)$ satisfies *quasiconcavity* if, for every two bundles $x, y \in X$, the utility of consuming the convex combination of these two bundles, $u(\alpha x + (1-\alpha)y)$, is *weakly* higher than the minimal utility from consuming each bundle separately, $\min\{u(x), u(y)\}$:

$$u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}$$

Quasiconcavity

- Quasiconcavity (second definition)



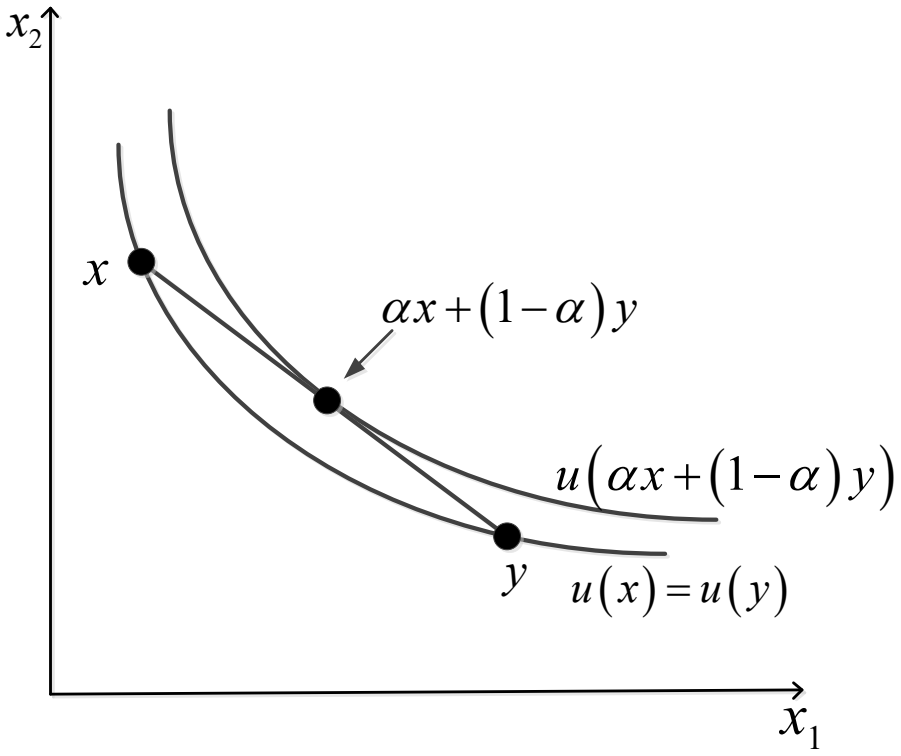
Quasiconcavity

- ***Strict quasiconcavity***:
  - A utility function $u(\cdot)$ satisfies *strict quasiconcavity* if, for every two bundles $x, y \in X$, the utility of consuming the convex combination of these two bundles, $u(\alpha x + (1-\alpha)y)$, is *strictly* higher than the minimal utility from consuming each bundle separately, $\min\{u(x), u(y)\}$:

$$u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\}$$

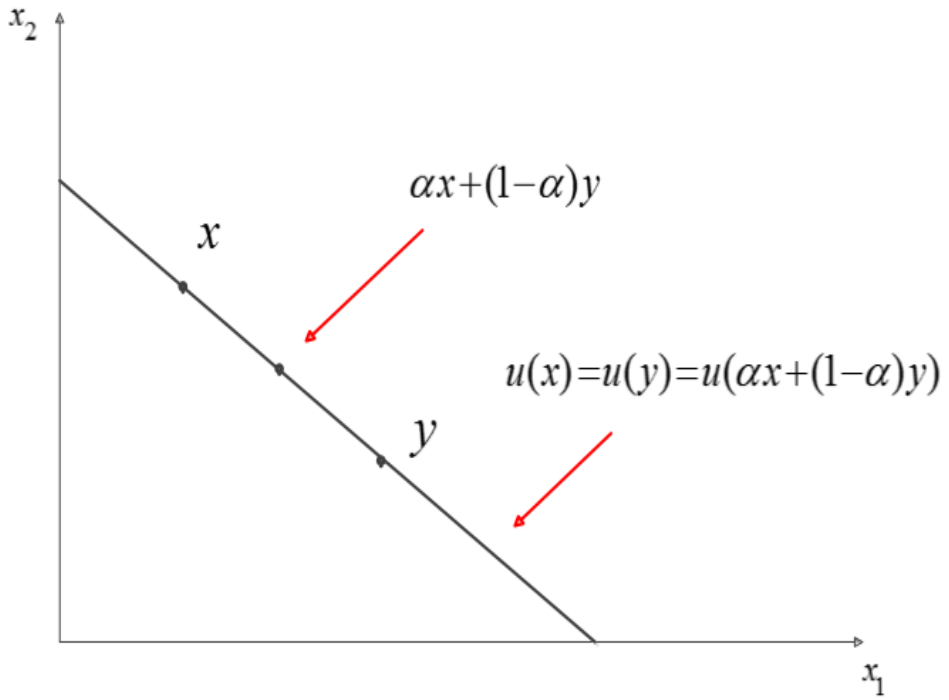
Quasiconcavity

- *What if bundles x and y lie on the same indifference curve?*
- Then, $u(x) = u(y)$.
- Since indifference curves are strictly convex, $u(\cdot)$ satisfies quasiconcavity.



Quasiconcavity

- *What if indifference curves are linear?*
- $u(\cdot)$ satisfies the definition of a quasiconcavity since
$$u(\alpha x + (1-\alpha)y) = \min\{u(x), u(y)\}$$
- But $u(\cdot)$ does not satisfy *strict* quasiconcavity.



Quasiconcavity

- ***Relationship between concavity and quasiconcavity***:

$$\text{Concavity} \underset{\not\Leftarrow}{\Rightarrow} \text{Quasiconcavity}$$

  - If a function $f(\cdot)$ is *concave*, then for any two points $x, y \in X$,

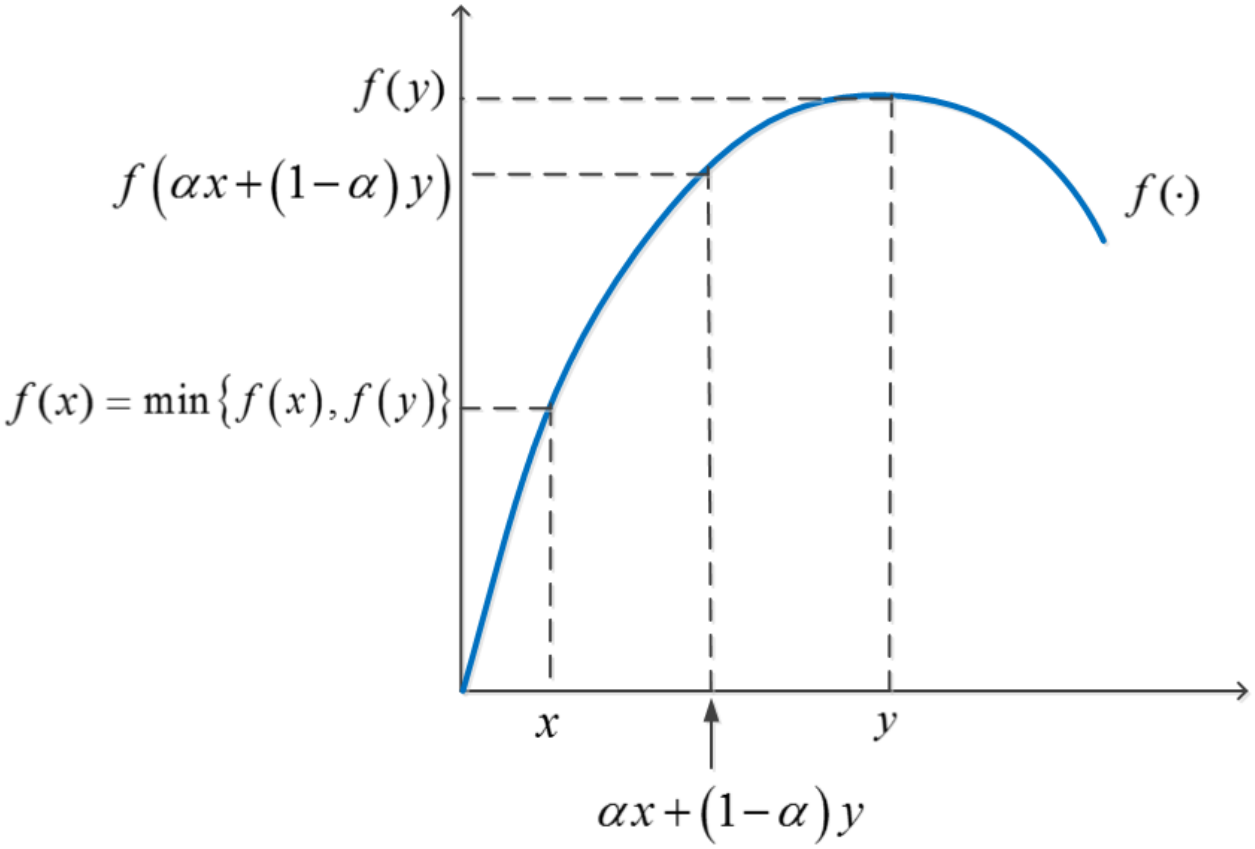
$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y)$$
$$\geq \min\{f(x), f(y)\}$$

    for all $\alpha \in (0,1)$.

    - The first inequality follows from the definition of concavity, while the second holds true for all concave functions.
    - Hence, quasiconcavity is a weaker condition than concavity.

Quasiconcavity

- Concavity implies quasiconcavity



Quasiconcavity

- To check curvature, we use Hessian matrix.

$$|H| = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

  - If the function is concave, then the first (second) leading principal minor is negative (positive).
  - If the function is convex, then both the first and second leading principal minors are positive.
  - If the second leading principal minor is negative, then the function is neither concave nor convex.

Quasiconcavity

- The k^{th} leading principal minor is obtained from the first k row and the first k column.
- In the case of a 2 x 2 matrix,
  - The first leading principal minor is u_{11}.
  - The second leading principal minor is

$$u_{11}u_{22} - u_{12}u_{21}$$

Quasiconcavity

- To check quasiconcavity or quasiconvexity, we use bordered Hessian matrix.

$$|BH| = \begin{bmatrix} 0 & u_1 & u_2 \\ u_1 & u_{11} & u_{12} \\ u_2 & u_{21} & u_{22} \end{bmatrix}$$

  - For a quasiconcave function, the second (third) leading principal minor is negative (positive).
  - For a quasiconvex function, both the second and third leading principal minors are negative.

Quasiconcavity

- In the case of a 3 x 3 bordered matrix,
  - The second leading principal minor is $-u_1^2 < 0$.
  - The third leading principal minor is

$$2u_1u_2u_{12} - u_2^2u_{11} - u_1^2u_{22}$$

Quasiconcavity

- For example, if $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$, then
  - $u_1 = \alpha x_1^{\alpha-1} x_2^{\beta}$ and $u_2 = \beta x_1^{\alpha} x_2^{\beta-1}$
  - $u_{11} = \alpha(\alpha - 1) x_1^{\alpha-2} x_2^{\beta}$
  - $u_{22} = \beta(\beta - 1) x_1^{\alpha} x_2^{\beta-2}$
  - $u_{21} = u_{12} = \alpha\beta x_1^{\alpha-1} x_2^{\beta-1}$

Quasiconcavity

- For $u(\cdot)$ to be concave (convex), we need $u_{11} \leq 0$ (≥ 0).
- Note that if $u_{11} \leq 0$ ($u_{11} \geq 0$), it must be that $u_{22} \leq 0$ ($u_{22} \geq 0$) in order for $u_{11}u_{22} - u_{12}^2 \geq 0$.
- If $\alpha \leq 1$ ($\alpha \geq 1$) and $\beta \leq 1$ ($\beta \geq 1$), we have the necessary condition for the utility function to be concave (convex).
- Since $u_{11}u_{22} - u_{12}^2 = \alpha\beta(1-\alpha-\beta)x_1^{2(\alpha-1)}x_2^{2(\beta-1)} \geq 0$ if and only if $\alpha + \beta \leq 1$. If $\alpha \geq 1$ and $\beta \geq 1$, it is impossible for $|H| \geq 0$. Hence, the Cobb-Douglas function cannot be convex.

Quasiconcavity

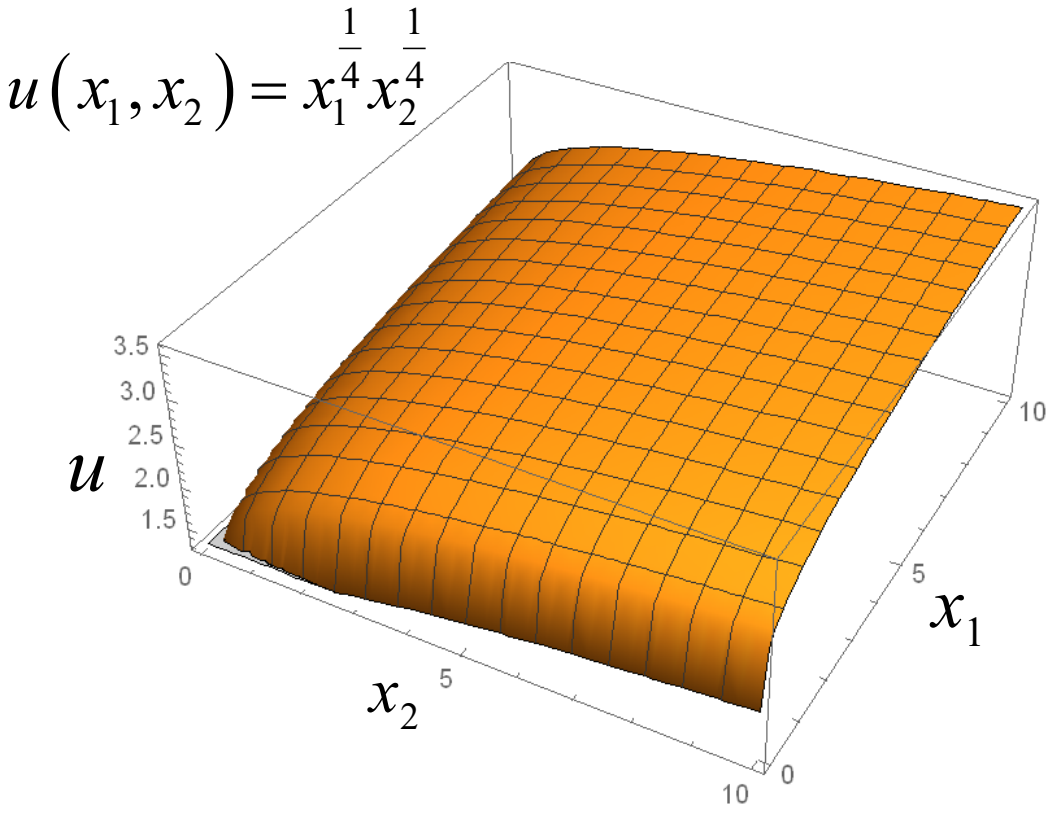
- For $u(x_1, x_2)$ to be quasiconcave, we require
- $2u_1u_2u_{12} - u_2^2u_{11} - u_1^2u_{22}$
- $= 2\alpha^2\beta^2x_1^{3\alpha-2}x_2^{3\beta-2} + \alpha(1-\alpha)\beta^2x_1^{3\alpha-2}x_2^{3\beta-2} + \alpha^2\beta(1-\beta)x_1^{3\alpha-2}x_2^{3\beta-2}$
- $= \alpha\beta(\alpha+\beta)x_1^{3\alpha-2}x_2^{3\beta-2} \geq 0$
- Therefore, the Cobb-Douglas function is always quasiconcave.
- For example, if $\alpha = \beta = 1/4$, then since $\alpha + \beta = 1/2 < 1$, $u(x_1, x_2) = x_1^{\frac{1}{4}}x_2^{\frac{1}{4}}$ is concave that implies it is quasiconcave.
- However, when $\alpha = \beta = 6/4$, then since $\alpha + \beta = 3 > 1$, $u(x_1, x_2) = x_1^{\frac{6}{4}}x_2^{\frac{6}{4}}$ is indeterminate though it is quasiconcave.

Quasiconcavity

- A concave $u(\cdot)$ exhibits diminishing marginal utility.
  - That is, for an increase in the consumption bundle, the increase in utility is *smaller* as we move away from the origin.
- The "jump" from one indifference curve to another requires:
  - a slight increase in the amount of x_1 and x_2 when we are close to the origin
  - a large increase in the amount of x_1 and x_2 as we get further away from the origin

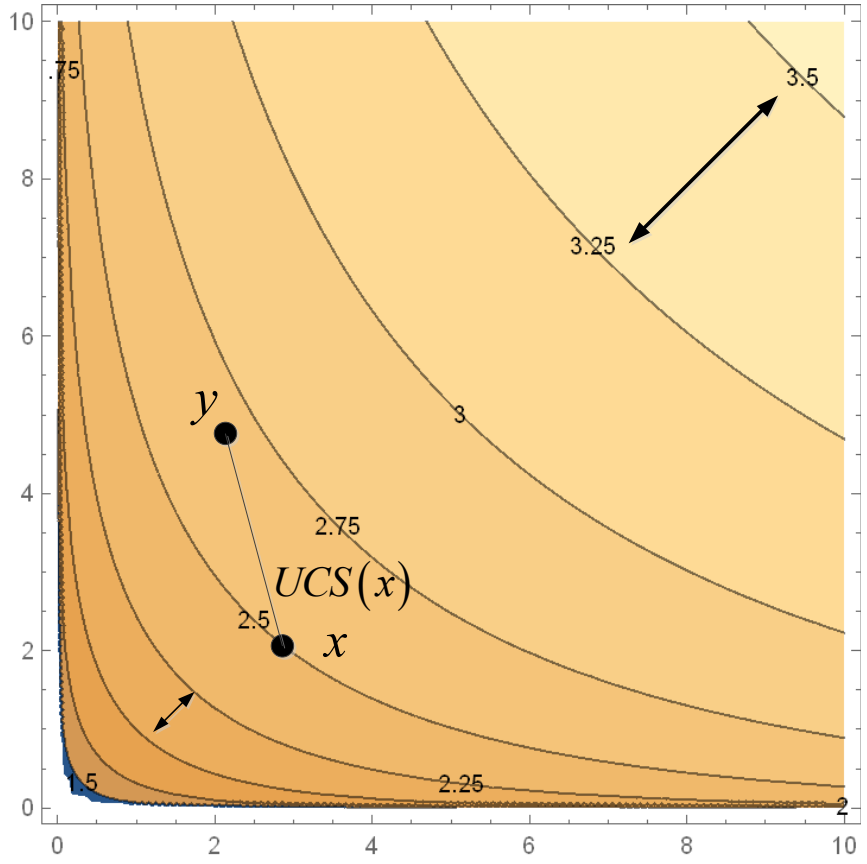
Quasiconcavity

- Concave and quasiconcave utility function (3D)



Quasiconcavity

- Concave and quasiconcave utility function (2D)

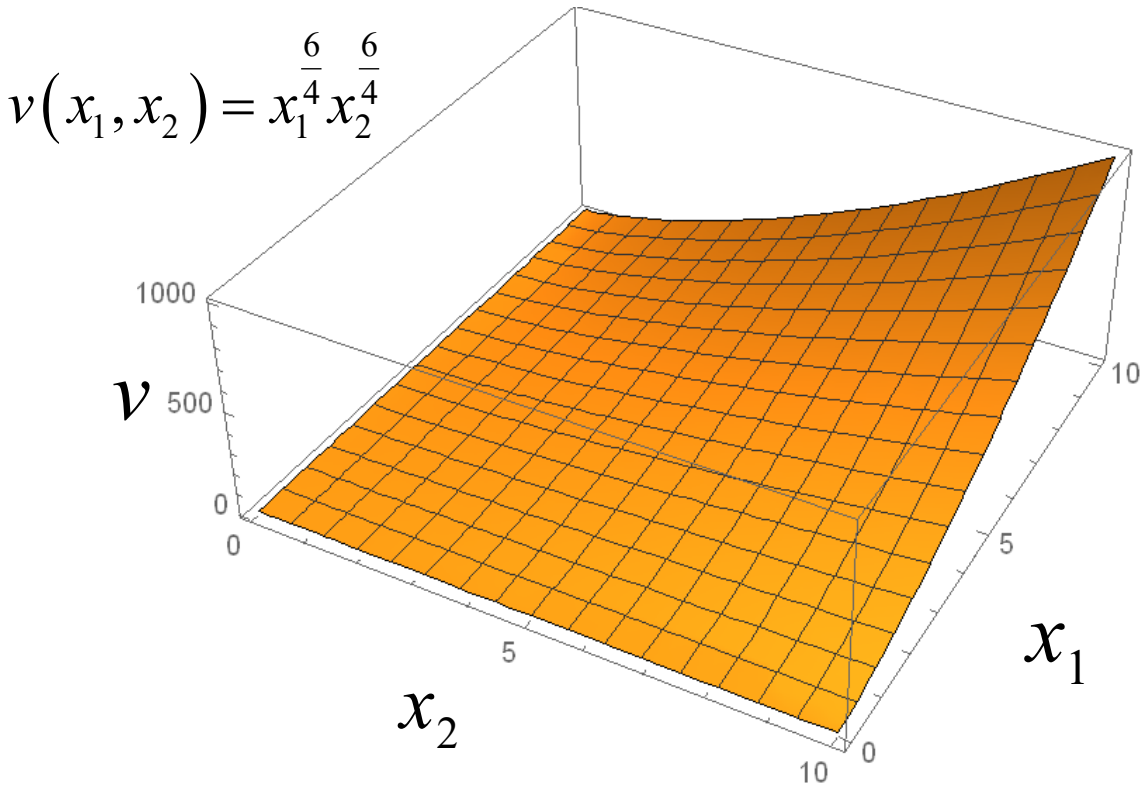


Quasiconcavity

- A convex $u(\cdot)$ exhibits increasing marginal utility.
  - That is, for an increase in the consumption bundle, the increase in utility is *larger* as we move away from the origin.
- The "jump" from one indifference curve to another requires:
  - a large increase in the amount of x_1 and x_2 when we are close to the origin, but…
  - a small increase in the amount of x_1 and x_2 as we get further away from the origin

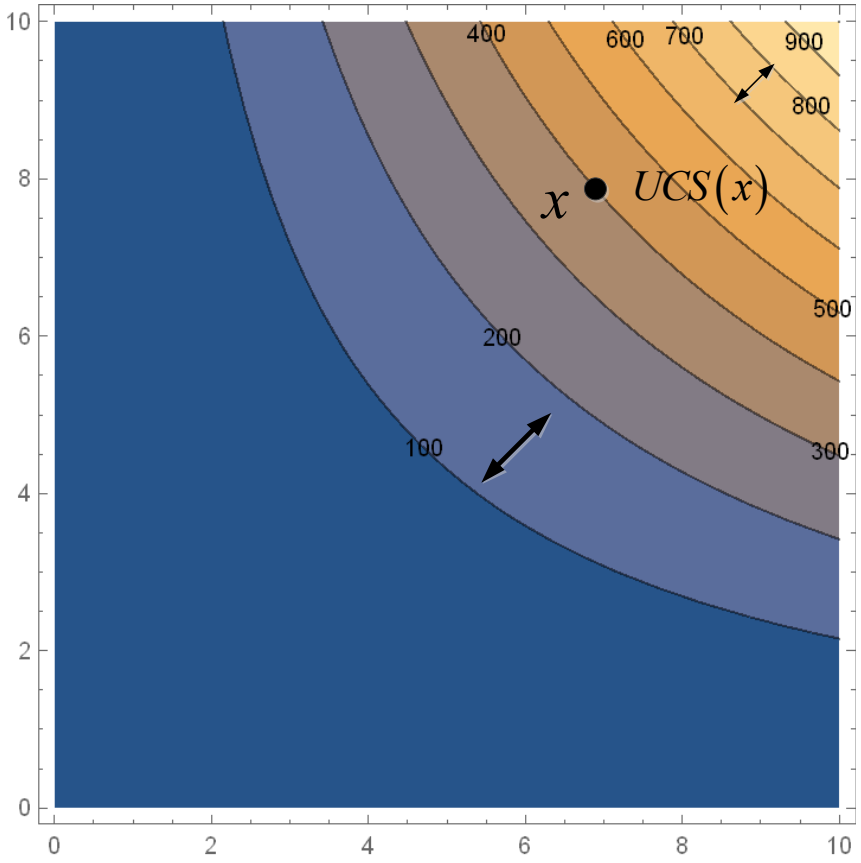
Quasiconcavity

- Non-convex and quasiconcave utility function (3D)



Quasiconcavity

- Non-convex and quasiconcave utility function (2D)



Quasiconcavity

- *Note*:
  - Utility function $v(x_1, x_2) = x_1^{\frac{6}{4}} x_2^{\frac{6}{4}}$ is a strictly monotonic transformation of $u(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$,
    - That is, $v(x_1, x_2) = f(u(x_1, x_2))$, where $f(u) = u^6$.
  - While both utility functions $u(x_1, x_2)$ and $v(x_1, x_2)$ are quasiconcave and represent the same preference relation, $u(x_1, x_2)$ is concave while $v(x_1, x_2)$ is neither concave nor convex (that is, indeterminate).
  - Hence, we normally require utility functions to satisfy quasiconcavity alone.

Quasiconcavity

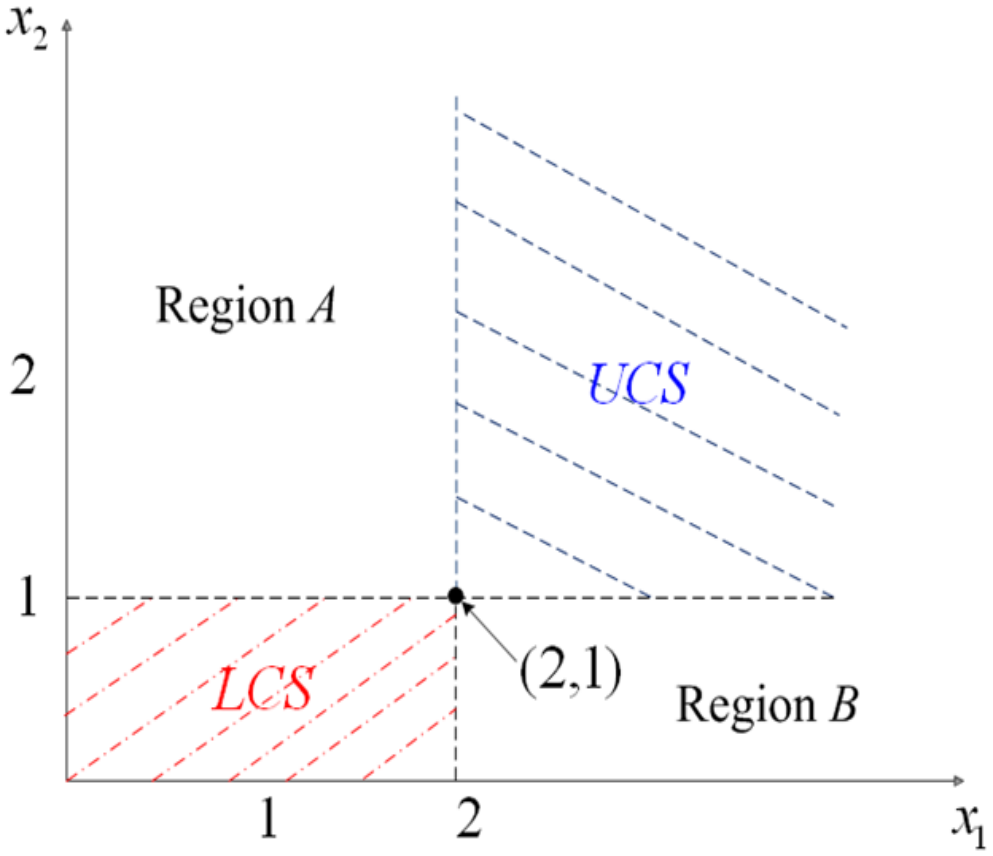
- ***Example 1.8*** (Testing properties of preference relations):
  - Consider an individual decision maker who consumes bundles in $\mathbb{R}_+^L$.
  - Informally, he "prefers more of everything"
  - Formally, for two bundles $x, y \in \mathbb{R}_+^L$, bundle x is weakly preferred to bundle y, $x \gtrsim y$, iff bundle x contains more units of every good than bundle y does, i.e., $x_k \geq y_k$ for every good k.
  - Let us check if this preference relation satisfies: (a) completeness, (b) transitivity, (c) strong monotonicity, (d) strict convexity, and (e) local non-satiation.

Quasiconcavity

- ***Example 1.8*** (continued):
  - Let us consider the case of only two goods, $L = 2$.
  - Then, an individual prefers a bundle $x = (x_1, x_2)$ to another bundle $y = (y_1, y_2)$ iff x contains more units of both goods than bundle y, i.e., $x_1 \geq y_1$ and $x_2 \geq y_2$.
  - For illustration purposes, let us take bundle such as $(2,1)$.

Quasiconcavity

- ***Example 1.8*** (continued):



Quasiconcavity

- ***Example 1.8*** (continued):

  1) UCS:

  – The upper contour set of bundle $(2,1)$ contains bundles (x_1, x_2) with weakly more than 2 units of good 1 and/or weakly more than 1 unit of good 2:

  $$UCS(2,1) = \{(x_1, x_2) \succsim (2,1) \Leftrightarrow x_1 \geq 2, x_2 \geq 1\}$$

  – The frontiers of the UCS region also represent bundles preferred to $(2,1)$.

Quasiconcavity

- ***Example 1.8*** (continued):

  2) LCS:

  – The bundles in the lower contour set of bundle $(2,1)$ contain fewer units of both goods:

  $$LCS(2,1) = \{(2,1) \gtrsim (x_1, x_2) \iff x_1 \leq 2, x_2 \leq 1\}$$

  – The frontiers of the LCS region also represent bundles with fewer unis of either good 1 or good 2.

Quasiconcavity

- ***Example 1.8*** (continued):

  3) IND:

  – The indifference set comprising bundles (x_1, x_2) for which the consumer is indifferent between (x_1, x_2) and $(2,1)$ is a singleton (itself):

  $$IND(2,1) = \{(2,1) \sim (x_1, x_2)\} = \{(2,1)\}$$

  – There is no other bundle making the consumer indifferent between (2,1) and such a bundle.

  – There is no region for which the UCS and LCS overlap. These sets only "touch" at bundle (2,1).

Quasiconcavity

- ***Example 1.8*** (continued):

  4) Regions A and B:

  - Region A contains bundles with more units of good 2 but fewer units of good 1 (the opposite argument applies to region B).
  - The consumer cannot compare bundles in either of these regions against bundle $(2,1)$.
  - For him to be able to rank one bundle against another, one of the bundles must contain the same or more units of all goods.

Quasiconcavity

- ***Example 1.8*** (continued):

  5) Preference relation is not complete:

  - Completeness requires for every pair x and y, either $x \succsim y$ or $y \succsim x$ (or both).
  - Consider two bundles $x, y \in \mathbb{R}^2_+$ with bundle x containing more units of good 1 than bundle y but fewer units of good 2, i.e., $x_1 > y_1$ and $x_2 < y_2$ (as in Region B)
  - Then, we have neither $x \succsim y$ (UCS) nor $y \succsim x$ (LCS).

Quasiconcavity

- ***Example 1.8*** (continued):

  6) Preference relation is transitive:

  - Transitivity requires that, for any three bundles x, y and z, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$.
  - Now $x \succsim y$ and $y \succsim z$ means $x_k \geq y_k$ and $y_k \geq z_k$ for all k goods.
  - Then, $x_k \geq z_k$ implies $x \succsim z$.

Quasiconcavity

- ***Example 1.8*** (continued):

  7) Preference relation is strongly monotone:

  - Strong monotonicity requires that if we increase one of the goods in a given bundle y, then the newly created bundle x must be strictly preferred to the original bundle.
  - Now $x \geq y$ and $x \neq y$ implies that $x_l \geq y_l$ for all good l and $x_k > y_k$ for at least one good k.
  - Thus, $x \geq y$ and $x \neq y$ implies $x \gtrsim y$ and not $y \gtrsim x$.
  - Thus, we can conclude that $x \succ y$.

Quasiconcavity

- ***Example 1.8*** (continued):

  8) Preference relation is strictly convex:

  – Strict convexity requires that if $x \gtrsim z$ and $y \gtrsim z$ and $x \neq y$, then $\alpha x + (1-\alpha)y \succ z$ for all $\alpha \in (0,1)$.

  – Now $x \gtrsim z$ and $y \gtrsim z$ implies that $x_l \geq z_l$ and $y_l \geq z_l$ for all good l.

  – $x \neq z$ implies, for some good k, we must have $x_k > z_k$.

Quasiconcavity

- ***Example 1.8*** (continued):
  - Hence, for any $\alpha \in (0,1)$, we must have that

$$\alpha x_l + (1-\alpha) y_l \geq z_l \text{ for every good } l$$

$$\alpha x_k + (1-\alpha) y_k > z_k \text{ for some } k$$

  - Thus, we have that $\alpha x + (1-\alpha) y \geq z$ and $\alpha x + (1-\alpha) y \neq z$, and so

$$\alpha x + (1-\alpha) y \gtrsim z$$

$$\text{and not } z \gtrsim \alpha x + (1-\alpha) y$$

  - Therefore, $\alpha x + (1-\alpha) y \succ z$.

Quasiconcavity

- ***Example 1.8*** (continued):

  9) Preference relation satisfies LNS:

  – Take any bundle (x_1, x_2) and a scalar $\varepsilon > 0$.

  – Let us define a new bundle (y_1, y_2) where

  $$(y_1, y_2) \equiv \left(x_1 + \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2}\right)$$

  so that $y_1 > x_1$ and $y_2 > x_2$.

  – Hence, $y \gtrsim x$ but not $x \gtrsim y$, which implies $y \succ x$.

Quasiconcavity

- ***Example 1.8*** (continued):
  - Let us know check if bundle y is within an ε-ball around x.
  - The Cartesian distance between x and y is

$$\|x-y\| = \sqrt{\left[x_1 - \left(x_1 + \frac{\varepsilon}{2}\right)\right]^2 + \left[x_2 - \left(x_2 + \frac{\varepsilon}{2}\right)\right]^2} = \frac{\varepsilon}{\sqrt{2}}$$

    which is smaller than ε for all $\varepsilon > 0$.

Common Utility Functions

Common Utility Functions

- ***Cobb-Douglas utility functions***:
  - In the case of two goods, x_1 and x_2,

$$u(x_1, x_2) = Ax_1^{\alpha} x_2^{\beta}$$

    where $A, \alpha, \beta > 0$.
  - Applying logs on both sides

$$\log u = \log A + \alpha \log x_1 + \beta \log x_2$$

  - Hence, the exponents in the original $u(\cdot)$ can be interpreted as *elasticities*:

$$\varepsilon_{u,x_1} = \frac{\partial u(x_1,x_2)}{\partial x_1} \cdot \frac{x_1}{u(x_1,x_2)} = \alpha A x_1^{\alpha-1} x_2^{\beta} \cdot \frac{x_1}{A x_1^{\alpha} x_2^{\beta}} = \alpha$$

Common Utility Functions

- Intuitively, a one-percent increase in the amount of good x_1 increases individual utility by α percent.
- Similarly, $\varepsilon_{u,x_2} = \beta$.
- Special cases:
  - $\alpha + \beta = 1$: $u(x_1, x_2) = Ax_1^{\alpha} x_2^{1-\alpha}$
  - $A = 1$: $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$
  - $A = \alpha = \beta = 1$: $u(x_1, x_2) = x_1 x_2$

Common Utility Functions

– Marginal utilities:

$$\frac{\partial u}{\partial x_1} > 0 \text{ and } \frac{\partial u}{\partial x_2} > 0$$

– Diminishing MRS, since

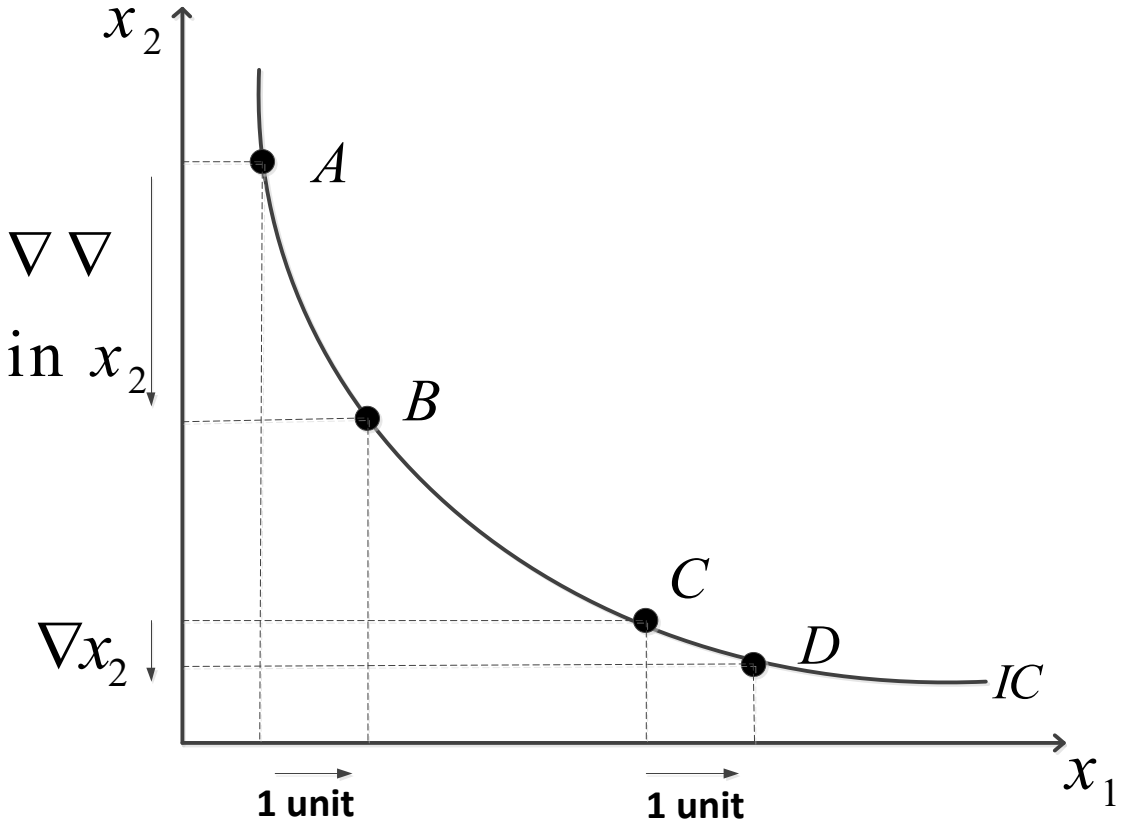
$$MRS_{x_1,x_2} = \frac{\alpha A x_1^{\alpha-1} x_2^{\beta}}{\beta A x_1^{\alpha} x_2^{\beta-1}} = \frac{\alpha x_2}{\beta x_1}$$

which is decreasing in x_1.

- Hence, indifference curves become flatter as x_1 increases.

Common Utility Functions

- Cobb-Douglas preference



Common Utility Functions

- ***Perfect substitutes***:
  - In the case of two goods, x_1 and x_2,

$$u(x_1, x_2) = Ax_1 + Bx_2$$

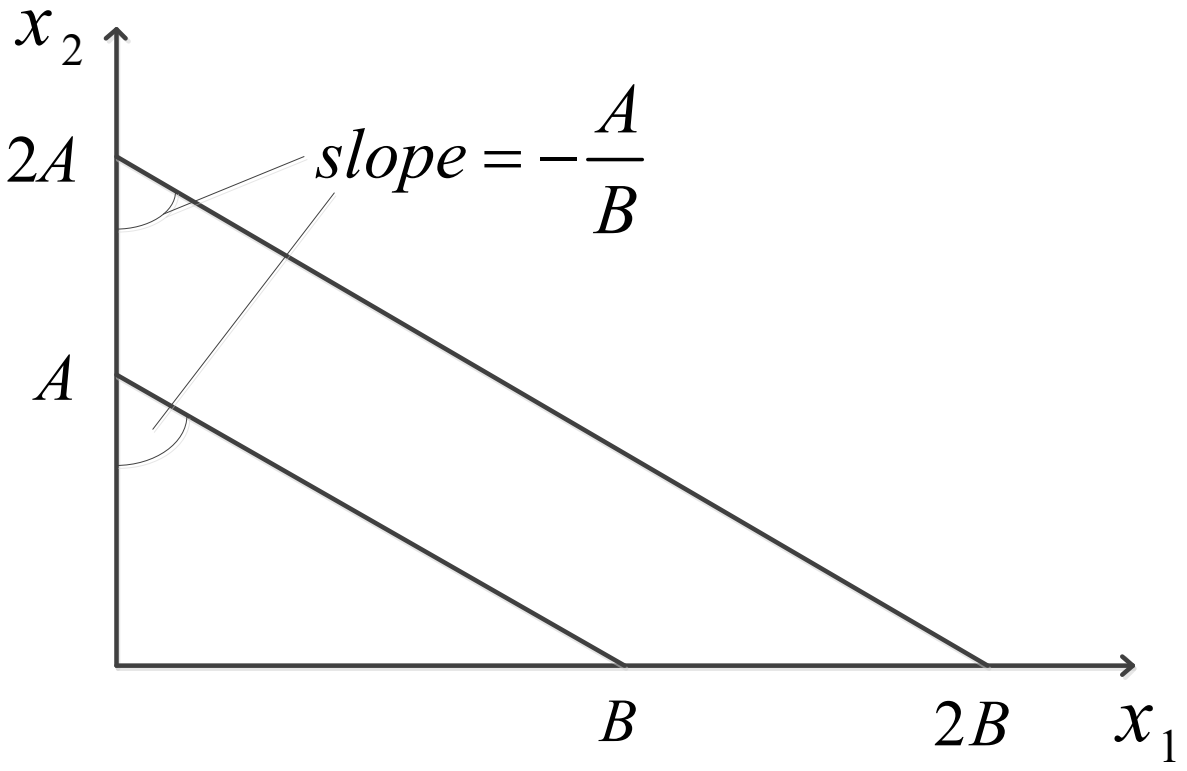
    where $A, B > 0$.
  - Hence, the marginal utility of every good is constant:

$$\frac{\partial u}{\partial x_1} = A \text{ and } \frac{\partial u}{\partial x_2} = B$$

  - MRS is also constant, i.e., $MRS_{x_1,x_2} = \frac{A}{B}$
    - Therefore, indifference curves are straight lines with a slope of $-\frac{A}{B}$.

Common Utility Functions

- Perfect substitutes



Common Utility Functions

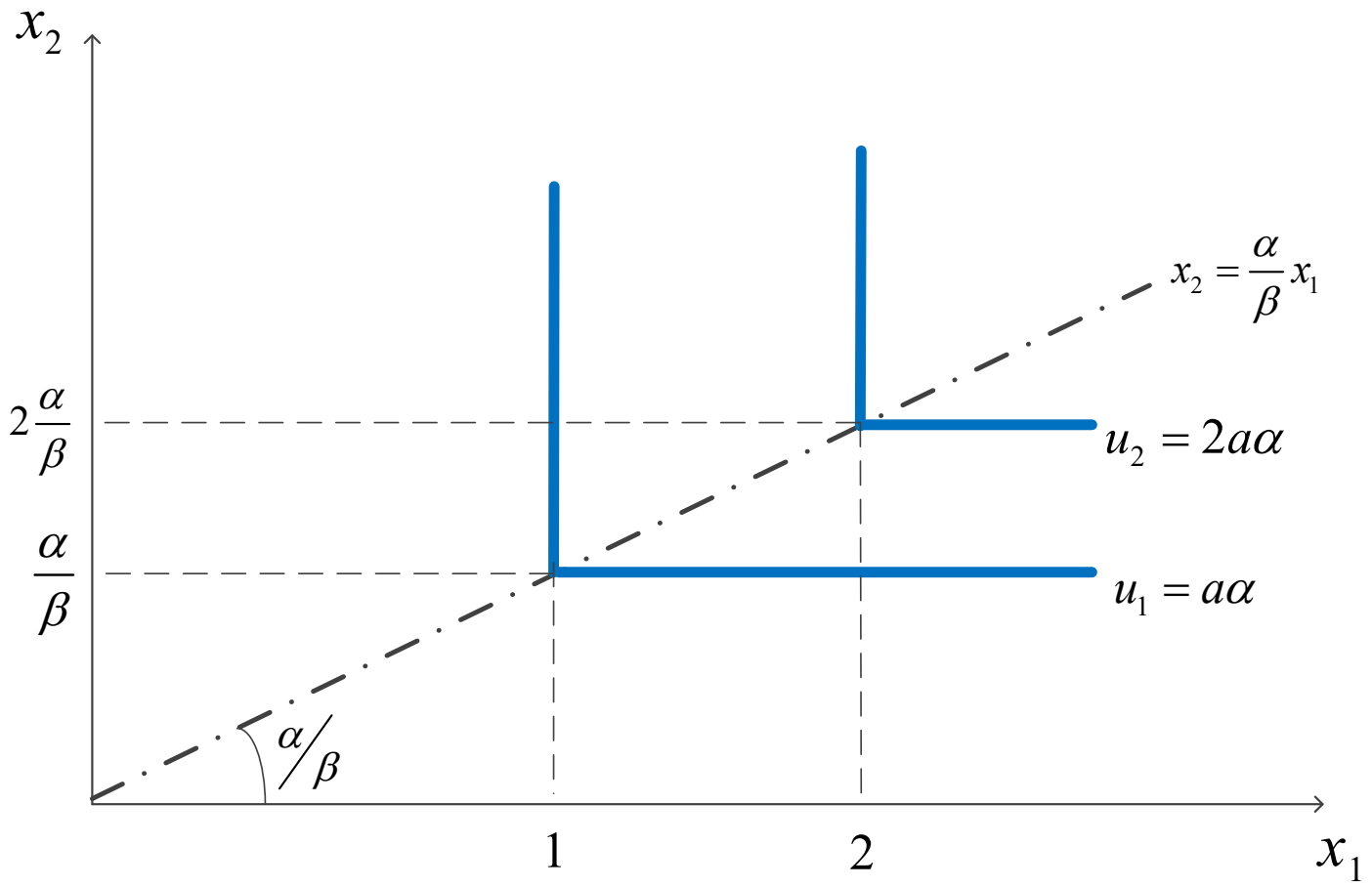
- Intuitively, the individual is willing to give up $\frac{A}{B}$ units of x_2 to obtain one more unit of x_1 and keep his utility level unaffected.
- Unlike in the Cobb-Douglas case, such willingness is independent in the relative abundance of the two goods.
- *Examples*: butter and margarine, coffee and black tea, or two brands of unflavored mineral water

Common Utility Functions

- ***Perfect Complements***:
  - In the case of two goods, x_1 and x_2,
    $$u(x_1, x_2) = A \cdot \min\{\alpha x_1, \beta x_2\}$$
    where $A, \alpha, \beta > 0$.
  - Intuitively, increasing one of the goods without increasing the amount of the other good entails *no* increase in utility.
    - The amounts of *both* goods must increase for the utility to go up.
  - The indifference curve is right angle with a kink at $\alpha x_1 = \beta x_2$.

Common Utility Functions

- Perfect complements



Common Utility Functions

– The slope of a ray $x_2 = \frac{\alpha}{\beta} x_1, \frac{\alpha}{\beta}$, indicates the rate at which goods x_1 and x_2 must be consumed to achieve utility gains.

– Special case: $\alpha = \beta$

$$\begin{aligned} u(x_1, x_2) &= A \cdot \min\{\alpha x_1, \alpha x_2\} \\ &= A\alpha \cdot \min\{x_1, x_2\} \\ &= B \cdot \min\{x_1, x_2\} \text{ if } B \equiv A\alpha \end{aligned}$$

– *Examples*: cars and gasoline, or peanut butter and jelly. Other food recipes, which often require the use of ingredients in a fixed proportion, are good examples as well.

Common Utility Functions

- ***CES utility function***:
  - In the case of two goods, x_1 and x_2,

$$u(x_1, x_2) = \left[a x_1^{\frac{\sigma-1}{\sigma}} + b x_2^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

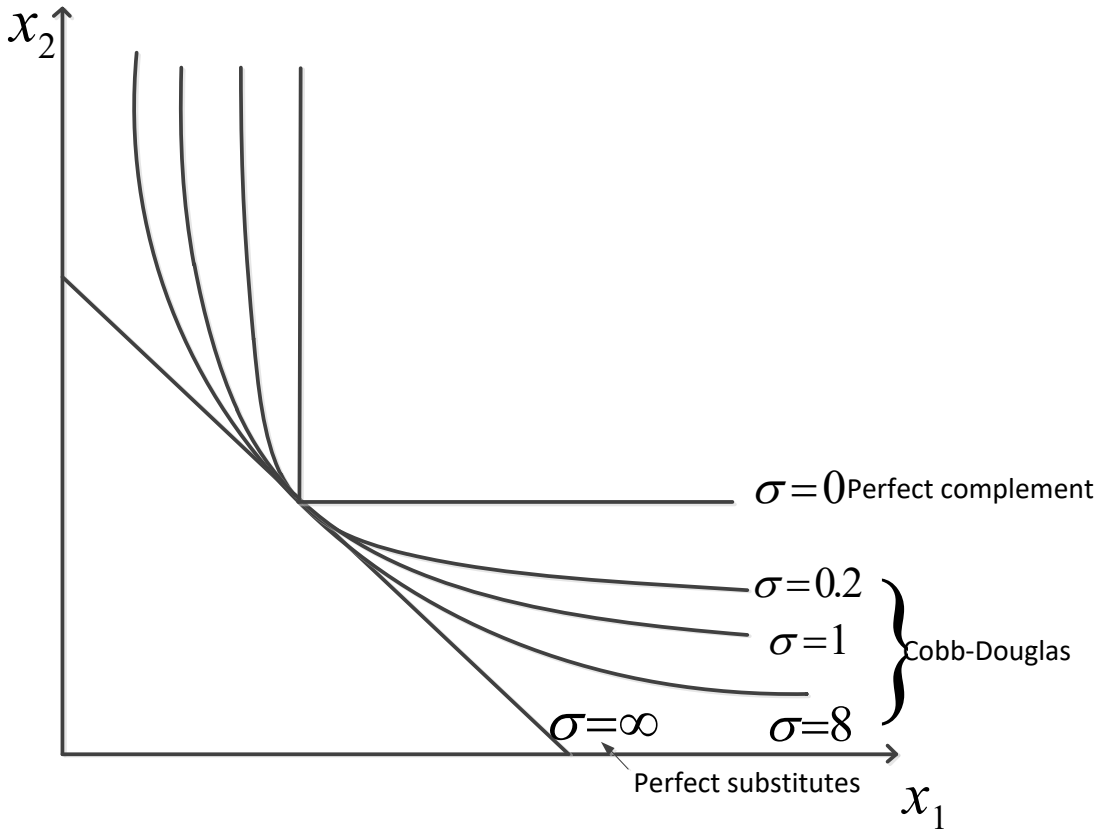
where σ measures the elasticity of substitution between goods x_1 and x_2.

  - In particular,

$$\sigma = \frac{\partial\left(\frac{x_2}{x_1}\right)}{\partial MRS_{1,2}} \cdot \frac{MRS_{1,2}}{\frac{x_2}{x_1}}$$

Common Utility Functions

- CES preferences



Common Utility Functions

– CES utility function is often presented as

$$u(x_1, x_2) = \left[a x_1^{\rho} + b x_2^{\rho}\right]^{\frac{1}{\rho}}$$

where $\rho \equiv \frac{\sigma - 1}{\sigma}$.

Common Utility Functions

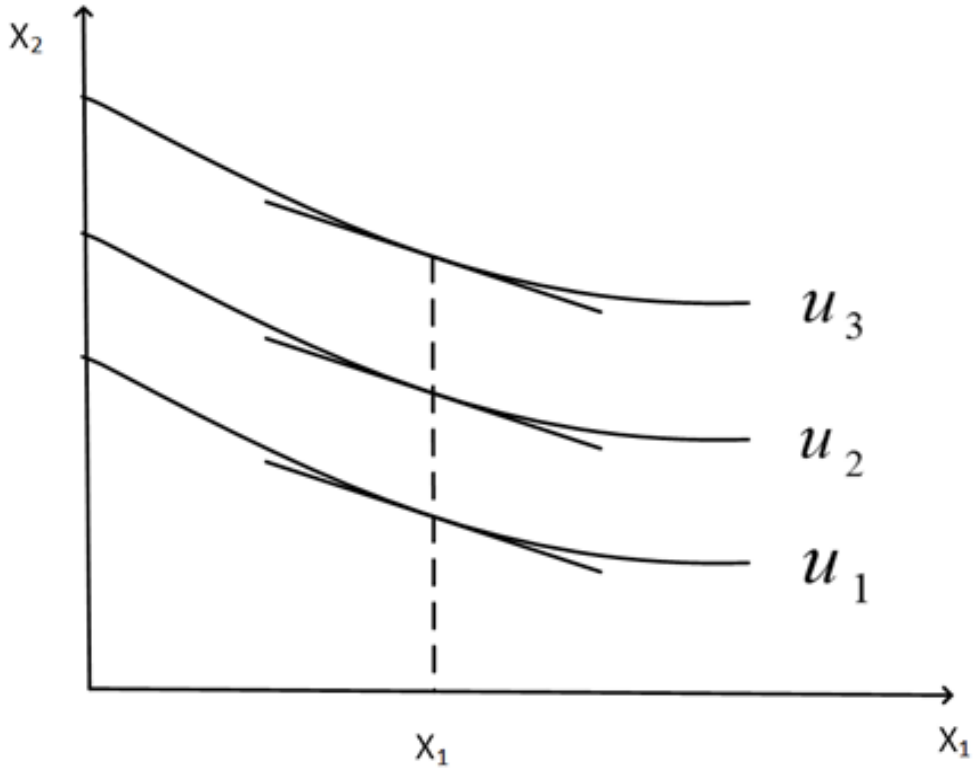
- ***Quasilinear utility function***:
  - In the case of two goods, x_1 and x_2,

$$u(x_1, x_2) = v(x_1) + bx_2$$

    where x_2 enters *linearly*, $b > 0$, and $v(x_1)$ is a *nonlinear* function of x_1.
    - For example, $v(x_1) = a \ln x_1$ or $v(x_1) = ax_1^{\alpha}$, where $a > 0$ and $\alpha \neq 1$.
  - The MRS is constant in the good that enters linearly in the utility function (x_2 in our case).

Common Utility Functions

- MRS of quasilinear preferences



Common Utility Functions

– For $u(x_1, x_2) = v(x_1) + bx_2$, the marginal utilities are

$$\frac{\partial u}{\partial x_2} = b \text{ and } \frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_1}$$

which implies

$$MRS_{x_1,x_2} = \frac{\frac{\partial v}{\partial x_1}}{b}$$

which is constant in the good entering linearly, x_2

– Quasilinear preferences are often used to represent the consumption of goods that are relatively insensitive to income.

– *Examples*: garlic, toothpaste, etc.

Properties of Preference Relations

Properties of Preference Relations

- ***Homogeneity***:
  - A utility function is *homogeneous of degree* k if varying the amounts of all goods by a common factor $\alpha > 0$ produces an increase in the utility level by α^k.
  - That is, for the case of two goods,

$$u(\alpha x_1, \alpha x_2) = \alpha^k u(x_1, x_2)$$

    where $\alpha > 0$. This allows for:
    - $\alpha > 1$ in the case of a common increase
    - $0 < \alpha < 1$ in the case of a common decrease

Properties of Preference Relations

– Three properties:

1) The first-order derivative of a function $u(x_1, x_2)$ which is homogeneous of degree k is homogeneous of degree $k-1$.

- Given $u(\alpha x_1, \alpha x_2) = \alpha^k u(x_1, x_2)$, we can show that

$$\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_i} \cdot \alpha = \alpha^k \cdot \frac{\partial u(x_1, x_2)}{\partial x_i}$$

or re-arranging

$$u'(\alpha x_1, \alpha x_2) = \alpha^{k-1} u'(x_1, x_2)$$

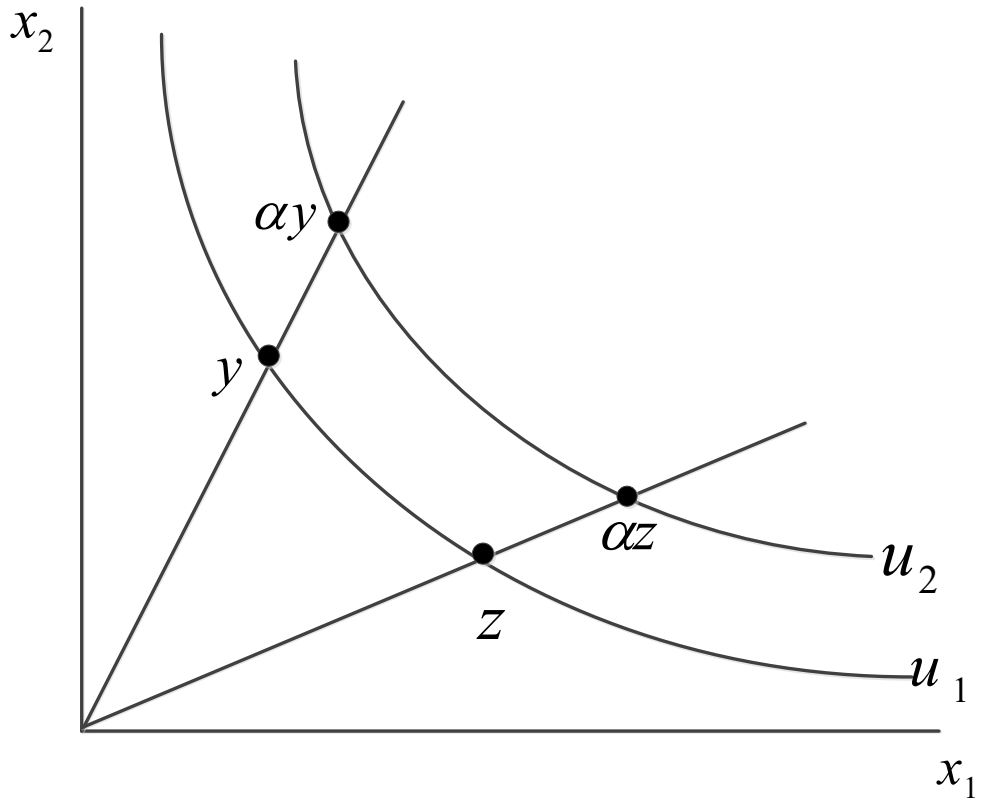
Properties of Preference Relations

2) The indifference curves of homogeneous functions are radial expansions of one another.

- That is, if two bundles y and z lie on the same indifference curve, i.e., $u(y) = u(z)$, bundles αy and αz also lie on the same indifference curve, i.e., $u(\alpha y) = u(\alpha z)$.

Properties of Preference Relations

- Homogenous preference



Properties of Preference Relations

3) *The MRS of a homogeneous function is constant for all points along each ray from the origin.*

- That is, the slope of the indifference curve at point y coincides with the slope at a "scaled-up version" of point y, αy, where $\alpha > 1$.
- The MRS at bundle $x = (x_1, x_2)$ is

$$MRS_{1,2}(x_1, x_2) = \frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}}$$

Properties of Preference Relations

- The MRS at $(\alpha x_1, \alpha x_2)$ is

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\dfrac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\dfrac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}}$$

$$= \frac{\alpha^{k-1}\dfrac{\partial u(x_1, x_2)}{\partial x_1}}{\alpha^{k-1}\dfrac{\partial u(x_1, x_2)}{\partial x_2}} = \frac{\dfrac{\partial u(x_1, x_2)}{\partial x_1}}{\dfrac{\partial u(x_1, x_2)}{\partial x_2}}$$

where the second equality uses the first property.

- Hence, the MRS is unaffected along all the points crossed by a ray from the origin.

Properties of Preference Relations

- ***Homotheticity***:
  - A utility function $u(x)$ is homothetic if it is a monotonic transformation of a homogeneous function.
  - That is, $u(x) = g(v(x))$, where
    - $g: \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, and
    - $v: \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree k.

Properties of Preference Relations

- Properties:
  - If $u(x)$ is homothetic, and two bundles y and z lie on the same indifference curve, i.e., $u(y) = u(z)$, bundles αy and αz also lie on the same indifference curve, i.e., $u(\alpha y) = u(\alpha z)$ for all $\alpha > 0$.
    - In particular,

$$u(\alpha y) = g(v(\alpha y)) = g(\alpha^k\, v(y))$$
$$u(\alpha z) = g(v(\alpha z)) = g(\alpha^k\, v(z))$$

Properties of Preference Relations

- The MRS of a homothetic function is homogeneous of degree zero.
- In particular,

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\frac{\partial g}{\partial v} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial g}{\partial v} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_2}}$$

where $u(x_1, x_2) \equiv g(v(x_1, x_2))$.

- Canceling the $\frac{\partial g}{\partial v}$ terms yields

$$\frac{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\alpha^{k-1} \cdot \frac{\partial v(x_1, x_2)}{\partial x_1}}{\alpha^{k-1} \cdot \frac{\partial v(x_1, x_2)}{\partial x_2}}$$

Properties of Preference Relations

– Canceling the α^{k-1} terms yields

$$\frac{\frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial v(x_1,x_2)}{\partial x_2}}$$

– In summary,

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} =$$

$$= \frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}} = MRS_{1,2}(x_1, x_2)$$

Properties of Preference Relations

- ***Homotheticity (graphical interpretation)***
  - A preference relation on $X = \mathbb{R}_+^L$ is homothetic if all indifference sets are related to proportional expansions along the rays.
  - That is, if the consumer is indifferent between bundles x and y, i.e., $x \sim y$, he must also be indifferent between a common scaling in these two bundles, i.e., $\alpha x \sim \alpha y$, for every scalar $\alpha > 0$.

Properties of Preference Relations

- For a given ray from the origin, the slope of the indifference curves (i.e., the MRS) that the ray crosses coincides.
  - The ratio between the two goods x_1/x_2 remains constant along all points in the ray.
- Intuitively, the rate at which a consumer is willing to substitute one good for another (his MRS) only depends on:
  - the rate at which he consumes the two goods, i.e., x_1/x_2, but does not depend on the utility level he obtains.
- But it is independent in the volume of goods he consumes, and in the utility he achieves.

Properties of Preference Relations

- ***Homogeneity and homotheticity***:
  - Homogeneous functions are homothetic.
    - We only need to apply a monotonic transformation $g(\cdot)$ on $v(x_1, x_2)$, i.e., $u(x_1, x_2) = g(v(x_1, x_2))$.
  - But homothetic functions are not necessarily homogeneous.
    - Take a homogeneous (of degree two) function $v(x_1, x_2) = x_1 x_2$.
    - Apply a monotonic transformation $g(y) = y + a$, where $a > 0$, to obtain homothetic function

$$u(x_1, x_2) = x_1 x_2 + a$$

Properties of Preference Relations

- This function is not homogeneous, since increasing all arguments by α yields

$$u(\alpha x_1, \alpha x_2) = (\alpha x_1)(\alpha x_2) + a$$
$$= \alpha^2 v(x_1, x_2) + a$$

- Other monotonic transformations yielding non-homogeneous utility functions are

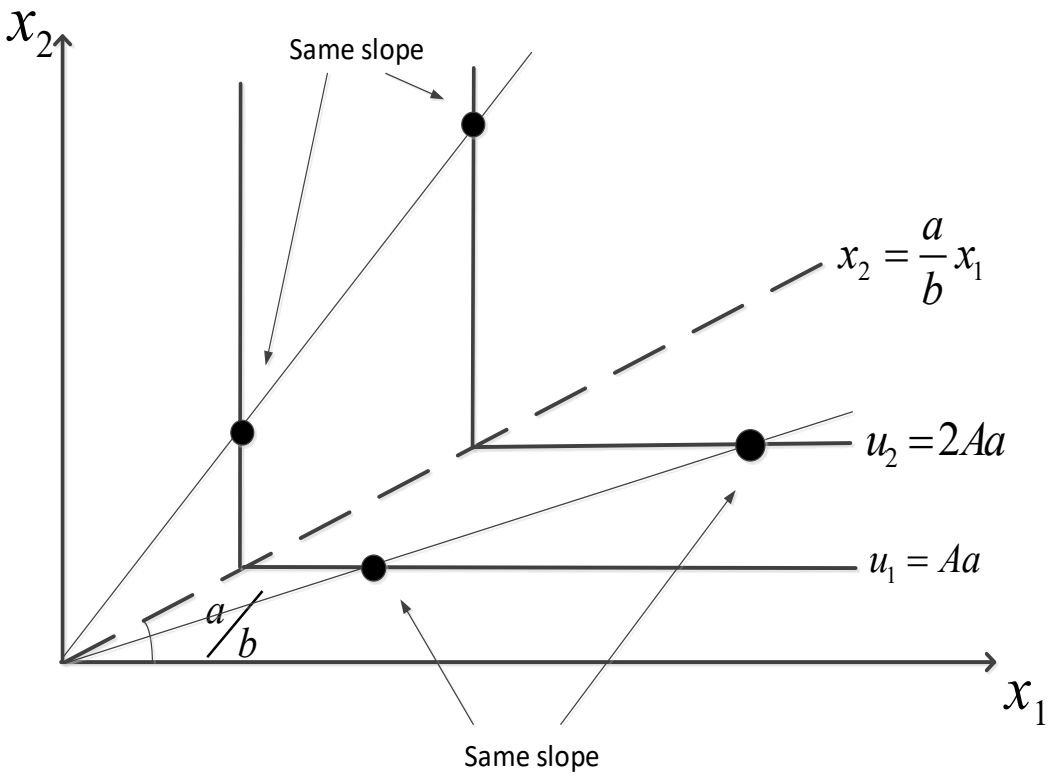
$$g(y) = ay^\gamma + by, \quad \text{where } a, b, \gamma > 0, \quad \text{or}$$
$$g(y) = a \ln y, \quad \text{where } a > 0$$

Properties of Preference Relations

- Utility functions that satisfy homotheticity:
  - Linear utility function $u(x_1, x_2) = ax_1 + bx_2$, where $a, b > 0$
    - Goods x_1 and x_2 are perfect substitutes
    - $MRS(x_1, x_2) = \frac{a}{b}$ and $MRS(tx_1, tx_2) = \frac{a}{b}$
  - The Leontief utility function $u(x_1, x_2) = A \cdot \min\{ax_1, bx_2\}$, where $A > 0$
    - Goods x_1 and x_2 are perfect complements
    - We cannot define the MRS along all the points of the indifference curves
    - However, the slope of the indifference curves coincide for those points where these curves are crossed by a ray from the origin.

Properties of Preference Relations

- Perfect complements and homotheticity



Properties of Preference Relations

- ***Example 1.9*** (Testing for quasiconcavity and homotheticity):
  - Let us determine if $u(x_1, x_2) = \ln(x_1^{0.3} x_2^{0.6})$ is quasiconcave, homothetic, both or neither.
  - *Quasiconcavity*:
    - Note that $\ln(x_1^{0.3} x_2^{0.6})$ is a monotonic transformation of the Cobb-Douglas function $x_1^{0.3} x_2^{0.6}$.
    - Since $x_1^{0.3} x_2^{0.6}$ is a Cobb-Douglas function, where $\alpha + \beta = 0.3 + 0.6 < 1$, it must be a concave function.
    - Hence, $x_1^{0.3} x_2^{0.6}$ is also quasiconcave, which implies $\ln(x_1^{0.3} x_2^{0.6})$ is quasiconcave (as quasiconcavity is preserved through a monotonic transformation).

Properties of Preference Relations

- ***Example 1.9*** (continued):
  - *Homogeneity*:
    - Increasing all arguments by a common factor α,

$$(\alpha x_1)^{0.3}(\alpha x_2)^{0.6} = \alpha^{0.3} x_1^{0.3} \alpha^{0.6} x_2^{0.6} = \alpha^{0.9} x_1^{0.3} x_2^{0.6}$$

    - Hence, $x_1^{0.3} x_2^{0.6}$ is homogeneous of degree 0.9

  - *Homotheticity:*
    - Therefore, $x_1^{0.3} x_2^{0.6}$ is also homothetic.
    - As a consequence, its transformation, $\ln(x_1^{0.3} x_2^{0.6})$, is also homothetic (as homotheticity is preserved through a monotonic transformation).

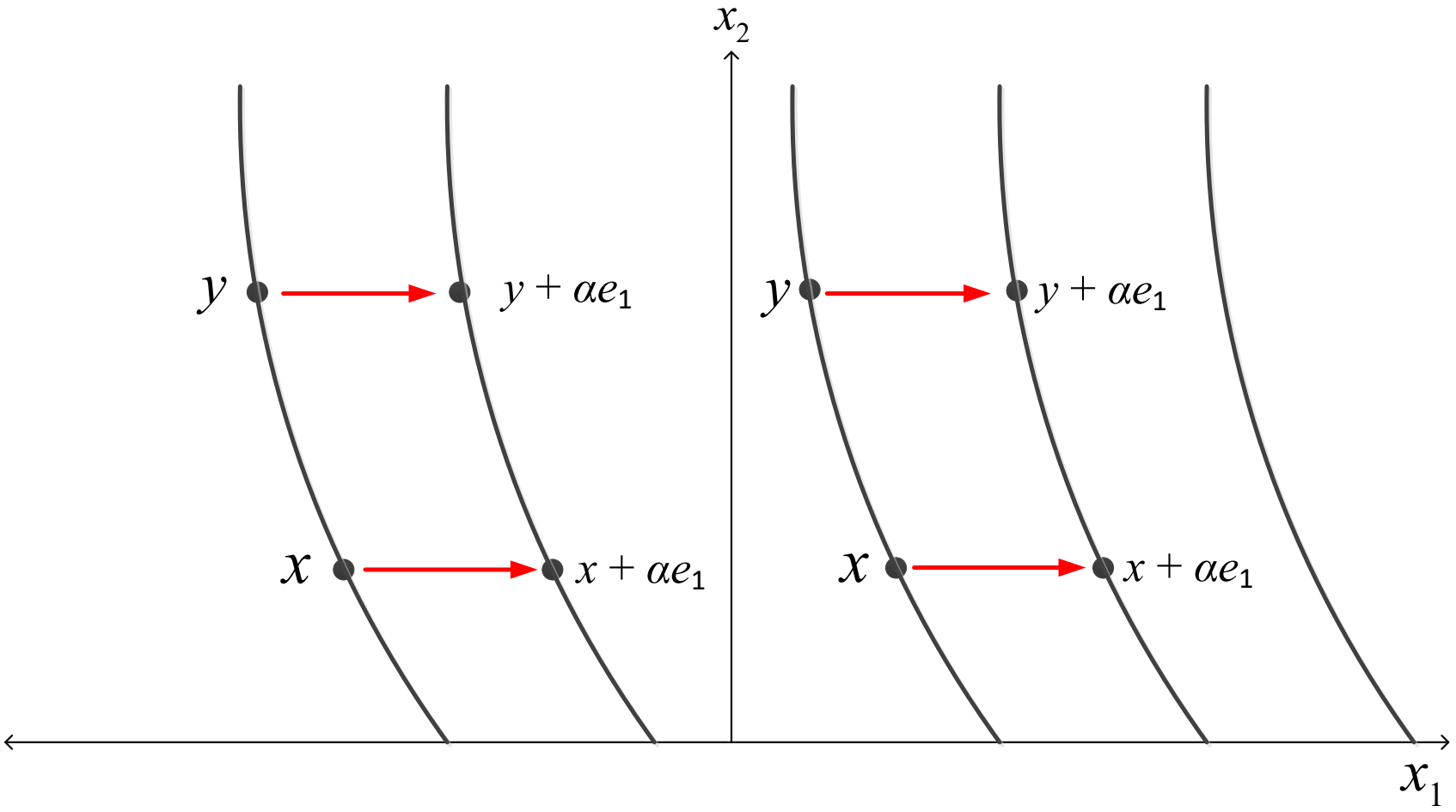
Properties of Preference Relations

- ***Quasilinear preference relations***:
  - The preference relation on $X = (-\infty, \infty) \times \mathbb{R}_{+}^{L-1}$ is *quasilinear* with respect to good 1 if:

    1) All indifference sets are parallel displacements of each other along the axis of good 1.
       - That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$, where $e_1 = (1, 0, \ldots, 0)$.
    2) Good 1 is desirable.
       - That is, $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

Properties of Preference Relations

- Quasilinear preference-I

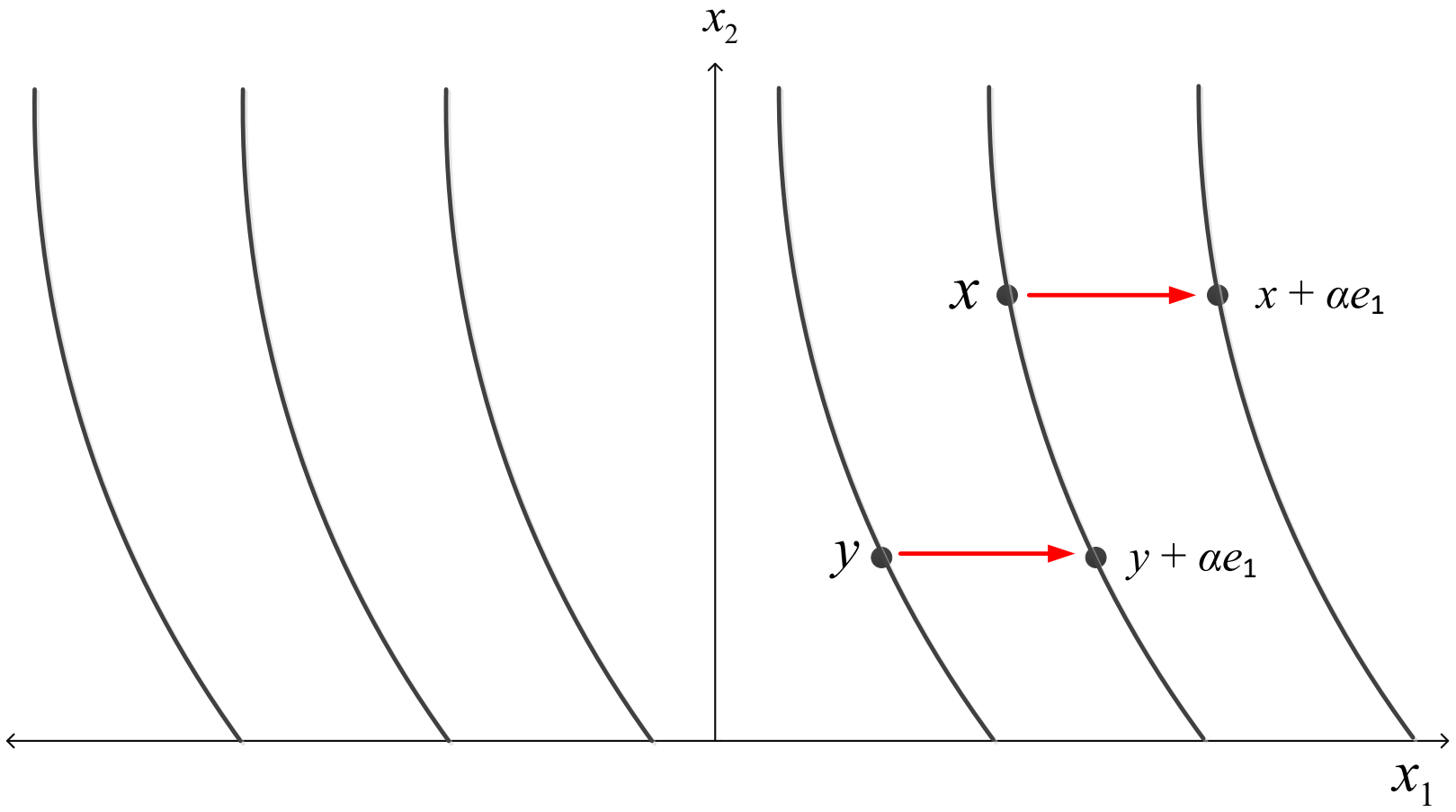


Properties of Preference Relations

- *Notes*:
  - No lower bound on the consumption of good 1, i.e., $x_1 \in (-\infty, \infty)$.
  - If $x \succ y$, then $(x + \alpha e_1) \succ (y + \alpha e_1)$.

Properties of Preference Relations

- Quasilinear preference-II



Properties of Preference Relations

- The properties we considered so far are not enough to guarantee that a preference relation can be represented by a utility function.
- *Example*:
  - Lexicographic preferences cannot be represented by a utility function.

Lexicographic Preferences

– A bundle $x = (x_1, x_2)$ is weakly preferred to another bundle $y = (y_1, y_2)$, i.e., $(x_1, x_2) \succsim (y_1, y_2)$, if and only if

$$\begin{cases} x_1 > y_1, & \text{or if} \\ x_1 = y_1 \text{ and } x_2 > y_2 \end{cases}$$

– *Intuition*:

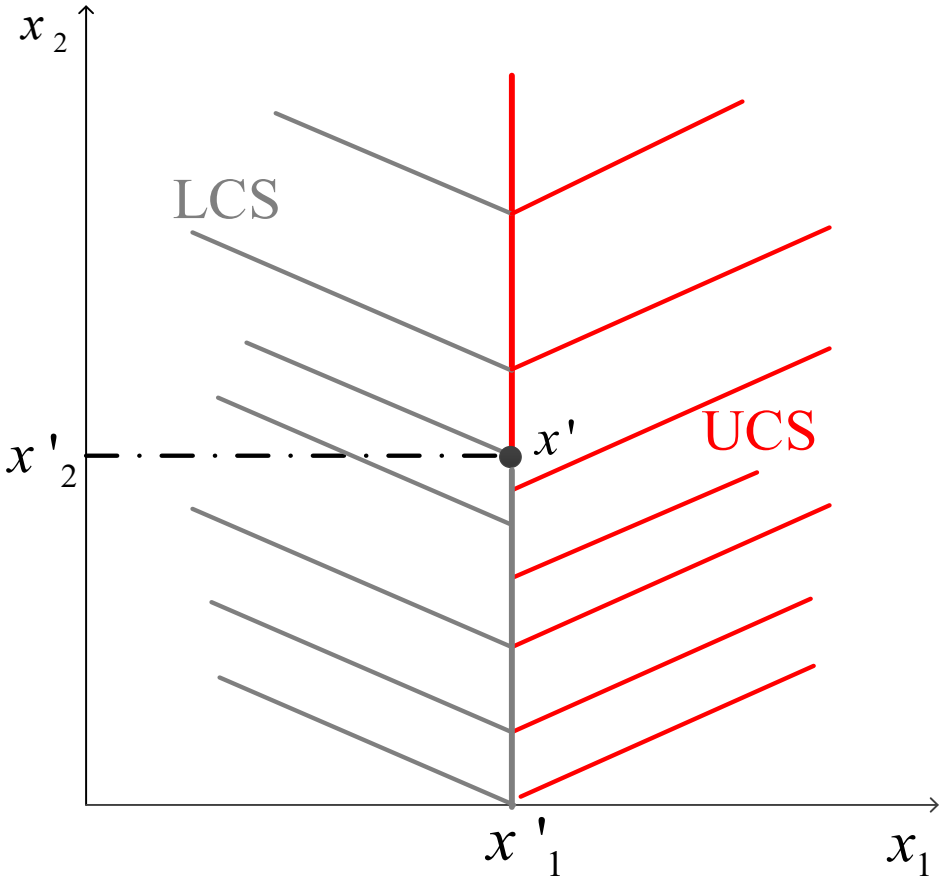
- The individual prefers bundle x if it contains more of good 1 than bundle y, i.e., $x_1 > y_1$.
- If, however, both bundles contain the same amount of good 1, $x_1 = y_1$, then the individual prefers bundle x if it contains more of the second good, i.e., $x_2 > y_2$.

Lexicographic Preferences

- Indifference set cannot be drawn as an indifference curve.
  - For a given bundle $x' = (x_1', x_2')$, there are no more bundles for which the consumer is indifferent.
  - Indifference sets are then *singletons* (sets containing only one element).

Lexicographic Preferences

- Lexicographic preference relation



Continuous Preferences

Continuous Preferences

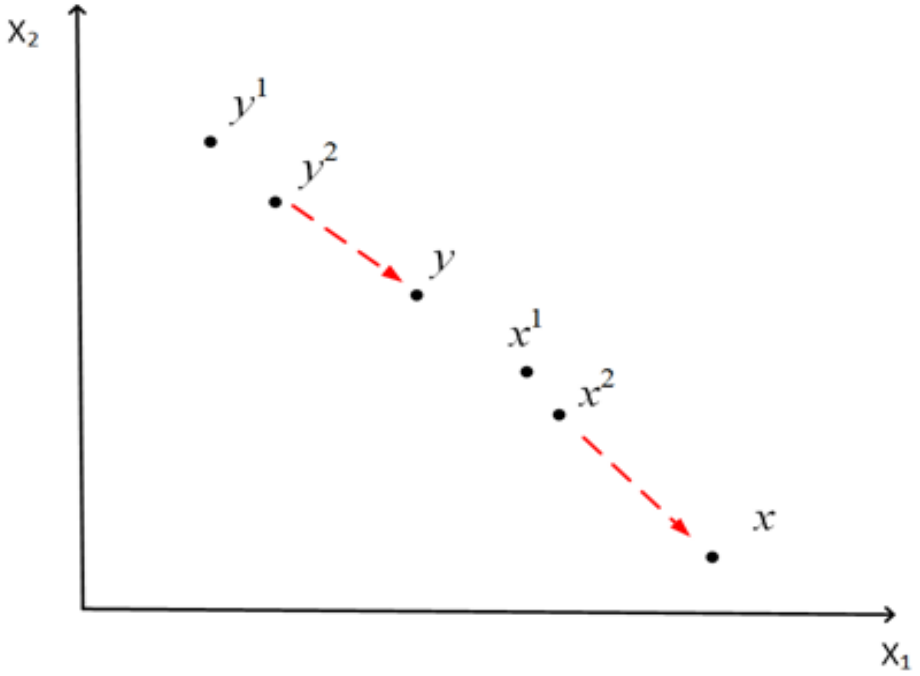
- In order to guarantee that preference relations can be represented by a utility function we need *continuity*.
- ***Continuity***: A preference relation defined on X is continuous if it is preserved under limits.
  - That is, for any sequence of pairs

    $$\{(x^n, y^n)\}_{n=1}^{\infty} \text{ with } x^n \succsim y^n \text{ for all } n$$

    and where $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = y$, the preference relation is maintained in the limiting points, i.e., $x \succsim y$.

Continuous Preferences

– Intuitively, there can be no sudden jumps (i.e., preference reversals) in an individual preference over a sequence of bundles.



Continuous Preferences

- *Lexicographic preferences are not continuous*
  - Consider the sequence $x^n = \left(\frac{1}{n}, 0\right)$ and $y^n = (0,1)$, where $n = \{1,2,3, \dots\}$.
  - The sequence $y^n = (0,1)$ is constant in n.
  - The sequence $x^n = \left(\frac{1}{n}, 0\right)$ is not:
    - It starts at $x^1 = (1,0)$, and moves leftwards to $x^2 = \left(\frac{1}{2}, 0\right)$, $x^3 = \left(\frac{1}{3}, 0\right)$, etc.

Continuous Preferences

- Thus, the individual prefers:

$$x^1 = (1,0) \succ (0,1) = y^1$$

$$x^2 = \left(\frac{1}{2}, 0\right) \succ (0,1) = y^2$$

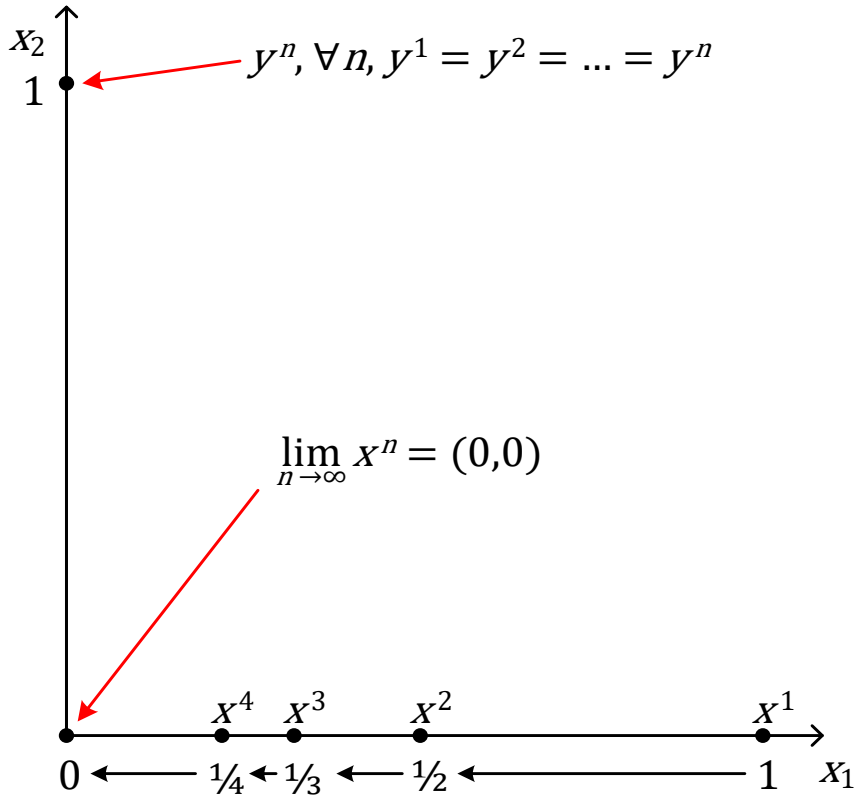
$$x^3 = \left(\frac{1}{3}, 0\right) \succ (0,1) = y^3$$

$$\vdots$$

- But,

$$\lim_{n\to\infty} x^n = (0,0) \prec (0,1) = \lim_{n\to\infty} y^n$$

- Preference reversal!



Existence of Utility Function

Existence of Utility Function

- *If a preference relation satisfies monotonicity and continuity, then there exists a utility function $u(\cdot)$ representing such preference relation.*
- *Proof*:
  - Take a bundle $x \neq 0$.
  - By monotonicity, $x \gtrsim 0$, where $0 = (0,0, \ldots, 0)$.
    - That is, if bundle $x \neq 0$, it contains positive amounts of at least one good and, it is preferred to bundle 0.

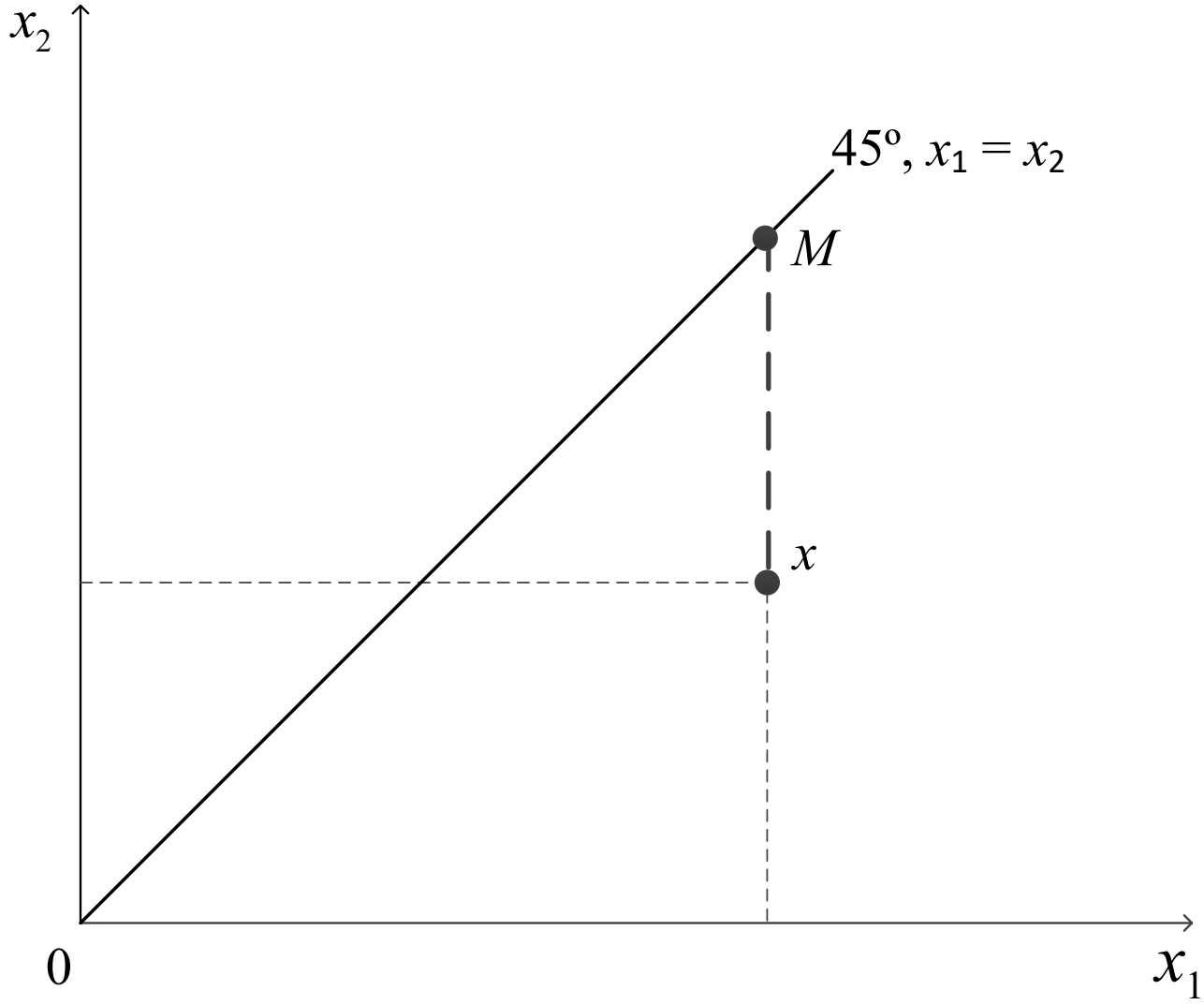
Existence of Utility Function

- Let $m \equiv \max\{x_1, x_2, \ldots, x_N\}$ be the number of units of the most abundant good in bundle x.
- Define bundle M as the bundle where all components coincide with the highest component of bundle x. That is,

$$M = (m, m, \ldots, m).$$

- Hence, by monotonicity, $M \gtrsim x$.
- Bundles 0 and M are both on the main diagonal, since each of them contains the same amount of good x_1 and x_2.

Existence of Utility Function



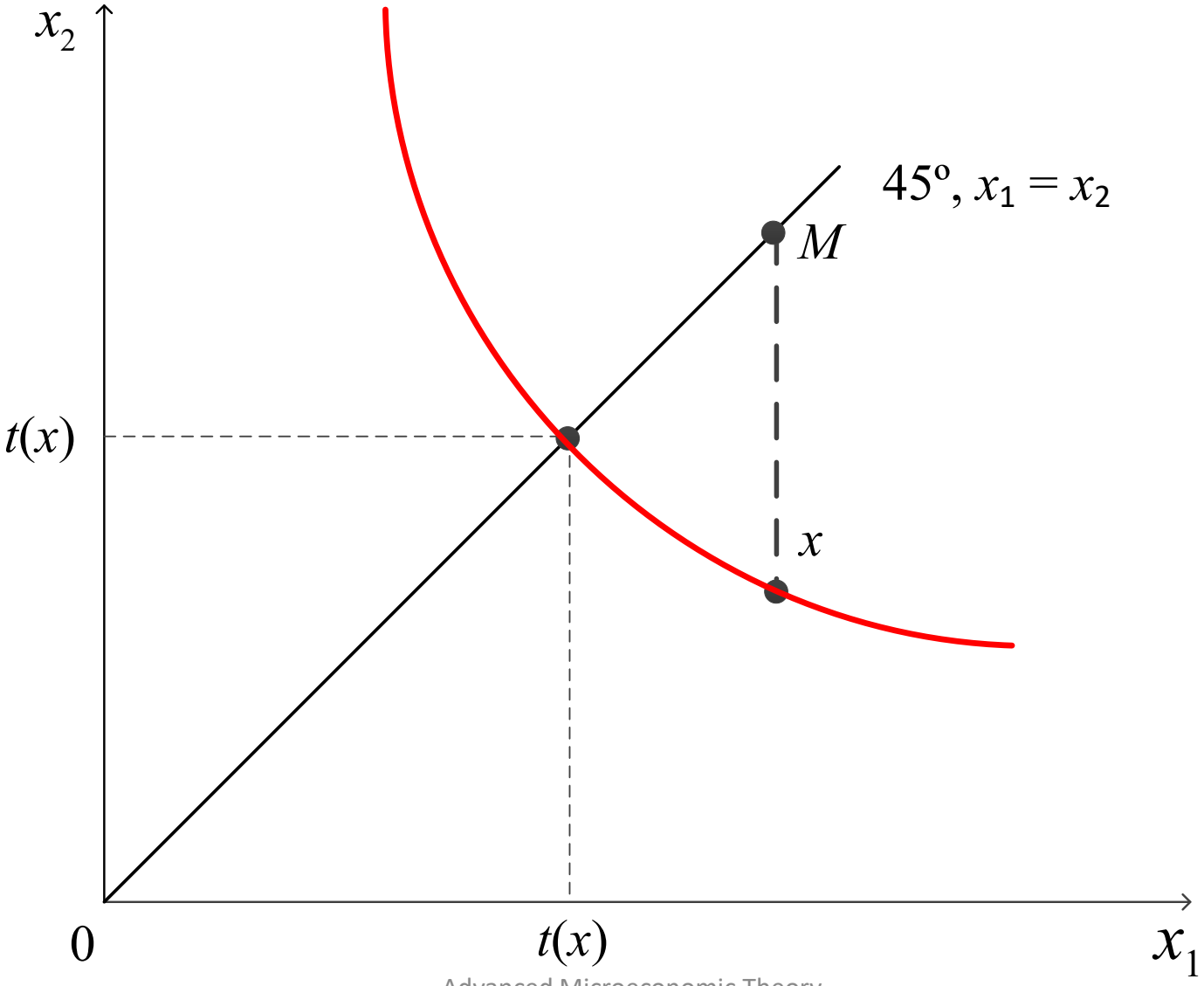
Existence of Utility Function

- By continuity and monotonicity, there exists a bundle that is indifferent to x and which lies on the main diagonal.
- By monotonicity, this bundle is unique
  - Otherwise, modifying any of its components would lead to higher/lower indifference curves.
- Denote such bundle as

$$(t(x), t(x), \dots, t(x))$$

- Let $u(x) = t(x)$, which is a real number.

Existence of Utility Function



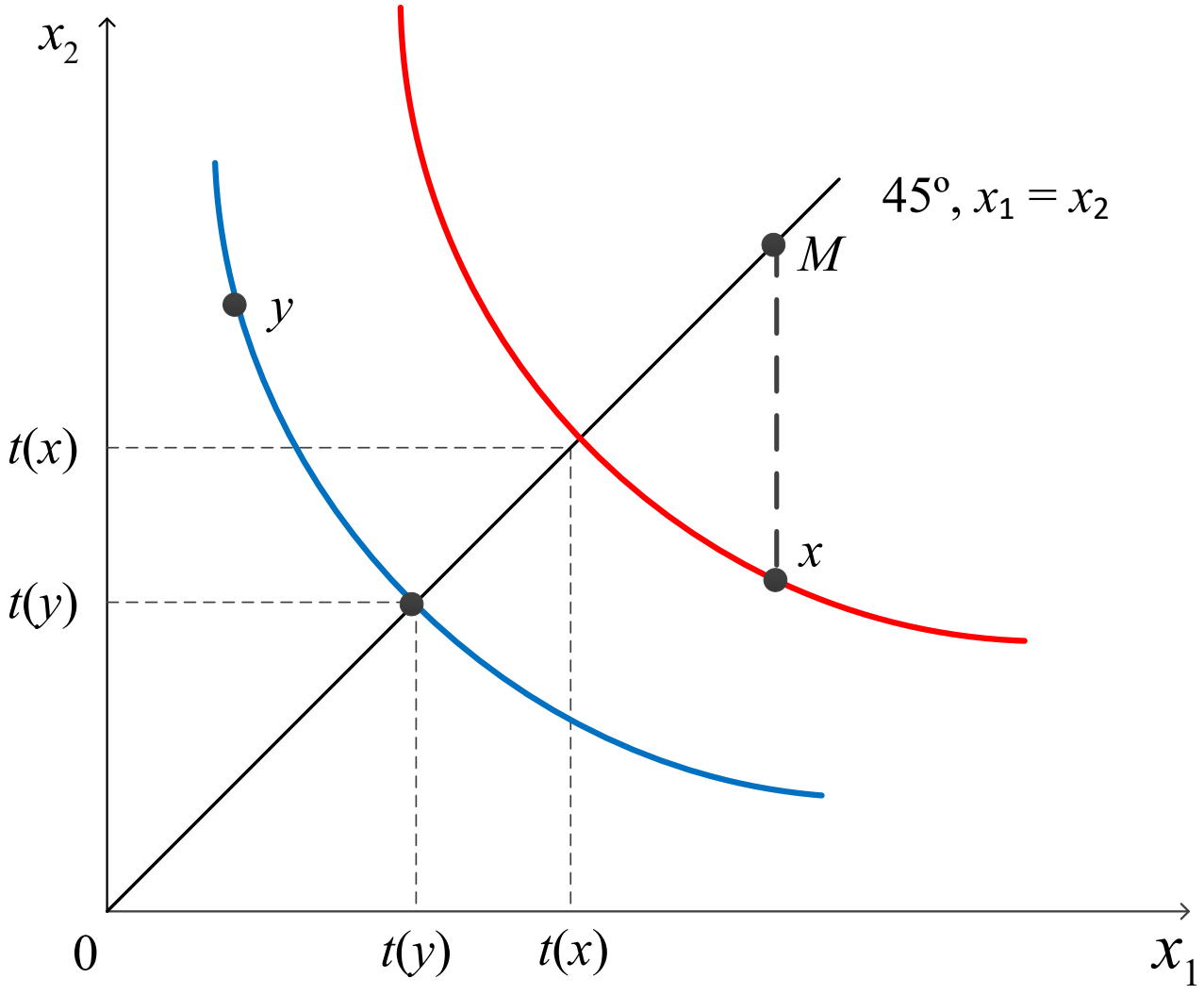
Existence of Utility Function

– Applying the same steps to another bundle $y \neq x$, we obtain

$$(t(y), t(y), \ldots, t(y))$$

and let $u(y) = t(y)$, which is also a real number.

Existence of Utility Function



Existence of Utility Function

– We know that

$$x \sim (t(x), t(x), \ldots, t(x))$$
$$y \sim (t(y), t(y), \ldots, t(y))$$
$$x \succsim y$$

– Hence, by transitivity, $x \succsim y$ iff

$$x \sim (t(x), t(x), \ldots, t(x)) \succsim \big(t(y), t(y), \ldots, t(y)\big) \sim y$$

– And by monotonicity,

$$x \succsim y \iff t(x) \geq t(y) \iff u(x) \geq u(y)$$

Existence of Utility Function

- *Note*: A utility function can satisfy continuity but still be non-differentiable.
  - For instance, the Leontief utility function, $A\min\{ax_1, bx_2\}$, is continuous but cannot be differentiated at the kink.

Social and Reference-Dependent Preferences

Social Preferences

- We now examine social, as opposed to individual, preferences.
- Consider additively separable utility functions of the form

$$u_i(x_i, x) = f(x_i) + g_i(x)$$

where

  - $f(x_i)$ captures individual i's utility from the monetary amount that he receives, x_i;
  - $g_i(x)$ measures the utility/disutility he derives from the distribution of payoffs $x = (x_1, x_2, \ldots, x_N)$ among all N individuals.

Social Preferences

- **Fehr and Schmidt (1999)**:
  - For the case of two players,

$$u_i(x_i, x_j) = x_i - \alpha_i \max\{x_j - x_i, 0\} - \beta_i \max\{x_i - x_j, 0\}$$

    where x_i is player i's payoff and $j \neq i$.

  - Parameter $\alpha_i \geq 0$ represents player i's disutility from envy
    - When $x_i < x_j$, $\max\{x_j - x_i, 0\} = x_j - x_i > 0$ but $\max\{x_i - x_j, 0\} = 0$.
    - Hence, $u_i(x_i, x_j) = x_i - \alpha_i(x_j - x_i)$.

Social Preferences

- Parameter $\beta_i \geq 0$ captures player i's disutility from guilt
  - When $x_i > x_j$, $\max\{x_i - x_j, 0\} = x_i - x_j > 0$ but $\max\{x_j - x_i, 0\} = 0$.
  - Hence, $u_i(x_i, x_j) = x_i - \beta_i(x_i - x_j)$.
- Players' envy is stronger than their guilt, i.e., $\alpha_i \geq \beta_i$ for $0 \leq \beta_i < 1$.
  - Intuitively, players (weakly) suffer more from inequality directed at them than inequality directed at others.

Social Preferences

- Thus players exhibit "concerns for fairness" (or "social preferences") in the distribution of payoffs.
- If $\alpha_i = \beta_i = 0$ for every player i, individuals only care about their material payoff $u_i(x_i, x_j) = x_i$.
  - Preferences coincide with the individual preferences.

Social Preferences

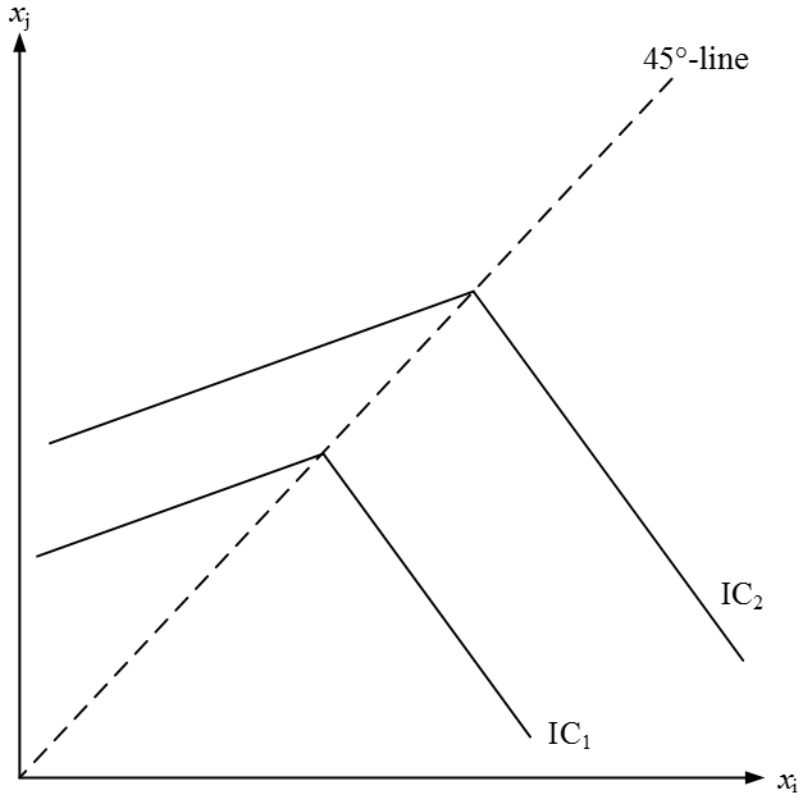
- Let's depict the indifference curves of this utility function by fixing the utility level at $u = \overline{u}$.
- When $x_i > x_j$,

  $\overline{u} = x_i - \beta_i(x_i - x_j)$

  which, solving for x_j, yields

  $$x_j = \frac{\overline{u}}{\beta_i} - \frac{1-\beta_i}{\beta_i} x_i$$

  with slope $-\frac{1-\beta_i}{\beta_i}$ for all points below the 45-degree line.
- See downward sloping segment of ICs.

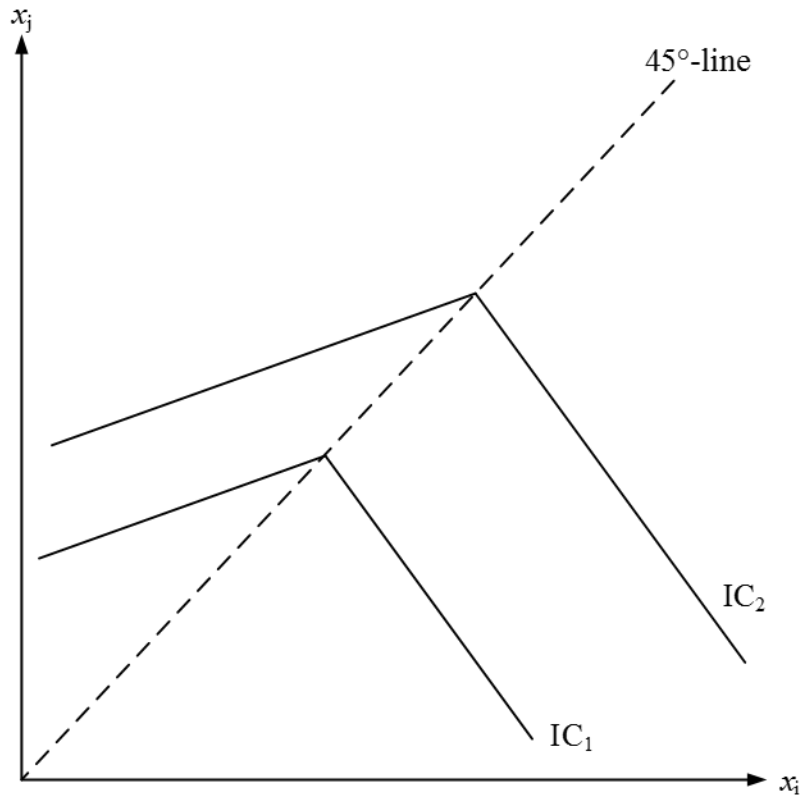


Social Preferences

- Similarly, when $x_i < x_j$, $\overline{u} = x_i - \alpha_i(x_j - x_i)$, which, solving for x_j, yields

  $$x_j = -\frac{\overline{u}}{\alpha_i} + \frac{1+\alpha_i}{\alpha_i} x_i$$

  with slope $\frac{1+\alpha_i}{\alpha_i}$ above 45-degree line.
- See upward sloping segment of the ICs.
- Note that (x_i, x_j)-pairs to the northeast yield larger utility levels for individual i.



Social Preferences

– ***Remark 1***: If

- the disutility from envy is positive, $\alpha_i \in [0,1]$;
- the disutility from guilt is negative, $\beta_i \in (-1,0]$; and
- the former dominates the latter in absolute value, $|\alpha_i| \geq |\beta_i|$;

then Fehr and Schmidt's (1999) specification would capture *concerns for status acquisition*.

Social Preferences

– ***Remark 2***: If

- the disutility from envy is negative, $\alpha_i \in \left(-\frac{1}{2}, 0\right]$;
- the disutility from guilt is positive, $\beta_i \in \left[0, \frac{1}{2}\right)$; and
- the latter dominates the former in absolute value, $|\alpha_i| < |\beta_i|$;

then Fehr and Schmidt's (1999) specification would now capture a *preference for efficiency*.

That is, a reduction in my own payoff is acceptable only if the payoff other individuals receive increases by a larger amount.

Social Preferences

- **Bolton and Ockenfels (2000)**:
  - Similar to Fehr and Schmidt (1999), but they allow for nonlinearities

$$u_i\left(x_i, \frac{x_i}{x_i+x_j}\right)$$

where $u_i(\cdot)$

- increases in x_i (i.e., selfish component)
- decreases in the share of total payoffs that individual i enjoys, $\frac{x_i}{x_i+x_j}$ (i.e., social preferences)

Social Preferences

– For instance,

$$u_i\left(x_i, \frac{x_i}{x_i+x_j}\right) = x_i - \alpha\left(\frac{x_i}{x_i+x_j}\right)^{\frac{1}{2}}$$

– Letting $u = \overline{u}$ and solving for x_j yields the indifference curve

$$x_j = \frac{x_i[\alpha^2 - (\overline{u} - x_i)^2]}{(\overline{u} - x_i)^2}$$

which produces nonlinear indifference curves (nonlinear in x_i).

Social Preferences

- **Charness and Rabin (2002)**:
  - Fehr and Schmidt's (1999) preferences might not explain individuals' reactions in strategic settings.
    - *Example*: inferring certain intentions from individuals who acted before them.
  - Utility function that rationalizes such behavior

$$u_i(x_i, x_j) = x_i - (\alpha_i - \theta\gamma_j)\max\{x_j - x_i, 0\} - (\beta_i + \theta\gamma_j)\max\{x_i - x_j, 0\}$$

where parameter γ_j only takes two possible values, i.e., $\gamma_j = \{-1, 0\}$.

Social Preferences

- If $\gamma_j = -1$, individual i interprets that j misbehaved, and thus increases its envy parameter by θ, or reduces his guilt parameter by θ.
- If $\gamma_j = 0$, individual i interprets that j is well behaved, implying that the utility function coincides with that in Fehr and Schmidt's (1999) specification.
- Intuitively, when individuals interpret that others misbehaved, the envy (guilt) concerns analyzed above are emphasized (attenuated, respectively).

Social Preferences

- **Andreoni and Miller (2002)**:
  - A CES utility function

$$u_i(x_i, x_j) = \left(\alpha x_i^{\rho} + (1-\alpha)x_j^{\rho}\right)^{\frac{1}{\rho}}$$

    where x_i and x_j are the monetary payoff of individual i rather than the amount of goods.

  - If individual i is completely selfish, i.e., $\alpha = 1$, $u(x_i) = x_i$

Social Preferences

- If $\alpha \in (0,1)$, parameter ρ captures the elasticity of substitution between individual i's and j's payoffs.
  - That is, if x_j decreases by one percent, x_i needs to be increased by ρ percent for individual i to maintain his utility level unaffected.

Hyperbolic and Quasi-Hyperbolic Discounting

Exponential discounting (standard)

- The discounted value of an amount of money $\$x$ received t periods from today is

$$\frac{1}{(1+r)^t}x$$

- We can find the "subjective discount rate" which measures how $\frac{1}{(1+r)^t}x$ varies along time, relative to its initial value,

$$\frac{\frac{\partial\left(\frac{1}{(1+r)^t}x\right)}{\partial t}}{\frac{1}{(1+r)^t}x} = \frac{-\ln(1+r)\frac{1}{(1+r)^t}x}{\frac{1}{(1+r)^t}x} = -\ln(1+r)$$

  which is constant in the time period t when it is evaluated.

- In other words, exponential discounting assumes that the comparison of $\$x$ between period 0 and k coincides with the comparison between period t and $t+k$ since k periods mediated.

Exponential discounting (standard)

- Not generally confirmed in controlled experiments.
- In particular, individuals exhibit *present bias*:
  - When asked to choose between $100 today or $110 tomorrow, most individuals prefer $100 today.
  - However, when the same individuals are asked between $100 in, for instance, 60 days or $110 in 61 days, some reveal a preference for the $110 in 61 days.
- Individuals show a large discount of future payoffs
- Preferences are time-inconsistent

Hyperbolic Discounting

- This approach assumes that the discounted value of an amount of money $\$x$ received t periods from today is

$$\frac{1}{(1+rt)^{\gamma/\alpha}}x$$

where $\gamma, \alpha > 0$. In this setting, the subjective discount rate is

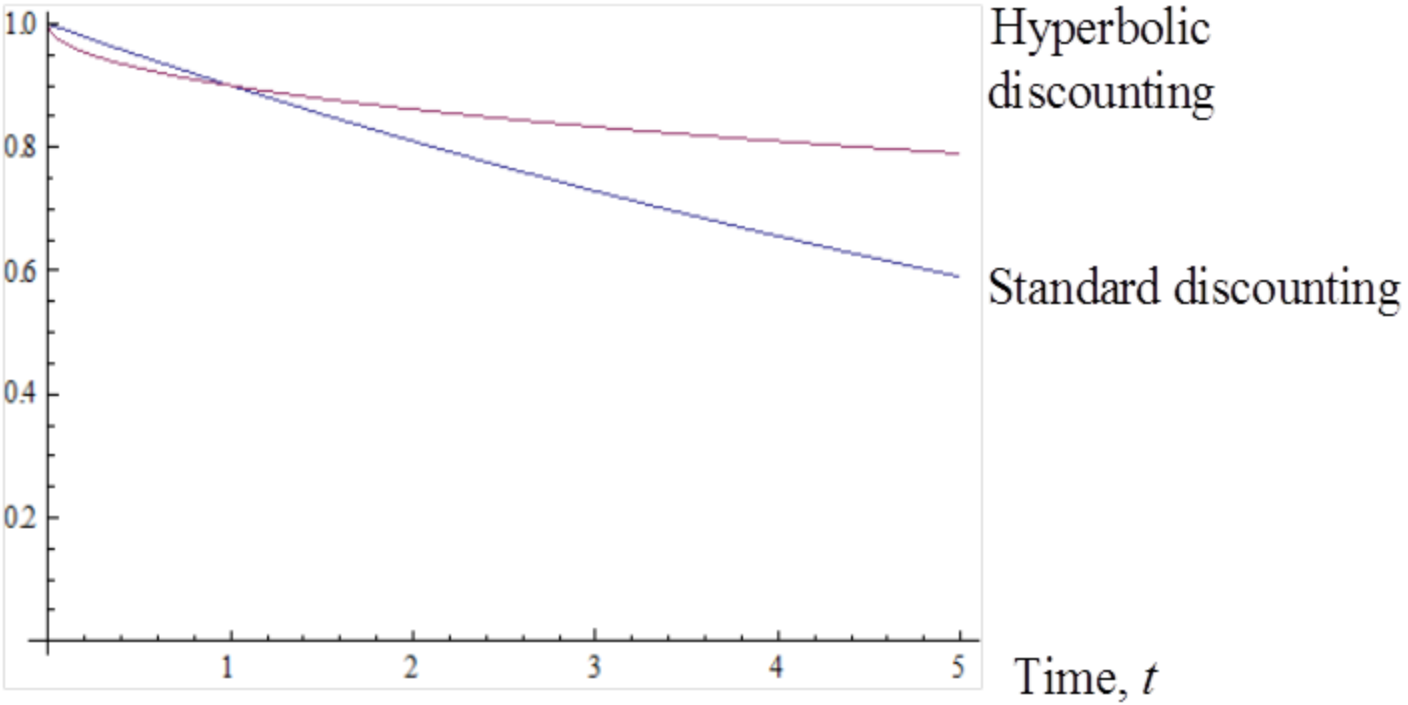
$$\frac{\frac{\partial\left(\frac{1}{(1+rt)^{\gamma/\alpha}}x\right)}{\partial t}}{\frac{1}{(1+rt)^{\gamma/\alpha}}x} = \frac{-\gamma r}{\alpha(1+rt)}$$

which is decreasing in t.

In most applications, $\gamma = \alpha$, yielding a subjective discount rate of

$$\frac{-r}{(1+rt)}$$

Hyperbolic Discounting



- Individuals with hyperbolic discounting exhibit present bias: relative to standard (exponential) discounting
  - They strongly discount payoffs in the nearby future, but
  - They do not significantly discount two distant payoffs that are close to each other.

Quasi-Hyperbolic Discounting

- In a discrete time context, individuals discount future payoffs according to

$$\beta\delta^t$$

for all $t \geq 1$, where parameter $\beta \leq 1$.

  - When $\beta = 1$, Quasi-hyperbolic discounting embodies exponential discounting.
  - The subjective discount rate is

$$\frac{\frac{\Delta(\beta\delta^t x)}{\Delta t}}{\beta\delta^t x} = \frac{\beta\delta^{t+1}x - \beta\delta^t x}{\beta\delta^t x} = \delta - 1$$

which is constant in time, but still allows for present bias to arise.

Quasi-Hyperbolic Discounting

- Consider an individual evaluating today whether to invest in a firm.
- He will need to incur a cost $c > 0$ in period t, and obtain a benefit $b > 0$ with certainty n periods into the future (in period $t + n$).
- Under exponential discounting, he would invest if

$$\delta^t c < \delta^{t+n} b \text{ or } c < \delta^n b$$

- If this individual is given the opportunity to reconsider his investment when period t arrives, he will ***not*** reconsider his decision since $c < \delta^n b$ still holds.

Quasi-Hyperbolic Discounting

- Under quasi-hyperbolic discounting, he would invest if
$$\beta\delta^t c < \beta\delta^{t+n} b \text{ or } c < \delta^n b$$
which is same decision rule as the above time-consistent individual.
- However, if this individual is given the opportunity to reconsider his investment when period t arrives, he will invest only if
$$c < \beta\delta^n b$$
which does not coincide with his decision rule t periods ago.
  - Hence, preference reversal occurs if
$$\beta\delta^n b < c < \delta^n b$$

Choice Based Approach

Choice Based Approach

- We now focus on the actual choice behavior rather than individual preferences.
  - From the alternatives in set B, which one would you choose?
- A choice structure $(\mathcal{B}, c(\cdot))$ contains two elements:

  1) $\mathcal{B}$ is a family of nonempty subsets of X, so that every element of $\mathcal{B}$ is a set $B \subset X$.

Choice Based Approach

- *Example 1*: In consumer theory, B is a particular set of all the affordable bundles for a consumer, given his wealth and market prices.
- *Example 2*: B is a particular list of all the universities where you were admitted, among all universities in the scope of your imagination X, i.e., $B \subset X$.

Choice Based Approach

2) $c(\cdot)$ is a choice rule that selects, for each budget set B, a subset of elements in B, with the interpretation that $c(B)$ are the chosen elements from B.

- *Example 1*: In consumer theory, $c(B)$ would be the bundles that the individual chooses to buy, among all bundles he can afford in budget set B;
- *Example 2*: In the example of the universities, $c(B)$ would contain the university that you choose to attend.

Choice Based Approach

– *Note*:

- If $c(B)$ contains a single element, $c(\cdot)$ is a function;
- If $c(B)$ contains more than one element, $c(\cdot)$ is a correspondence.

Choice Based Approach

- ***Example 1.11*** (Choice structures):
  - Define the set of alternatives as
$$X = \{x, y, z\}$$
  - Consider two different budget sets
$$B_1 = \{x, y\} \text{ and } B_2 = \{x, y, z\}$$
  - Choice structure one $(\mathcal{B}, c_1(\cdot))$
$$c_1(B_1) = c_1(\{x, y\}) = \{x\}$$
$$c_1(B_2) = c_1(\{x, y, z\}) = \{x\}$$

Choice Based Approach

- ***Example 1.11*** (continued):
  - Choice structure two $(\mathcal{B}, c_2(\cdot))$

$$c_2(B_1) = c_2(\{x, y\}) = \{x\}$$
$$c_2(B_2) = c_2(\{x, y, z\}) = \{y\}$$

  - Is such a choice rule consistent?
    - We need to impose a consistency requirement on the choice-based approach, similar to rationality assumption on the preference-based approach.

Consistency on Choices: the Weak Axiom of Revealed Preference (WARP)

WARP

- ***Weak Axiom of Revealed Preference (WARP)***: The choice structure $(\mathcal{B}, c(\cdot))$ satisfies the WARP if:
  1) for some budget set $B \in \mathcal{B}$ with $x, y \in B$, we have that element x is chosen, $x \in c(B)$, then
  2) for any other budget set $B' \in \mathcal{B}$ where alternatives x and y are also available, $x, y \in B'$, and where alternative y is chosen, $y \in c(B')$, then we must have that alternative x is chosen as well, $x \in c(B')$.

WARP

- ***Example 1.12*** (Checking WARP in choice structures):
  - Take budget set $B = \{x, y\}$ with the choice rule of $c(\{x, y\}) = x$.
  - Then, for budget set $B' = \{x, y, z\}$, the "legal" choice rules are either:

$$c(\{x, y, z\}) = \{x\}, \text{ or}$$

$$c(\{x, y, z\}) = \{z\}, \text{ or}$$

$$c(\{x, y, z\}) = \{x, z\}$$

WARP

- ***Example 1.12*** (continued):
  - This implies that the individual decision-maker cannot select

$$c(\{x, y, z\}) \neq \{y\}$$
$$c(\{x, y, z\}) \neq \{y, z\}$$
$$c(\{x, y, z\}) \neq \{x, y\}$$
$$c(\{x, y, z\}) \neq \{x, y, z\}$$

WARP

- ***Example 1.13*** (More on choice structures satisfying/violating WARP):
  - Take budget set $B = \{x, y\}$ with the choice rule of $c(\{x, y\}) = \{x, y\}$.
  - Then, for budget set $B' = \{x, y, z\}$, the "legal" choices according to WARP are either:

$$c(\{x, y, z\}) = \{x, y\}, \text{ or}$$

$$c(\{x, y, z\}) = \{z\}, \text{ or}$$

$$c(\{x, y, z\}) = \{x, y, z\}$$

WARP

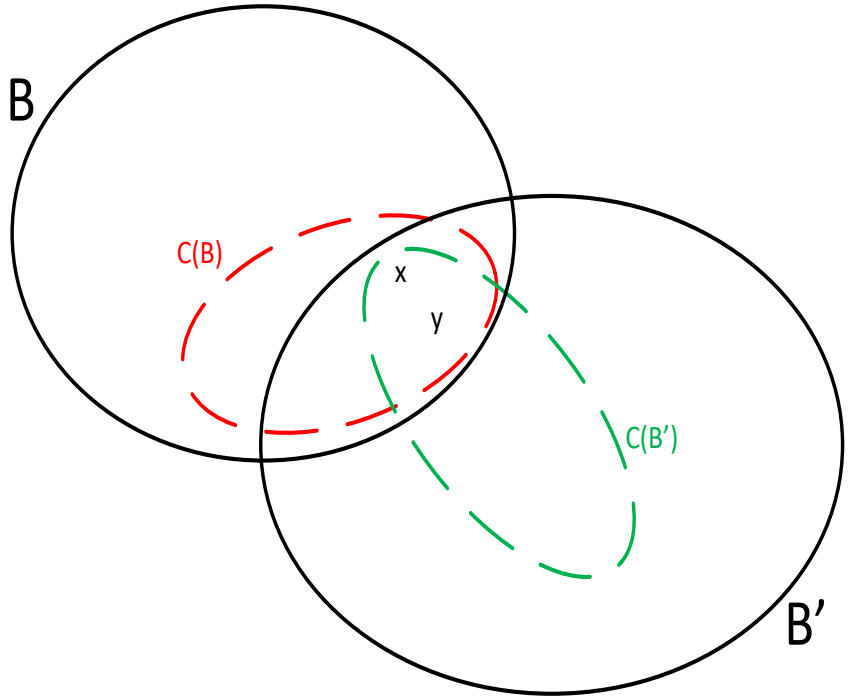
- ***Example 1.13*** (continued):
  - This implies that the decision-maker cannot select:

$$c(\{x, y, z\}) \neq \{x\}$$
$$c(\{x, y, z\}) \neq \{y\}$$
$$c(\{x, y, z\}) \neq \{x, z\}$$
$$c(\{x, y, z\}) \neq \{y, z\}$$

  - In summary, when both x and y are available in B and B', as long as they are chosen in B, both of them must be chosen in B'.

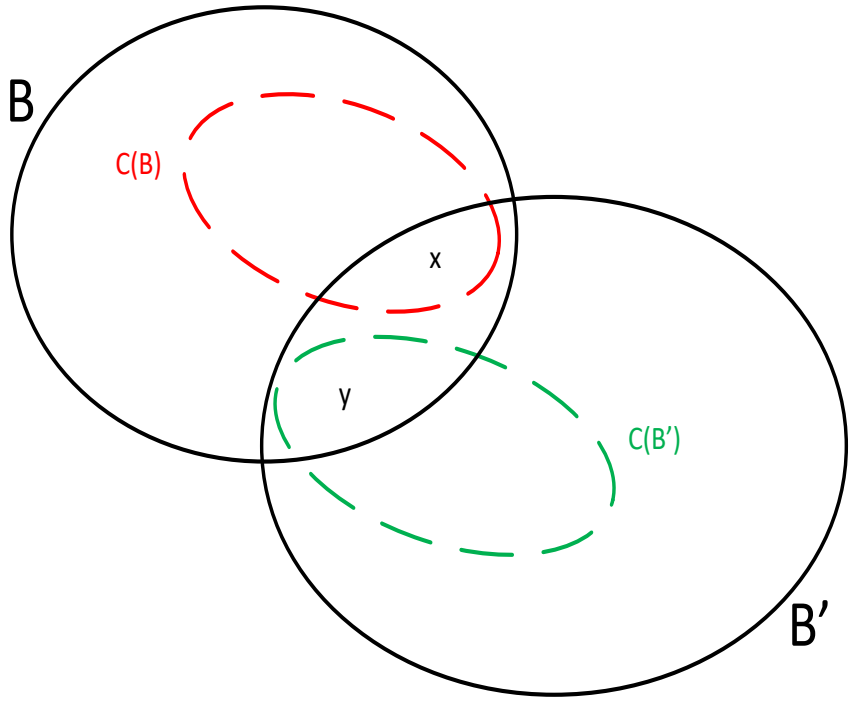
WARP

- ***Example 1.13*** (continued):
  - Choice rule satisfying WARP



WARP

- ***Example 1.13*** (continued):
  - Choice rule violating WARP



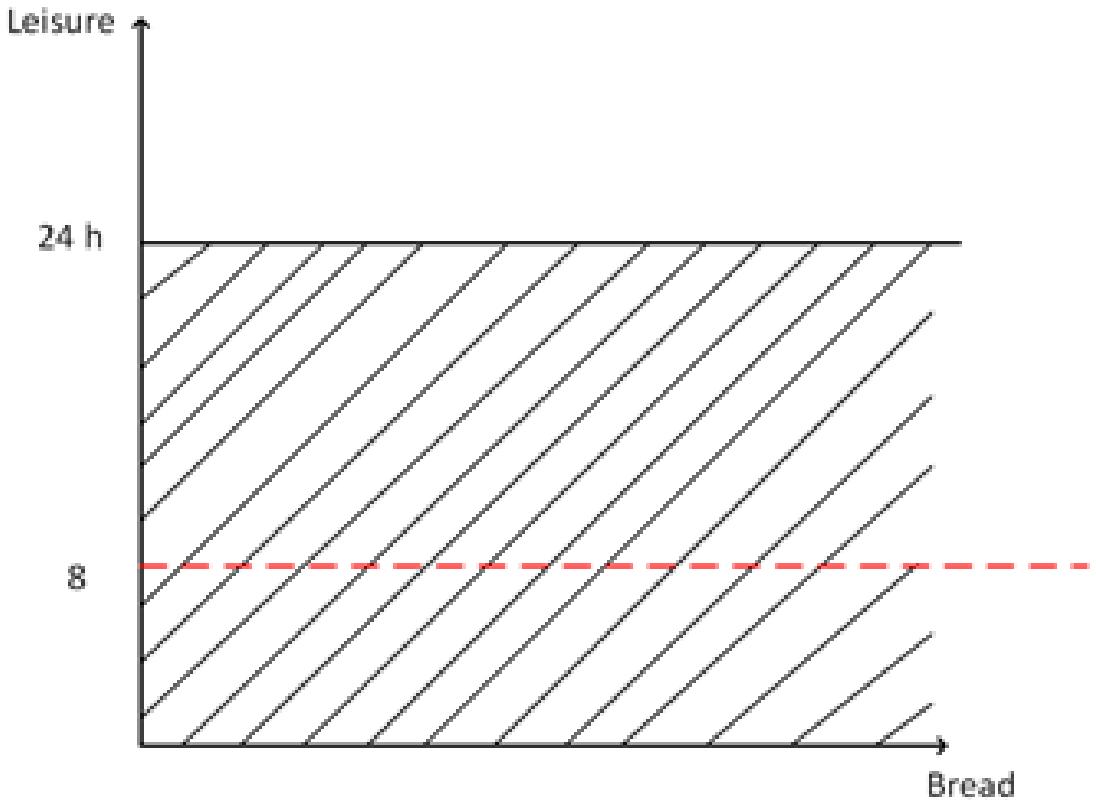
Consumption Sets

Consumption Sets

- ***Consumption set***: a subset of the commodity space $\mathbb{R}^N$, denoted by $x \subset \mathbb{R}^N$, whose elements are the consumption bundles that the individual can conceivably consume, given the physical constraints imposed by his environment.
- Let us denote a commodity bundle x as a vector of N components.

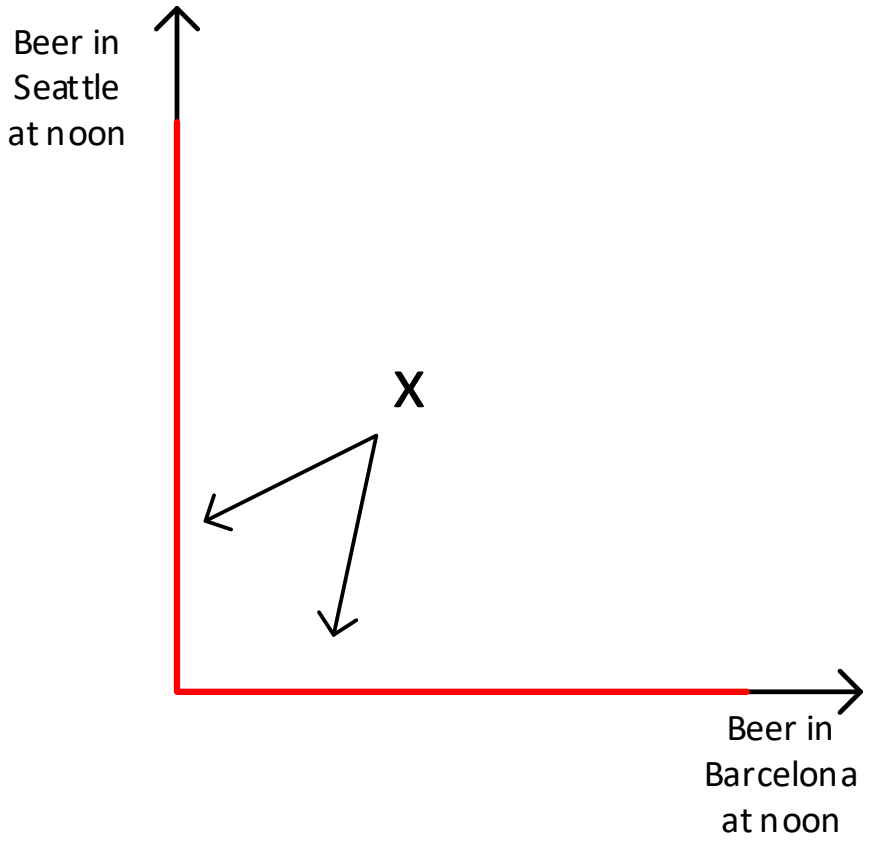
Consumption Sets

- Physical constraint on the labor market



Consumption Sets

- Consumption at two different locations



Consumption Sets

- ***Convex consumption sets***:
  - A consumption set X is convex if, for two consumption bundles $x, x' \in X$, the bundle

$$x'' = \alpha x + (1 - \alpha)x'$$

    is also an element of X for any $\alpha \in (0,1)$.

  - Intuitively, a consumption set is convex if, for any two bundles that belong to the set, we can construct a straight line connecting them that lies completely within the set.

Consumption Sets: Economic Constraints

- Assumptions on the price vector in $\mathbb{R}^N$:

  1) All commodities can be traded in a market, at prices that are publicly observable.
     - This is the principle of completeness of markets
     - It discards the possibility that some goods cannot be traded, such as pollution.
  2) Prices are strictly positive for all N goods, i.e., $p \gg 0$ for every good k.
     - Some prices could be negative, such as pollution.

Consumption Sets: Economic Constraints

3) Price taking assumption: a consumer's demand for all N goods represents a small fraction of the total demand for the good.

Consumption Sets: Economic Constraints

- Bundle $x \in \mathbb{R}_+^N$ is affordable if
$$p_1 x_1 + p_2 x_2 + \cdots + p_N x_N \leq w$$
or, in vector notation, $p \cdot x \leq w$.
- Note that $p \cdot x$ is the total cost of buying bundle $x = (x_1, x_2, \ldots, x_N)$ at market prices $p = (p_1, p_2, \ldots, p_N)$, and w is the total wealth of the consumer.
- When $x \in \mathbb{R}_+^N$ then the set of feasible consumption bundles consists of the elements of the set:
$$B_{p,w} = \{x \in \mathbb{R}_+^N : p \cdot x \leq w\}$$

Consumption Sets: Economic Constraints

- ***Example for two goods***:

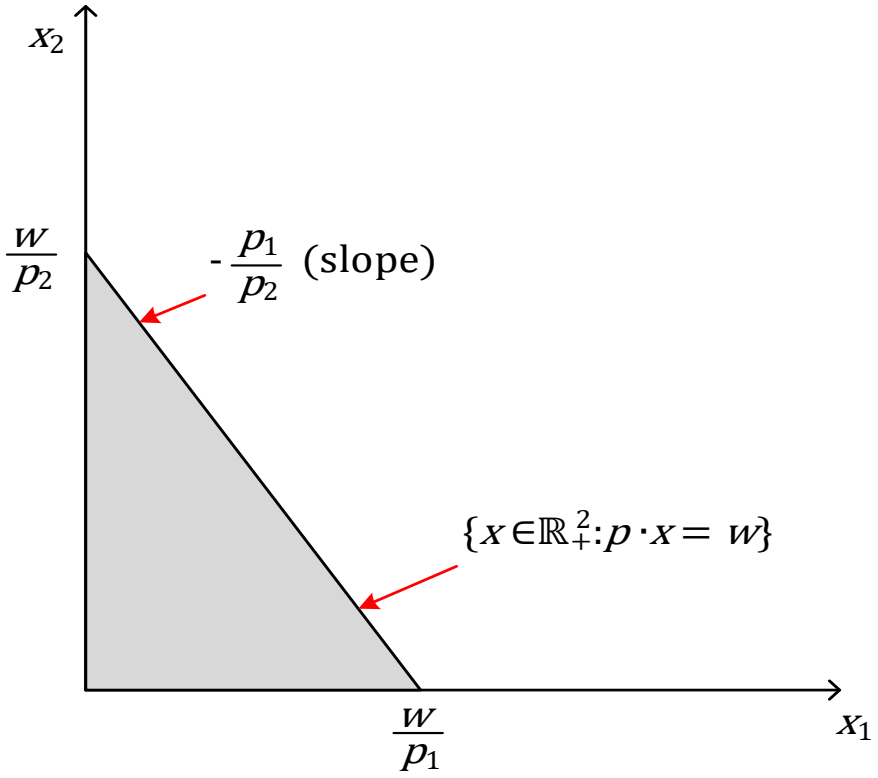
$$B_{p,w} = \{x \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \leq w\}$$

The budget line is

$p_1 x_1 + p_2 x_2 = w.$

Hence, solving for the good on the vertical axis, x_2, we obtain

$$x_2 = \frac{w}{p_2} - \frac{p_1}{p_2} x_1$$

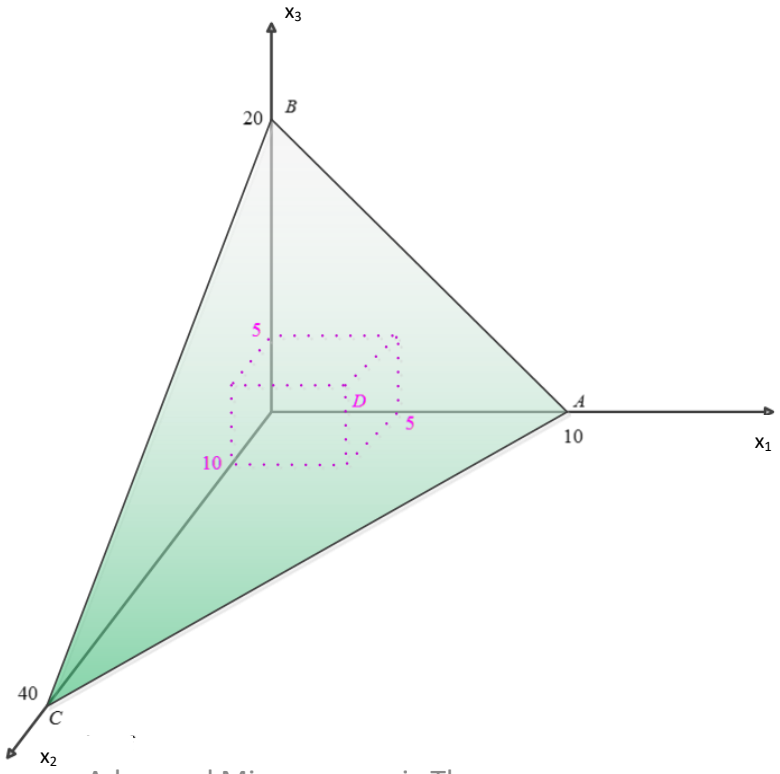


Consumption Sets: Economic Constraints

- ***Example for three goods***:

$$B_{p,w} = \{x \in \mathbb{R}^3_+ : p_1x_1 + p_2x_2 + p_3x_3 \leq w\}$$

  - The surface $p_1x_1 + p_2x_2 + p_3x_3 = w$ is referred to as the "Budget hyperplane"



Consumption Sets: Economic Constraints

- *Price vector p is orthogonal (perpendicular) to the budget line $B_{p,w}$.*
  - Note that $p \cdot x = w$ holds for any bundle x on the budget line.
  - Take any other bundle x' which also lies on $B_{p,w}$. Hence, $p \cdot x' = w$.
  - Then,

$$p \cdot x' = p \cdot x = w$$

$$p \cdot (x' - x) = 0 \text{ or } p \cdot \Delta x = 0$$

Consumption Sets: Economic Constraints

– Since this is valid for any two bundles on the budget line, then p must be perpendicular to Δx on $B_{p,w}$.

– This implies that the price vector is perpendicular (orthogonal) to $B_{p,w}$.

Consumption Sets: Economic Constraints

- *The budget set $B_{p,w}$ is convex.*
  - We need that, for any two bundles $x, x' \in B_{p,w}$, their convex combination
$$x'' = \alpha x + (1-\alpha)x'$$
also belongs to the $B_{p,w}$, where $\alpha \in (0,1)$.
  - Since $p \cdot x \leq w$ and $p \cdot x' \leq w$, then
$$\begin{aligned} p \cdot x'' &= p\alpha x + p(1-\alpha)x' \\ &= \alpha p x + (1-\alpha)px' \leq w \end{aligned}$$

Appendix 1.1: Rational Preference Relations Satisfy the WARP

Rational Preferences and WARP

- We can construct the preferences that the individual "reveals" in his actual choices when he is confronted to choose an element(s) from different budget sets.

  1) If there is some budget set B for which the individual chooses $x \in c(B)$, where $x, y \in B$, then we can say that alternative x is ***revealed at least as good as*** alternative y, and denote it as $x \succsim^* y$.
  2) If there is some budget set B for which the individual chooses $x \in c(B)$ but $y \notin c(B)$, where $x, y \in B$, then we can say that alternative x is ***revealed preferred to*** alternative y, and denote it as $x \succ^* y$.

Rational Preferences and WARP

- Let $C^*(B, \succsim)$ be the set of optimal choices generated by the preference relation $\succsim$ when facing a budget set B.
- Using this notation, we can restate the WARP as follows:
  - If alternative x is revealed at least as good as y, then y cannot be revealed preferred to x.
  - That is, if $x \succsim^* y$, then we cannot have $y \succ^* x$.

Rational Preferences and WARP

- Let us next show that:

$$\text{Rational preference relation} \Longrightarrow$$
$$\text{Choice structure satisfying WARP}$$

- *Proof*:
  - Suppose that for some budget set $B \in \mathcal{B}$, we have that $x, y \in B$ and $x \in C^*(B, \succsim)$.
    - That is, x belongs to the set of optimal choices given the preference relation $\succsim$ when the decision maker faces a budget set B.
  - Hence, $x \in C^*(B, \succsim) \implies x \succsim y$ for all $y \in B$.

Rational Preferences and WARP

– In order to check WARP, assume some other budget set $B' \in \mathcal{B}$ with $x, y \in B'$ and $y \in C^*(B', \succsim)$.

  - That is, y belongs to the set of optimal choices given the preference relation $\succsim$ when the decision maker faces budget set B'.

– Thus, $y \in C^*(B', \succsim) \implies y \succsim z$ for all $z \in B'$.

Rational Preferences and WARP

– Combining the conclusions from the previous two points, $x \succsim y$ and $y \succsim z$, we can apply transitivity (because the preference relation is rational), and we obtain $x \succsim z$.

– Then $x \in C^*(B', \succsim)$, and we find that

$$x, y \in C^*(B', \succsim)$$

which proves that WARP is satisfied.

Rational Preferences and WARP

- Regarding:

$$\text{Choice structure satisfying WARP} \Longrightarrow$$
$$\text{Rational preference relation}$$

- It is only true if the budget set B contains three or fewer elements (See MWG for a proof based on Arrow (1959)).

Advanced Microeconomic Theory: *An Intuitive Approach with Examples*

by Felix Munoz-Garcia

Chapter 2: Demand Theory

Outline

- Utility maximization problem (UMP)
- Walrasian demand and indirect utility function
- WARP and Walrasian demand
- Income and substitution effects (Slutsky equation)
- Duality between UMP and expenditure minimization problem (EMP)
- Hicksian demand and expenditure function
- Connections

Utility Maximization Problem

Utility Maximization Problem

- Consumer maximizes his utility level by selecting a bundle x (where x can be a vector) subject to his budget constraint:

$$\max_{x \geq 0} u(x)$$
$$\text{s.t. } p \cdot x \leq w$$

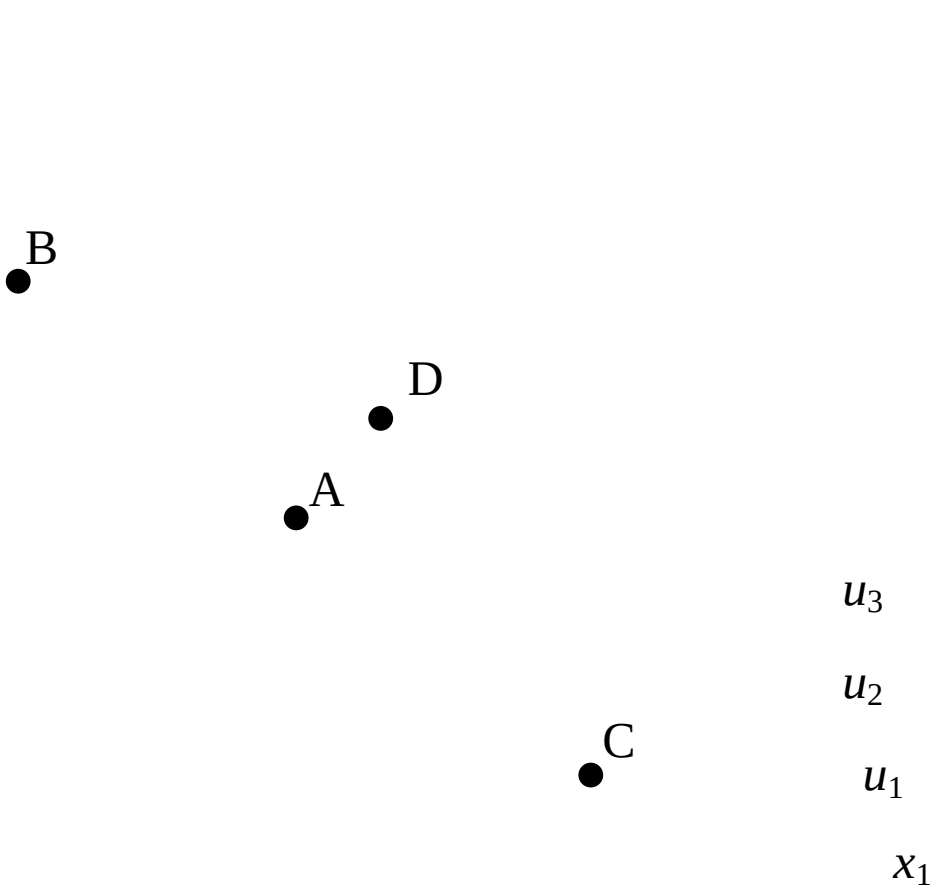
- ***Weierstrass Theorem***: for optimization problems defined on the reals, if the objective function is continuous and constraints define a closed and bounded set, then the solution to such optimization problem exists.

Utility Maximization Problem

- ***Existence***: if $p \gg 0$ and $w > 0$ (i.e., if $B_{p,w}$ is closed and bounded), and if $u(\cdot)$ is continuous, then there exists at least one solution to the UMP.
  - If, in addition, preferences are strictly convex, then the solution to the UMP is unique.
- We denote the solution of the UMP as the **argmax** of the UMP (the argument, x, that solves the optimization problem), and we denote it as $x(p, w)$.
  - $x(p, w)$ is the ***Walrasian demand*** correspondence, which specifies a demand for every good in $\mathbb{R}_+^L$ for every possible price vector, p, and every possible wealth level, w.

Utility Maximization Problem

- Walrasian demand $x(p, w)$ at bundle *A* is optimal, as the consumer reaches a utility level of u_2 by exhausting all his wealth.
- Bundles *B* and *C* are not optimal, despite exhausting the consumer's wealth. They yield a lower utility level u_1, where $u_1 < u_2$.
- Bundle *D* is unaffordable and, hence, it cannot be the argmax of the UMP given a wealth level of w.



Properties of Walrasian Demand

- If the utility function is continuous and preferences satisfy LNS over the consumption set $X = \mathbb{R}^L_+$, then the Walrasian demand $x(p, w)$ satisfies:

1) Homogeneity of degree zero:

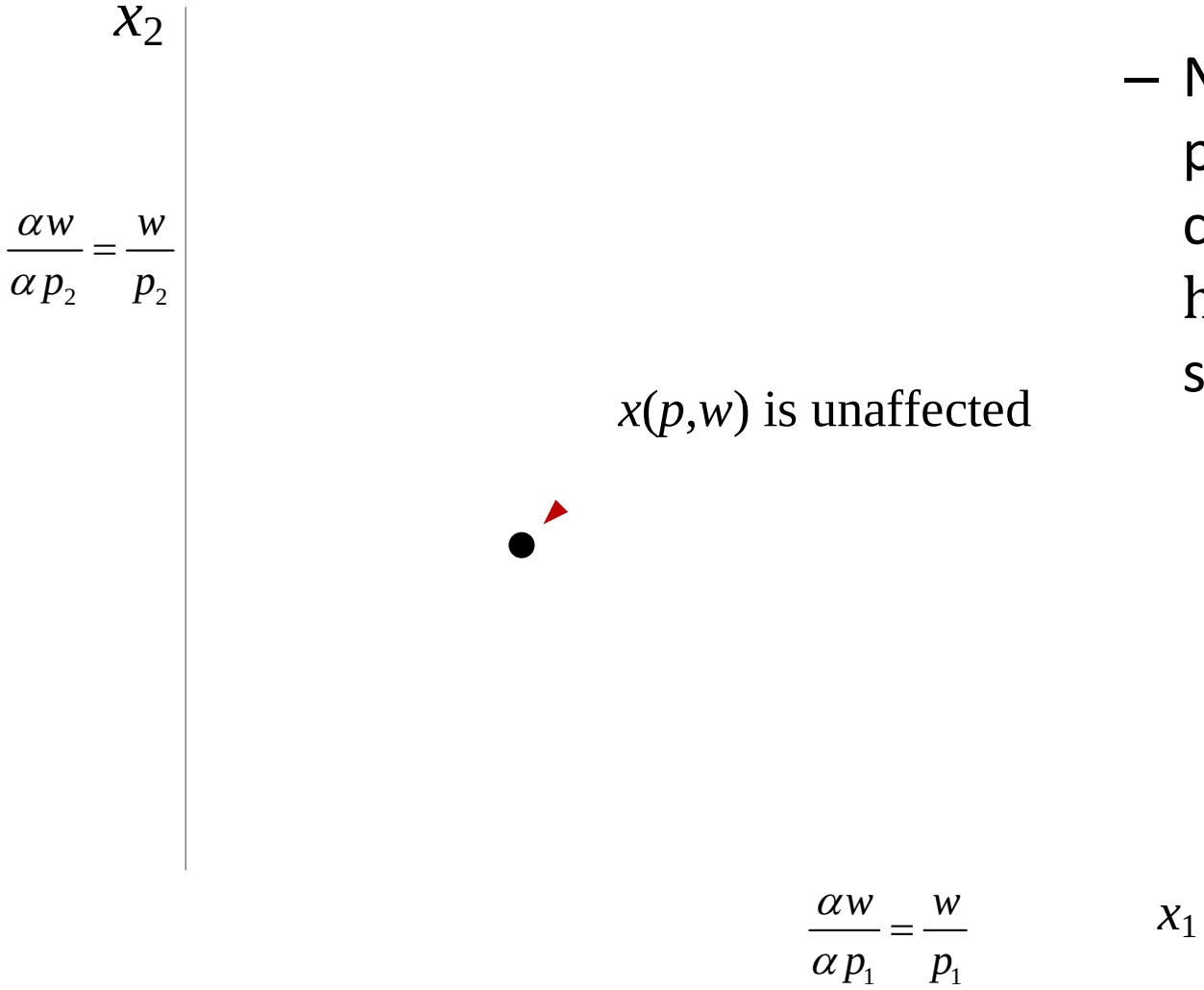
$x(p, w) = x(\alpha p, \alpha w)$ for all p, w, and for all $\alpha > 0$

That is, the budget set is unchanged!

$$\{x \in \mathbb{R}^L_+ : \ p \cdot x \leq w \ \} = \{x \in \mathbb{R}^L_+ : \ \alpha p \cdot x \leq \alpha w\}$$

Note that we don't need any assumption on the preference relation to show this. We only rely on the budget set being unaffected.

Properties of Walrasian Demand



– Note that the preference relation can be linear, and homog(0) would still hold.

Properties of Walrasian Demand

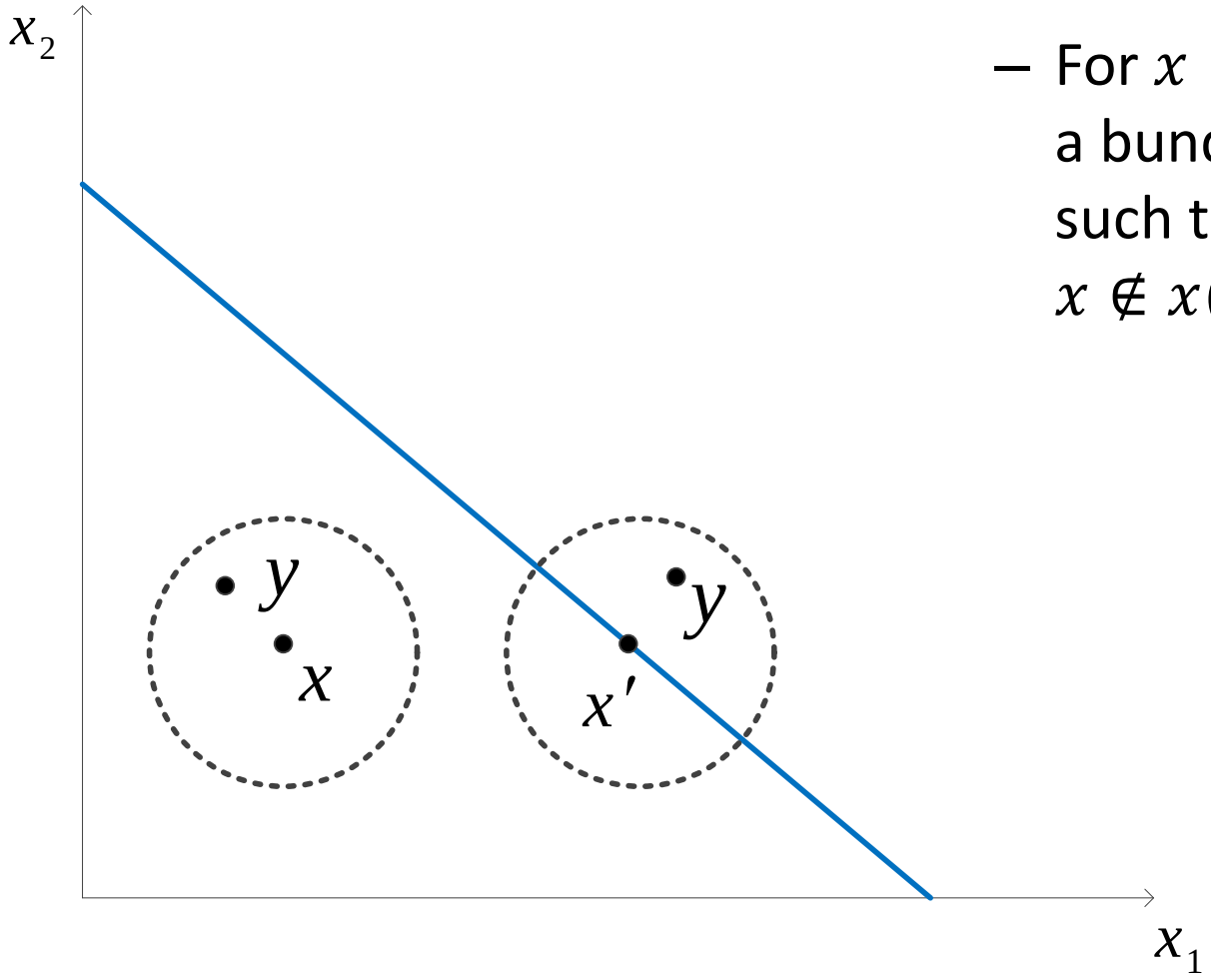
2) Walras' Law:

$$p \cdot x = w \quad \text{for all } x = x(p, w)$$

It follows from LNS: if the consumer selects a Walrasian demand $x \in x(p, w)$, where $p \cdot x < w$, then it means we can still find other bundle y, which is ε–close to x, where consumer can improve his utility level.

If the bundle the consumer chooses lies on the budget line, i.e., $p \cdot x' = w$, we could then identify bundles that are *strictly* preferred to x', but these bundles would be unaffordable to the consumer.

Properties of Walrasian Demand



– For $x \in x(p, w)$, there is a bundle y, ε–close to x, such that $y \succ x$. Then, $x \notin x(p, w)$.

Properties of Walrasian Demand

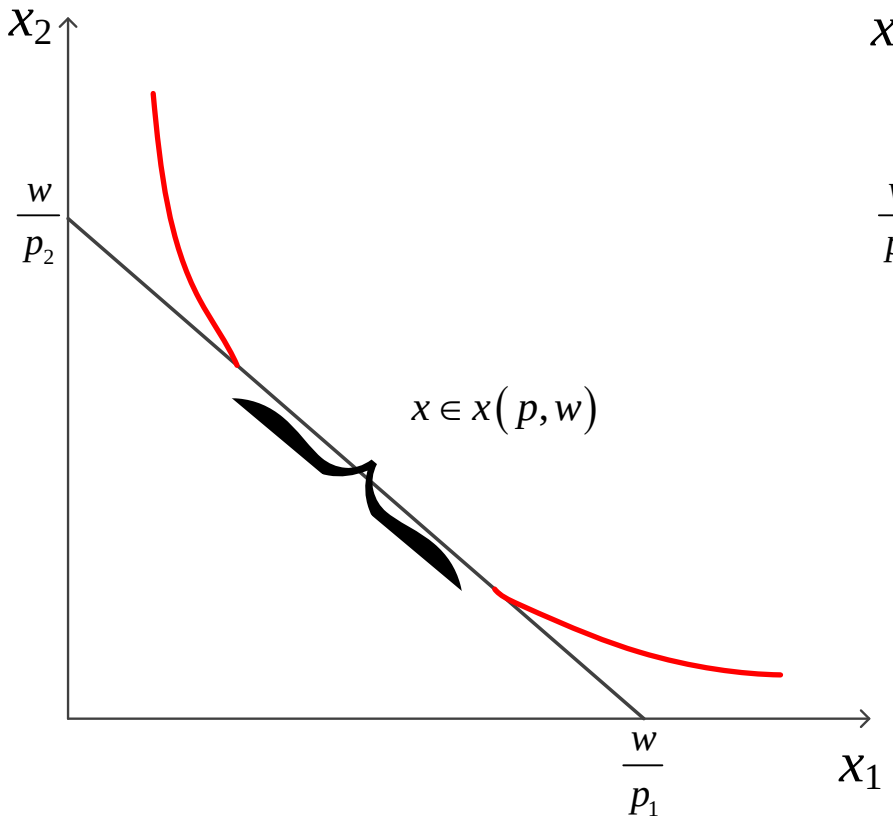
3) Convexity/Uniqueness:

a) If the preferences are convex, then the Walrasian demand correspondence $x(p, w)$ defines a convex set, i.e., a continuum of bundles are utility maximizing.

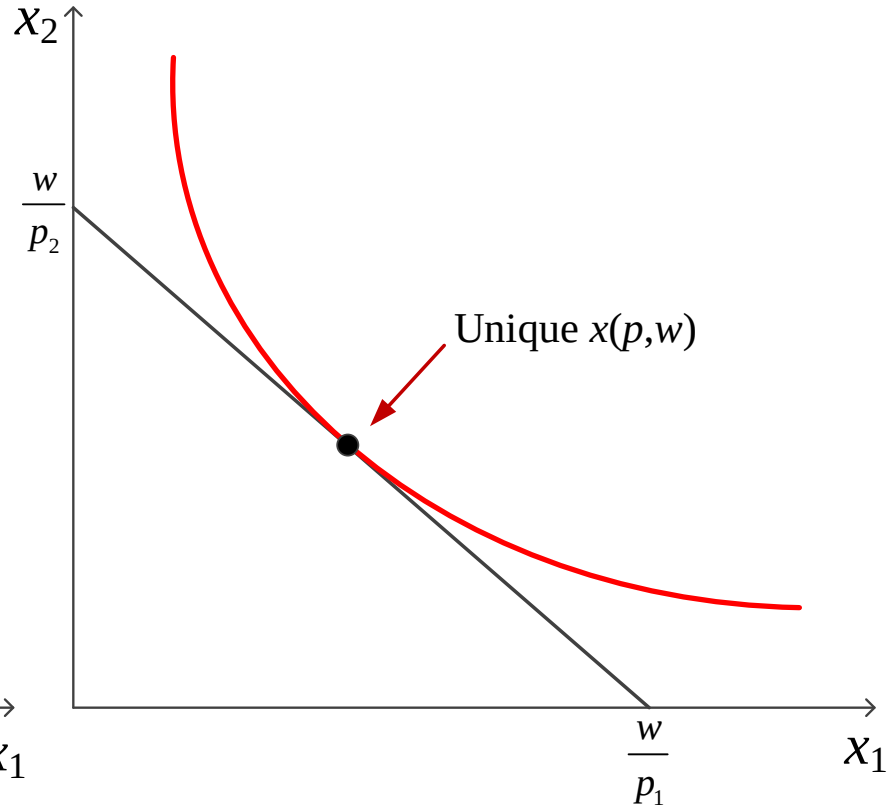
b) If the preferences are strictly convex, then the Walrasian demand correspondence $x(p, w)$ contains a single element.

Properties of Walrasian Demand

Convex preferences



Strictly convex preferences



UMP: Necessary Condition

$$\max_{x \geq 0} u(x) \quad \text{s.t. } p \cdot x \leq w$$

- We solve it using Kuhn-Tucker conditions over the Lagrangian $L = u(x) + \lambda(w - p \cdot x)$,

$$\frac{\partial L}{\partial x_k} = \frac{\partial u(x^*)}{\partial x_k} - \lambda p_k \leq 0 \text{ for all } k, \ = 0 \text{ if } x_k^* > 0$$

$$\frac{\partial L}{\partial \lambda} = w - p \cdot x^* = 0$$

- That is, in an *interior* optimum, $\frac{\partial u(x^*)}{\partial x_k} = \lambda p_k$ for every good k, which implies

$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_l}{p_k} \Leftrightarrow MRS_{l,k} = \frac{p_l}{p_k} \Leftrightarrow \frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$$

UMP: Sufficient Condition

- When are Kuhn-Tucker (necessary) conditions, also sufficient?
  - That is, when can we guarantee that $x(p, w)$ is the $\max$ of the UMP and not the $\min$?

UMP: Sufficient Condition

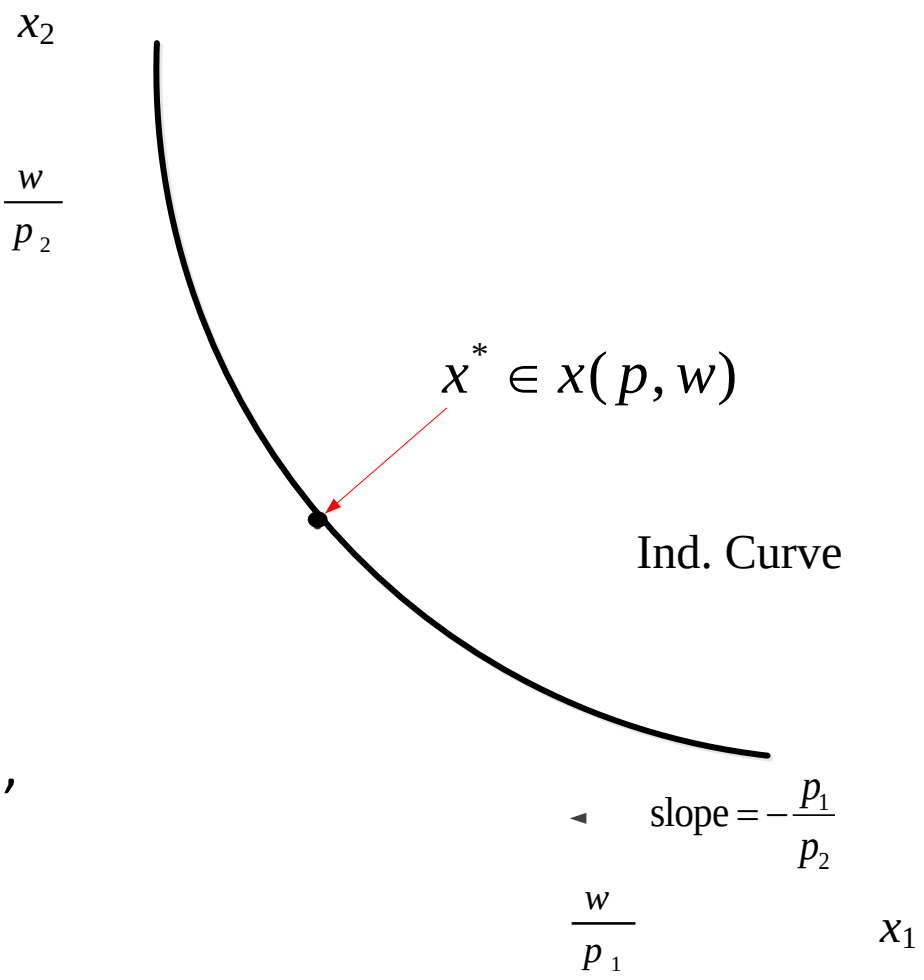
- Kuhn-Tucker conditions are sufficient for a max if:

  1) $u(x)$ is quasiconcave, i.e., convex upper contour set (UCS).

  2) $u(x)$ is monotone.

  3) $\nabla u(x) \neq 0$ for $x \in \mathbb{R}^L_+$.

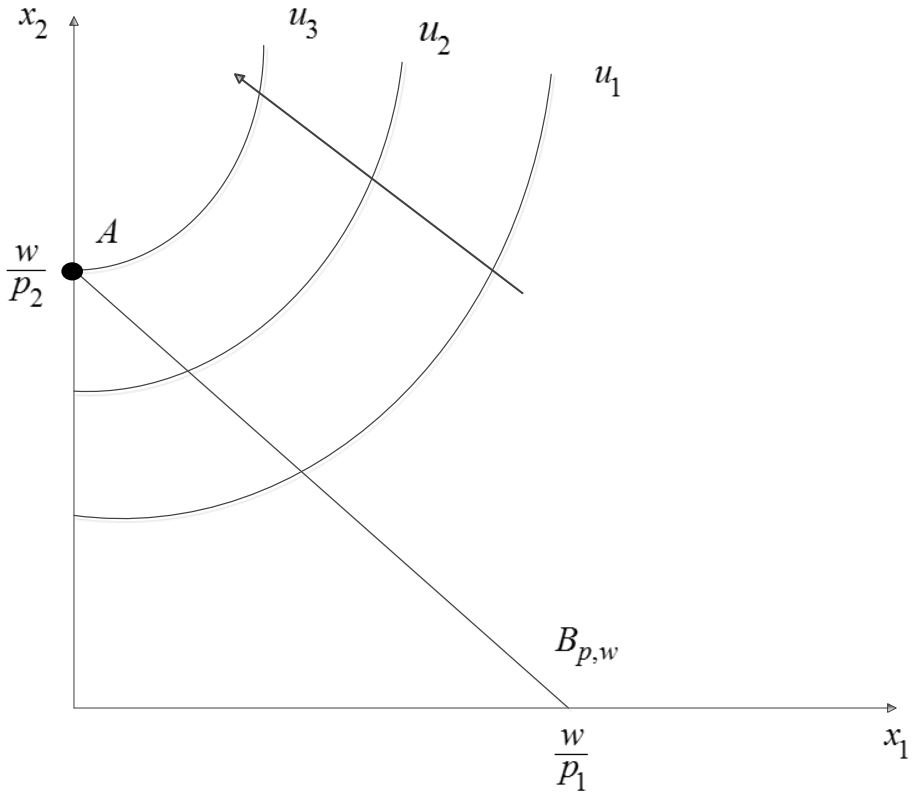
  – If $\nabla u(x) = 0$ for some x, then we would be at the "top of the mountain" (i.e., bliss point), which violates both LNS and monotonicity.



UMP: Violations of Sufficient Condition

1) $u(\cdot)$ is non-monotone:

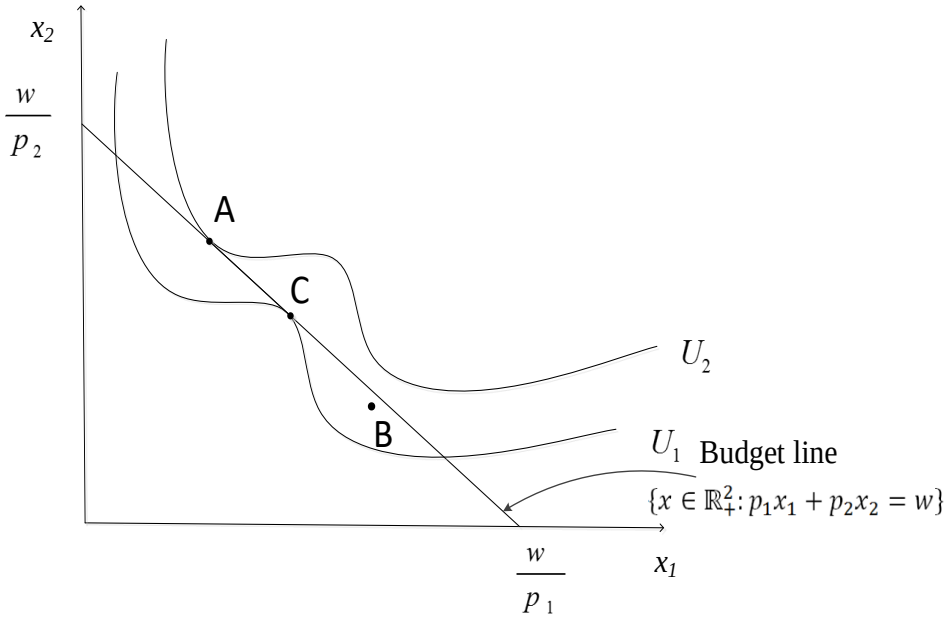
- The consumer chooses bundle *A* (at a corner) since it yields the highest utility level given his budget constraint.
- At point *A*, however, the tangency condition $MRS_{1,2} = \frac{p_1}{p_2}$ does not hold.



UMP: Violations of Sufficient Condition

2) $u(\cdot)$ is not quasiconcave:

- The upper contour sets (UCS) are not convex.
- $MRS_{1,2} = \frac{p_1}{p_2}$ is not a sufficient condition for a max.
- A point of tangency (*C*) gives a lower utility level than a point of non-tangency (*B*).
- True maximum is at point *A*.



UMP: Corner Solution

- Analyzing differential changes in x_l and x_k that keep individual's utility unchanged, $du = 0$,

$$\frac{du(x)}{dx_l} dx_l + \frac{du(x)}{dx_k} dx_k = 0 \text{ (total diff.)}$$

- Rearranging,

$$\frac{dx_k}{dx_l} = -\frac{\frac{du(x)}{dx_l}}{\frac{du(x)}{dx_k}} = -MRS_{l,k}$$

- ***Corner Solution***: $MRS_{l,k} > \frac{p_l}{p_k}$, or alternatively, $\frac{\frac{du(x^*)}{dx_l}}{p_l} > \frac{\frac{du(x^*)}{dx_k}}{p_k}$, i.e., the consumer prefers to consume more of good l.

UMP: Corner Solution

- In the FOCs, this implies:
  - *a)* $\frac{\partial u(x^*)}{\partial x_k} \leq \lambda p_k$ for the goods whose consumption is zero, $x_k^* = 0$, and
  - *b)* $\frac{\partial u(x^*)}{\partial x_l} = \lambda p_l$ for the good whose consumption is positive, $x_l^* > 0$.
- *Intuition*: the marginal utility per dollar spent on good l is still larger than that on good k.

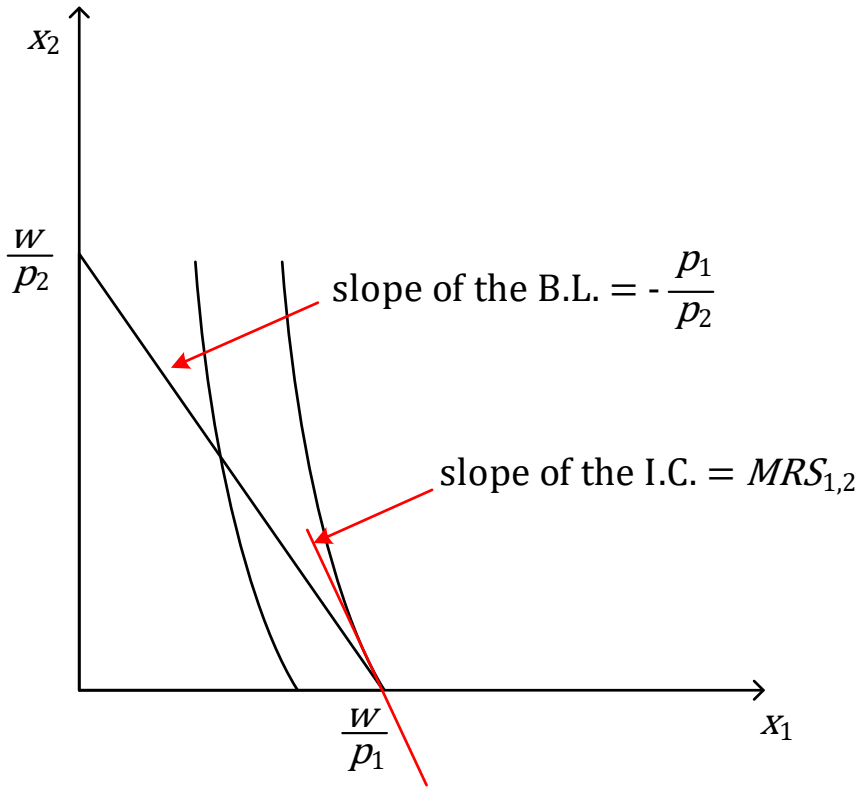
$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} = \lambda \geq \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$$

UMP: Corner Solution

- Consumer seeks to consume good 1 alone.
- At the corner solution, the indifference curve is steeper than the budget line, i.e.,

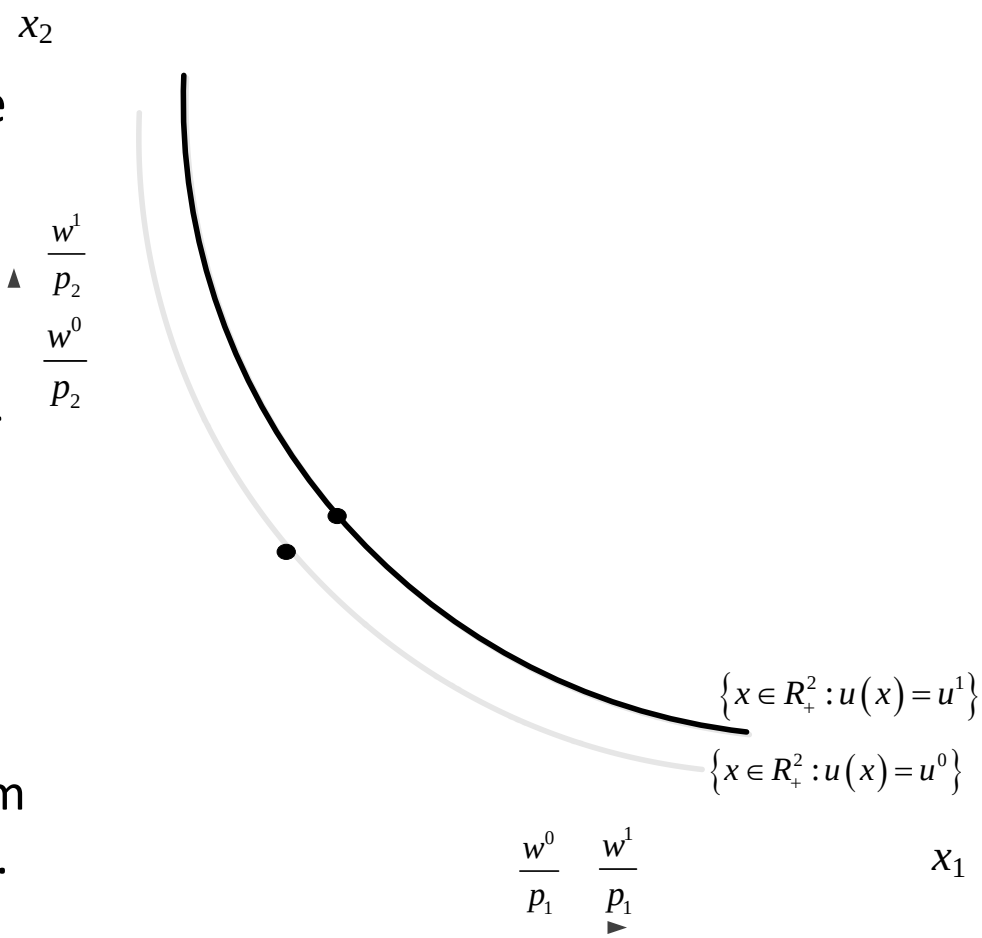
$$MRS_{1,2} > \frac{p_1}{p_2} \text{ or } \frac{MU_1}{p_1} > \frac{MU_2}{p_2}$$

- Intuitively, the consumer would like to consume more of good 1, even after spending his entire wealth on good 1 alone.



UMP: Lagrange Multiplier

- λ is referred to as the "marginal value of relaxing the constraint" in the UMP (a.k.a. "***shadow price of wealth***").
- If we provide more wealth to the consumer, he is capable of reaching a higher indifference curve and, as a consequence, obtaining a higher utility level.
  - We want to measure the change in utility resulting from a marginal increase in wealth.



UMP: Lagrange Multiplier

- Let us take $u(x(p,w))$, and analyze the change in utility from the change in wealth. Using chain rule yields,

$$\nabla u(x(p,w)) \cdot D_w x(p,w)$$

- Substituting $\nabla u(x(p,w)) = \lambda p$ (in interior solutions),

$$\lambda p \cdot D_w x(p,w)$$

UMP: Lagrange Multiplier

- From Walras' Law, $p \cdot x(p,w) = w$, the change in expenditure from an increase in wealth is given by

$$D_w[p \cdot x(p,w)] = p \cdot D_w x(p,w) = 1$$

- Hence,

$$\nabla u(x(p,w)) \cdot D_w x(p,w) = \lambda \underbrace{p \cdot D_w x(p,w)}_{1} = \lambda$$

- *Intuition*: If $\lambda = 5$, then a \$1 increase in wealth implies an increase in 5 units of utility.

UMP: Lagrange Multiplier — Cobb-Douglas Example

- Take $u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$ with prices (p_1, p_2) and wealth w.
- The UMP first-order conditions are

$$\tfrac{1}{2} x_1^{-1/2} x_2^{1/2} = \lambda p_1 \quad \text{and} \quad \tfrac{1}{2} x_1^{1/2} x_2^{-1/2} = \lambda p_2.$$

- Dividing the two conditions gives $x_2/x_1 = p_1/p_2$, and combining with the budget constraint $p_1x_1 + p_2x_2 = w$ yields the Walrasian demand

$$x_1(p, w) = w / (2 p_1), \quad x_2(p, w) = w / (2 p_2).$$

- Substituting back into either FOC,

$$\lambda(p, w) = 1 / (2 \sqrt{(p_1 p_2)}).$$

- ***Reading λ.*** If $p_1 = p_2 = 1$ and $w = 10$, then $x_1 = x_2 = 5$, indirect utility $v = 5$, and $\lambda = \tfrac{1}{2}$. An extra \$1 of wealth raises utility by ½ a util.

Walrasian Demand: Wealth Effects

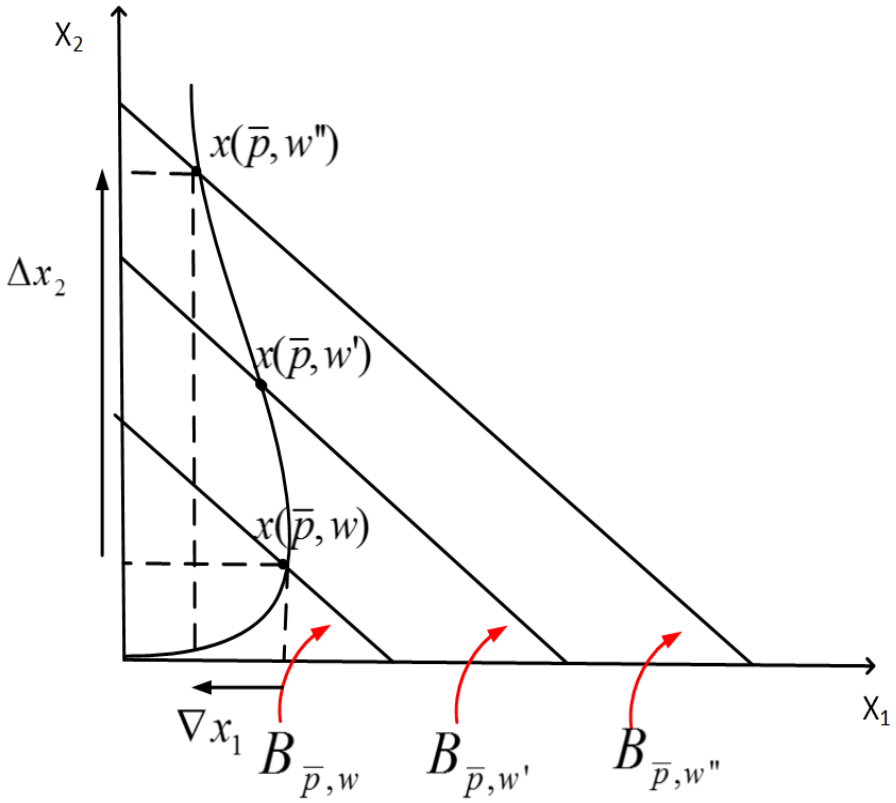
- Normal vs. Inferior goods

$$\frac{\partial x(p,w)}{\partial w} \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \quad \begin{Bmatrix} \text{normal} \\ \text{inferior} \end{Bmatrix}$$

- Examples of inferior goods:
  - Two-buck chuck (a really cheap wine)
  - Walmart during the economic crisis

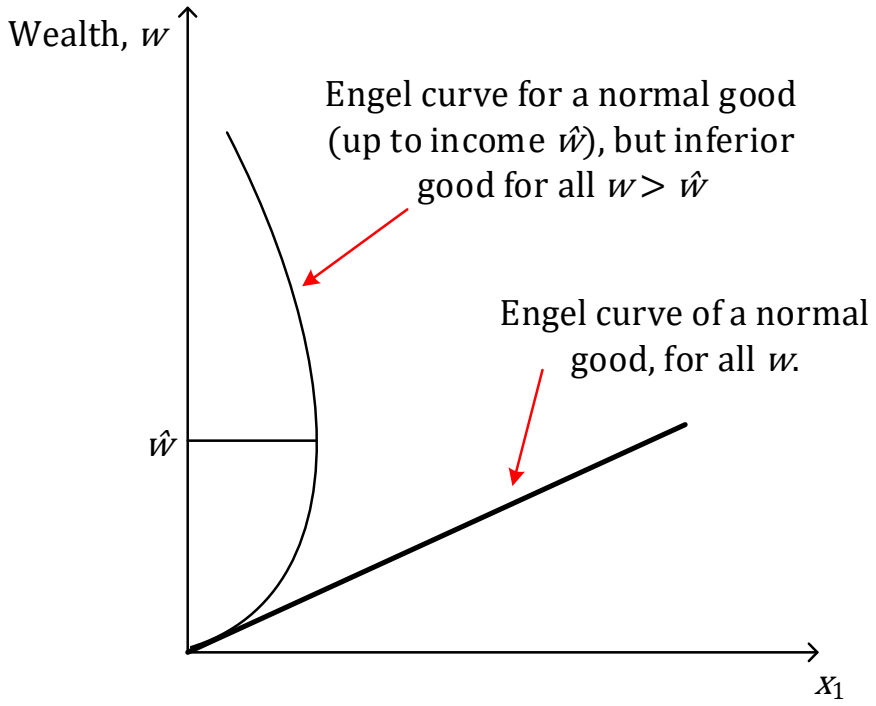
Walrasian Demand: Wealth Effects

- An increase in the wealth level produces an outward shift in the budget line.
- x_2 is normal as $\frac{\partial x_2(p,w)}{\partial w} > 0$, while x_1 is inferior as $\frac{\partial x_1(p,w)}{\partial w} < 0$.
- ***Wealth expansion path:***
  - connects the optimal consumption bundle for different levels of wealth
  - indicates how the consumption of a good changes as a consequence of changes in the wealth level



Walrasian Demand: Wealth Effects

- ***Engel curve*** depicts the consumption of a particular good on the horizontal axis and wealth on the vertical axis.
- The slope of the Engel curve is:
  - positive if the good is normal
  - negative if the good is inferior
- Engel curve can be positively sloped for low wealth levels and becomes negatively sloped afterwards.



Walrasian Demand: Price Effects

- Own price effect:

$$\frac{\partial x_k(p,w)}{\partial p_k} \begin{Bmatrix} < \\ > \end{Bmatrix} 0 \quad \begin{Bmatrix} \text{Usual} \\ \text{Giffen} \end{Bmatrix}$$

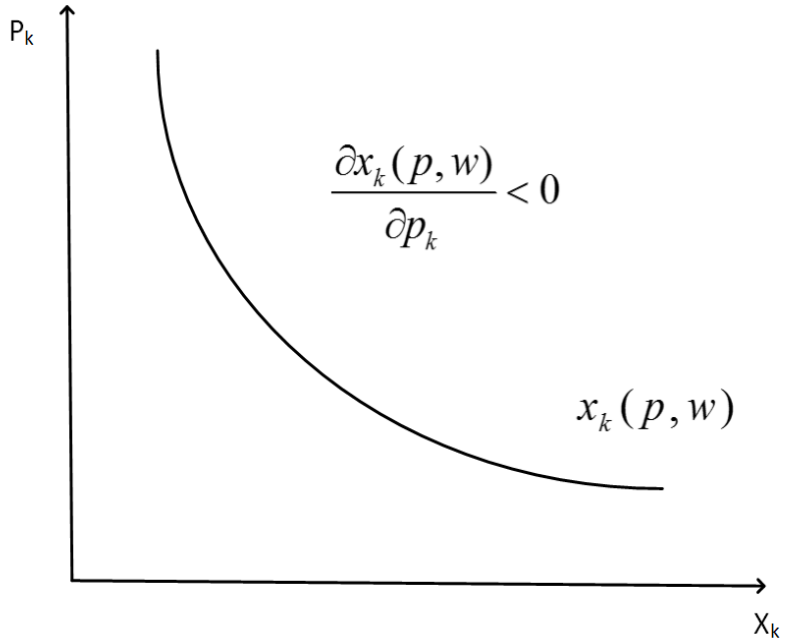
- Cross-price effect:

$$\frac{\partial x_k(p,w)}{\partial p_l} \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \quad \begin{Bmatrix} \text{Substitutes} \\ \text{Complements} \end{Bmatrix}$$

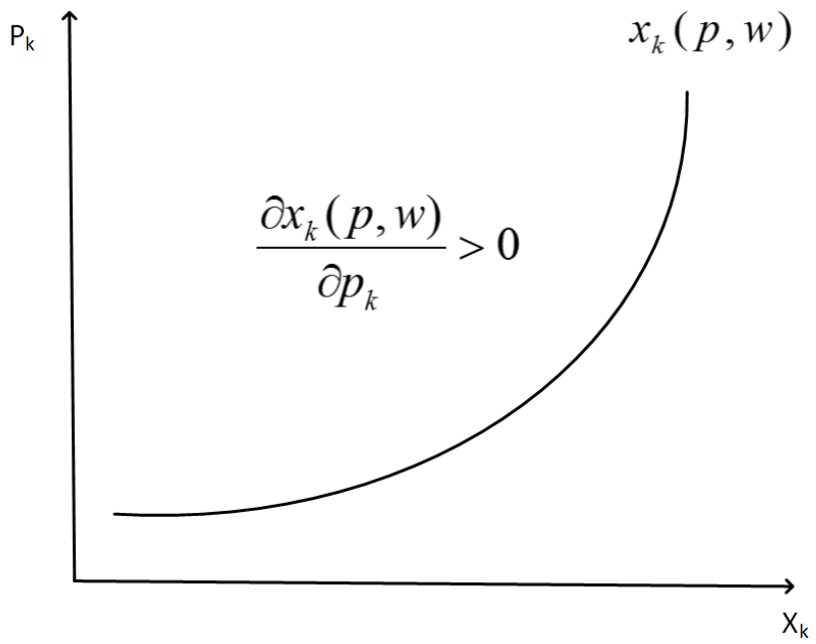
  - *Examples of Substitutes*: two brands of mineral water, such as Aquafina vs. Poland Springs.
  - *Examples of Complements*: cars and gasoline.

Walrasian Demand: Price Effects

- Own price effect



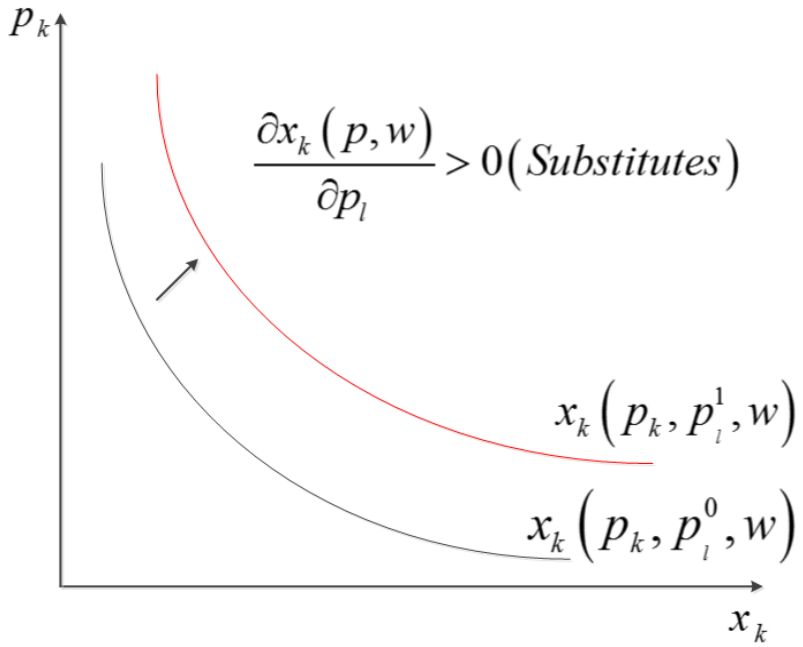
Usual good



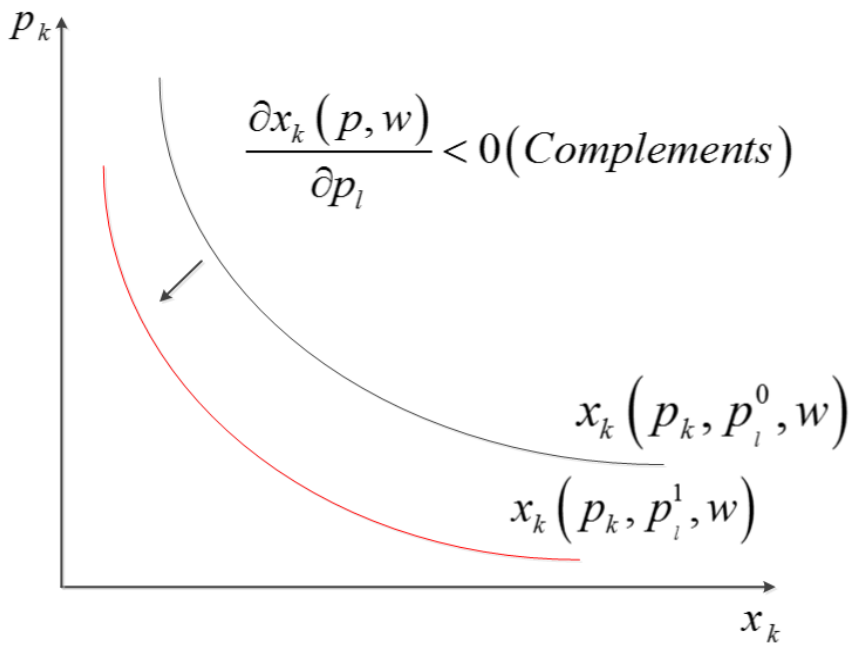
Giffen good

Walrasian Demand: Price Effects

- Cross-price effect



Substitutes



Complements

Indirect Utility Function

- The Walrasian demand function, $x(p, w)$, is the solution to the UMP (i.e., argmax).

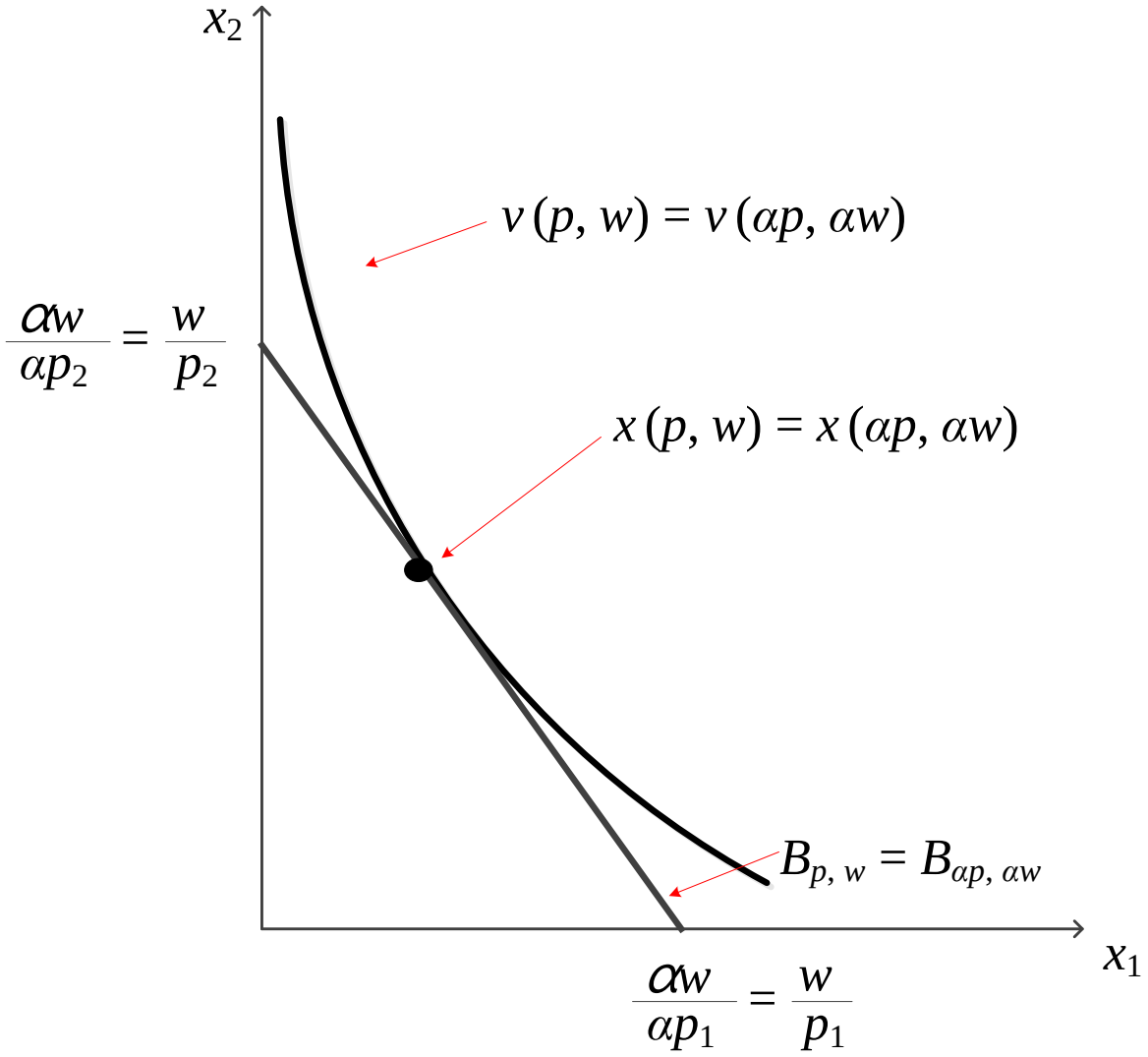
- What would be the utility function evaluated at the solution of the UMP, i.e., $x(p, w)$?
  - This is the ***indirect utility function*** (i.e., the highest utility level), $v(p, w) \in \mathbb{R}$, associated with the UMP.
  - It is the "value function" of this optimization problem.

Properties of Indirect Utility Function

- If the utility function is continuous and preferences satisfy LNS over the consumption set $X = \mathbb{R}^L_+$, then the indirect utility function $v(p, w)$ satisfies:

1) ***Homogenous of degree zero:*** Increasing p and w by a common factor $\alpha > 0$ does not modify the consumer's optimal consumption bundle, $x(p, w)$, nor his maximal utility level, measured by $v(p, w)$.

Properties of Indirect Utility Function

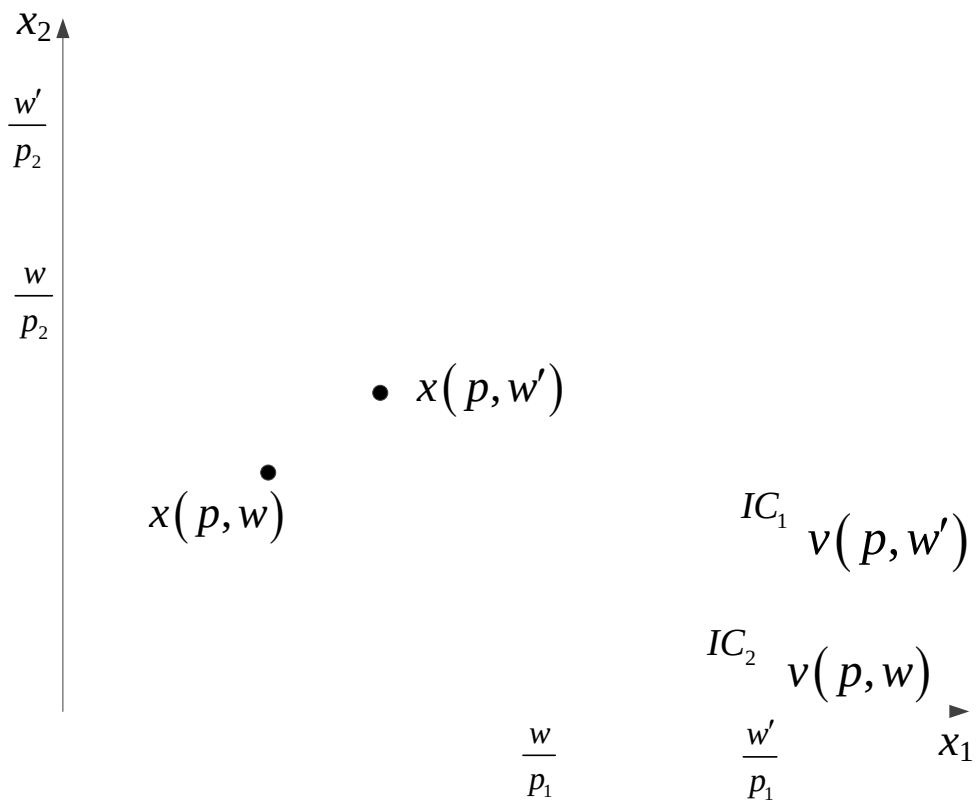


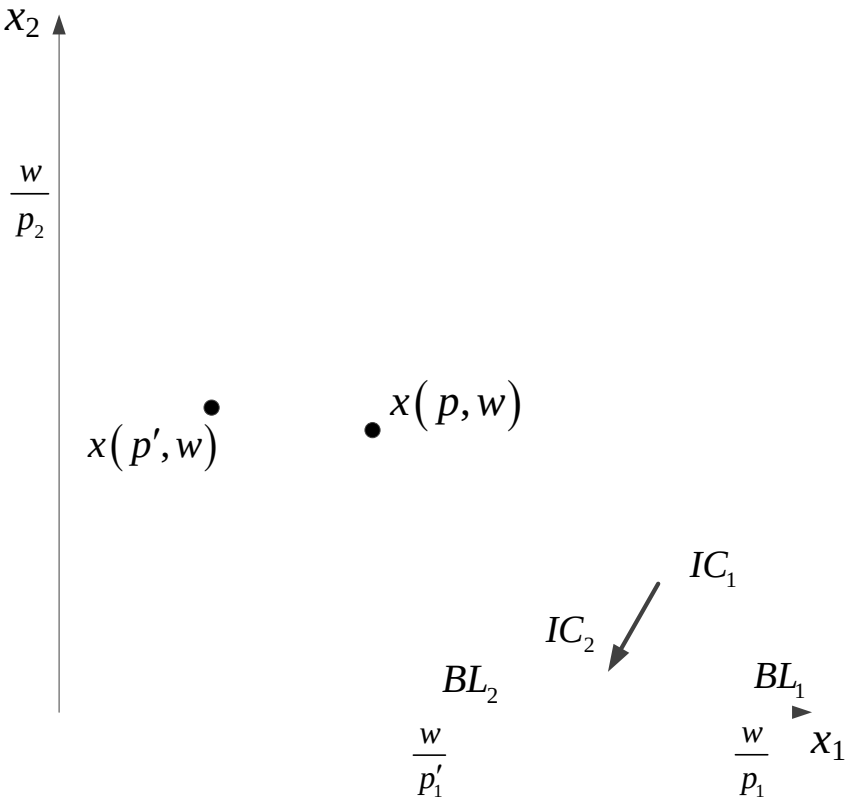
Properties of Indirect Utility Function

2) Strictly increasing in w:

$v(p, w') > v(p, w)$ for $w' > w$.

3) non-increasing (i.e., weakly decreasing) in p_k

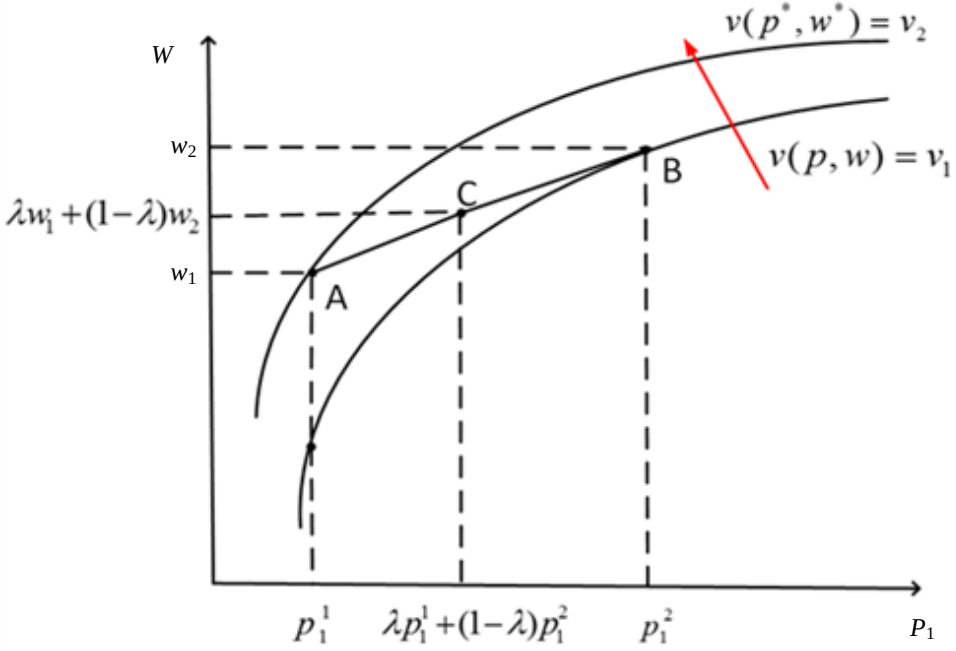




Properties of Indirect Utility Function

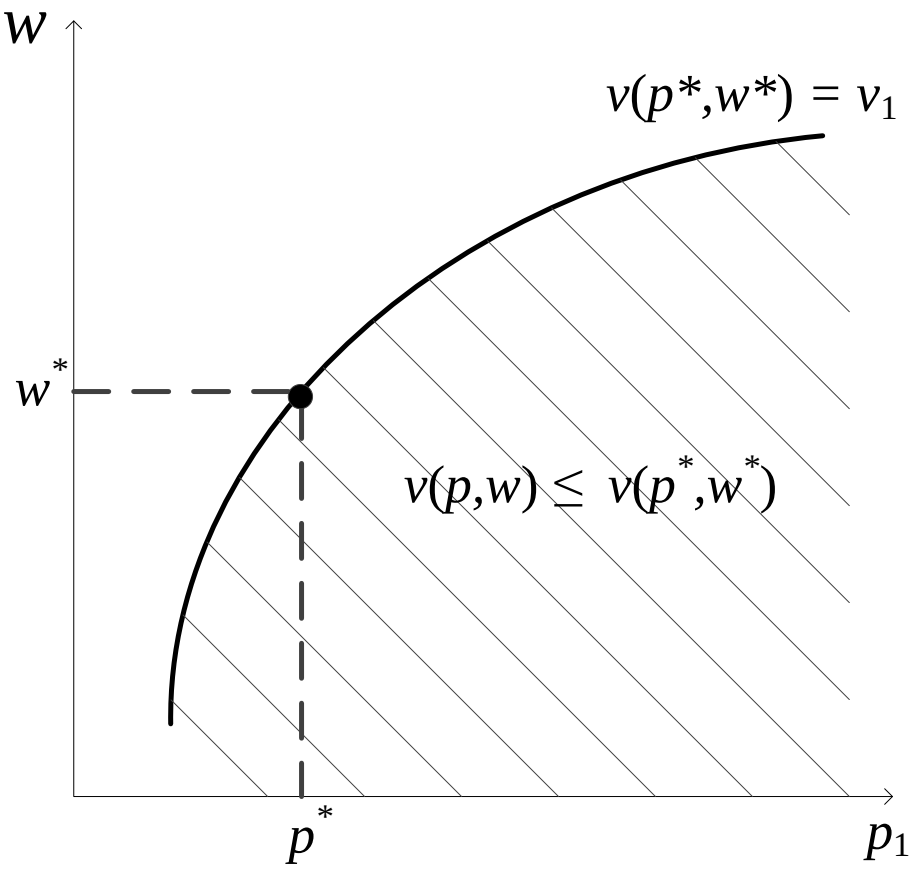
4) Quasiconvex: The set $\{(p, w): v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$.

- ***Interpretation I***: If $(p^1, w^1) \succsim^* (p^2, w^2)$, then $(p^1, w^1) \succsim^* (\lambda p^1 + (1-\lambda)p^2,\ \lambda w^1 + (1-\lambda)w^2)$; i.e., if $A \succsim^* B$, then $A \succsim^* C$.



Properties of Indirect Utility Function

- ***Interpretation II***: $v(p,w)$ is quasiconvex if the set of (p,w) pairs for which $v(p,w) \leq v(p^*,w^*)$ is convex.



Properties of Indirect Utility Function

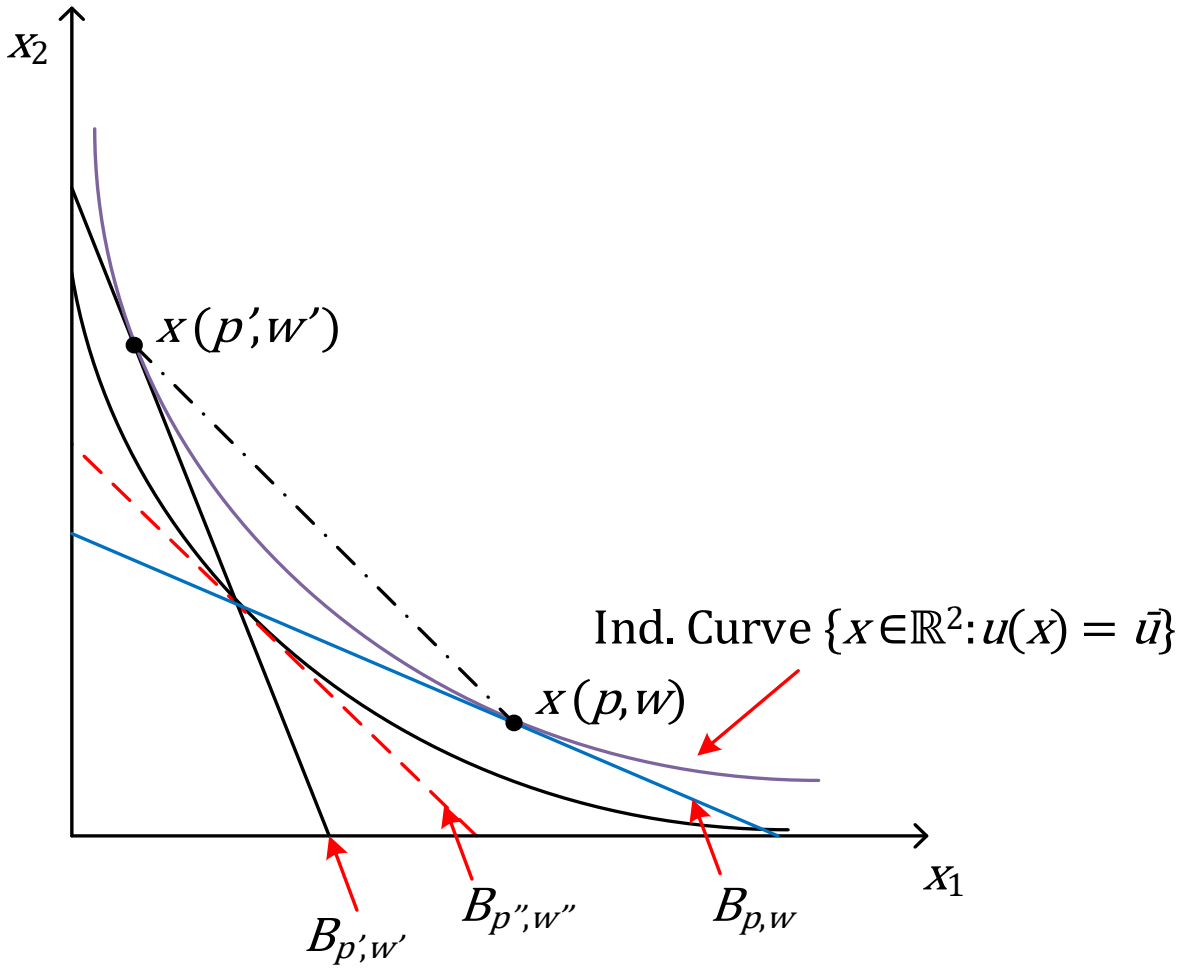
- ***Interpretation III***: Using x_1 and x_2 in the axis, perform following steps:
  1) When $B_{p,w}$, then $x(p,w)$
  2) When $B_{p',w'}$, then $x(p',w')$
  3) Both $x(p,w)$ and $x(p',w')$ induce an indirect utility of $v(p,w\ ) = v(p',w') = \bar{u}$
  4) Construct a linear combination of prices and wealth:

$$\left.\begin{aligned} p'' &= \alpha p + (1-\alpha)p' \\ w'' &= \alpha w + (1-\alpha)w' \end{aligned}\right\} \; B_{p'',w''}$$

  5) Any solution to the UMP given $B_{p'',w''}$ must lie on a lower indifference curve (i.e., lower utility)

$$v(p'',w'') \leq \bar{u}$$

Properties of Indirect Utility Function



WARP and Walrasian Demand

- Relation between Walrasian demand $x(p, w)$ and WARP
  - How does the WARP restrict the set of optimal consumption bundles that the individual decision-maker can select when solving the UMP?

WARP and Walrasian Demand

- Take two different consumption bundles $x(p, w)$ and $x(p', w')$, both being affordable (p, w), i.e.,

$$p \cdot x(p, w) \leq w \text{ and } p \cdot x(p', w') \leq w$$

- When prices and wealth are (p, w), the consumer chooses $x(p, w)$ despite $x(p', w')$ is also affordable.
- Then he "reveals" a preference for $x(p, w)$ over $x(p', w')$ when both are affordable.
- Hence, we should expect him to choose $x(p, w)$ over $x(p', w')$ when both are affordable. (Consistency)
- Therefore, bundle $x(p, w)$ must not be affordable at (p', w') because the consumer chooses $x(p', w')$. That is, $p' \cdot x(p, w) > w'$.

WARP and Walrasian Demand

- In summary, Walrasian demand satisfies WARP, if, for two different consumption bundles,
  $x(p,w) \neq x(p',w')$,
  $$p \cdot x(p',w') \leq w \;\; \Rightarrow \;\; p' \cdot x(p,w) > w'$$

- *Intuition*: if $x(p',w')$ is affordable under budget set $B_{p,w}$, then $x(p,w)$ cannot be affordable under $B_{p',w'}$.

Checking for WARP

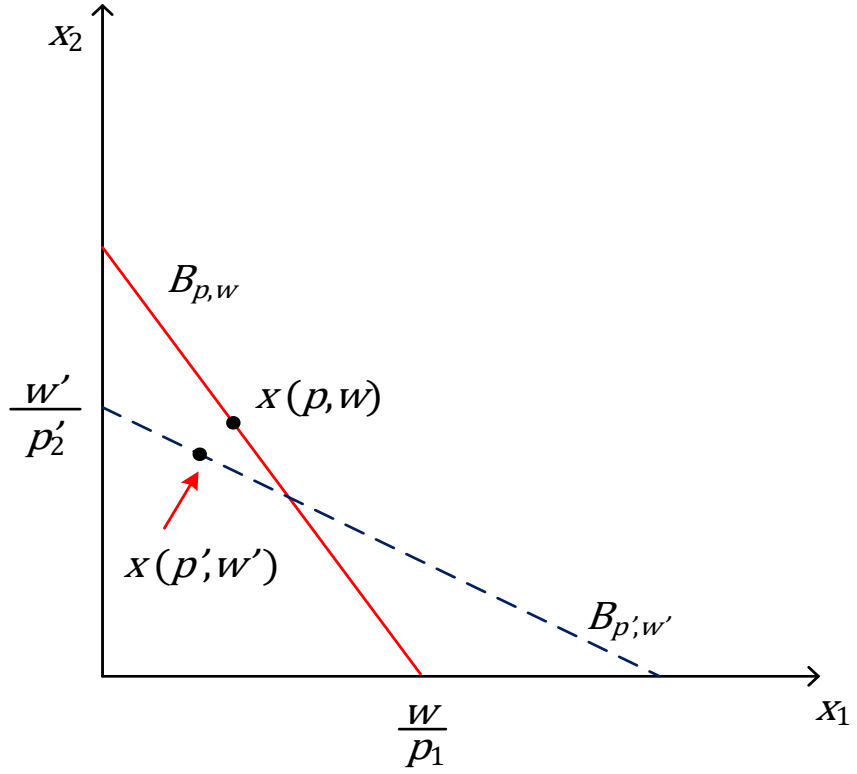
- A systematic procedure to check if Walrasian demand satisfies WARP:
  - ***Step 1:*** Check if bundles $x(p, w)$ and $x(p', w')$ are both affordable under $B_{p,w}$.
    - That is, graphically $x(p, w)$ and $x(p', w')$ have to lie on or below budget line $B_{p,w}$.
    - If step 1 is satisfied, then move to step 2.
    - Otherwise, the premise of WARP does not hold, which does not allow us to continue checking if WARP is violated or not. In this case, we can only say that "*WARP is not violated*".

Checking for WARP

- ***Step 2:*** Check if bundles $x(p, w)$ is affordable under $B_{p',w'}$.
  - That is, graphically $x(p, w)$ must lie on or below budget line $B_{p',w'}$.
  - If step 2 is satisfied, then this Walrasian demand violates WARP.
  - Otherwise, the Walrasian demand satisfies WARP.

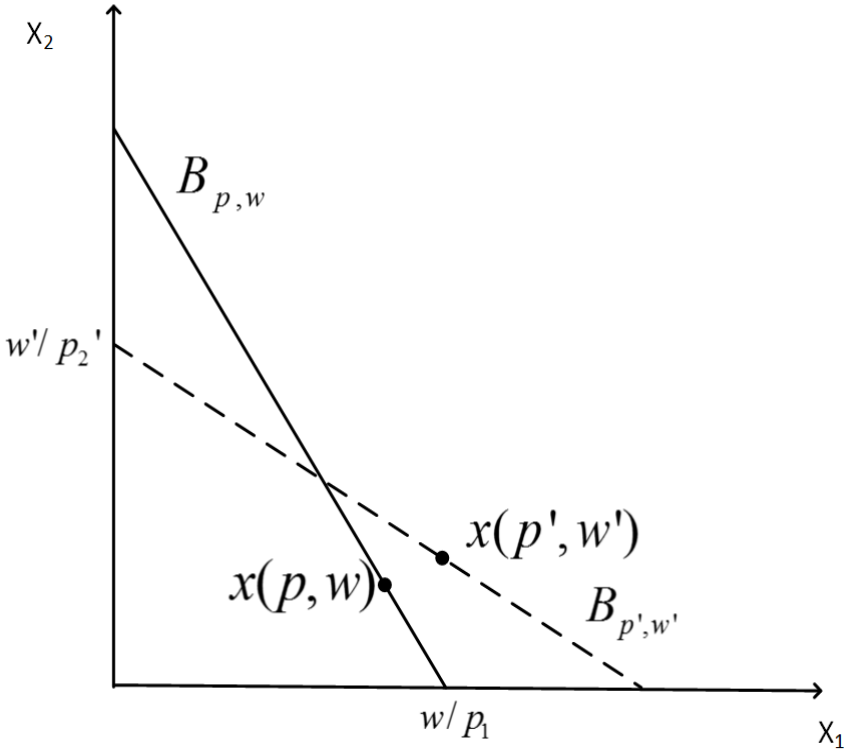
Checking for WARP: Example 1

- First, $x(p, w)$ and $x(p', w')$ are both affordable under $B_{p,w}$.
- Second, $x(p, w)$ is not affordable under $B_{p',w'}$.
- Hence, WARP is *satisfied*!



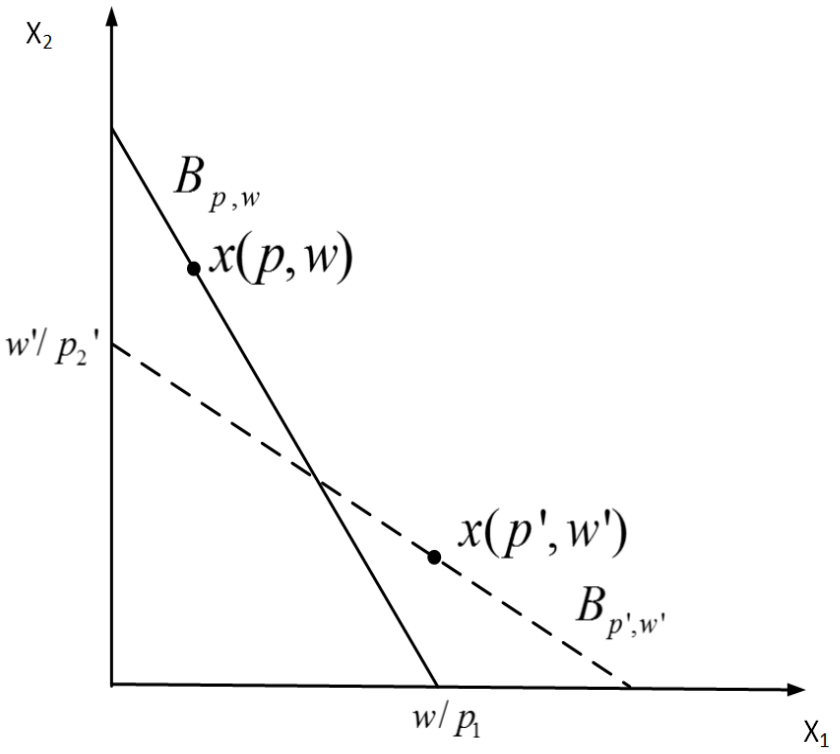
Checking for WARP: Example 2

- The demand $x(p', w')$ under final prices and wealth is not affordable under initial prices and wealth, i.e., $p \cdot x(p', w') > w$.
  - The premise of WARP does not hold.
  - Violation of Step 1!
  - WARP is *not violated*.



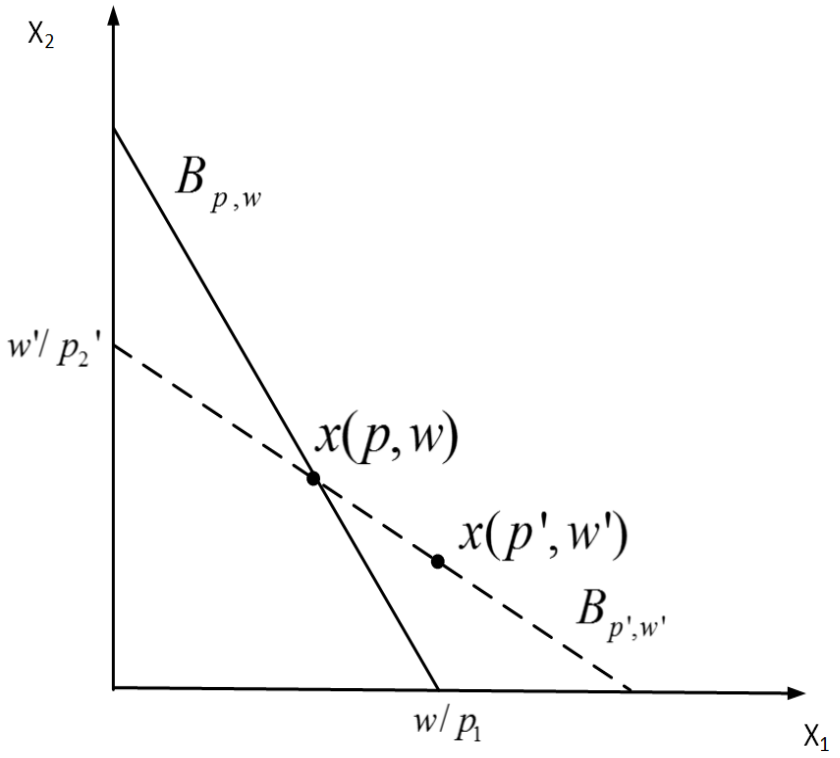
Checking for WARP: Example 3

- The demand $x(p', w')$ under final prices and wealth is not affordable under initial prices and wealth, i.e., $p \cdot x(p', w') > w$.
  - The premise of WARP does not hold.
  - Violation of Step 1!
  - WARP is *not violated*.



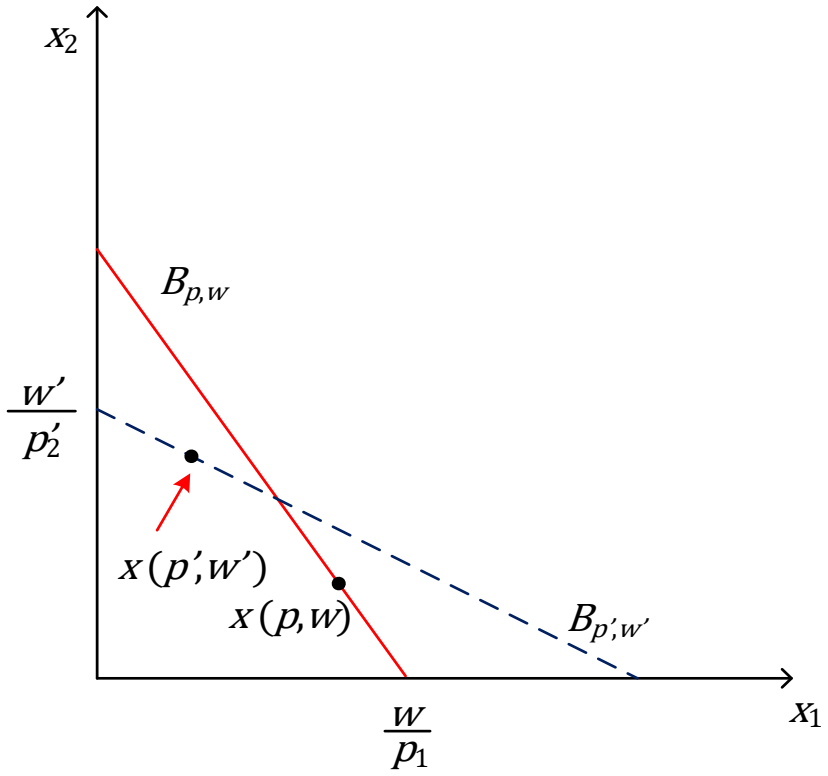
Checking for WARP: Example 4

- The demand $x(p', w')$ under final prices and wealth is not affordable under initial prices and wealth, i.e., $p \cdot x(p', w') > w$.
  - The premise of WARP does not hold.
  - Violation of Step 1!
  - WARP is *not violated*.



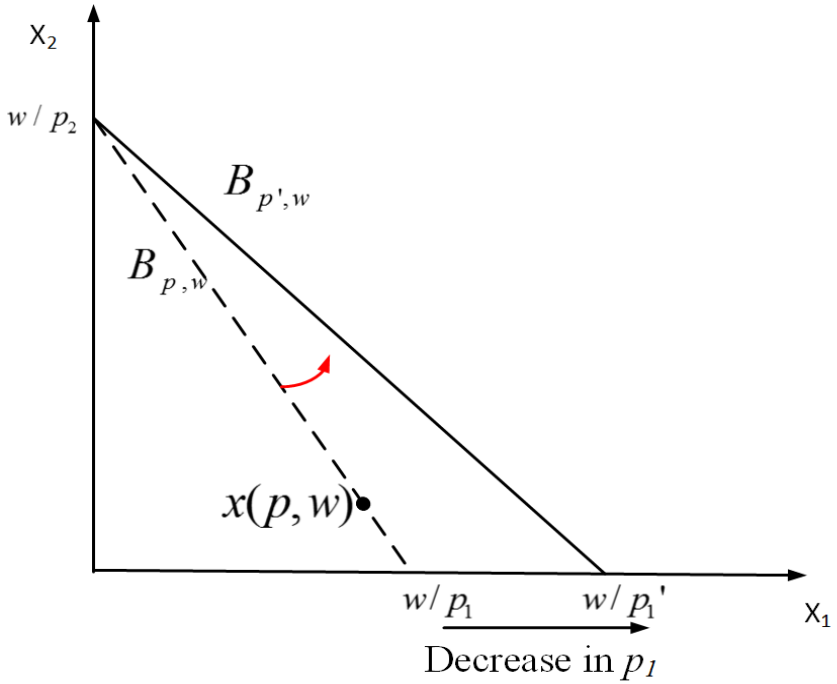
Checking for WARP: Example 5

- First, $x(p, w)$ and $x(p', w')$ are both affordable under $B_{p,w}$.
- Second, $x(p, w)$ is affordable under $B_{p',w'}$, i.e., $p' \cdot x(p, w) < w'$
- Hence, WARP is *NOT satisfied*!



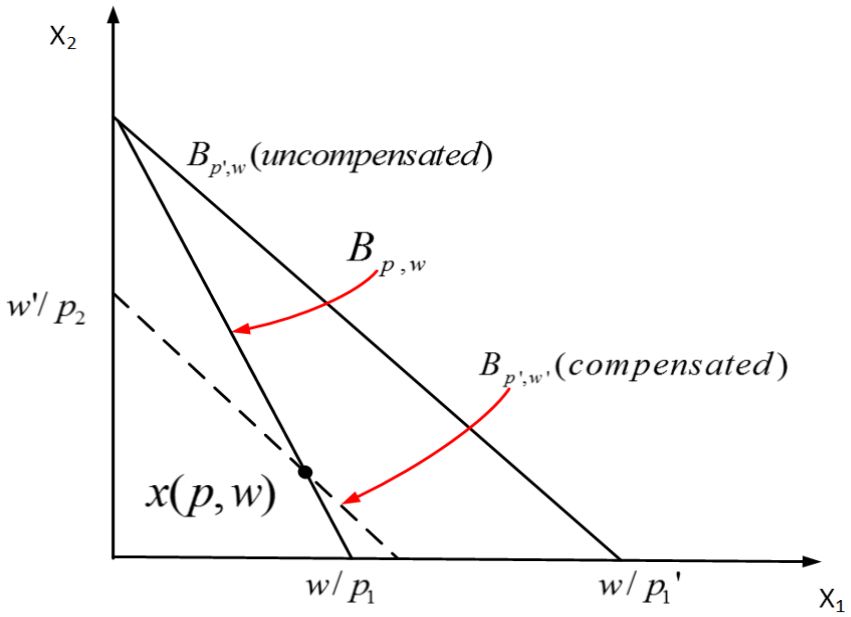
Implications of WARP

- How do price changes affect the WARP predictions?
- Assume a reduction in p_1
  - the consumer's budget line rotates (***uncompensated price change***)



Implications of WARP

- Adjust the consumer's wealth level so that he can consume his initial demand $x(p, w)$ at the new prices.
  - shift the final budget line inwards until the point at which we reach the initial consumption bundle $x(p, w)$ (***compensated price change***)



Implications of WARP

- What is the wealth adjustment?

$$w = p \cdot x(p, w) \text{ under } (p, w)$$
$$w' = p' \cdot x(p, w) \text{ under } (p', w')$$

- Then,

$$\Delta w = \Delta p \cdot x(p, w)$$

where $\Delta w = w' - w$ and $\Delta p = p' - p$.

- This is the ***Slutsky wealth compensation:***
  - the increase (decrease) in wealth, measured by Δw, that we must provide to the consumer so that he can afford the same consumption bundle as before the price increase (decrease), $x(p, w)$.

Implications of WARP: Law of Demand

- Suppose that the Walrasian demand $x(p, w)$ satisfies $\text{homog}(0)$ and Walras' Law. Then, $x(p, w)$ satisfies WARP iff:

$$\Delta p \cdot \Delta x \leq 0$$

where

  - $\Delta p = p' - p$ and $\Delta x = x(p', w') - x(p, w)$
  - w' is the wealth level that allows the consumer to buy the initial demand at the new prices, $w' = p' \cdot x(p, w)$

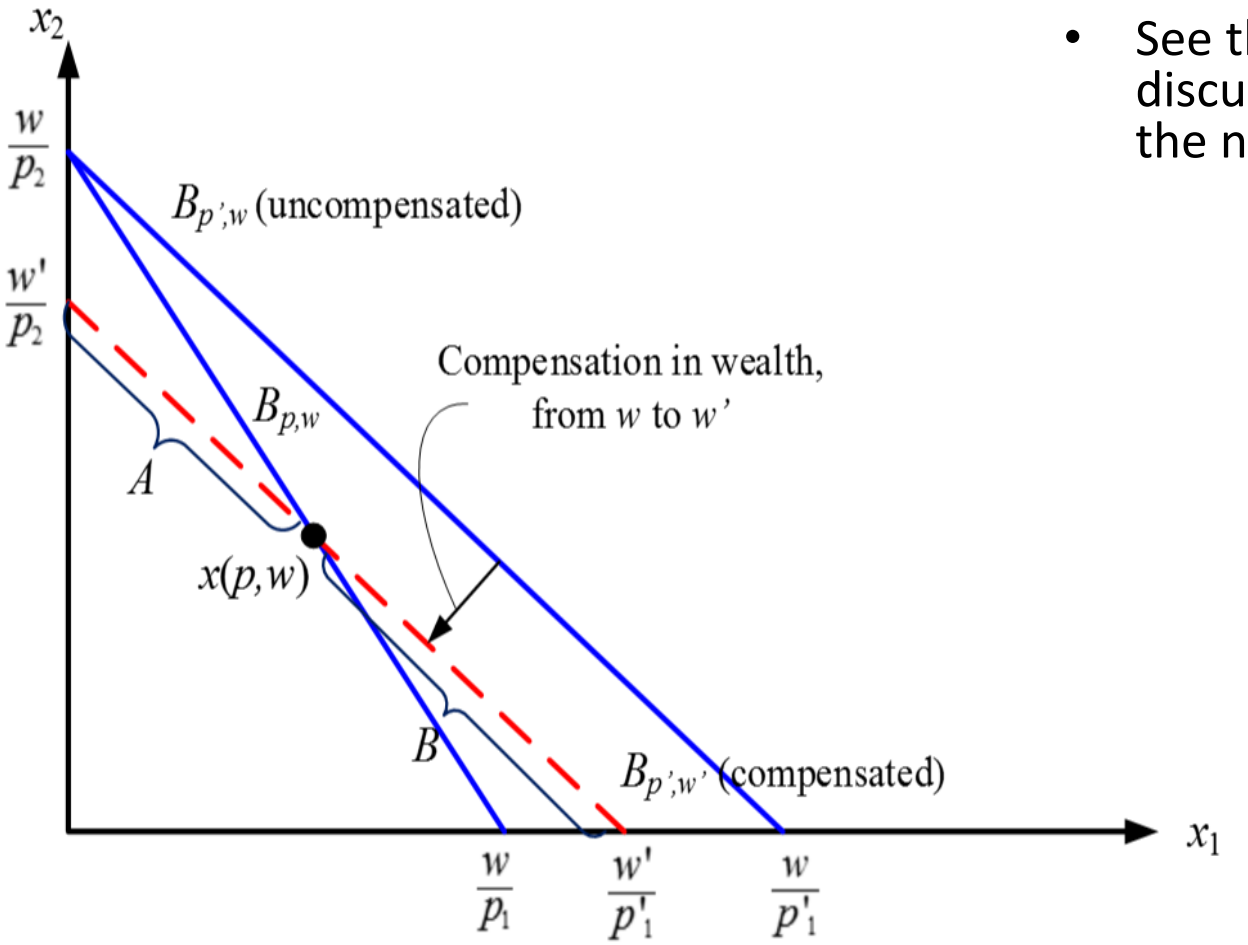
- This is the ***Law of Demand***: quantity demanded and price move in opposite directions.

Implications of WARP: Law of Demand

- Does WARP restrict behavior when we apply Slutsky wealth compensations?
  - Yes!

- What if we were not applying the Slutsky wealth compensation, would WARP impose any restriction on allowable locations for $x(p', w')$?
  - No!

Implications of WARP: Law of Demand



- See the discussions on the next slide

Implications of WARP: Law of Demand

- Can $x(p', w')$ lie on segment *A*?
  1) $x(p, w)$ and $x(p', w')$ are both affordable under $B_{p,w}$.
  2) $x(p, w)$ is affordable under $B_{p',w'}$.
     ⇒ WARP is violated if $x(p', w')$ lies on segment *A*

- Can $x(p', w')$ lie on segment *B*?
  1) $x(p, w)$ is affordable under $B_{p,w}$, but $x(p', w')$ is not.
     ⇒ The premise of WARP does not hold
     ⇒ WARP is not violated if $x(p', w')$ lies on segment *B*

Implications of WARP: Law of Demand

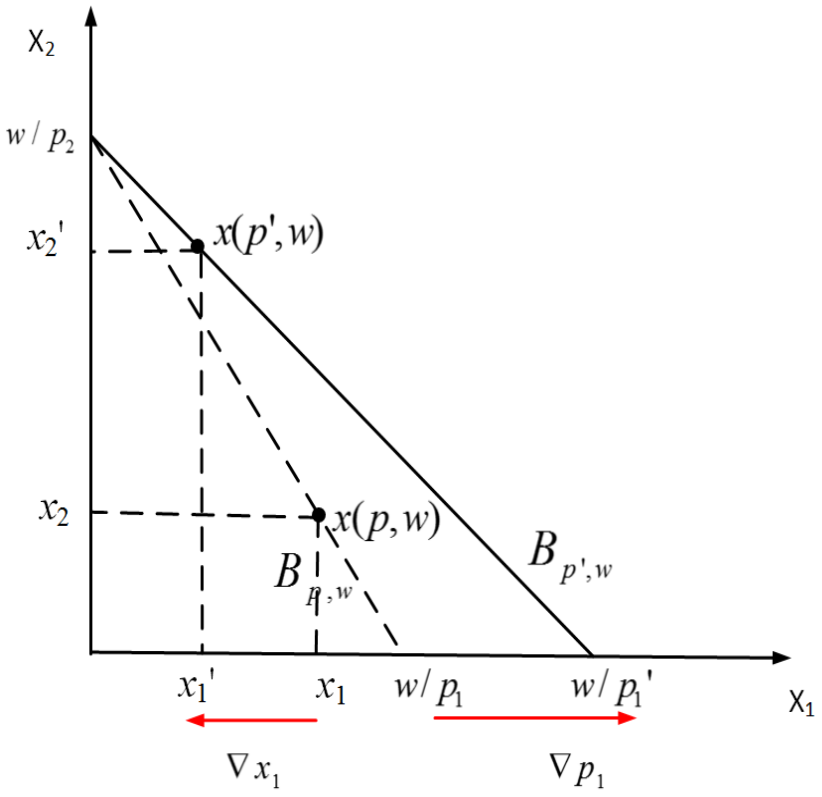
- What did we learn from this figure?
  1) We started from ∇p_1, and compensated the wealth of this individual (reducing it from w to w') so that he could afford his initial bundle $x(p, w)$ under the new prices.
     - From this wealth compensation, we obtained budget line $B_{p',w'}$.
  2) From WARP, we know that $x(p', w')$ must contain more of good 1.
     - That is, graphically, segment *B* lies to the right-hand side of $x(p, w)$.
  3) Then, a price reduction, ∇p_1, when followed by an appropriate wealth compensation, leads to an increase in the quantity demanded of this good, Δx_1.
     - This is the ***compensated law of demand*** (CLD).

Implications of WARP: Law of Demand

- *Practice problem*:
  - Can you repeat this analysis but for an increase in the price of good 1?
    - First, pivot budget line $B_{p,w}$ inwards to obtain $B_{p',w}$.
    - Then, increase the wealth level (wealth compensation) to obtain $B_{p',w'}$.
    - Identify the two segments in budget line $B_{p',w'}$, one to the right-hand side of $x(p,w)$ and other to the left.
    - In which segment of budget line $B_{p',w'}$ can the Walrasian demand $x(p',w')$ lie?

Implications of WARP: Law of Demand

- Is WARP satisfied under the ***uncompensated law of demand*** (ULD)?
  - $x(p, w)$ is affordable under budget line $B_{p,w}$, but $x(p', w)$ is not.
    - Hence, the premise of WARP is not satisfied. As a result, WARP is not violated.
  - But, is this result implying something about whether ULD must hold?
    - No!
    - Although WARP is not violated, ULD is: a decrease in the price of good 1 yields a decrease in the quantity demanded.



Implications of WARP: Law of Demand

- Distinction between the uncompensated and the compensated law of demand:
  - quantity demanded and price can move in the same direction, when wealth is left uncompensated, i.e.,

$$\Delta p_1 \cdot \Delta x_1 > 0$$

$$\text{as } \Delta p_1 \Rightarrow \Delta x_1$$

- Hence, WARP is not sufficient to yield law of demand for price changes that are uncompensated, i.e.,

$$\text{WARP} \not\Leftrightarrow \text{ULD, but}$$

$$\text{WARP} \Leftrightarrow \text{CLD}$$

Implications of WARP: Slutsky Matrix

- Let us focus now on the case in which $x(p, w)$ is differentiable.
- First, note that the law of demand is $dp \cdot dx \leq 0$ (equivalent to $\Delta p \cdot \Delta x \leq 0$).
- Totally differentiating $x(p, w)$,

$$dx = D_p x(p, w) dp + D_w x(p, w) dw$$

- And since the consumer's wealth is compensated, $dw = x(p, w) \cdot dp$ (this is the differential analog of $\Delta w = \Delta p \cdot x(p, w)$).
  - Recall that $\Delta w = \Delta p \cdot x(p, w)$ was obtained from the Slutsky wealth compensation.

Implications of WARP: Slutsky Matrix

- Substituting,

$$dx = D_p x(p,w)dp + D_w x(p,w)\underbrace{[x(p,w)\cdot dp]}_{dw}$$

or equivalently,

$$dx = \left[D_p x(p,w) + D_w x(p,w)x(p,w)^T\right]dp$$

Implications of WARP: Slutsky Matrix

- Hence the law of demand, $dp \cdot dx \leq 0$, can be expressed as

$$dp \cdot \underbrace{\left[D_p x(p,w) + D_w x(p,w) x(p,w)^T\right] dp}_{dx} \leq 0$$

where the term in brackets is the ***Slutsky (or substitution) matrix***

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix}$$

where each element in the matrix is

$$s_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

Implications of WARP: Slutsky Matrix

- ***Proposition:*** If $x(p, w)$ is differentiable, satisfies Walras' law, $\text{homog}(0)$, and WARP, then $S(p, w)$ is negative semi-definite,

$$v \cdot S(p, w)v \leq 0 \text{ for any } v \in \mathbb{R}^L$$

- Implications:
  - $s_{ll}(p, w)$: substitution effect of good l with respect to its own price is non-positive (own-price effect)
  - Negative semi-definiteness does not imply that $S(p, w)$ is symmetric (except when $L = 2$).
    - Usual confusion: "then $S(p, w)$ is not symmetric", NO!

Implications of WARP: Slutsky Matrix

- ***Proposition:*** If preferences satisfy LNS and strict convexity, and they are represented with a continuous utility function, then the Walrasian demand $x(p, w)$ generates a Slutsky matrix, $S(p, w)$, which is symmetric.

- The above assumptions are really common.
  - Hence, the Slutsky matrix will then be symmetric.

- However, the above assumptions are not satisfied in the case of preferences over perfect substitutes (i.e., preferences are convex, but not strictly convex).

Implications of WARP: Slutsky Matrix

- Non-positive substitution effect, $s_{ll} \leq 0$:

$$\underbrace{s_{ll}(p,w)}_{\text{substitution effect } (-)} = \underbrace{\frac{\partial x_l(p,w)}{\partial p_l}}_{\substack{\text{Total effect:} \\ (-)\text{ usual good} \\ (+)\text{ Giffen good}}} + \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_l(p,w)}_{\substack{\text{Income effect:} \\ (+)\text{ normal good} \\ (-)\text{ inferior good}}}$$

- Substitution Effect = Total Effect + Income Effect

  $\Rightarrow$ Total Effect = Substitution Effect - Income Effect

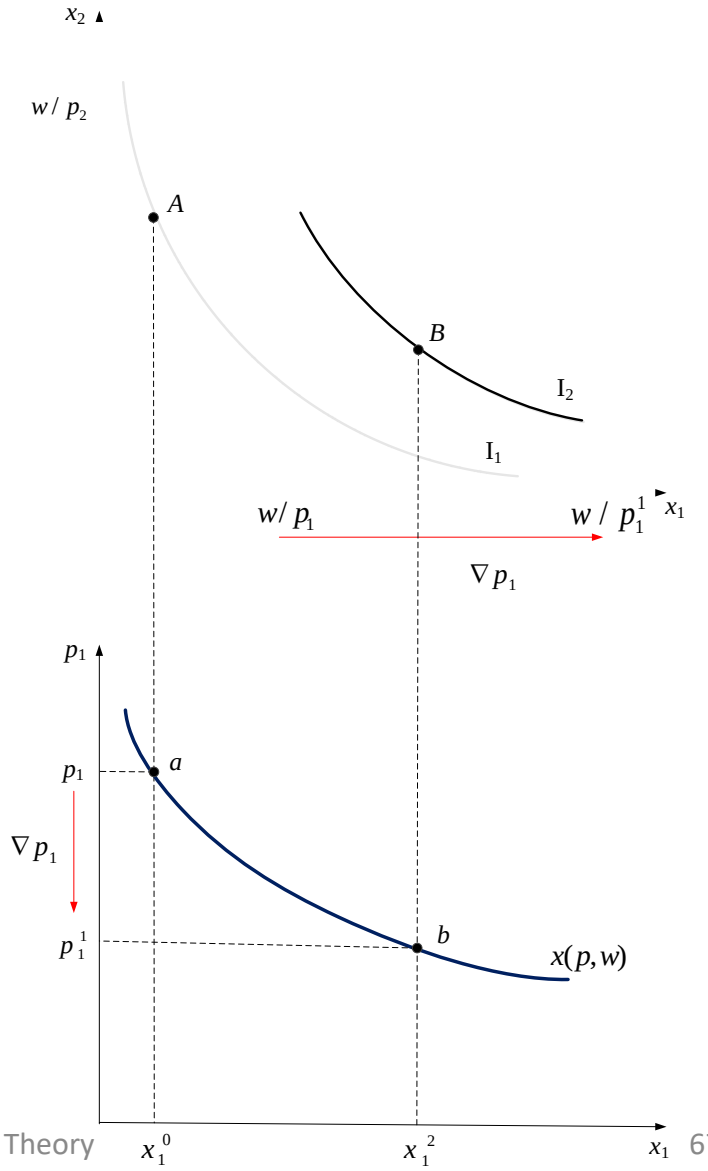
Implications of WARP: Slutsky Equation

$$\underbrace{s_{ll}(p,w)}_{\text{substitution effect } (-)} = \underbrace{\frac{\partial x_l(p,w)}{\partial p_l}}_{\text{Total effect}} + \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_l(p,w)}_{\text{Income effect}}$$

- ***Total Effect:*** measures how the quantity demanded is affected by a change in the price of good l, when we leave the wealth uncompensated.
- ***Income Effect:*** measures the change in the quantity demanded as a result of the wealth adjustment.
- ***Substitution Effect***: measures how the quantity demanded is affected by a change in the price of good l, after the wealth adjustment.
  - That is, the substitution effect only captures the change in demand due to variation in the price ratio, but abstracts from the larger (smaller) purchasing power that the consumer experiences after a decrease (increase, respectively) in prices.

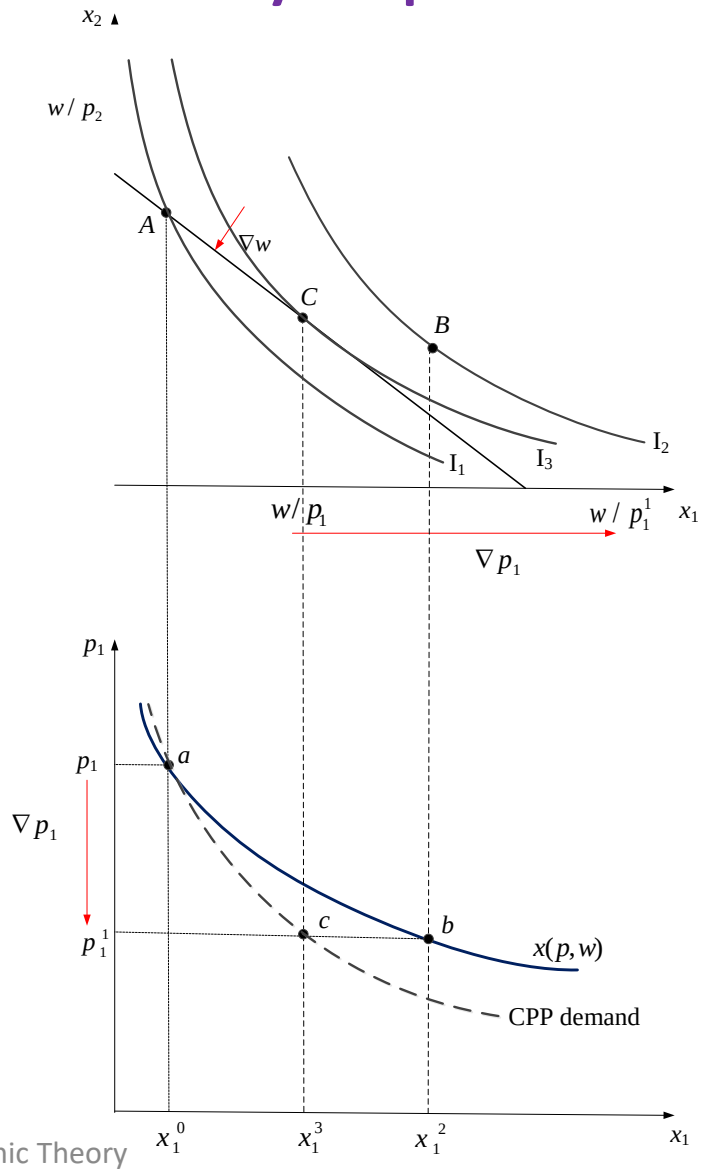
Implications of WARP: Slutsky Equation

- Reduction in the price of x_1.
  - It enlarges consumer's set of feasible bundles.
  - He can reach an indifference curve further away from the origin.
- The Walrasian demand curve indicates that a decrease in the price of x_1 leads to an increase in the quantity demanded.
  - This induces a negatively sloped Walrasian demand curve (so the good is "usual").
- The increase in the quantity demanded of x_1 as a result of a decrease in its price represents the ***total effect (TE)***.



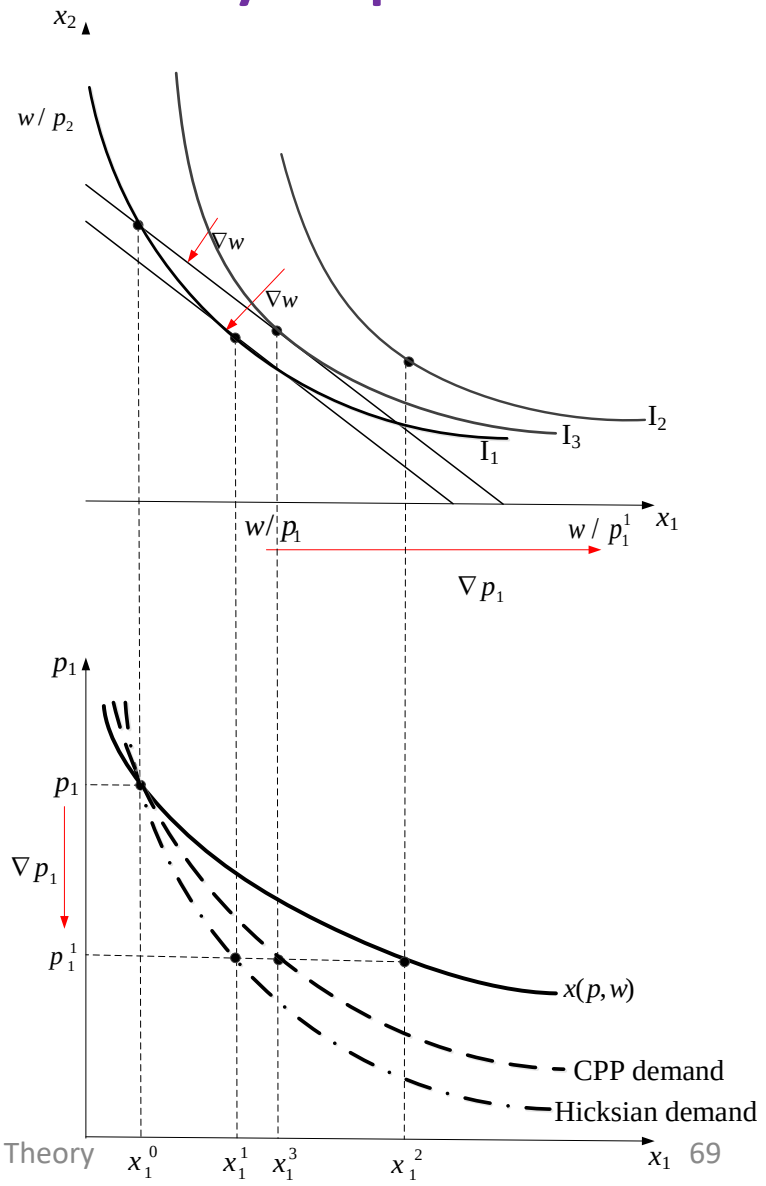
Implications of WARP: Slutsky Equation

- Reduction in the price of x_1.
  - Disentangle the total effect into the substitution and income effects
  - ***Slutsky wealth compensation?***
- Reduce the consumer's wealth so that he can afford the same consumption bundle as the one before the price change (i.e., *A*).
  - Shift the budget line after the price change inwards until it "crosses" through the initial bundle *A*.
  - "Constant purchasing power" demand curve (CPP curve) results from applying the Slutsky wealth compensation.
  - The quantity demanded for x_1 increases from x_1^0 to x_1^3.
- When we do not hold the consumer's purchasing power constant, we observe relatively large increase in the quantity demanded for x_1 (i.e., from x_1^0 to x_1^2).



Implications of WARP: Slutsky Equation

- Reduction in the price of x_1.
  - ***Hicksian wealth compensation*** (i.e., "constant utility" demand curve)?
- The consumer's wealth level is adjusted so that he can still reach his initial utility level (i.e., the same indifference curve I_1 as before the price change).
  - A more significant wealth reduction than when we apply the Slutsky wealth compensation.
  - The Hicksian demand curve reflects that, for a given decrease in p_1, the consumer slightly increases his consumption of good one.
- In summary, a given decrease in p_1 produces:
  - A small increase in the Hicksian demand for the good, i.e., from x_1^0 to x_1^1.
  - A larger increase in the CPP demand for the good, i.e., from x_1^0 to x_1^3.
  - A substantial increase in the Walrasian demand for the product, i.e., from x_1^0 to x_1^2.

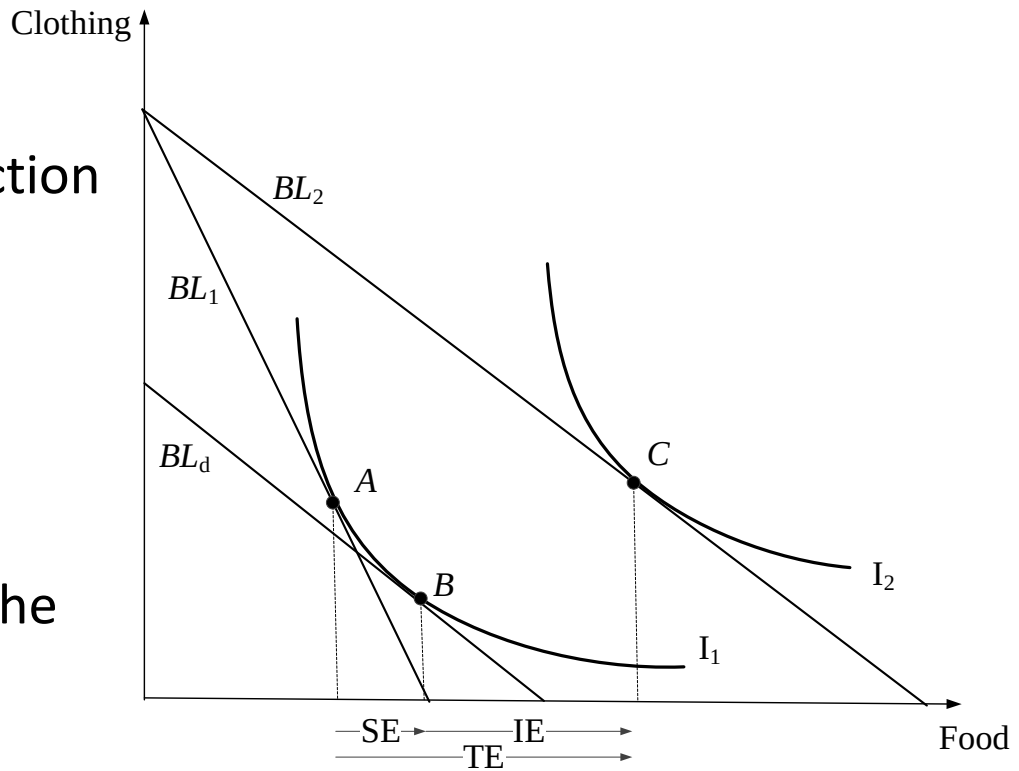


Implications of WARP: Slutsky Equation

- A decrease in price of x_1 leads the consumer to increase his consumption of this good, Δx_1, but:
  - The Δx_1 which is solely due to the price effect (either measured by the Hicksian demand curve or the CPP demand curve) is smaller than the Δx_1 measured by the Walrasian demand, $x(p, w)$, which also captures wealth effects.
  - The wealth compensation (a reduction in the consumer's wealth in this case) that maintains the original utility level (as required by the Hicksian demand) is larger than the wealth compensation that maintains his purchasing power unaltered (as required by the Slutsky wealth compensation, in the CPP curve).

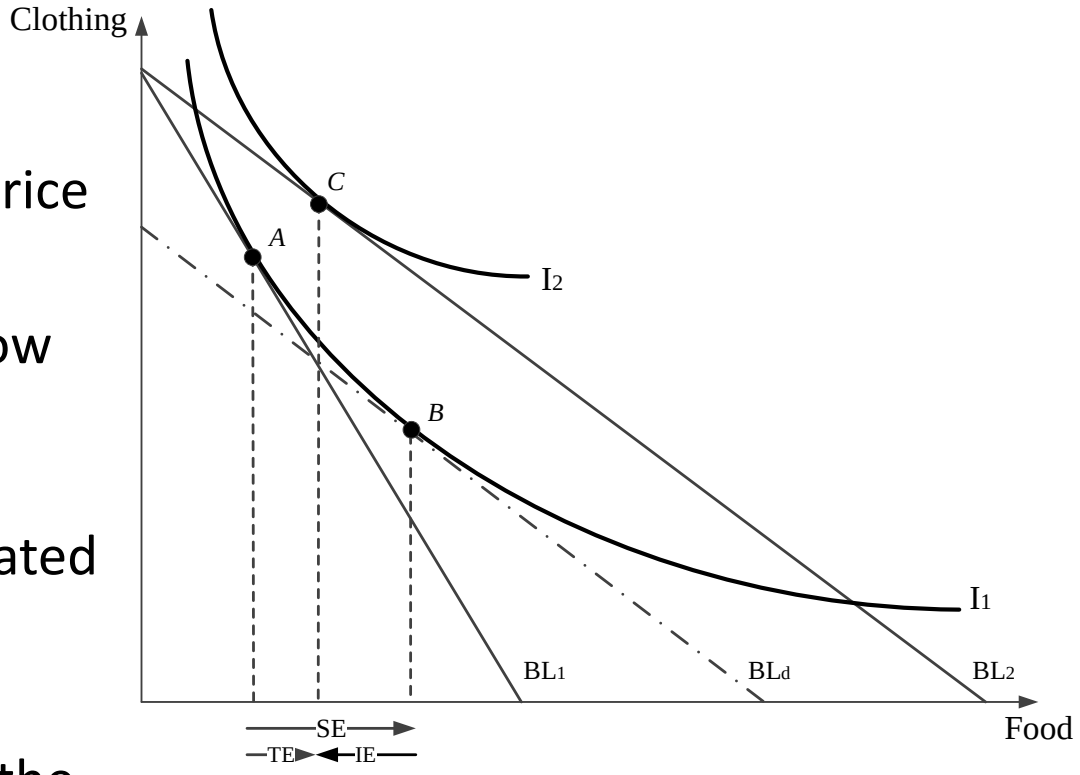
Substitution and Income Effects: Normal Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The substitution effect (SE) moves in the opposite direction as the price change.
  - A reduction in the price of food implies a positive substitution effect.
- The income effect (IE) is positive (thus it reinforces the SE).
  - The good is normal.



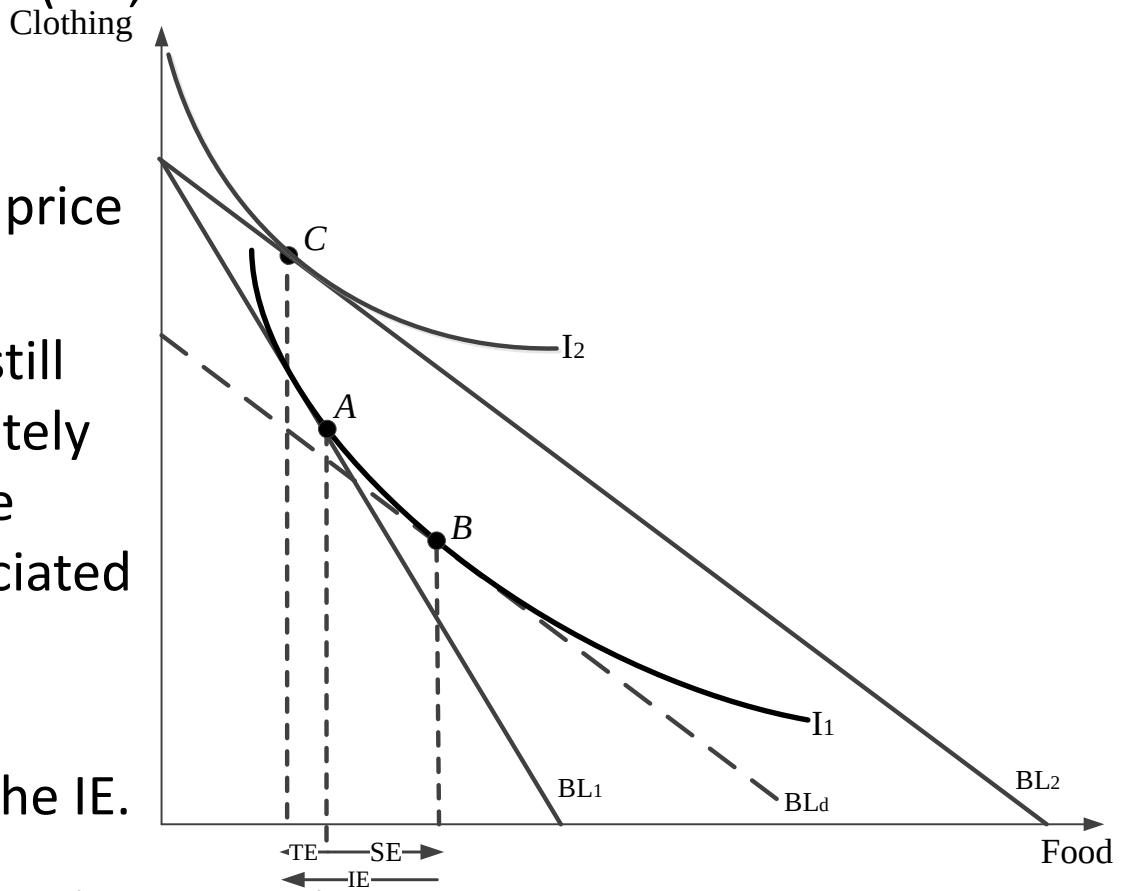
Substitution and Income Effects: Inferior Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The SE still moves in the opposite direction as the price change.
- The income effect (IE) is now negative (which partially offsets the increase in the quantity demanded associated with the SE).
  - The good is inferior.
- *Note*: the SE is larger than the IE.



Substitution and Income Effects: Giffen Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The SE still moves in the opposite direction as the price change.
- The income effect (IE) is still negative but now completely offsets the increase in the quantity demanded associated with the SE.
  - The good is Giffen good.
- *Note*: the SE is less than the IE.



Substitution and Income Effects

	SE	IE	TE
Normal Good	+	+	+
Inferior Good	+	-	+
Giffen Good	+	-	-

- Not Giffen: Demand curve is negatively sloped (as usual)
- Giffen: Demand curve is positively sloped

Substitution and Income Effects

- ***Summary:***
  1) SE is negative (since $\downarrow p_1 \Rightarrow \uparrow x_1$)
     - SE < 0 does not imply $\downarrow x_1$
  2) If good is inferior, IE < 0. Then,

$$\text{TE} = \underbrace{\text{SE}}_{-} - \underbrace{\text{IE}}_{+} \quad \Rightarrow \quad \text{if } |\text{IE}| \begin{Bmatrix} > \\ < \end{Bmatrix} |\text{SE}|, \text{ then } \begin{Bmatrix} \text{TE}(-) \\ \text{TE}(+) \end{Bmatrix}$$

For a price decrease, this implies

$$\begin{Bmatrix} \text{TE}(-) \\ \text{TE}(+) \end{Bmatrix} \Rightarrow \begin{Bmatrix} \downarrow x_1 \\ \uparrow x_1 \end{Bmatrix} \quad \begin{matrix} \text{Giffen good} \\ \text{Non-Giffen good} \end{matrix}$$

  3) Hence,
     a) A good can be inferior, but not necessarily be Giffen
     b) But all Giffen goods must be inferior.

Expenditure Minimization Problem

Expenditure Minimization Problem

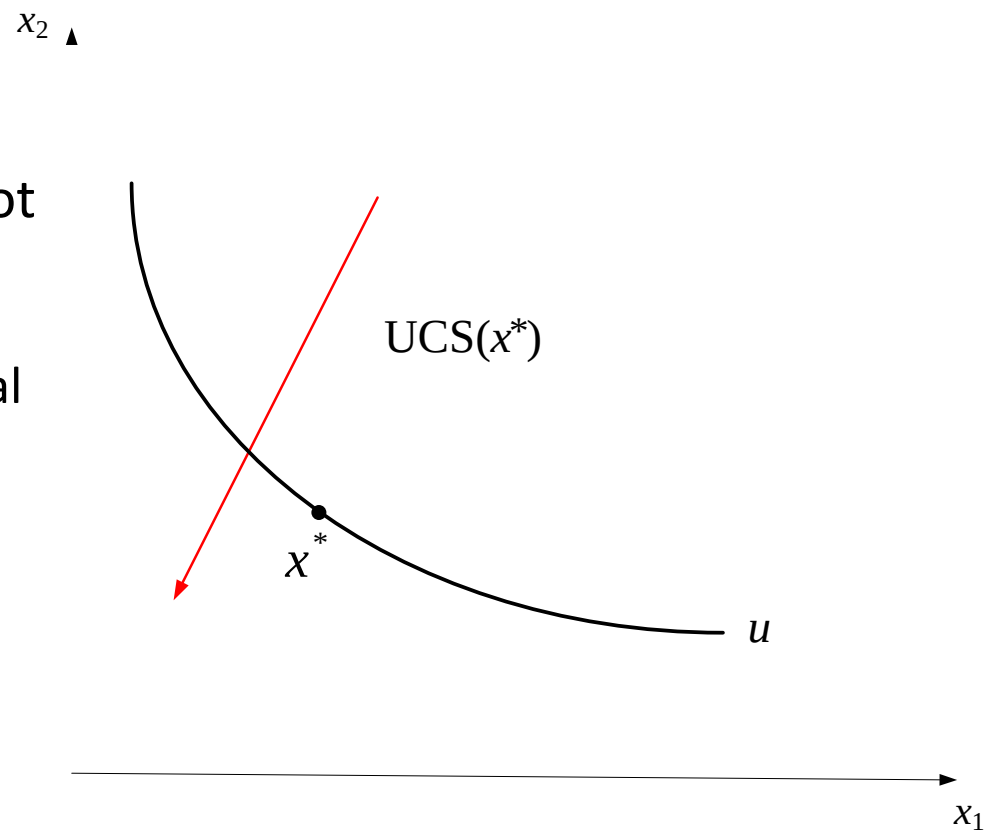
- Expenditure minimization problem (EMP):

$$\min_{x \geq 0} p \cdot x$$
$$\text{s.t. } u(x) \geq u$$

- Alternative to utility maximization problem

Expenditure Minimization Problem

- Consumer seeks a utility level associated with a particular indifference curve, while spending as little as possible.
- Bundles strictly above x^* cannot be a solution to the EMP:
  - They reach the utility level u
  - But, they do not minimize total expenditure
- Bundles on the budget line strictly below x^* cannot be the solution to the EMP problem:
  - They are cheaper than x^*
  - But, they do not reach the utility level u



Expenditure Minimization Problem

- Lagrangian

$$L = p \cdot x + \mu[u - u(x)]$$

- FOCs (necessary conditions)

$$\frac{\partial L}{\partial x_k} = p_k - \mu \frac{\partial u(x^*)}{\partial x_k} \geq 0$$

[= 0 for interior solutions]

$$\frac{\partial L}{\partial \mu} = u - u(x^*) = 0$$

Expenditure Minimization Problem

- For interior solutions,

$$p_k = \mu \frac{\partial u(x^*)}{\partial x_k} \quad \text{or} \quad \frac{1}{\mu} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$$

for any good k. This implies,

$$\frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k} = \frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} \quad \text{or} \quad \frac{p_k}{p_l} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{\frac{\partial u(x^*)}{\partial x_l}}$$

- The consumer allocates his consumption across goods until the point in which the marginal utility per dollar spent on each good is equal across all goods (i.e., same "bang for the buck").
- That is, the slope of indifference curve is equal to the slope of the budget line.

EMP: Hicksian Demand

- The bundle $x^* \in \operatorname{argmin} p \cdot x$ (the argument that solves the EMP) is the ***Hicksian demand***, which depends on p and u,

$$x^* \in h(p, u)$$

- Recall that if such bundle x^* is unique, we denote it as $x^* = h(p, u)$.

Properties of Hicksian Demand

- Suppose that $u(\cdot)$ is a continuous function, satisfying LNS defined on $X = \mathbb{R}^L_+$. Then for $p \gg 0$, $h(p, u)$ satisfies:
  1) ***Homog(0)*** in p, i.e., $h(p, u) = h(\alpha p, u)$ for any p, u, and $\alpha > 0$.
     - If $x^* \in h(p, u)$ is a solution to the problem

       $$\min_{x \geq 0} p \cdot x$$

       then it is also a solution to the problem

       $$\min_{x \geq 0} \alpha p \cdot x$$

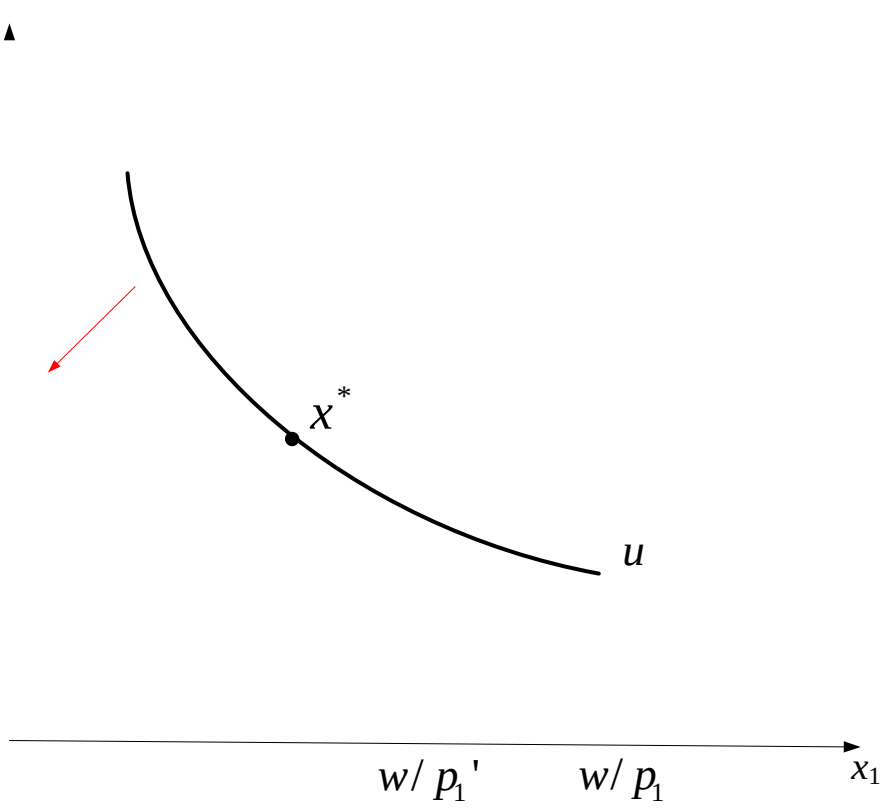
     - *Intuition*: a common change in all prices does not alter the slope of the consumer's budget line.

Properties of Hicksian Demand

- x^* is a solution to the EMP when the price vector is $p = (p_1, p_2)$.
- Increase all prices by factor α

  $$p' = (p_1', p_2') = (\alpha p_1, \alpha p_2)$$
- Downward (parallel) shift in the budget line, i.e., the slope of the budget line is unchanged.
- But I have to reach utility level u to satisfy the constraint of the EMP!
- Spend more to buy bundle $x^*(x_1^*, x_2^*)$, i.e.,

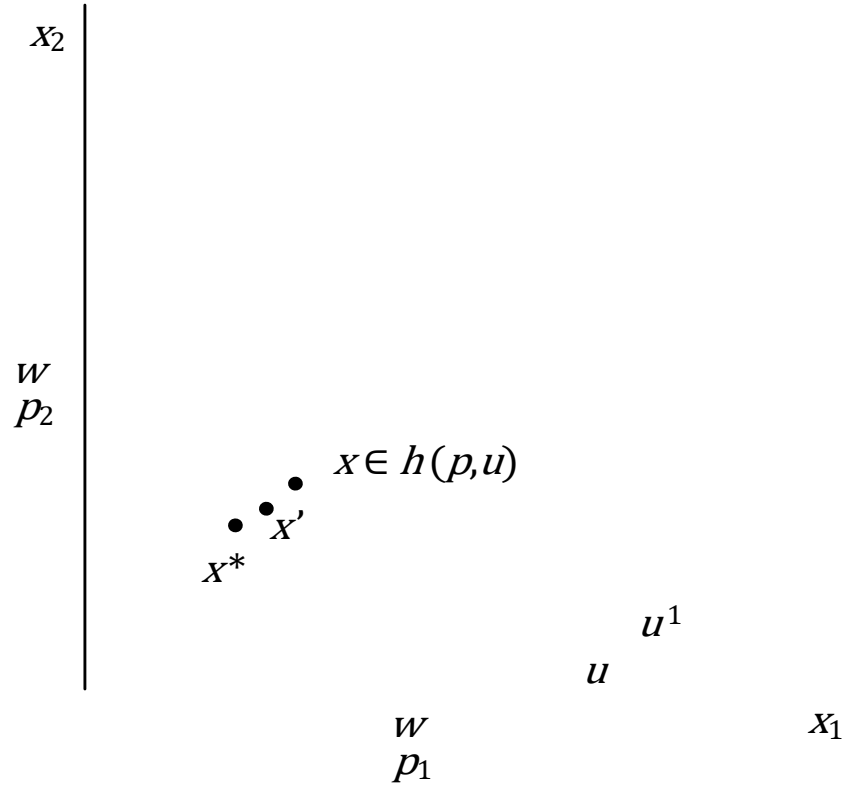
  $$p_1' x_1^* + p_2' x_2^* > p_1 x_1^* + p_2 x_2^*$$
- Hence, $h(p, u) = h(\alpha p, u)$



Properties of Hicksian Demand

2) No excess utility:

for any optimal consumption bundle $x \in h(p, u)$, utility level satisfies $u(x) = u$.



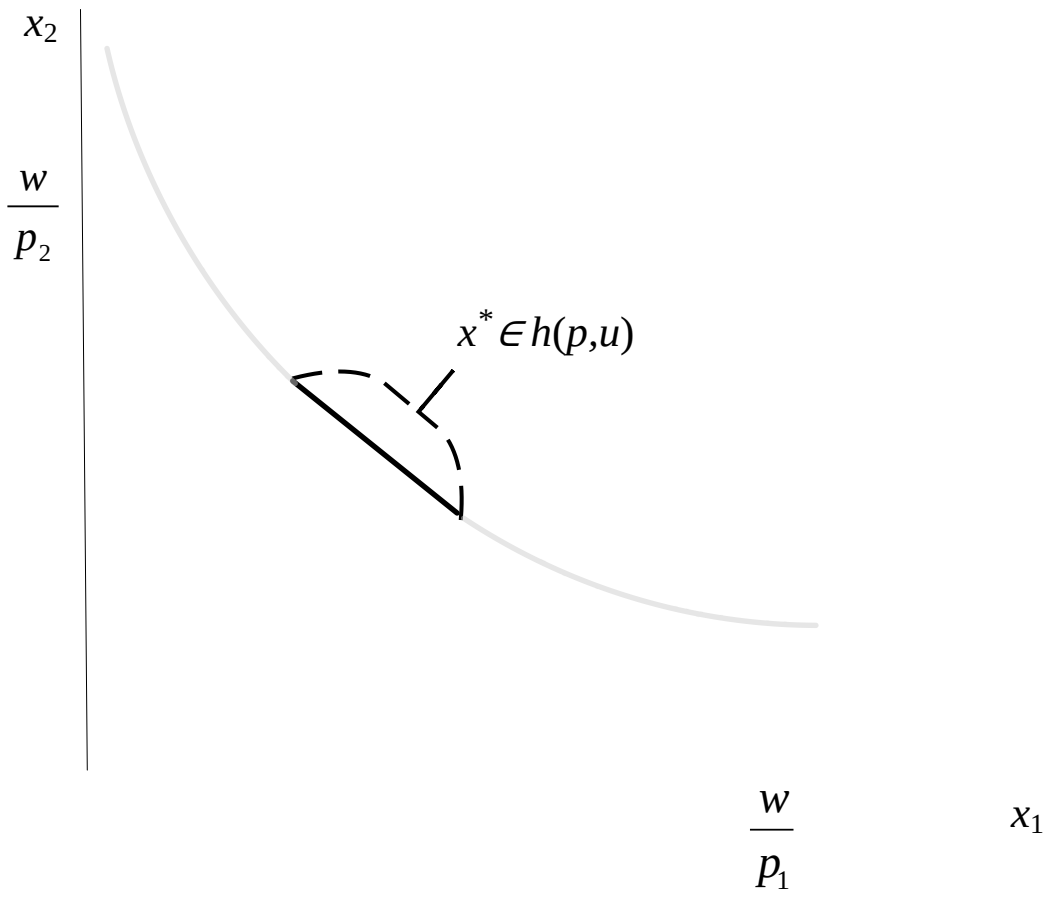
Properties of Hicksian Demand

- *Intuition*: Suppose there exists a bundle $x \in h(p,u)$ for which the consumer obtains a utility level $u(x) = u^1 > u$, which is higher than the utility level u he must reach when solving EMP.
- But we can then find another bundle $x' = x\alpha$, where $\alpha \in (0,1)$, very close to x ($\alpha \to 1$), for which $u(x') > u$.
- Bundle x':
  - is cheaper than x since it contains fewer units of all goods; and
  - exceeds the minimal utility level u that the consumer must reach in his EMP.
- We can repeat that argument until reaching bundle x^*.
- In summary, for a given utility level u that you seek to reach in the EMP, bundle $h(p,u)$ does not exceed u. Otherwise you can find a cheaper bundle that exactly reaches u.

Properties of Hicksian Demand

3) Convexity:

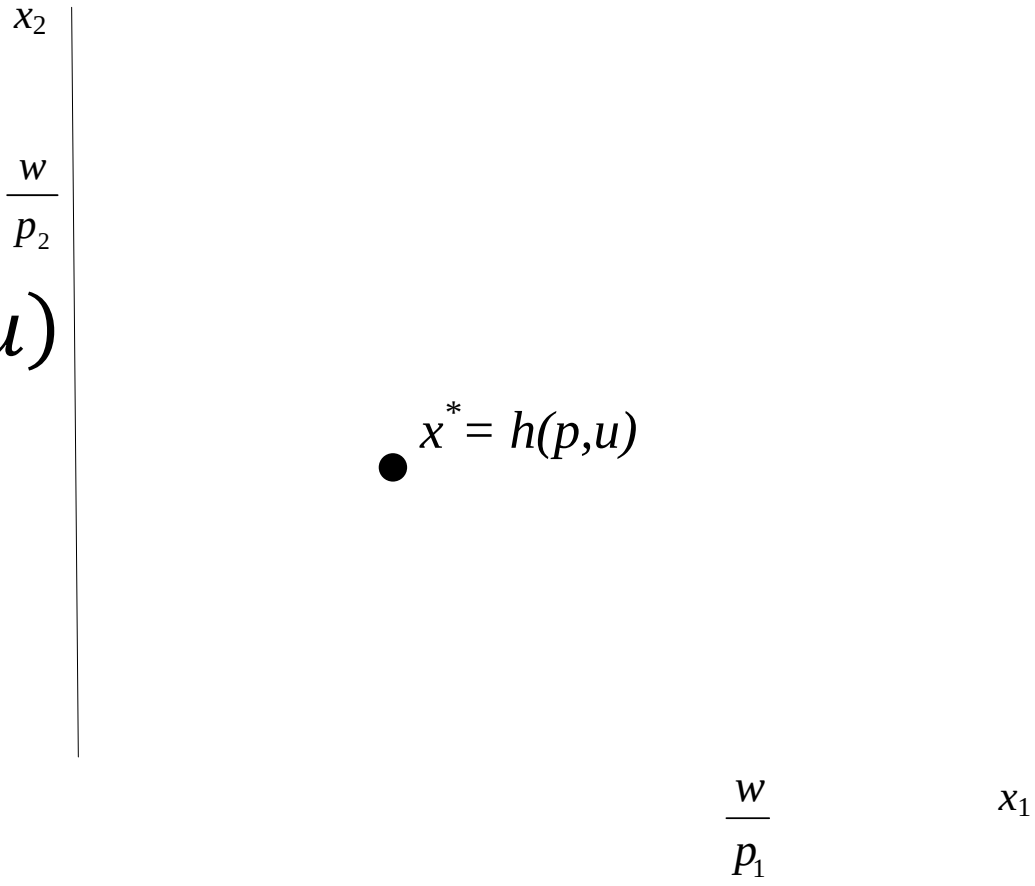
If the preference relation is convex, then $h(p,u)$ is a convex set.



Properties of Hicksian Demand

4) Uniqueness:

If the preference relation is strictly convex, then $h(p,u)$ contains a single element.



Properties of Hicksian Demand

- ***Compensated Law of Demand:*** for any change in prices p and p',

$$(p'-p) \cdot [h(p', u) - h(p, u)] \leq 0$$

  - *Implication*: for every good k,

$$(p'_k - p_k) \cdot [h_k(p', u) - h_k(p, u)] \leq 0$$

  - This is true for compensated demand, but not necessarily true for Walrasian demand (which is uncompensated):
    - Recall the figures on Giffen goods, where a decrease in p_k in fact decreases $x_k(p, u)$ when wealth was left uncompensated.

The Expenditure Function

- Plugging the result from the EMP, $h(p,u)$, into the objective function, $p \cdot x$, we obtain the value function of this optimization problem,

$$p \cdot h(p,u) = e(p,u)$$

where $e(p,u)$ represents the ***minimal expenditure*** that the consumer needs to incur in order to reach utility level u when prices are p.

Properties of Expenditure Function

- Suppose that $u(\cdot)$ is a continuous function, satisfying LNS defined on $X = \mathbb{R}^L_+$. Then for $p \gg 0$, $e(p,u)$ satisfies:

1) ***Homog(1)*** in p,

$$e(\alpha p, u) = \alpha \underbrace{p \cdot x^*}_{e(p,u)} = \alpha \cdot e(p,u)$$

for any p, u, and $\alpha > 0$.

- We know that the optimal bundle is not changed when all prices change, since the optimal consumption bundle in $h(p,u)$ satisfies homogeneity of degree zero.
- Such a price change just makes it more or less expensive to buy the same bundle.

Properties of Expenditure Function

2) Strictly increasing in u:

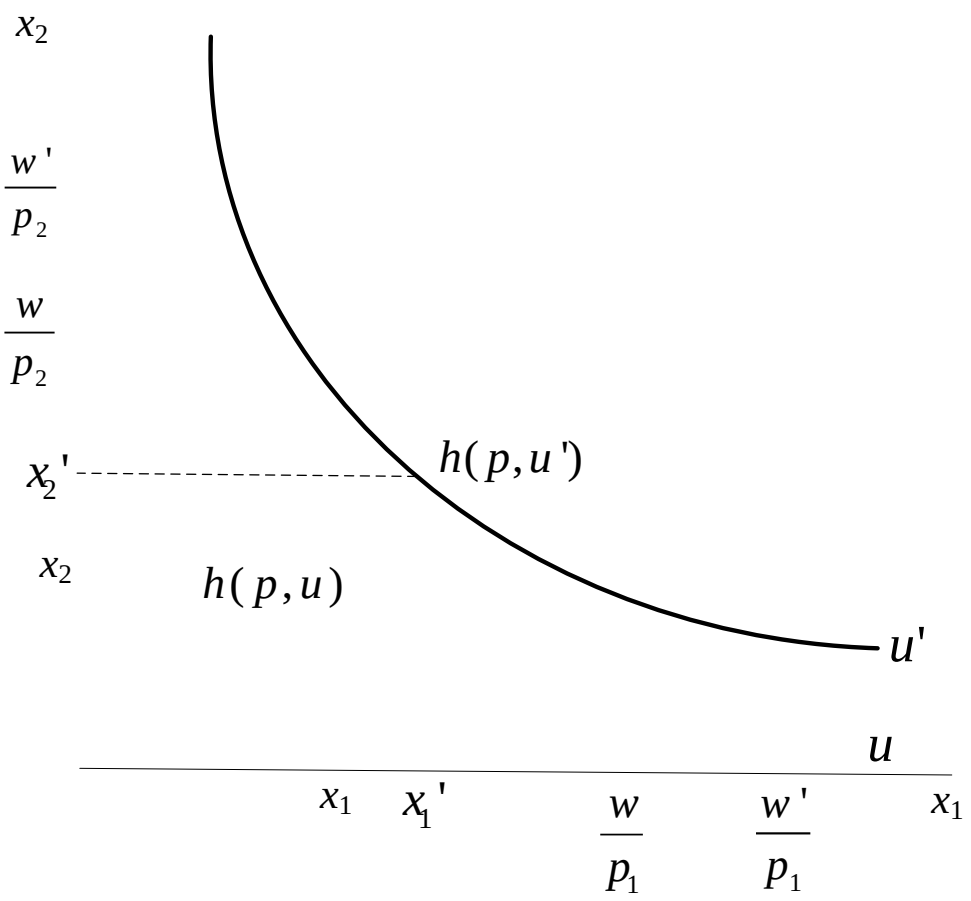
For a given price vector, reaching a higher utility requires higher expenditure:

$$p_1 x_1' + p_2 x_2' > p_1 x_1 + p_2 x_2$$

where $(x_1, x_2) = h(p, u)$ and $(x_1', x_2') = h(p, u')$.

Then,

$$e(p, u') > e(p, u)$$



Properties of Expenditure Function

3) Non-decreasing in p_k for every good k:

Higher prices mean higher expenditure to reach a given utility level.

- Let $p' = (p_1, p_2, \dots, p_k', \dots, p_L)$ and $p = (p_1, p_2, \dots, p_k, \dots, p_L)$, where $p_k' > p_k$.
- Let $x' = h(p', u)$ and $x = h(p, u)$ from EMP under prices p' and p, respectively.
- Then, $p' \cdot x' = e(p', u)$ and $p \cdot x = e(p, u)$.

$$e(p', u) = p' \cdot x' \geq p \cdot x' \geq p \cdot x = e(p, u)$$

- 1st inequality due to $p' \geq p$
- 2nd inequality: at prices p, bundle x minimizes EMP.

Properties of Expenditure Function

4) Concave in p:

Take $x^* \in h(\bar{p}, u)$, the argmin at the average price

$$\bar{p} = \alpha \cdot p' + (1-\alpha) \cdot p''.$$

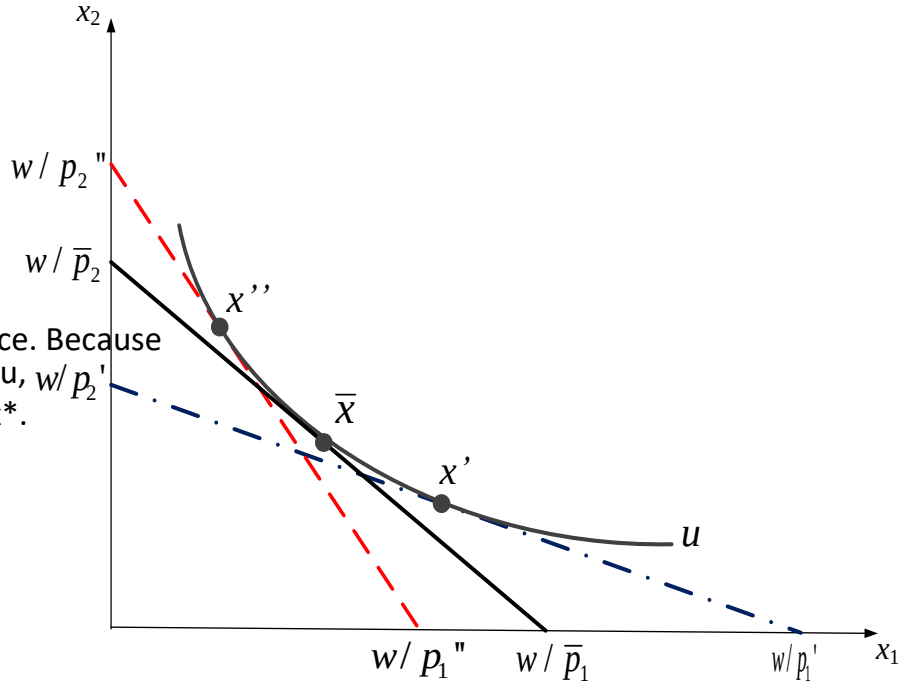
Since $u(x^*) \geq u$, bundle x^* is feasible for the EMP at every price. Because x' minimizes $p' \cdot x$ s.t. $u(x) \geq u$ and x'' minimizes $p'' \cdot x$ s.t. $u(x) \geq u$,

$$e(p', u) = p' \cdot x' \leq p' \cdot x^* \quad \text{and} \quad e(p'', u) = p'' \cdot x'' \leq p'' \cdot x^*.$$

Multiplying by α and $(1-\alpha)$ and summing:

$$\alpha \cdot e(p', u) + (1-\alpha) \cdot e(p'', u) \leq \alpha \cdot p' \cdot x^* + (1-\alpha) \cdot p'' \cdot x^*$$
$$= [\alpha \cdot p' + (1-\alpha) \cdot p''] \cdot x^* = \bar{p} \cdot x^* = e(\bar{p}, u),$$

as required by concavity.



Connections

Relationship between the Expenditure and Hicksian Demand

- Let's assume that $u(\cdot)$ is a continuous function, representing preferences that satisfy LNS and are strictly convex and defined on $X = \mathbb{R}^L_+$. For all p and u,

$$\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u) \text{ for every good } k$$

  This identity is "***Shephard's lemma***": if we want to find $h_k(p,u)$ and we know $e(p,u)$, we just have to differentiate $e(p,u)$ with respect to prices.

- *Proof*: three different approaches
  1) the support function
  2) first-order conditions
  3) the envelope theorem (See Appendix 2.2)

Relationship between the Expenditure and Hicksian Demand

- The relationship between the Hicksian demand and the expenditure function can be further developed by taking first order conditions again. That is,

$$\frac{\partial^2 e(p,u)}{\partial p_k^2} = \frac{\partial h_k(p,u)}{\partial p_k}$$

or

$$D_p^2 e(p,u) = D_p h(p,u)$$

- Since $D_p h(p,u)$ provides the Slutsky matrix, $S(p,w)$, then

$$S(p,w) = D_p^2 e(p,u)$$

where the Slutsky matrix can be obtained from the observable Walrasian demand.

Relationship between the Expenditure and Hicksian Demand

- There are three other important properties of $D_p h(p,u)$, where $D_p h(p,u)$ is $L \times L$ derivative matrix of the Hicksian demand, $h(p,u)$:
  1) $D_p h(p,u)$ is negative semidefinite
     - Hence, $S(p,w)$ is negative semidefinite.
  2) $D_p h(p,u)$ is a symmetric matrix
     - Hence, $S(p,w)$ is symmetric.
  3) $D_p h(p,u)p = 0$, which implies $S(p,w)p = 0$.
     - Not all goods can be net complements $\left(\frac{\partial h_l(p,u)}{\partial p_k} < 0\right)$. Otherwise, multiplying this vector of derivatives with the (positive) price vector $p \gg 0$ would yield a non-zero result.

Relationship between the Expenditure and Hicksian Demand

- To show $D_p h(p,u)p = 0$, recall that $h(p,u)$ is HD(0), so for all $\alpha > 0$, we have

$$h(\alpha p, u) = h(p,u)$$

- Differentiating both sides with respect to α, and by Chain rule, we find

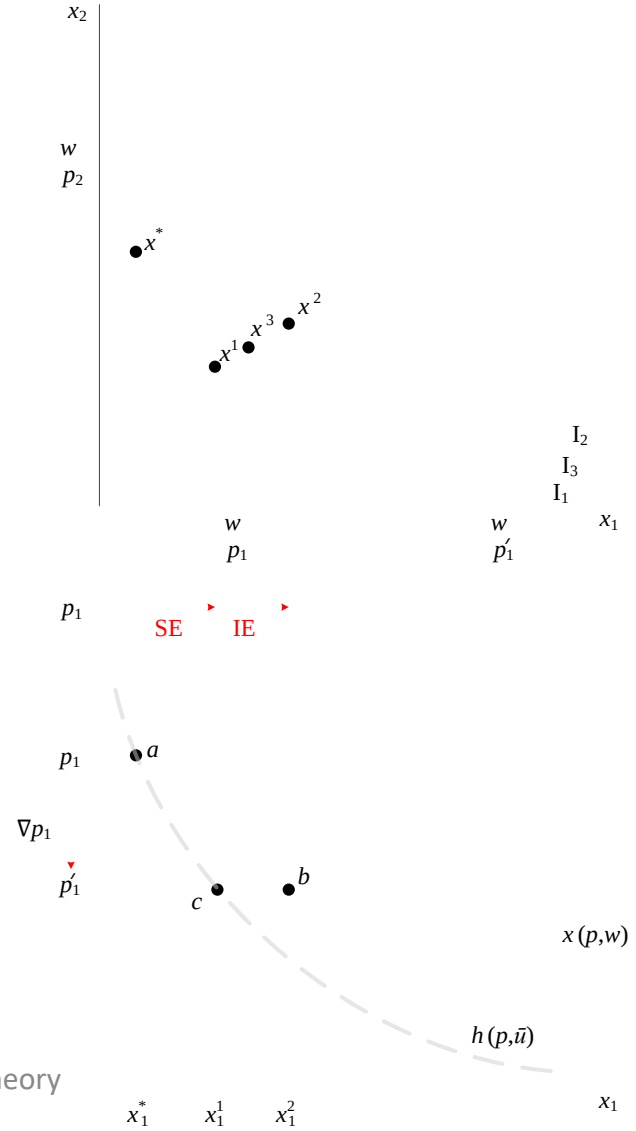
$$D_{\alpha p} h(\alpha p, u)p = 0$$

- Setting $\alpha = 1$, we obtain

$$D_p h(p,u)p = 0$$

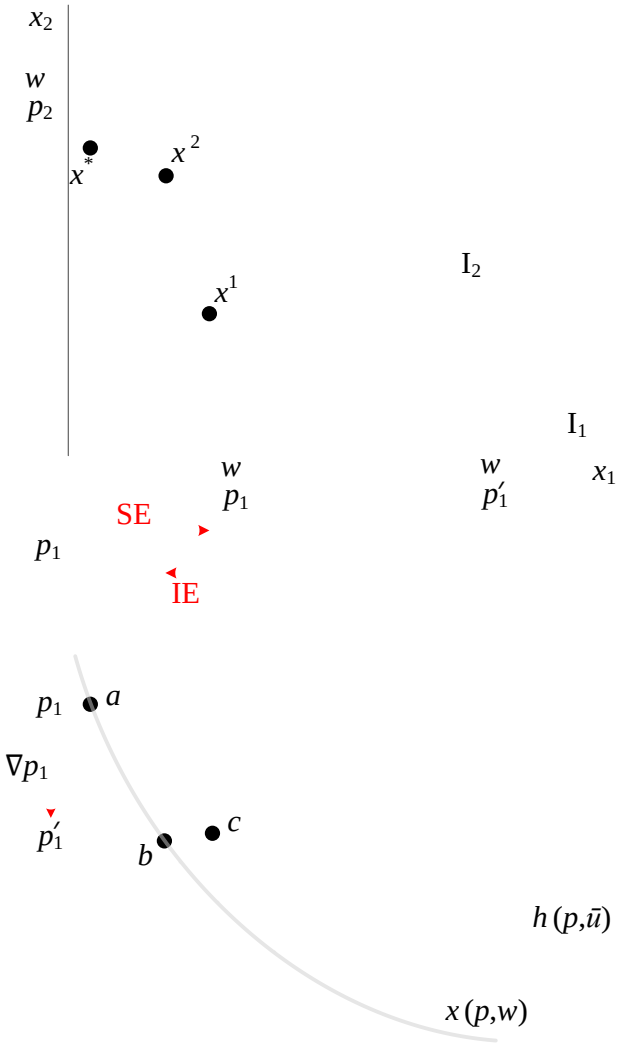
Relationship between Hicksian and Walrasian Demand

- When income effects are positive (*normal goods*), then the Walrasian demand $x(p, w)$ is **above** the Hicksian demand $h(p, u)$.
  - The Hicksian demand is *steeper* than the Walrasian demand.



Relationship between Hicksian and Walrasian Demand

- When income effects are negative (*inferior goods*), then the Walrasian demand $x(p, w)$ is **below** the Hicksian demand $h(p, u)$.
  - The Hicksian demand is *flatter* than the Walrasian demand.



Relationship between Hicksian and Walrasian Demand

- We can formally relate the Hicksian and Walrasian demand as follows:
  - Consider $u(\cdot)$ is a continuous function, representing preferences that satisfy LNS and are strictly convex and defined on $X = \mathbb{R}^L_+$.
  - Consider a consumer facing $(\bar{p}, \bar{w})$ and attaining utility level $\bar{u}$.
  - Note that $\bar{w} = e(\bar{p}, \bar{u})$. In addition, we know that for any (p, u), $h_l(p, u) = x_l(p, \underbrace{e(p, u)}_{w})$. Differentiating this expression with respect to p_k, and evaluating it at $(\bar{p}, \bar{u})$, we get:

$$\frac{\partial h_l(\bar{p}, \bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial w} \frac{\partial e(\bar{p}, \bar{u})}{\partial p_k}$$

Relationship between Hicksian and Walrasian Demand

- Using the fact that $\frac{\partial e(\bar{p},\bar{u})}{\partial p_k} = h_k(\bar{p},\bar{u})$,

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial w} h_k(\bar{p},\bar{u})$$

- Finally, since $\bar{w} = e(\bar{p},\bar{u})$ and $h_k(\bar{p},\bar{u}) = x_k\big(\bar{p},e(\bar{p},\bar{u})\big) = x_k(\bar{p},\bar{w})$, then

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},\bar{w})}{\partial p_k} + \frac{\partial x_l(\bar{p},\bar{w})}{\partial w} x_k(\bar{p},\bar{w})$$

Relationship between Hicksian and Walrasian Demand

- But this coincides with $s_{lk}(p, w)$ that we discussed in the Slutsky equation.
  - Hence, we have $\underbrace{D_p h(p, u)}_{L \times L} = S(p, w)$.
  - Or, more compactly, $SE = TE + IE$.

Relationship between Walrasian Demand and Indirect Utility Function

- Let's assume that $u(\cdot)$ is a continuous function, representing preferences that satisfy LNS and are strictly convex and defined on $X = \mathbb{R}^L_+$. Suppose also that $v(p, w)$ is differentiable at any $(p, w) \gg 0$. Then,

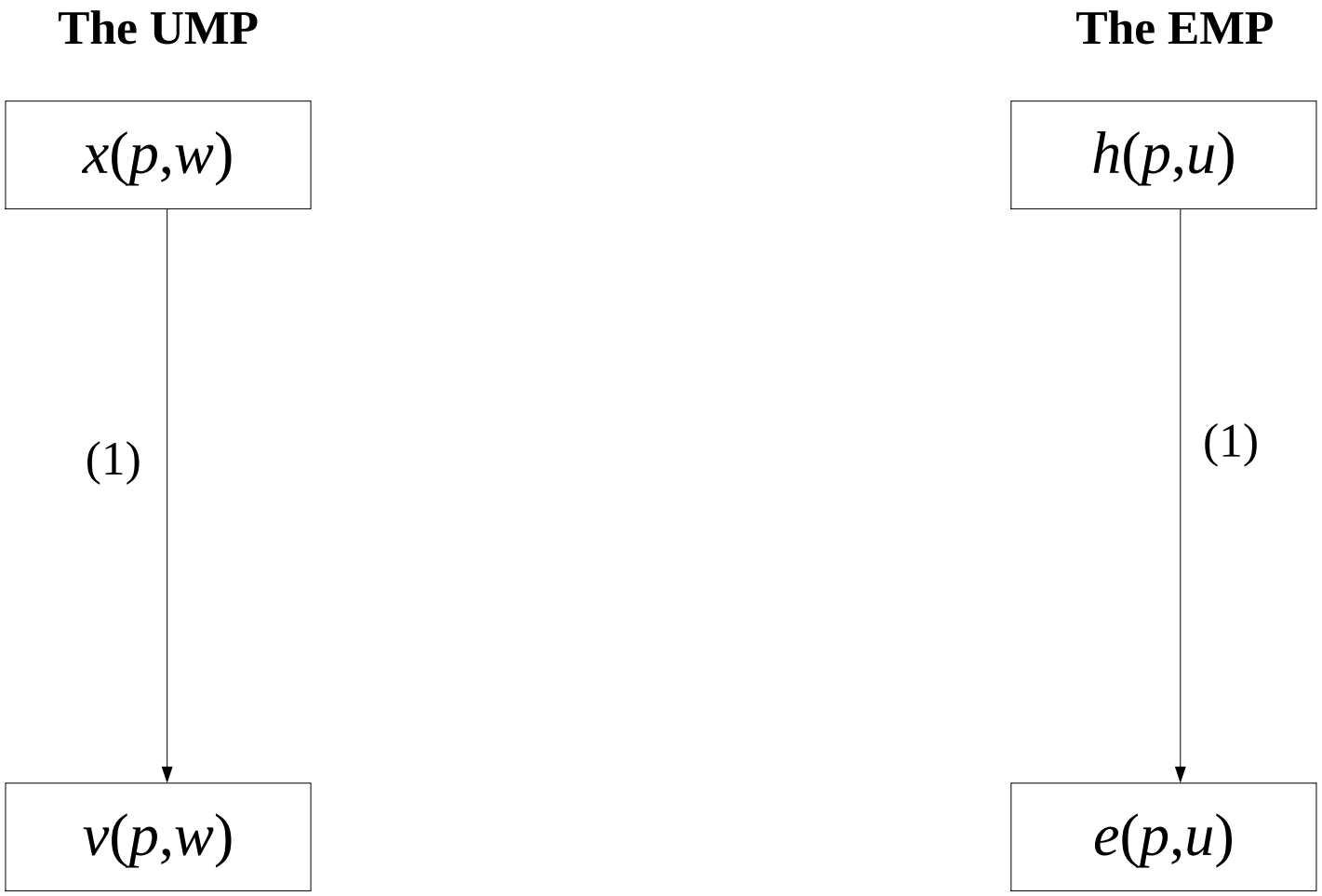
$$-\frac{\frac{\partial v(p,w)}{\partial p_k}}{\frac{\partial v(p,w)}{\partial w}} = x_k(p, w) \text{ for every good } k$$

- This is ***Roy's identity***.
- Powerful result, since in many cases it is easier to compute the derivatives of $v(p, w)$ than solving the UMP with the system of FOCs.

Summary of Relationships

- The Walrasian demand, $x(p, w)$, is the solution of the UMP.
  - Its value function is the indirect utility function, $v(p, w)$.
- The Hicksian demand, $h(p, u)$, is the solution of the EMP.
  - Its value function is the expenditure function, $e(p, u)$.

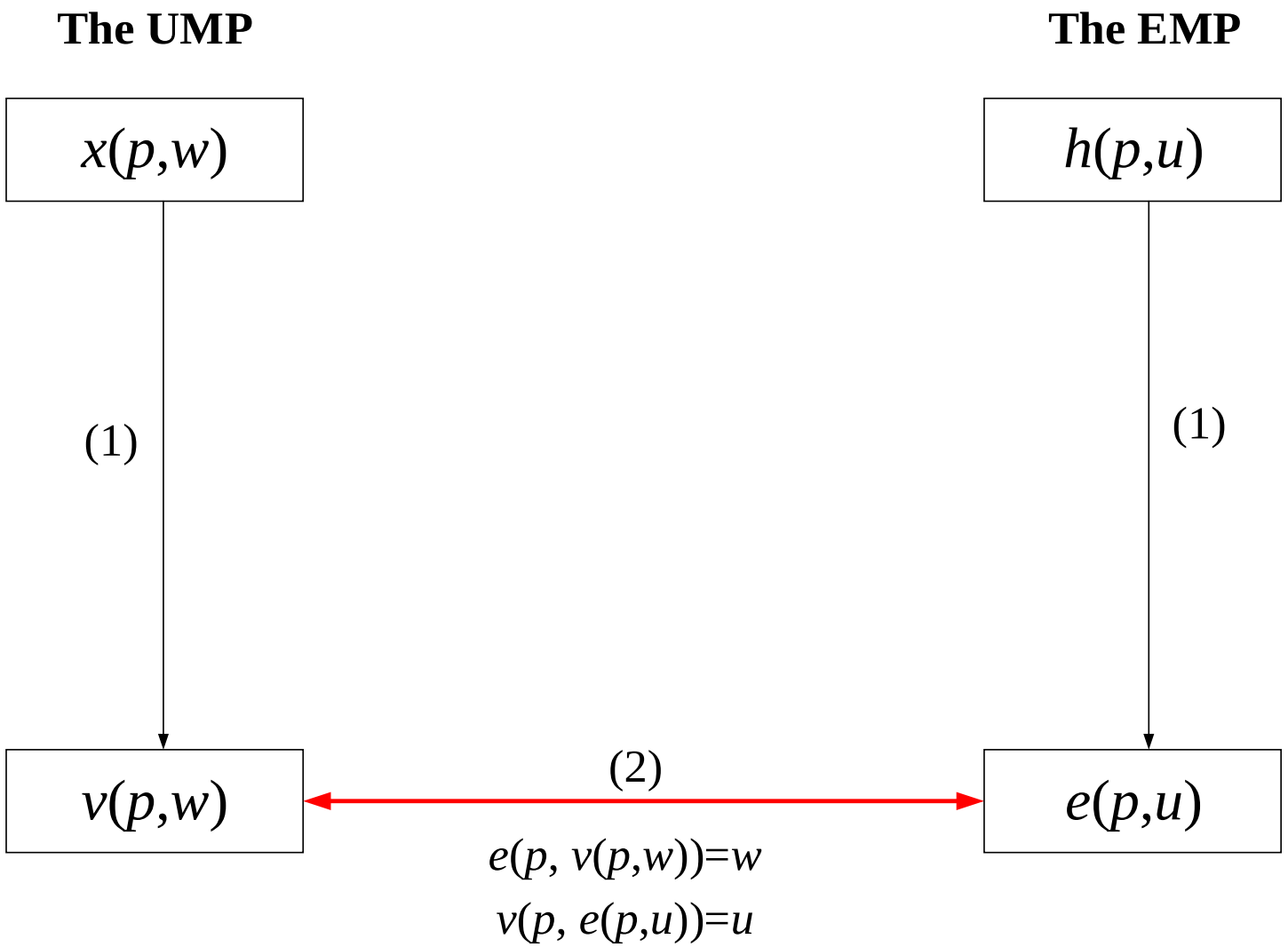
Summary of Relationships



Summary of Relationships

- Relationship between the value functions of the UMP and the EMP (lower part of figure):
  - $e\big(p, v(p,w)\big) = w$, i.e., the minimal expenditure needed in order to reach a utility level equal to the maximal utility that the individual reaches at his UMP, $u = v(p,w)$, must be w.
  - $v(p, e(p,u)) = u$, i.e., the indirect utility that can be reached when the consumer is endowed with a wealth level *w* equal to the minimal expenditure he optimally uses in the EMP, i.e., $w = e(p,u)$, is exactly u.

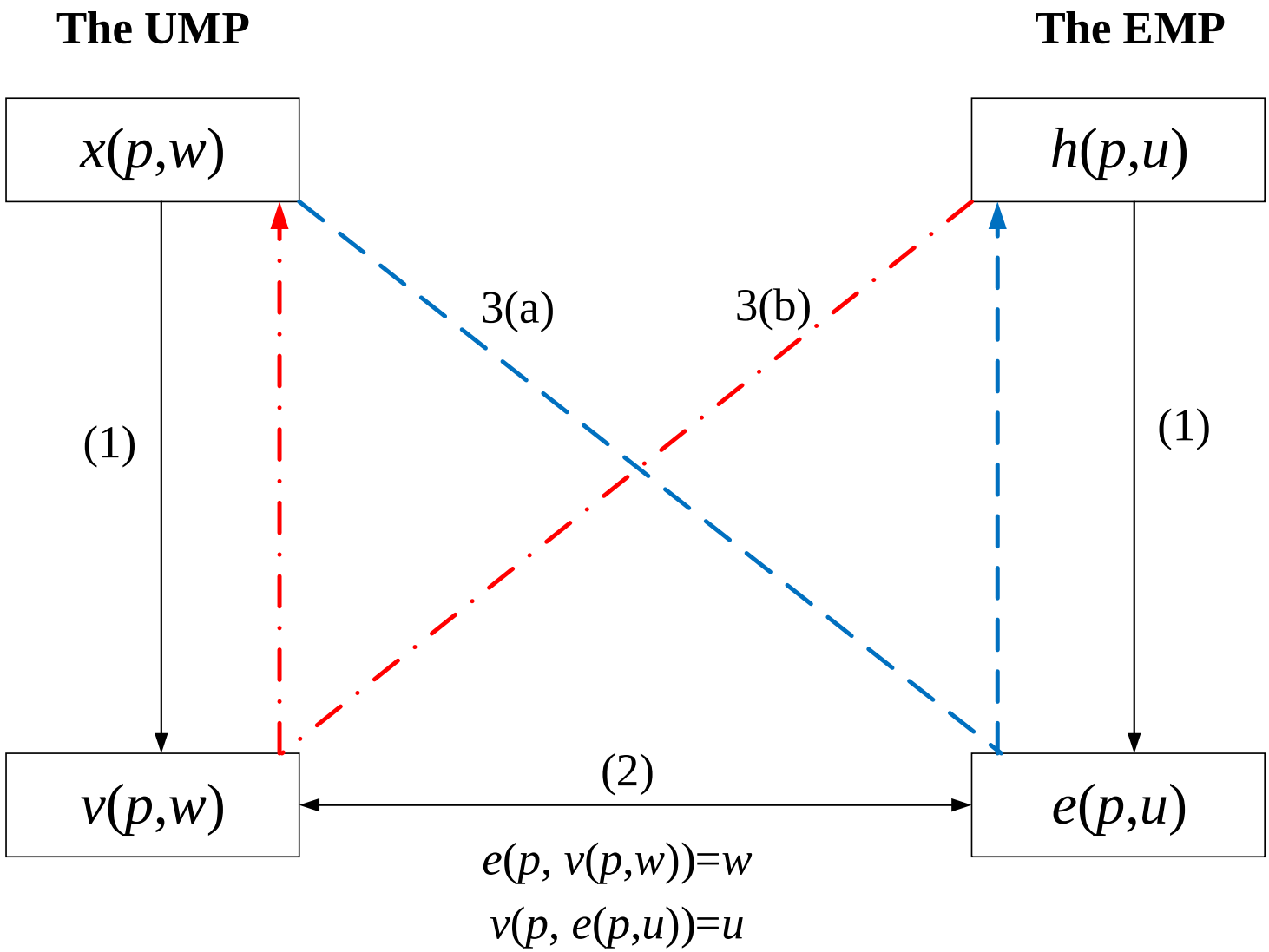
Summary of Relationships



Summary of Relationships

- Relationship between the argmax of the UMP (the Walrasian demand) and the argmin of the EMP (the Hicksian demand):
  - $x(p, e(p, u)) = h(p, u)$, i.e., the (uncompensated) Walrasian demand of a consumer endowed with an adjusted wealth level w (equal to the expenditure he optimally uses in the EMP), $w = e(p, u)$, coincides with his Hicksian demand, $h(p, u)$.
  - $h(p, v(p, w)) = x(p, w)$, i.e., the (compensated) Hicksian demand of a consumer reaching the maximum utility of the UMP, $u = v\,(p, w)$, coincides with his Walrasian demand, $x(p, w)$.

Summary of Relationships



Summary of Relationships

- Finally, we can also use:
  - The ***Slutsky equation***:

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

    to relate the derivatives of the Hicksian and the Walrasian demand.
  - ***Shephard's lemma***:

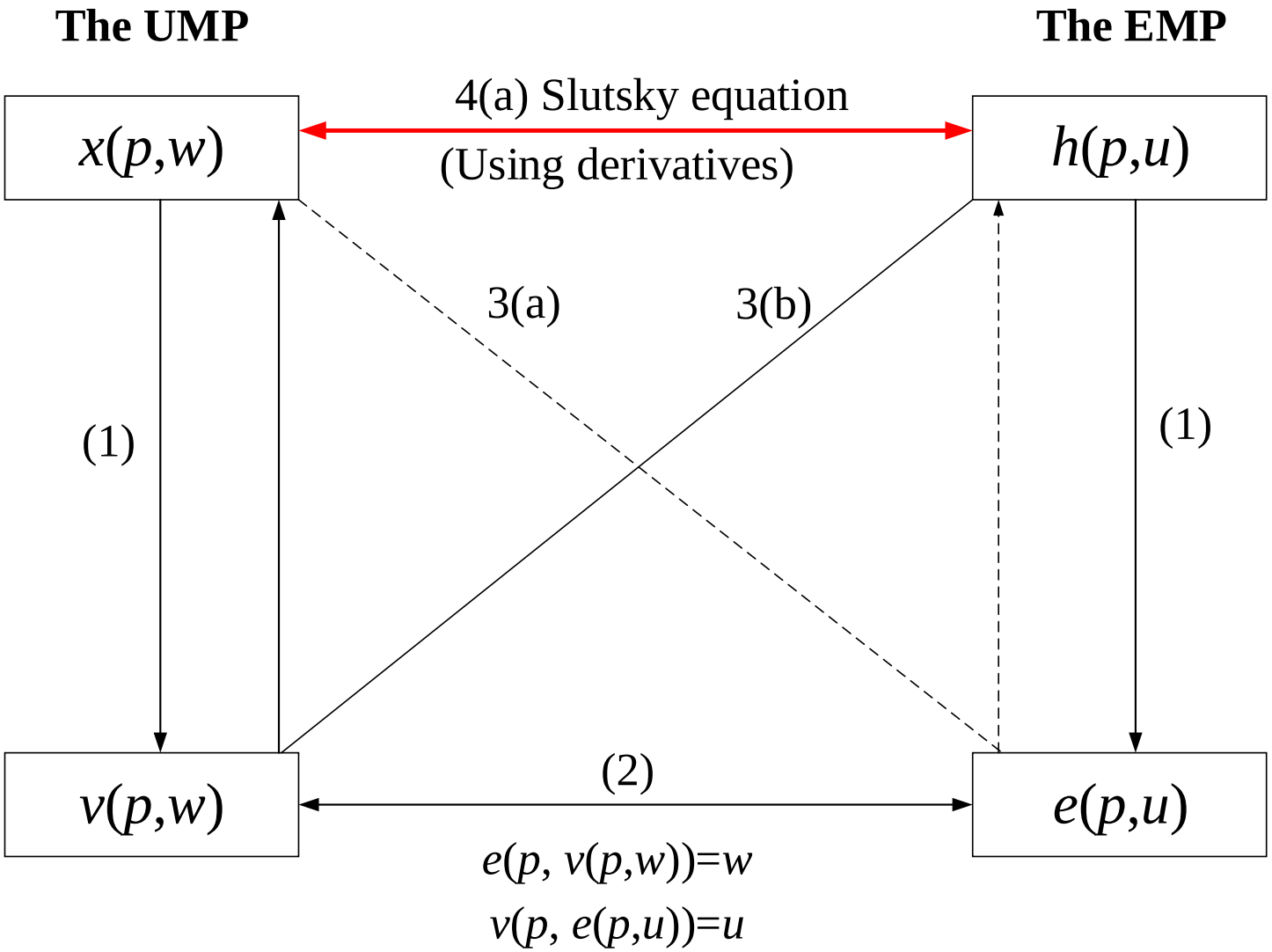
$$\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u)$$

    to obtain the Hicksian demand from the expenditure function.
  - ***Roy's identity***:

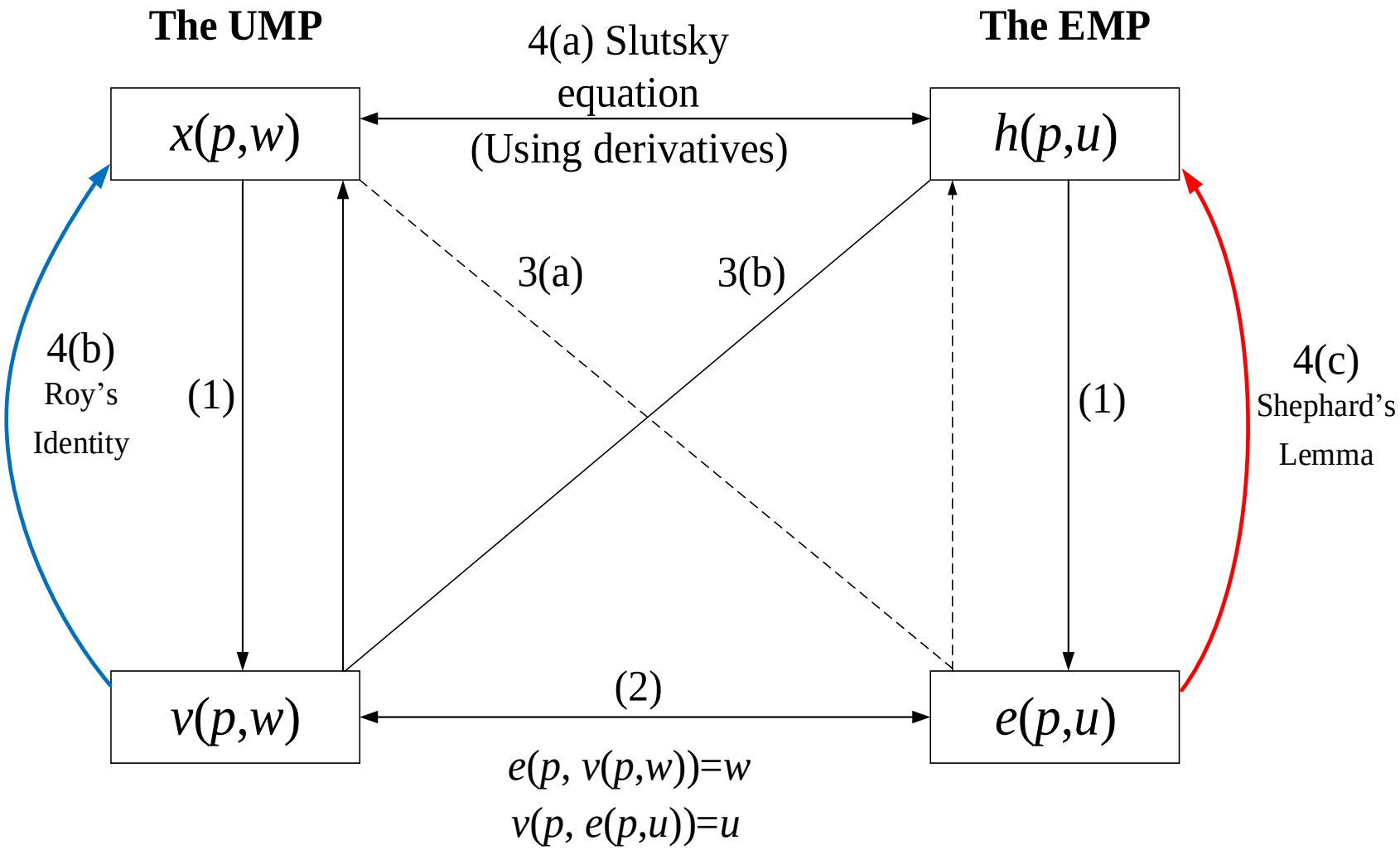
$$-\frac{\frac{\partial v(p,w)}{\partial p_k}}{\frac{\partial v(p,w)}{\partial w}} = x_k(p,w)$$

    to obtain the Walrasian demand from the indirect utility function.

Summary of Relationships



Summary of Relationships



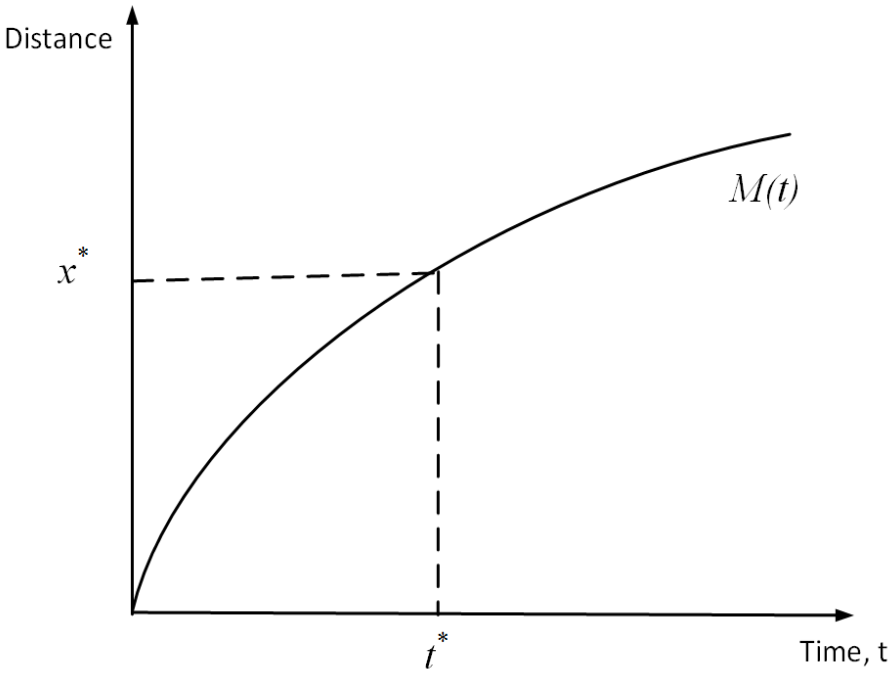
Appendix 2.1:
Duality in Consumption

Duality in Consumption

- We discussed the utility maximization problem (UMP) – the so-called *primal* problem describing the consumer's choice of optimal consumption bundles – and its *dual*: the expenditure minimization problem (EMP).
- When can we guarantee that the solution x^* to both problems coincide?

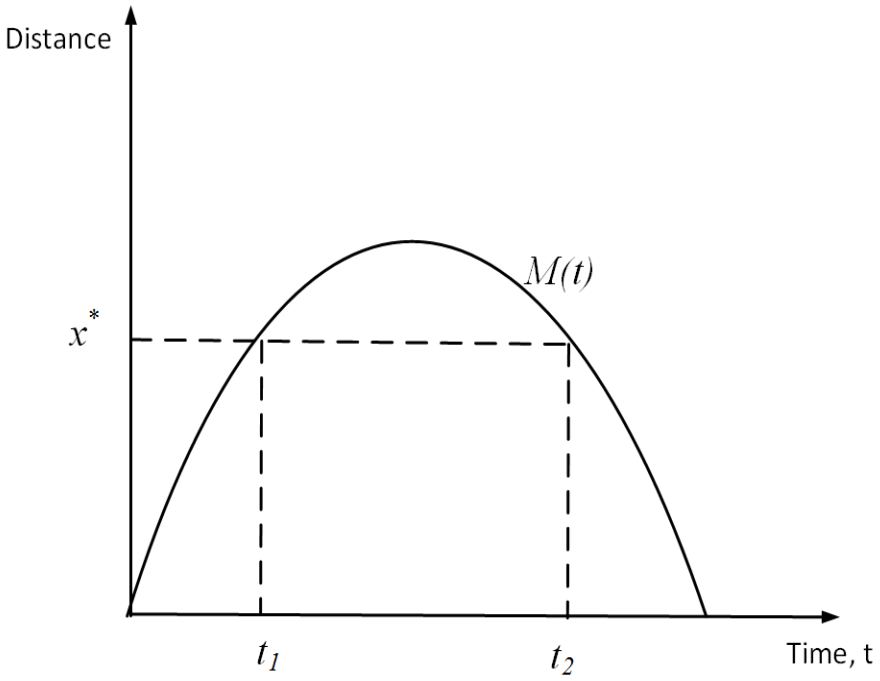
Duality in Consumption

- The maximal distance that a turtle can travel in t^* time is x^*.
  - From time (t^*) to distance (x^*)
- The minimal time that a turtle needs to travel x^* distance is t^*.
  - From distance (x^*) to time (t^*)
- In this case the primal and dual problems would provide us with the same answer to both of the above questions



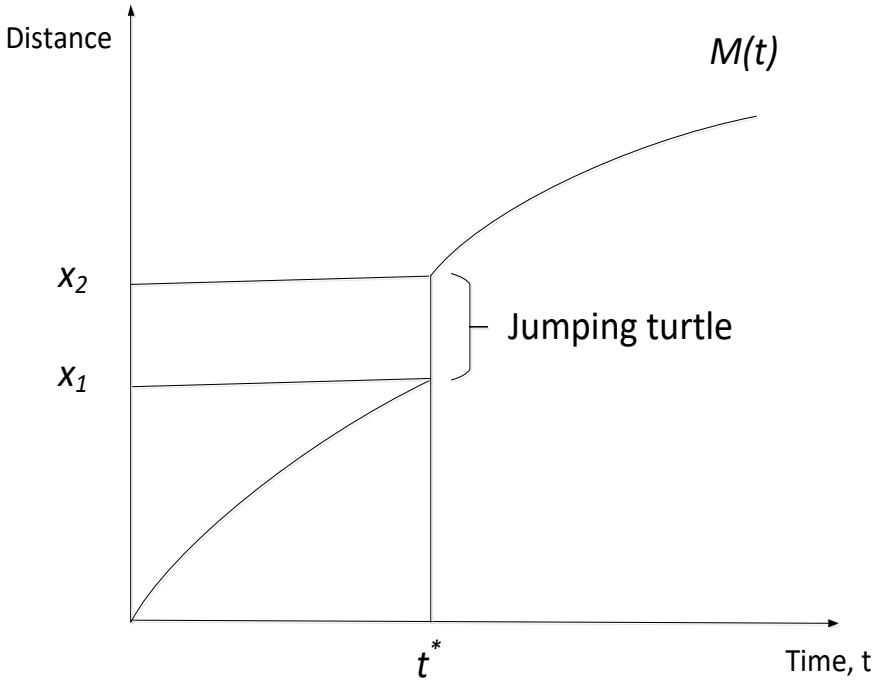
Duality in Consumption

- In order to obtain the same answer from both questions, we critically need that the function $M(t)$ satisfies ***monotonicity***.
- Otherwise: the maximal distance traveled at both t_1 and t_2 is x^*, but the minimal time that a turtle needs to travel x^* distance is t_1.
- Hence, a non-monotonic function cannot guarantee that the answers from both questions are compatible.



Duality in Consumption

- Similarly, we also require that the function $M(t)$ satisfies ***continuity***.
- Otherwise: the maximal distance that the turtle can travel in time t^* is x_2 , whereas the minimum time required to travel distance x_1 and x_2 is t^* for both distances.
- Hence, a non-continuous function does not guarantee that the answers to both questions are compatible.



Duality in Consumption

- Given $u(\cdot)$ is monotonic and continuous, then if x^* is the solution to the problem

$$\max_{x \geq 0} u(x) \quad \text{s.t.} \quad p \cdot x \leq w \qquad \text{(UMP)}$$

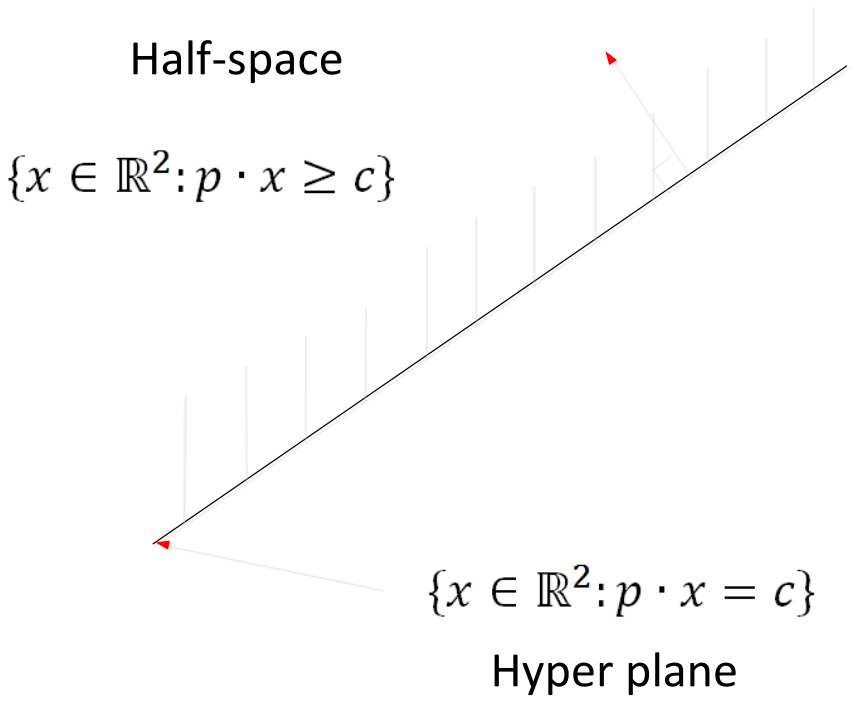
it must also be a solution to the problem

$$\min_{x \geq 0} p \cdot x \quad \text{s.t.} \quad u(x) \geq u \qquad \text{(EMP)}$$

Hyperplane Theorem

- ***Hyperplane***: for some $p \in \mathbb{R}^L_+$ and $c \in \mathbb{R}$, the set of points in $\mathbb{R}^L$ such that
$\{x \in \mathbb{R}^L : p \cdot x = c\}$

- ***Half-space***: the set of bundles x for which $p \cdot x \geq c$. That is,
$\{x \in \mathbb{R}^L : p \cdot x \geq c\}$



Hyperplane Theorem

- ***Separating hyperplane theorem***: For every convex and closed set K, there is a half-space containing K and excluding any point $\bar{x} \notin K$ outside of this set.
  - That is, there exist $p \in \mathbb{R}_+^L$ and $c \in \mathbb{R}$ such that

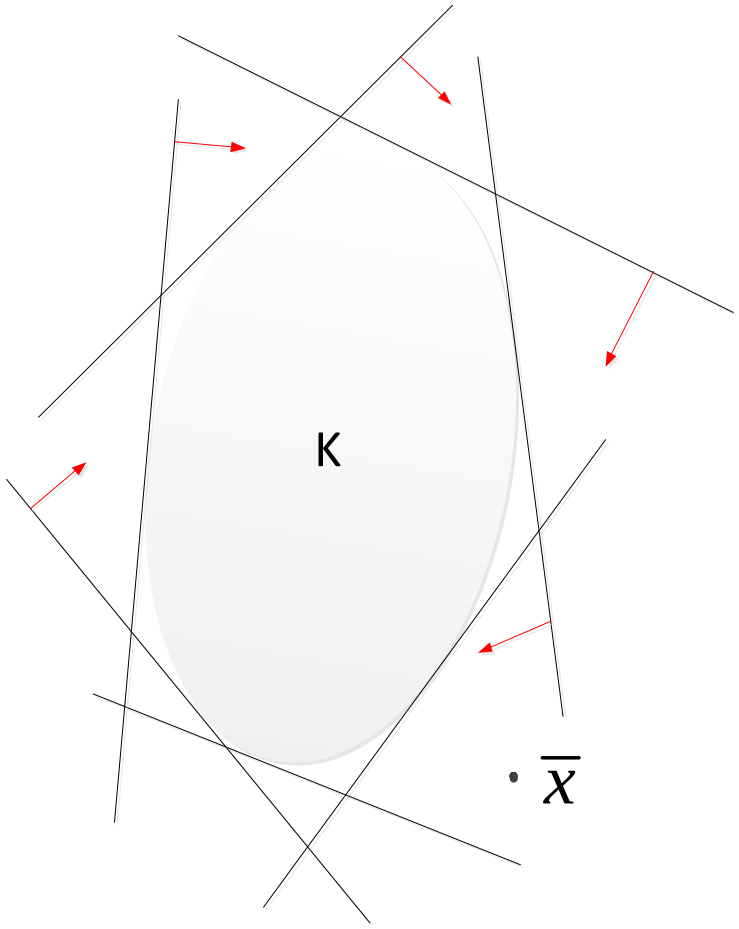
    $$p \cdot x \geq c \quad \text{for all elements in the set, } x \in K$$

    $$p \cdot \bar{x} < c \quad \text{for all elements outside the set, } \bar{x} \notin K$$

  - *Intuition*: every convex and closed set K can be equivalently described as the intersection of the half-spaces that contain it.
    - As we draw more and more half spaces, their intersection becomes the set K.

Hyperplane Theorem

- If K is a closed and convex set, we can then construct half-spaces for all the elements in the set $(x_1, x_2, \dots)$ such that their intersections coincide with the ("equivalently describes") set K.
  - We construct a cage (or hull) around the convex set K that exactly coincides with set K.
- Bundle like $\bar{x} \notin K$ lies outside the intersection of half-spaces.



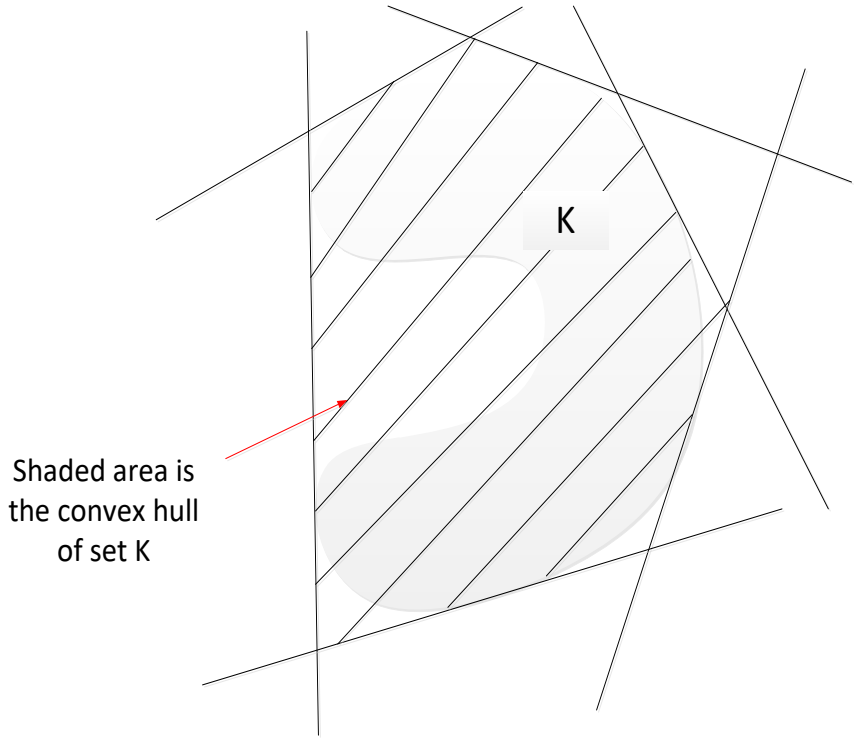
Hyperplane Theorem

- What if the set we are trying to "equivalently describe" by the use of half-spaces is *non-convex*?
  - The intersection of half-spaces does not coincide with set K (it is, in fact, larger, since it includes points outside set K). Hence, we cannot use several half-spaces to "equivalently describe" set K.
  - Then the intersection of half-spaces that contain K is the smallest, convex set that contains K, known as the closed, convex hull of K.

Hyperplane Theorem

- The convex hull of set K is often denoted as $\overline{K}$, and it is convex (unlike set K, which might not be convex).

- The convex hull $\overline{K}$ is convex, both when set K is convex and when it is not.



Support Function

- For every nonempty closed set $K \subset \mathbb{R}^L$, its ***support function*** is defined by

$$\mu_K(p) = \inf_x \{p \cdot x\} \text{ for all } x \in K \text{ and } p \in \mathbb{R}^L_+$$

that is, the support function finds, for a given price vector p, the bundle x that minimizes $p \cdot x$.

  - Recall that $\inf$ coincides with the argmin when the constraint includes the boundary.

- From the support function of K, we can reconstruct K.
  - In particular, for every $p \in \mathbb{R}^L_+$, we can define half-spaces whose boundary is the support function of set K.
    - That is, we define the set of bundles for which $p \cdot x \geq \mu_K(p)$. Note that all bundles x in such half-space contains elements in the set K, but does not contain elements outside K, i.e., $\bar{x} \notin K$.

Support Function

- Thus, the intersection of the half-spaces generated by all possible values of p describes ("reconstructs") the set K. That is, set K can be described by all those bundles $x \in \mathbb{R}^L$ such that

$$K = \{x \in \mathbb{R}^L : p \cdot x \geq \mu_K(p) \text{ for every } p\}$$

- By the same logic, if K is not convex, then the set

$$\{x \in \mathbb{R}^L : p \cdot x \geq \mu_K(p) \text{ for every } p\}$$

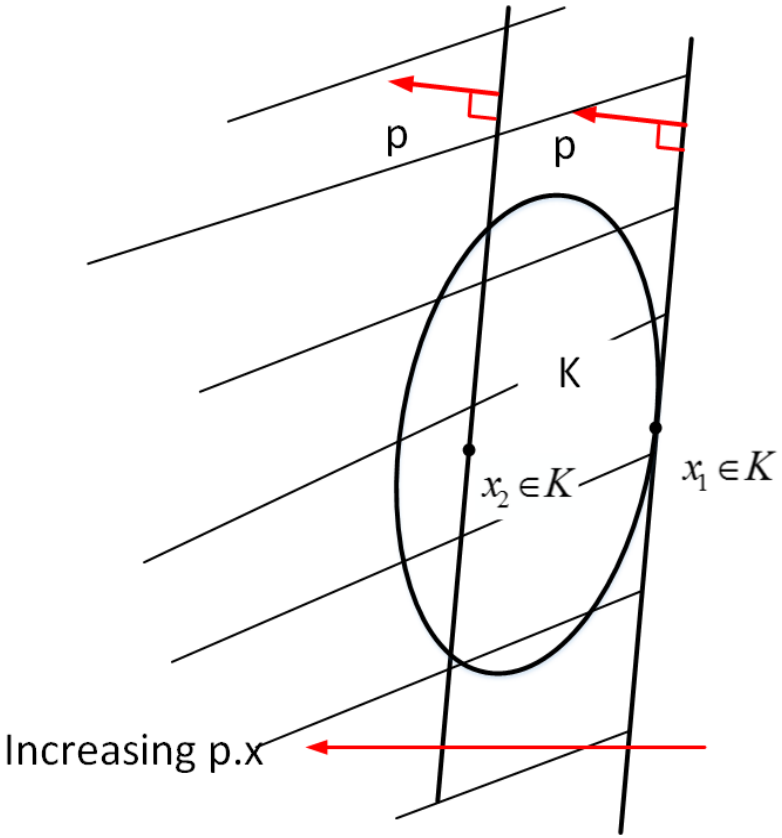
defines the smallest closed, convex set containing K (i.e., the convex hull of set K).

Support Function

- For a given p, the support $\mu_K(p)$ selects the element in K that minimizes $p \cdot x$ (i.e., x_1 in this example).
- Then, we can define the half-space of that hyperplane as:

$$p \cdot x \geq \underbrace{p \cdot x_1}_{\mu_K(p)}$$

  - The above inequality identifies all bundles x to the left of hyperplane $p \cdot x_1$.
- We can repeat the same procedure for any other price vector p.



Support Function

- The above definition of the support function provides us with a useful ***duality theorem***:
  - Let K be a nonempty, closed set, and let $\mu_K(\cdot)$ be its support function. Then there is a unique element in K, $\bar{x} \in K$, such that, for all price vector $\bar{p}$,

    $\bar{p} \cdot \bar{x} = \mu_K(\bar{p}) \Leftrightarrow \mu_K(\cdot)$ is differentiable at $\bar{p}$

    $p \cdot h(p,u) = e(p,u) \Leftrightarrow e(\cdot,u)$ is differentiable at p
  - Moreover, in this case, such derivative is

$$\frac{\partial(\bar{p} \cdot \bar{x})}{\partial p} = \bar{x}$$

    or in matrix notation

$$\nabla \mu_K(\bar{p}) = \bar{x}.$$

Appendix 2.2: Relationship between the Expenditure Function and Hicksian Demand

Proof I (Using Duality Theorem)

- The expenditure function is the support function for the set of all bundles in $\mathbb{R}_+^L$ for which utility reaches at least a level of u. That is,

$$\{x \in \mathbb{R}_+^L : u(x) \geq u\}$$

Using the Duality theorem, we can then state that there is a unique bundle in this set, $h(p,u)$, such that

$$p \cdot h(p,u) = e(p,u)$$

where the right-hand side is the support function of this problem.

- Moreover, from the duality theorem, the derivative of the support function coincides with this unique bundle, i.e.,

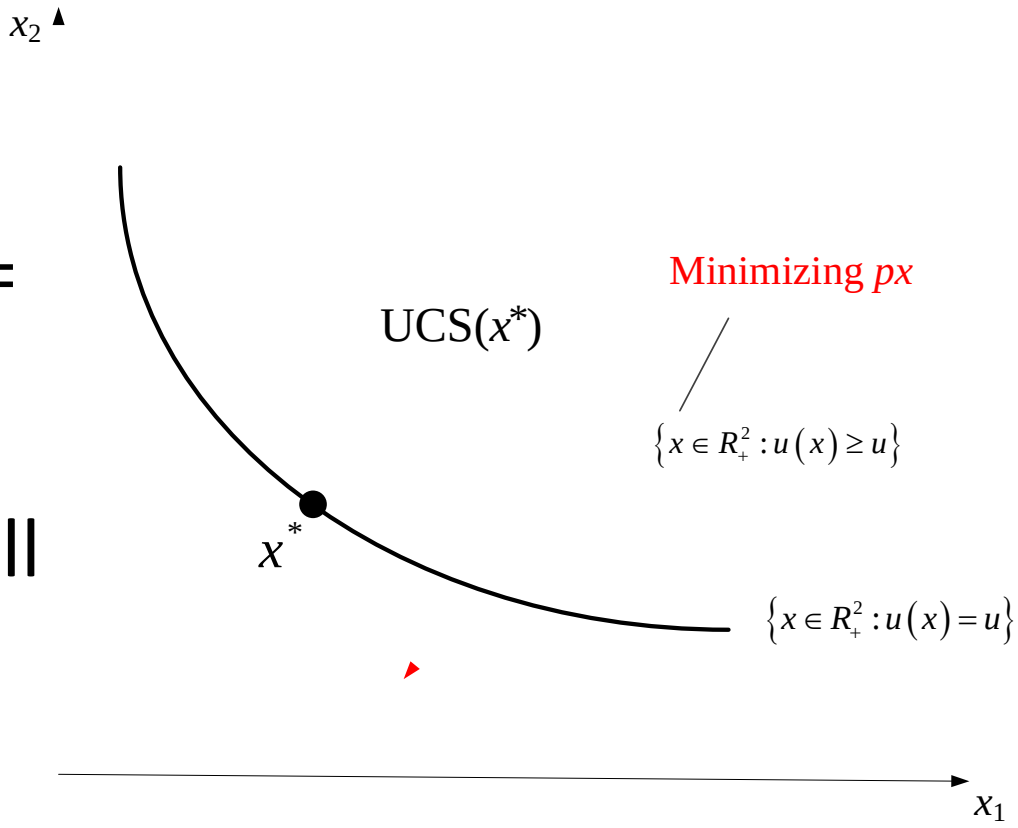
$$\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u) \text{ for every good } k$$

or

$$\nabla_p e(p,u) = h(p,u)$$

Proof I (Using Duality Theorem)

- UCS is a convex and closed set.
- Hyperplane $p \cdot x^* = e(p,u)$ provides us with the minimal expenditure that still reaches utility level u.



Proof II (Using First Order Conditions)

- Totally differentiating the expenditure function $e(p,u)$,

$$\nabla_p e(p,u) = \nabla_p [p \cdot h(p,u)] = h(p,u) + \left[p \cdot D_p h(p,u)\right]^T$$

- And, from the FOCs in interior solutions of the EMP, we know that $p = \lambda \nabla u(h(p,u))$. Substituting,

$$\nabla_p e(p,u) = h(p,u) + \lambda [\nabla u(h(p,u)) \cdot D_p h(p,u)]^T$$

- But, since $u\big(h(p,u)\big) = u$ for all p, differentiating both sides w.r.t. p (chain rule) gives $\nabla u(h) \cdot D_p h = 0$. Since $p = \lambda \cdot \nabla u(h)$ at an EMP interior optimum, the bracketed term $p \cdot D_p h(p,u) = 0$, which implies

$$\nabla_p e(p,u) = h(p,u)$$

That is, $\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u)$ for every good k.

Proof III (Using the Envelope Theorem)

- Using the envelope theorem in the expenditure function, we obtain

$$\frac{\partial e(p,u)}{\partial p_k} = \frac{\partial [p \cdot h(p,u)]}{\partial p_k} = h(p,u) + p\frac{\partial h(p,u)}{\partial p_k}$$

- And, since u(h(p,u)) = u for all p, the chain rule gives $\nabla u(h)\cdot \partial h/\partial p_k = 0$. From the EMP FOC $p = \lambda\cdot\nabla u(h)$, the last term is then $p\cdot\partial h/\partial p_k = \lambda\cdot\nabla u(h)\cdot\partial h/\partial p_k = 0$, implying

$$\frac{\partial e(p,u)}{\partial p_k} = h(p,u)$$

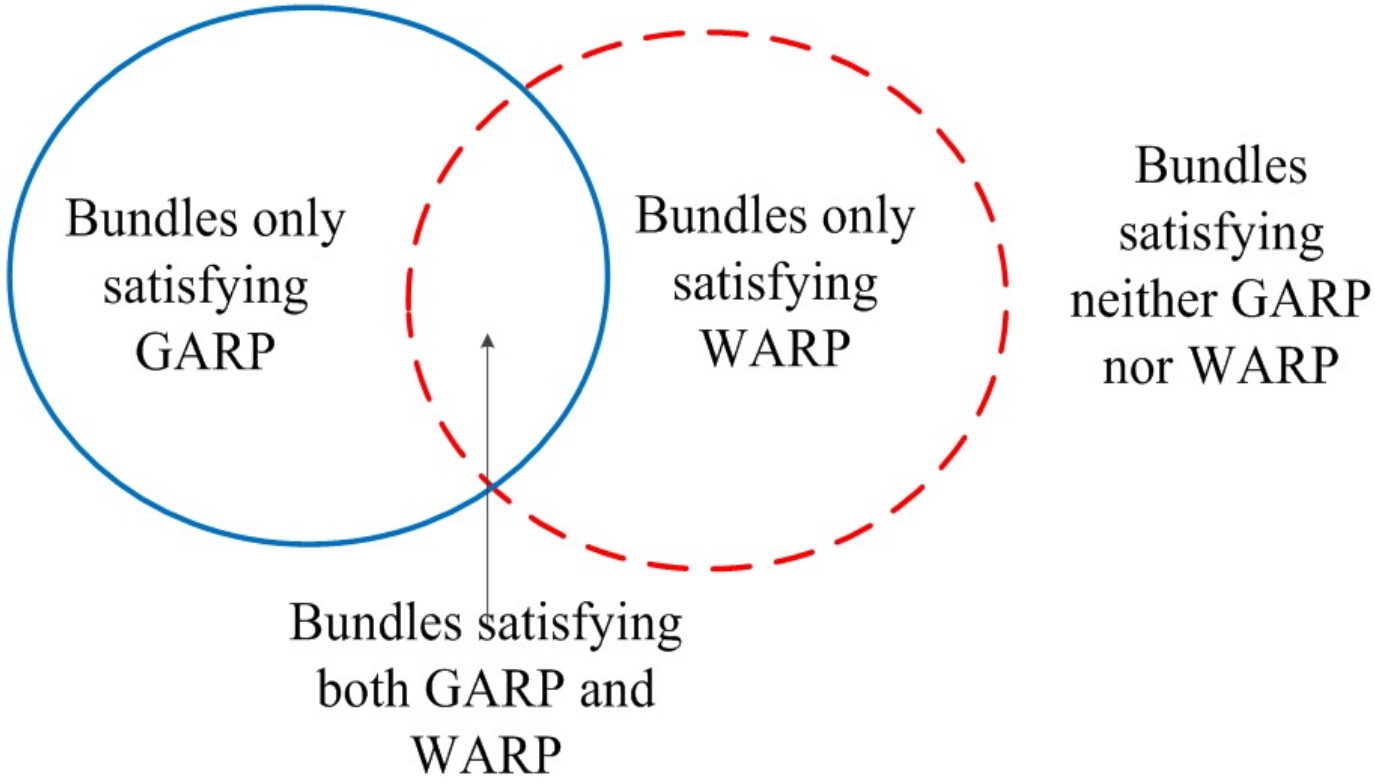
Appendix 2.3: Generalized Axiom of Revealed Preference (GARP)

GARP

- Consider a sequence of prices p^t where $t = 1,2,\ldots,T$ with an associated sequence of chosen bundles x^t.
- ***GARP***: If bundle x^t is revealed preferred to x^{t+1} for all $t = 1,2,\ldots,T$, i.e., $x^1 \gtrsim x^2, x^2 \gtrsim x^3, \ldots, x^{T-1} \gtrsim x^T$, then x^T is *not* strictly revealed preferred to x^1, i.e., $x^T \succ x^1$.
  - More general axiom of revealed preference
  - Neither GARP nor WARP implies one another
  - Some choices satisfy GARP, some WARP, choices for which both axioms hold, and some for which none do.

GARP

- Choices satisfying GARP, WARP, both, or none



GARP

- ***Example A2.1*** (GARP holds, but WARP does not):
  - Consider the following sequence of price vectors and their corresponding demanded bundles

$$p_1 = (1,1,2) \quad x_1 = (1,0,0)$$
$$p_2 = (1,2,1) \quad x_2 = (0,1,0)$$
$$p_3 = (1,3,1) \quad x_3 = (0,0,1)$$

  - The change from $t = 1$ to $t = 2$ violates WARP, but not GARP.

GARP

- ***Example A2.1*** (continued):
  - Let us compare bundles x_1 with x_2.
  - For the premise of WARP to hold:
    - Since bundle x_1 is revealed preferred to x_2 in $t = 1$, i.e., $x_1 \gtrsim x_2$, it must be that $p_1 \cdot x_2 \leq w_1 = p_1 \cdot x_1$.
    - That is, bundle x_2 is affordable under bundle x_1's prices.
  - Substituting our values, we find that

$$(1 \times 0) + (1 \times 1) + (2 \times 0) = 1 \leq 1$$
$$= (1 \times 1) + (1 \times 0) + (2 \times 0)$$

  - Hence, the premise of WARP holds.

GARP

- ***Example A2.1*** (continued):
  - For WARP to be satisfied:
    - We must also have that $p_2 \cdot x_1 > w_2 = p_2 \cdot x_2$.
    - That is, bundle x_1 is unaffordable under the new prices and wealth.
  - Plugging our values yields

$$p_2 \cdot x_1 = (1 \times 1) + (2 \times 0) + (1 \times 0) = 1$$

$$p_2 \cdot x_2 = w_2 = (1 \times 0) + (2 \times 1) + (1 \times 0) = 2$$

  - That is, $p_2 \cdot x_1 < w_2$, i.e., bundle x_1 is still affordable under period two's prices.
  - Thus WARP is violated.

GARP

- ***Example A2.1*** (continued):
  - Let us check GARP.
  - Assume that bundle x_1 is revealed preferred to x_2, $x_1 \succsim x_2$, and that x_2 is revealed preferred to x_3, $x_2 \succsim x_3$.
  - It is easy to show that $x_3 \not\succ x_1$ as bundle x_3 costs the same as bundle x_1 at prices p_3 (both equal 1), hence x_1 is not strictly cheaper than x_3 at t=3.
  - Hence, x_3 cannot be revealed preferred to x_1,
  - Thus, GARP is not violated.

GARP

- ***Example A2.2*** (WARP holds, GARP does not):
  - Consider the following sequence of price vectors and their corresponding demanded bundles

$$p_1 = (1,1,2) \quad x_1 = (1,0,0)$$
$$p_2 = (2,1,1) \quad x_2 = (0,1,0)$$
$$p_3 = (1,2,1+\varepsilon) \quad x_3 = (0,0,1)$$

where $0 < \varepsilon < 1$.

GARP

- ***Example A2.2*** (continued):
  - For the premise of WARP to hold:
    - Since bundle x_1 is revealed preferred to x_2 in $t = 1$, i.e., $x_1 \gtrsim x_2$, it must be that $p_1 \cdot x_2 \leq w_1 = p_1 \cdot x_1$.
    - That is, bundle x_2 is affordable under bundle x_1's prices.
  - Substituting our values, we find that

$$(1 \times 0) + (1 \times 1) + (2 \times 0) = 1 \leq 1$$
$$= (1 \times 1) + (1 \times 0) + (2 \times 0)$$

  - Hence, the premise of WARP holds.

GARP

- ***Example A2.2*** (continued):
  - For WARP to be satisfied:
    - We must also have that $p_2 \cdot x_1 > w_2 = p_2 \cdot x_2$.
    - That is, bundle x_1 is unaffordable under the new prices and wealth.
  - Plugging our values yields

$$(2 \times 1) + (1 \times 0) + (1 \times 0) = 2 > 1$$
$$= (2 \times 0) + (1 \times 1) + (1 \times 0)$$

  - Thus, WARP is satisfied.

GARP

- ***Example A2.2*** (continued):
  - A similar argument applies to the comparison of the choices in $t = 2$ and $t = 3$.
  - Bundle x_3 is affordable under the prices at $t = 2$,
$$p_2 \cdot x_3 = 1 \leq 1 = w_2 = p_2 \cdot x_2$$
  - But x_2 is unaffordable under the prices of $t = 3$, i.e., $p_3 \cdot x_2 > w_3 = p_3 \cdot x_3$, since
$$(1 \times 0) + (2 \times 1) + [(1 + \varepsilon) \times 0] = 2 >$$
$$(1 \times 0) + (2 \times 0) + [(1 + \varepsilon) \times 1] = 1 + \varepsilon$$
  - Hence, WARP also holds in this case.

GARP

- ***Example A2.2*** (continued):
  - Furthermore, bundle x_3 is unaffordable under bundle x_1's prices, i.e., $p_1 \cdot x_3 \leq w_1 = p_1 \cdot x_1$,

$$p_1 \cdot x_3 = (1 \times 0) + (1 \times 0) + (2 \times 1) = 2$$
$$p_1 \cdot x_1 = (1 \times 1) + (1 \times 0) + (2 \times 0) = 1$$

  - Thus, the premise for WARP does not hold when comparing bundles x_1 and x_3.
  - Hence, WARP is not violated.

GARP

- ***Example A2.2*** (continued):
  - Let us check GARP.
  - Assume that bundle x_1 is revealed preferred to x_2, and that x_2 is revealed preferred to x_3, i.e., $x_1 \succsim x_2$ and $x_2 \succsim x_3$.
  - Comparing bundles x_1 and x_3, we can see that bundle x_1 is affordable under bundle x_3's prices, that is

$$p_3 \cdot x_1 = (1 \times 1) + (2 \times 0) + [(1 + \varepsilon) \times 0] = 1$$
$$\succ 1 + \varepsilon = w_3 = p_3 \cdot x_3$$

GARP

- ***Example A2.2*** (continued):
  - In other words, both x_1 and x_3 are affordable at $t = 3$ but only x_3 is chosen.
  - Then, the consumer is revealing a preference for x_3 over x_1, i.e., $x_3 \succsim x_1$.
  - This violates GARP.

GARP

- ***Example A2.3*** (Both WARP and GARP hold):
  - Consider the following sequence of price vectors and their corresponding demanded bundles

$$p_1 = (1,1,1) \quad x_1 = (1,0,0)$$
$$p_2 = (2,1,1) \quad x_2 = (0,1,0)$$
$$p_3 = (3,2,1) \quad x_3 = (0,0,1)$$

GARP

- ***Example A2.3*** (continued):
  - Let us check WARP.
  - Note that bundle x_2 is affordable under the prices at $t = 1$, i.e., $p_1 \cdot x_2 \leq w_1 = p_1 \cdot x_1$, since
    $$(1 \times 0) + (1 \times 1) + (1 \times 0) = 1 \leq 1$$
    $$= (1 \times 1) + (1 \times 0) + (1 \times 0)$$
  - However, bundle x_1 is unaffordable under the prices at $t = 2$, i.e., $p_2 \cdot x_1 > w_2 = p_2 \cdot x_2$, since
    $$(2 \times 1) + (1 \times 0) + (1 \times 0) = 2 > 1$$
    $$= (2 \times 0) + (1 \times 1) + (1 \times 0)$$
  - Thus WARP is satisfied.

GARP

- ***Example A2.3*** (continued):
  - A similar argument applies to the comparison of the choices in $t = 2$ and $t = 3$.
  - Bundle x_3 is affordable under the prices at $t = 2$,
$$p_2 \cdot x_3 = 1 \leq 1 = w_2 = p_2 \cdot x_2$$
  - But x_2 is unaffordable under the prices of $t = 3$, i.e., $p_3 \cdot x_2 > w_3 = p_3 \cdot x_3$, since
$$(3 \times 0) + (2 \times 1) + (1 \times 0) = 2 > (3 \times 0) + (2 \times 0) + (1 \times 1) = 1$$
  - Thus, WARP also holds in this case.

GARP

- ***Example A2.3*** (continued):
  - A similar argument applies to the comparison of the choices in $t = 1$ and $t = 3$.
  - Bundle x_3 is affordable under the prices at $t = 1$, i.e., $p_1 \cdot x_3 \leq w_1 = p_1 \cdot x_1$, since
    $$p_1 \cdot x_3 = 1 \leq 1 = w_1 = p_1 \cdot x_1$$
  - But bundle x_1 is unaffordable under the prices of $t = 3$, i.e., $p_3 \cdot x_1 > w_3 = p_3 \cdot x_3$, since
    $$p_3 \cdot x_1 = (3 \times 1) + (2 \times 0) + (1 \times 0) = 3$$
    $$p_3 \cdot x_3 = (3 \times 0) + (2 \times 0) + (1 \times 1) = 1$$
  - Hence, WARP is also satisfied

GARP

- ***Example A2.3*** (continued):
  - Let us check GARP.
  - We showed above that x_1 is unaffordable under bundle x_3's prices, i.e., $p_3 \cdot x_1 > w_3 = p_3 \cdot x_3$.
  - We cannot establish bundle x_3 being strictly preferred to bundle x_1, i.e., $x_3 \nsucc x_1$.
  - Hence, GARP is satisfied.

GARP

- ***Example A2.4*** (Neither WARP nor GARP hold):
  - Consider the following sequence of price vectors and their corresponding demanded bundles

$$p_1 = (1,1,1) \quad x_1 = (1,0,0)$$
$$p_2 = (1,2,1) \quad x_2 = (0,1,0)$$
$$p_3 = (1,2,1+\varepsilon) \quad x_3 = (0,0,1)$$

  - This is actually a combination of the first two examples.

GARP

- ***Example A2.4*** (continued):
  - At $t = 1$, WARP will be violated by the same method as in our first example.
  - At $t = 3$, GARP will be violated by the same method as in our second example.
  - Hence, neither WARP nor GARP hold in this case.

GARP

- GARP constitutes a sufficient condition for utility maximization.
- That is, if the sequence of price-bundle pairs (p^t, x^t) satisfies GARP, then it must originate from a utility maximizing consumer.
- We refer to (p^t, x^t) as a set of "data."

GARP

- ***Afriat's theorem***. For a sequence of price-bundle pairs (p^t, x^t), the following statements are equivalent:
  1) The data satisfies GARP.
  2) The data can be rationalized by a utility function satisfying LNS.
  3) There exist positive numbers (u^t, λ^t) for all $t = 1,2, \dots, T$ that satisfy the Afriat inequalities

$$u^t \leq u^s + \lambda^s p^s (x^t - x^s) \text{ for all } s, t$$

  4) The data can rationalized by a continuous, concave, and strongly monotonic utility function.

GARP

- A data set is "rationalized" by a utility function if:
  - for every pair (p^t, x^t), bundle x^t yields a higher utility than any other feasible bundle x, i.e., $u(x^t) \geq u(x)$ for all x in budget set $B(p^t, p^t x^t)$.
- Hence, if a data set satisfies GARP, there exists a well-behaved utility function rationalizing such data.
  - That is, the utility function satisfies LNS, continuity, concavity, and strong monotonicity.

GARP

- Condition (3) in Afriat's Theorem has a concavity interpretation:
  - From FOCs, we have $Du(x^t) = \lambda^t p^t$.
  - By concavity,

$$\frac{u(x^t)-u(x^s)}{x^t-x^s} \leq u'(x^s)$$

  - Or, re-arranging,

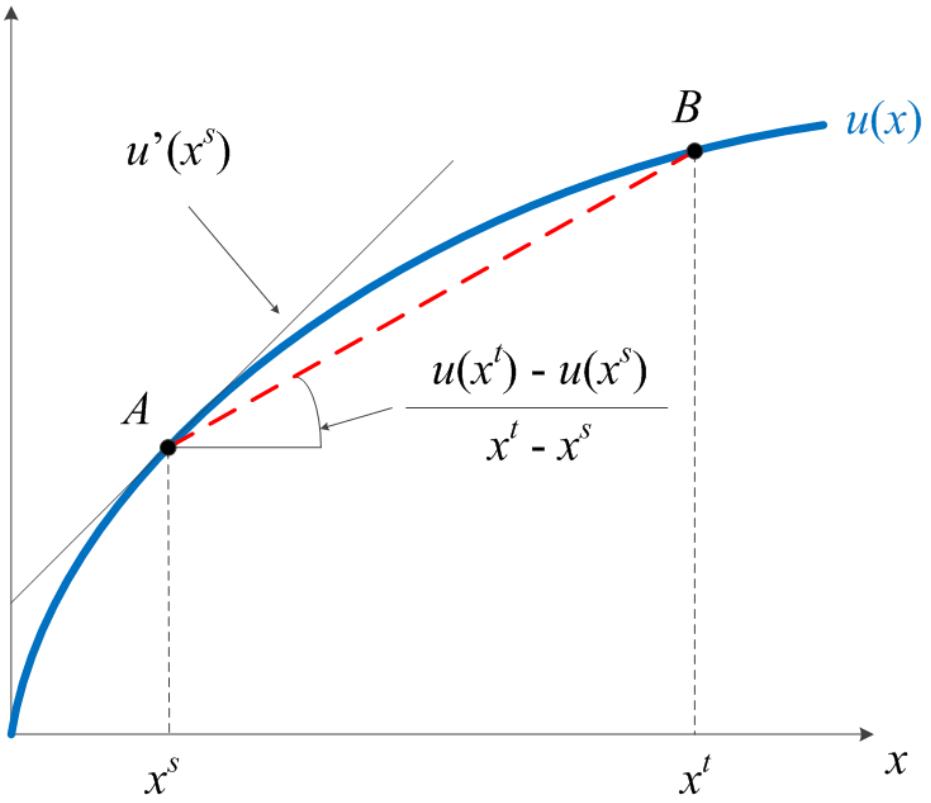
$$u(x^t) \leq u(x^s) + u'(x^s)(x^t - x^s)$$

  - Since $u'(x^s) = \lambda^s p^s$, it can be expressed as Afriat's inequality

$$u(x^t) \leq u(x^s) + \lambda^s p^s(x^t - x^s)$$

GARP

- Concave utility function
  - The utility function at point x^s is steeper than the ray connecting points *A* and *B*.



Advanced Microeconomic Theory

Chapter 3: Demand Theory Applications

Outline

- Welfare evaluation
  - Compensating variation
  - Equivalent variation
- Quasilinear preferences
- Slutsky equation revisited
- Income and substitution effects in labor markets
- Gross and net substitutability
- Aggregate demand

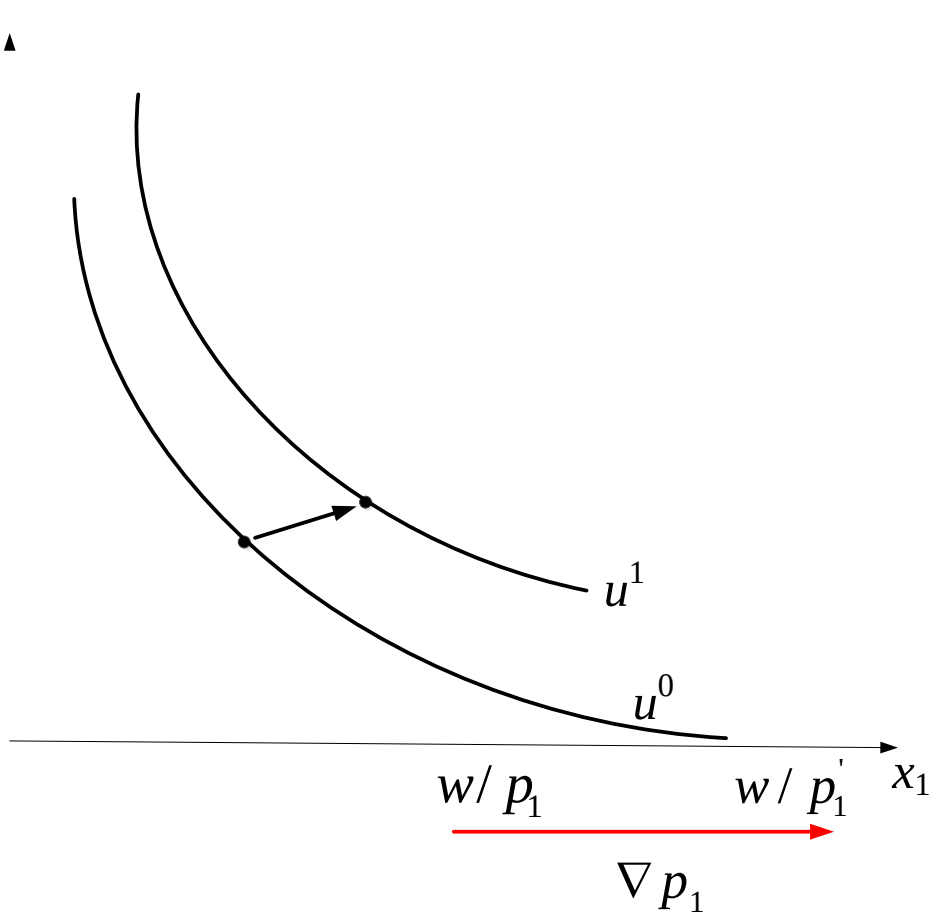
Measuring the Welfare Effects of a Price Change

Measuring the Welfare Effects of a Price Change

- How can we measure the welfare effects of:
  - a price decrease/increase
  - the introduction of a tax/subsidy
- Why not use the difference in the individual's utility level, i.e., from u^0 to u^1?
  - Two problems:
    1) *Within-subject criticism*: Only ranking matters (ordinality), not the difference;
    2) *Between-subject criticism*: Utility measures would not be comparable among different individuals.
- Instead, we will pursue monetary evaluations of such price/tax changes.

Measuring the Welfare Effects of a Price Change

- Consider a price decrease from p_1^0 to p_1^1.
- We cannot compare u^0 to u^1.
- Instead, we will find a money-metric measure of the consumer's welfare change due to the price change.



Measuring the Welfare Effects of a Price Change

- ***Compensating Variation (CV)***:
  - How much money a consumer would be willing to give up *after* a reduction in prices to be just as well off as *before* the price decrease.
- ***Equivalent Variation (EV)***:
  - How much money a consumer would need *before* a reduction in prices to be just as well off as *after* the price decrease.

Measuring the Welfare Effects of a Price Change

- Two approaches:
  1) Using expenditure function
  2) Using the Hicksian demand

CV using Expenditure Function

- $CV(p^0, p^1, w)$ using $e(p, u)$:

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0)$$

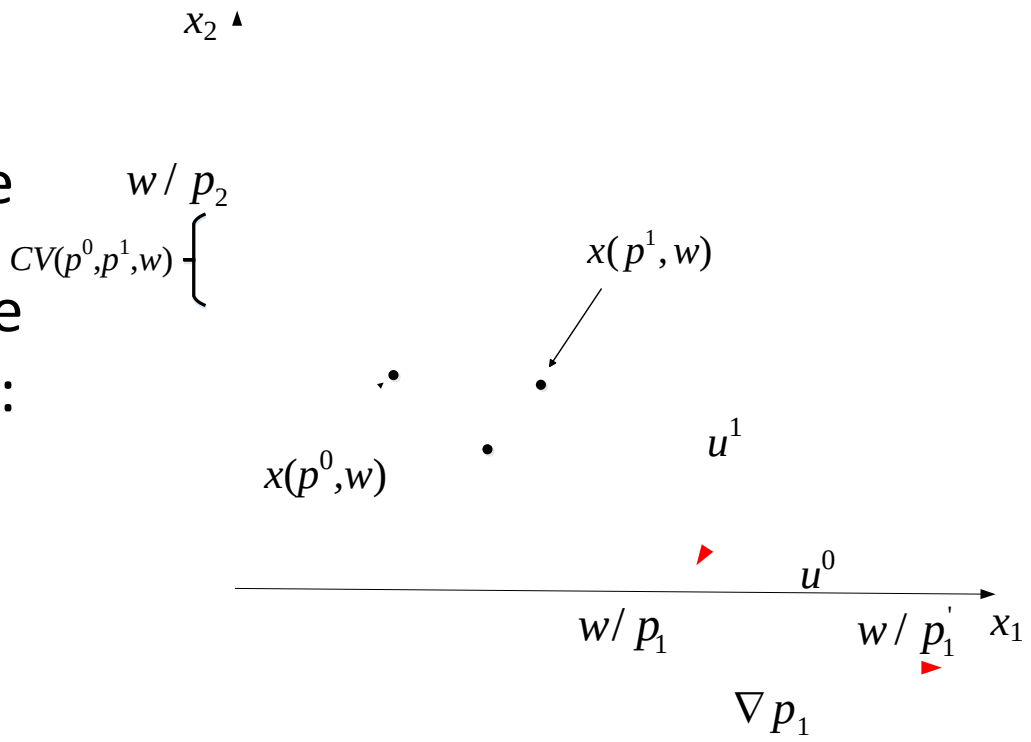
- The amount of money the consumer is willing to give up *after* the price decrease (after price level is p^1 and her utility level has improved to u^1) to be just as well off as *before* the price decrease (reaching utility level u^0).

CV using Expenditure Function

1) When $B_{p^0,w}$, $x(p^0, w)$
2) ∇p_1 and $x(p^1, w)$ under $B_{p^1,w}$
3) Adjust final wealth (*after* the price change) to make the consumer as well off as *before* the price change
4) Difference in expenditure:

$$CV(p^0, p^1, w) = \underbrace{e(p^1, u^1)}_{\text{at } B_{p^1,w}} - \underbrace{e(p^1, u^0)}_{\text{dashed line}}$$

This is Hicksian wealth compensation!



EV using Expenditure Function

- $EV(p^0, p^1, w)$ using $e(p, u)$:

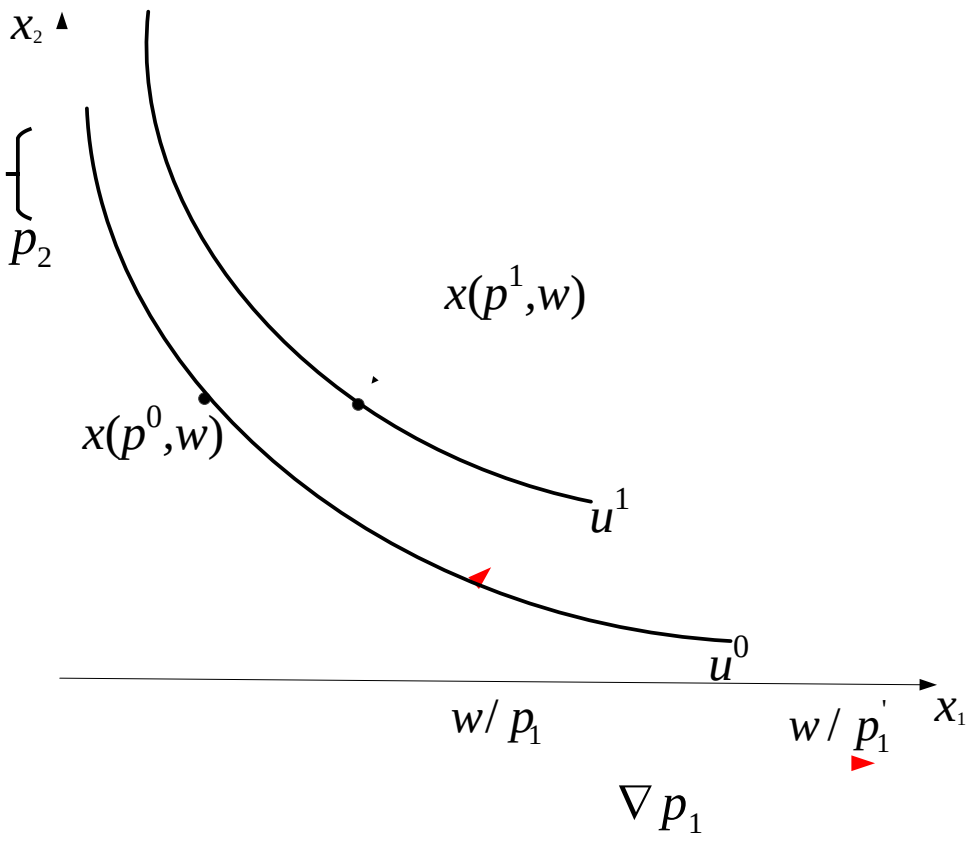
$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0)$$

- The amount of money the consumer needs to receive *before* the price decrease (at the initial price level p^0 when her utility level is still u^0) to be just as well off as *after* the price decrease (reaching utility level u^1).

EV using Expenditure Function

1) When $B_{p^0,w}$, $x(p^0, w)$
2) ∇p_1 and $x(p^1, w)$ under $B_{p^1,w}$
3) Adjust initial wealth (*before* the price change) to make the consumer as well off as *after* the price change
4) Difference in expenditure:

$$EV(p^0, p^1, w) = \underbrace{e(p^0, u^1)}_{\text{dashed line}} - \underbrace{e(p^0, u^0)}_{\text{at } B_{p^0,w}}$$



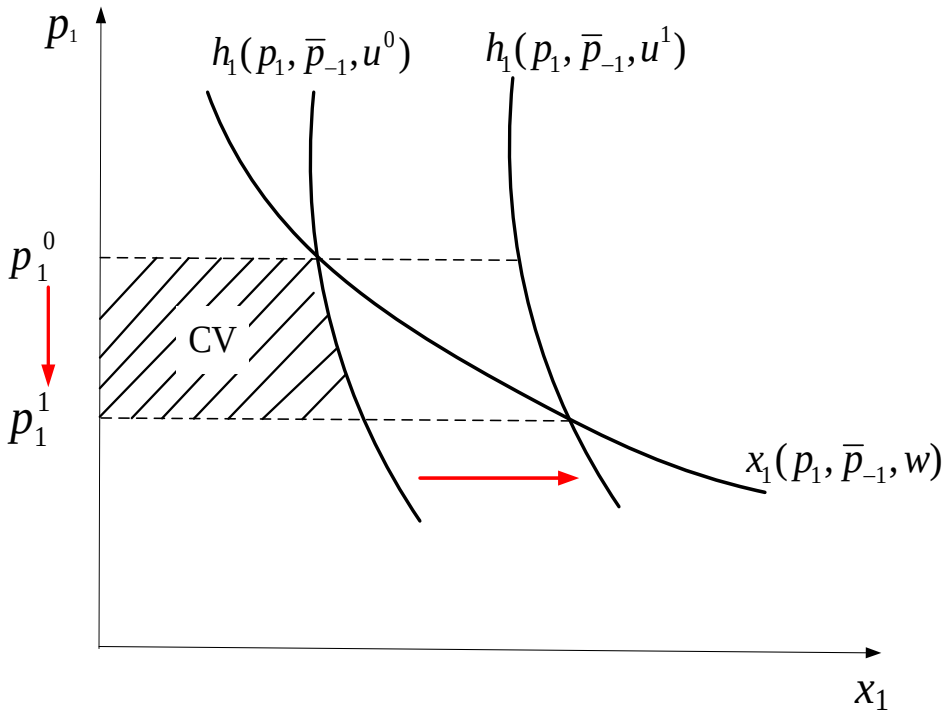
CV using Hicksian Demand

- From the previous definitions we know that, if $p_1^1 < p_1^0$ and $p_k^1 = p_k^0$ for all $k \neq 1$, then

$$\begin{aligned} CV(p^0, p^1, w) &= e(p^1, u^1) - e(p^1, u^0) \\ &= w - e(p^1, u^0) \\ &= e(p^0, u^0) - e(p^1, u^0) \\ &= \int_{p_1^1}^{p_1^0} \frac{\partial e(p_1, \bar{p}_{-1}, u^0)}{\partial p_1} dp_1 \\ &= \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^0)\, dp_1 \end{aligned}$$

CV using Hicksian Demand

- The case is:
  - Normal good
  - Price decrease
- Graphically, CV is represented by the area to the left of the Hicksian demand curve for good 1 associated with utility level u^0, and lying between prices p_1^1 and p_1^0.
- The welfare gain is represented by the shaded region.



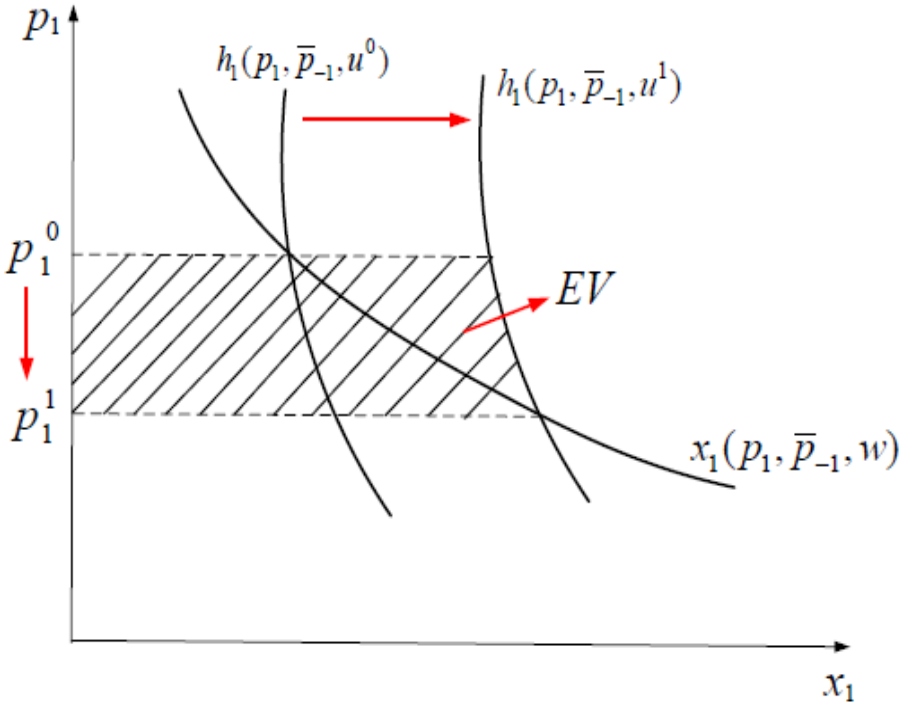
EV using Hicksian Demand

- From the previous definitions we know that, if $p_1^1 < p_1^0$ and $p_k^1 = p_k^0$ for all $k \neq 1$, then

$$\begin{aligned} EV(p^0, p^1, w) &= e(p^0, u^1) - e(p^0, u^0) \\ &= e(p^0, u^1) - w \\ &= e(p^0, u^1) - e(p^1, u^1) \\ &= \int_{p_1^1}^{p_1^0} \frac{\partial e(p_1, \bar{p}_{-1}, u^1)}{\partial p_1} dp_1 \\ &= \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1 \end{aligned}$$

EV using Hicksian Demand

- The case is:
  - Normal good
  - Price decrease
- Graphically, EV is represented by the area to the left of the Hicksian demand curve for good 1 associated with utility level u^1, and lying between prices p_1^1 and p_1^0.
- The welfare gain is represented by the shaded region.



What about a price increase?

- The Hicksian demand associated with initial utility level u^0 (before the price increase, or before the introduction of a tax) experiences an inward shift when the price increases, or when the tax is introduced, since the consumer's utility level is now u^1, where $u^0 > u^1$. Hence,

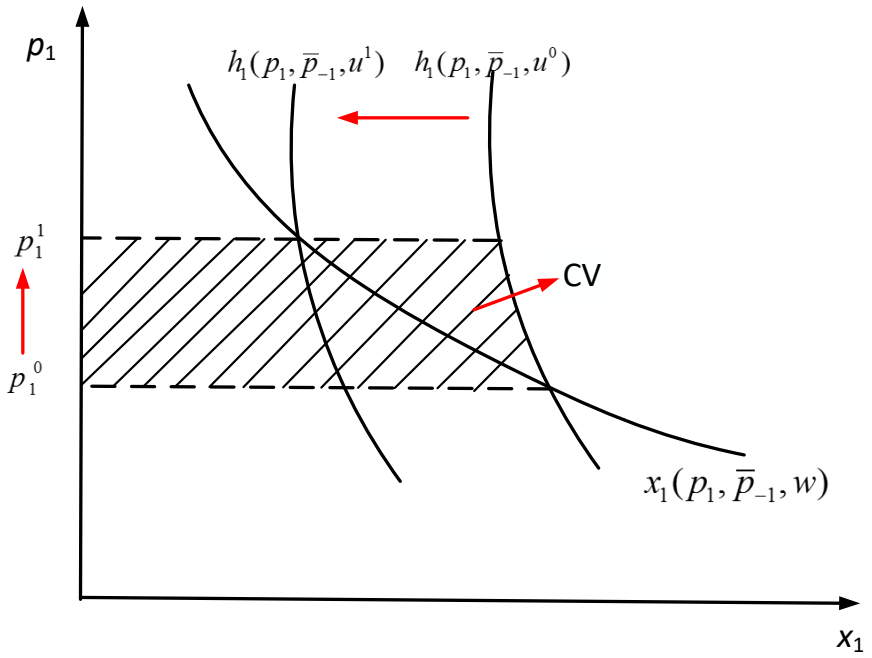
$$h_1(p_1, \bar{p}_{-1}, u^0) > h_1(p_1, \bar{p}_{-1}, u^1)$$

What about a price increase?

- The definitions of CV and EV would now be:
  - CV: the amount of money that a consumer would need *after* a price increase to be as well off as *before* the price increase.
  - EV: the amount of money that a consumer would be willing to give up *before* a price increase to be as well off as *after* the price increase.
- Graphically, it looks like the CV and EV areas have been reversed:
  - CV is associated to the area below $h_1(p_1, \bar{p}_{-1}, u^0)$ as usual
  - EV is associated with the area below $h_1(p_1, \bar{p}_{-1}, u^1)$.

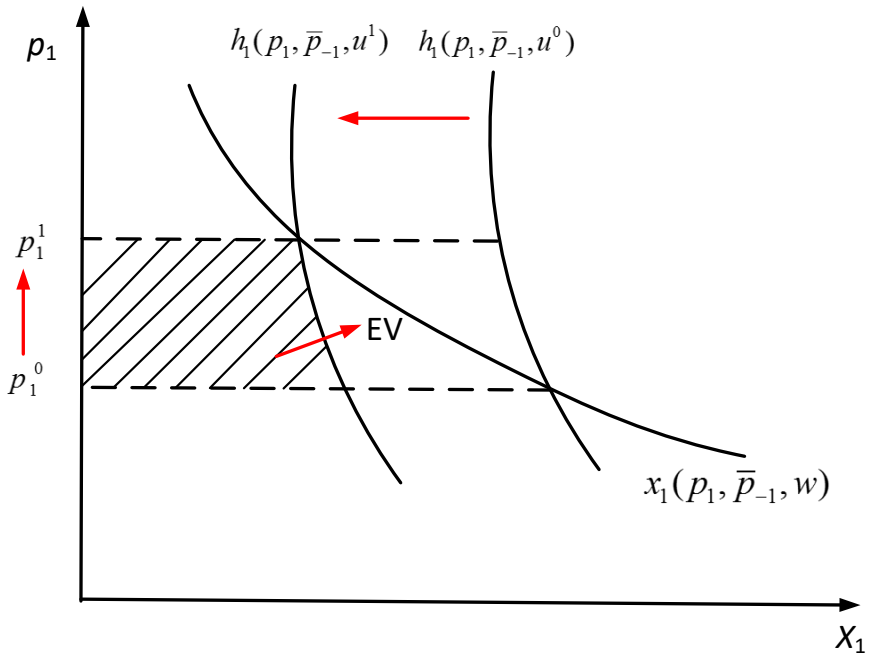
What about a price increase?

- CV is always associated with $h_1(p_1, \bar{p}_{-1}, u^0)$
- $CV(p^0, p^1, w) = \int_{p_1^0}^{p_1^1} h_1(p_1, \bar{p}_{-1}, u^0)\, dp_1$



What about a price increase?

- EV is always associated with $h_1(p_1, \bar{p}_{-1}, u^1)$
- $EV(p^0, p^1, w) = \int_{p_1^0}^{p_1^1} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1$



Introduction of a Tax

- The introduction of a tax can be analyzed as a price increase.
- *The main difference*: we are interested in the area of CV and EV that is *not* related to tax revenue.
- Tax revenue is:

$$T = \underbrace{[(p_1^0 + t) - p_1^0]}_{t} \cdot h(p_1, \bar{p}_{-1}, u^0) \quad \text{(using CV)}$$

$$T = \underbrace{[(p_1^0 + t) - p_1^0]}_{t} \cdot h(p_1, \bar{p}_{-1}, u^1) \quad \text{(using EV)}$$

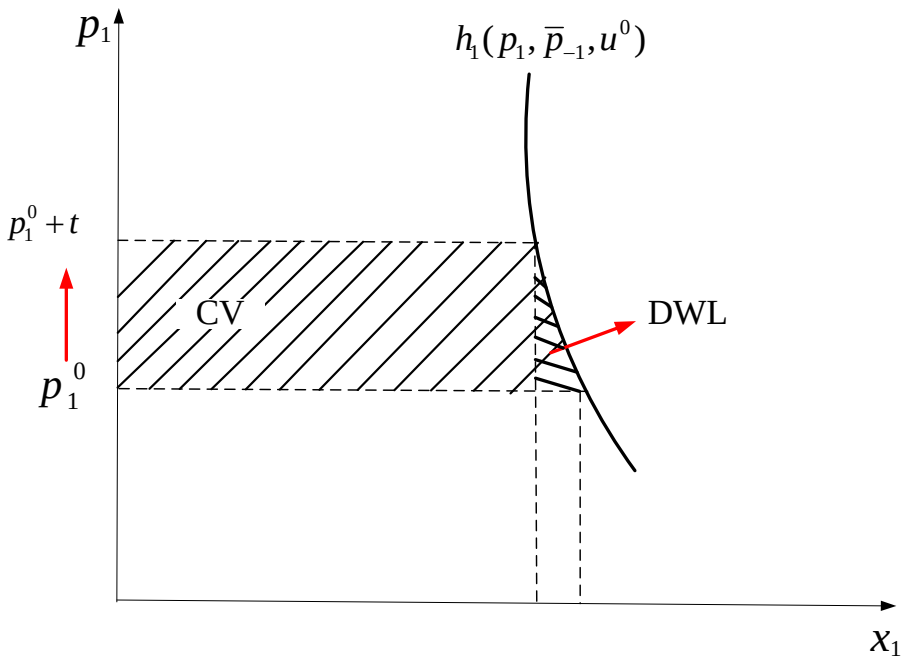
Introduction of a Tax

- CV is measured by the large shaded area to the left of $h(p_1, \bar{p}_{-1}, u^0)$:

$$CV(p^0, p^1, w) = \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^0)\, dp_1$$

- Welfare loss (DWL) is the area of the CV not transferred to the government via tax revenue:

$$DWL = CV - T$$



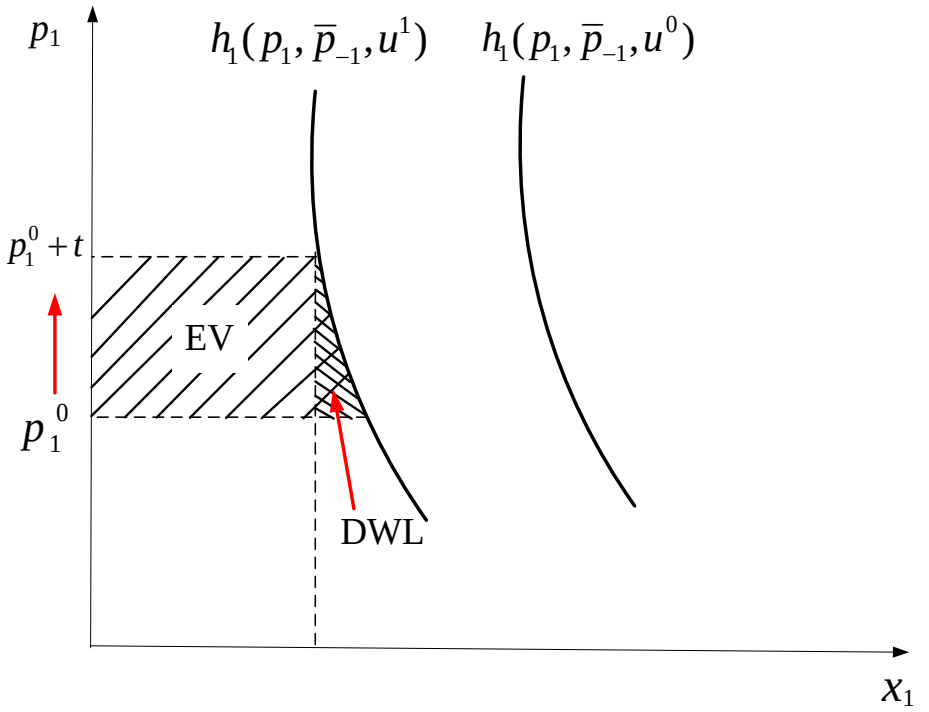
Introduction of a Tax

- EV is measured by the large shaded area to the left of $h(p_1, \bar{p}_{-1}, u^1)$:

$$EV(p^0, p^1, w) = \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1$$

- Welfare loss (DWL) is the area of the EV not transferred to the government via tax revenue:

$$DWL = EV - T$$

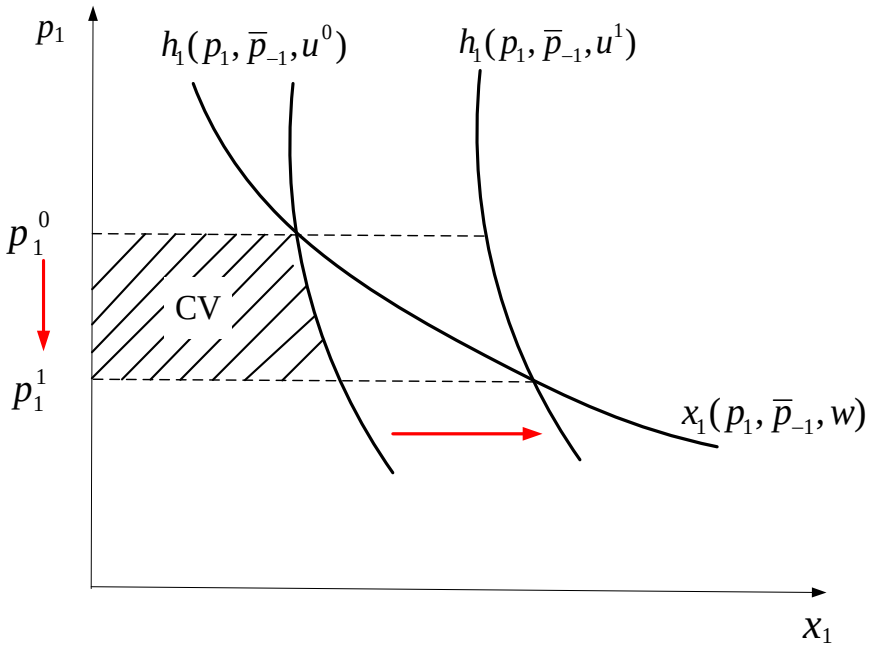


Why not use the Walrasian demand?

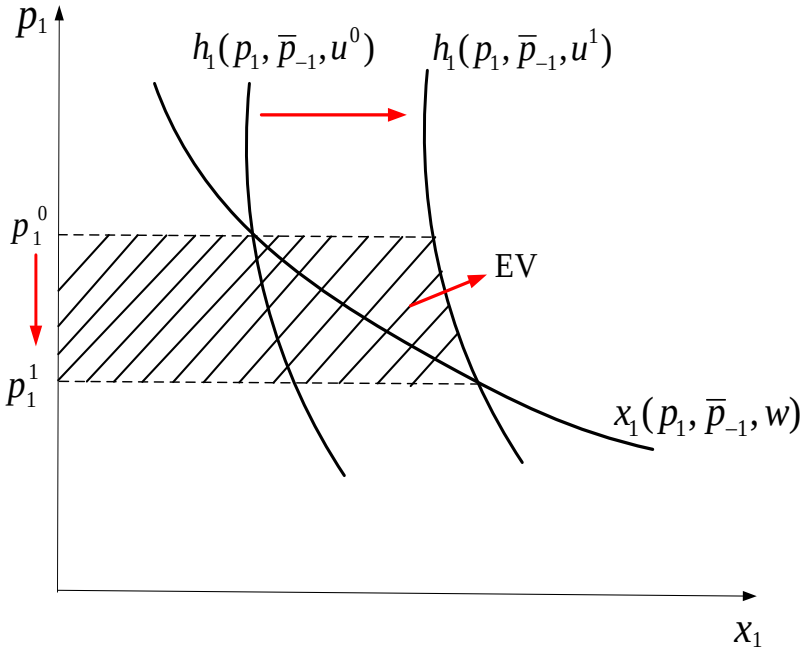
- Walrasian demand is easier to observe, so we could use the variation in consumer's surplus as an approximation of welfare changes.
- This is only valid when income effects are *zero*:
  - Recall that the Walrasian demand measures both income and substitution effects resulting from a price change, while
  - The Hicksian demand measures only substitution effects from such a price change.
- Hence, there will be a difference between CV and CS, and between EV and CS.

Why not use the Walrasian demand?

- Normal goods:



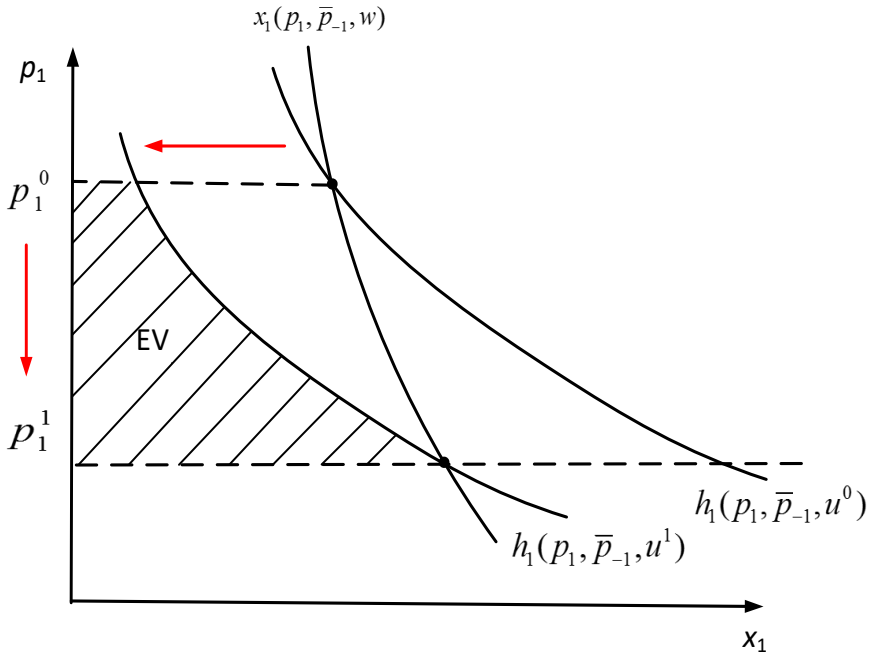
$$CV < CS$$



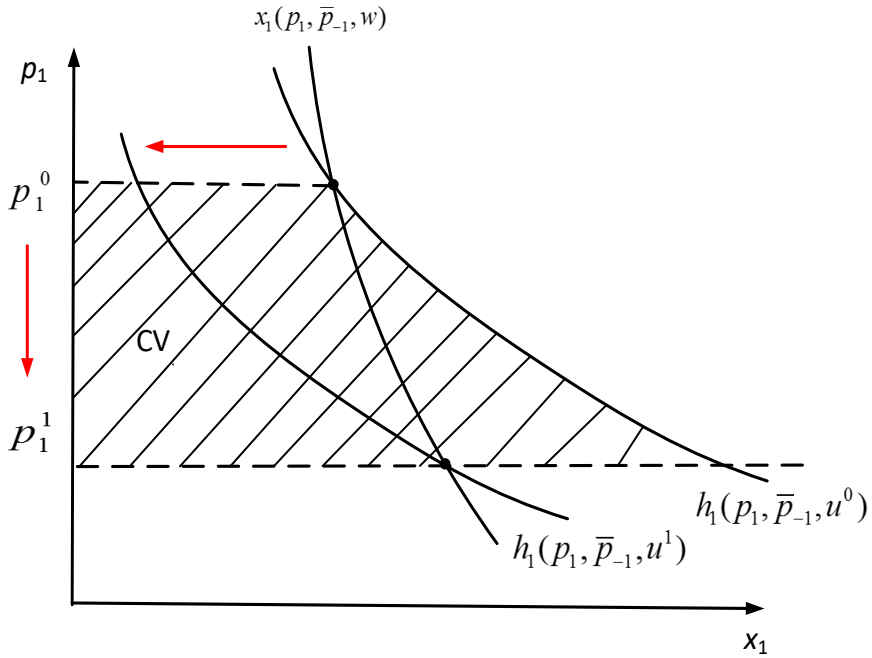
$$CS < EV$$

Why not use the Walrasian demand?

- Inferior goods:



$EV < CS$



$CS < CV$

Why not use the Walrasian demand?

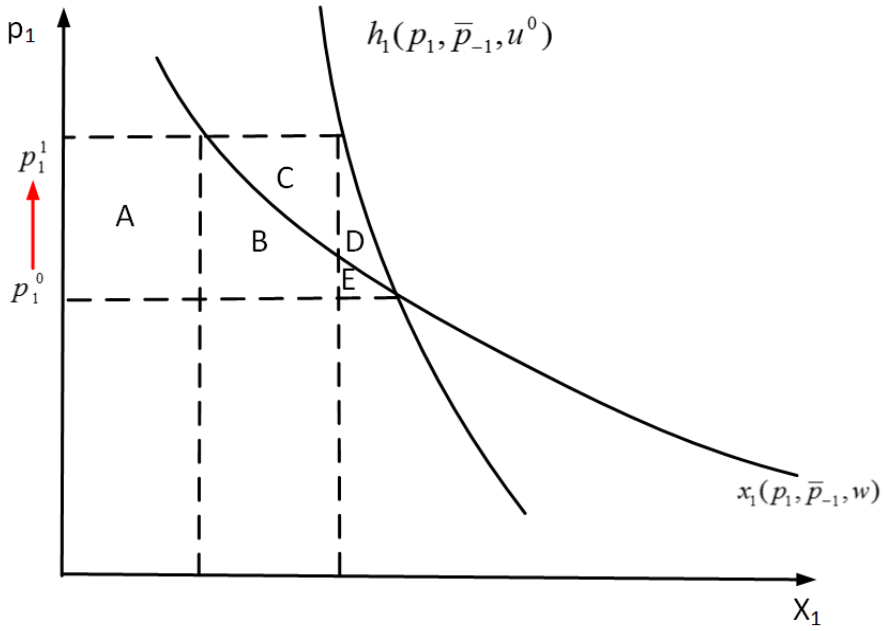
- For normal goods:
  - Price decrease: $CV < CS < EV$
  - Price increase: $CV > CS > EV$

- For inferior goods we find the opposite ranking:
  - Price decrease: $CV > CS > EV$
  - Price increase: $CV < CS < EV$

- NOTE: consumer surplus is also referred to as the ***area variation*** (AV).

When can we use the Walrasian demand?

- When the price change is small (using AV):
  - $CV = A + B + C + D + E$
  - $CS = A + B + E$
  - Measurement error from using CS (or AV) is $C + D$



When can we use the Walrasian demand?

- The measurement difference between CV (and EV) and CS, $C + D$, is relatively small:
  1) When income effects are small:
     - Graphically, $x(p, w)$ and $h(p, u)$ almost coincide.
     - The welfare change using the CV and EV coincide too.
  2) When the price change is very small:
     - The error involved in using *AV*, i.e., areas $C + D$, as a fraction of the true welfare change, becomes small. That is,

$$\lim_{(p_1^1 - p_1^0) \to 0} \frac{C + D}{CV} = 0$$

When can we use the Walrasian demand?

- However, if we measure the approximation error by $\frac{C+D}{DW}$, where $DW = D + E$, then

$$\lim_{(p_1^1 - p_1^0) \to 0} \frac{C + D}{DW}$$

does not necessarily converge to zero.

When can we use the Walrasian demand?

- Another possibility when the price change is relatively small:
  - Take a first-order Taylor approximation of $h(p, u^0)$ at p^0,

$$\tilde{h}(p, u^0) = h(p^0, u^0) + D_p h(p^0, u^0)\,(p - p^0)$$

and then calculate

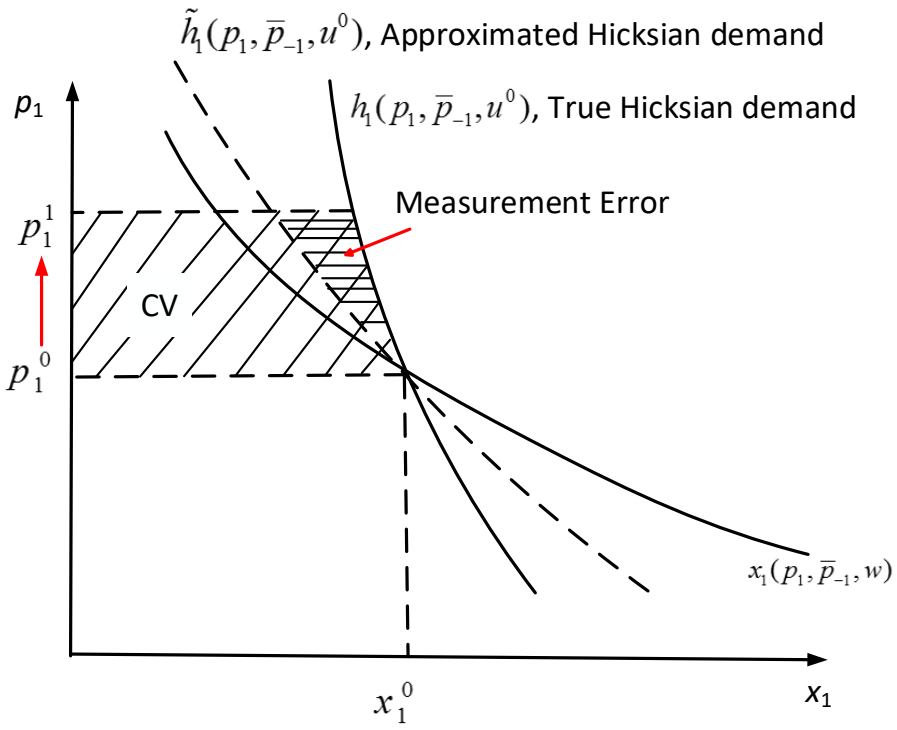
$$CV = \int_{p_1^0}^{p_1^1} \tilde{h}_1(p_1, \bar{p}_{-1}, u^0) dp_1$$

where, since $h(p^0, u^0) = x(p^0, w)$ and $D_p h(p^0, u^0) = S(p^0, w)$, we can rewrite the "approximated" Hicksian as

$$\tilde{h}(p, u^0) = x(p^0, w) + S(p^0, w)(p - p^0)$$

When can we use the Walrasian demand?

- The "approximated" Hicksian demand $\tilde{h}_1(p_1, \bar{p}_{-1}, u^0)$ lies between the true Hicksian demand, $h_1(p_1, \bar{p}_{-1}, u^0)$, and the Walrasian demand, $x(p^0, w)$.
- Since $\tilde{h}(p, u^0)$ has the same slope as $h(p, u^0)$ at price p^0, and a small change in prices will not imply a big error.



Application of IE and SE

- From the Slutsky equation, we know

$$\frac{\partial h_1(p,u)}{\partial p_1} = \frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w)$$

- Multiplying both terms by $\frac{p_1}{x_1}$,

$$\frac{\partial h_1(p,u)}{\partial p_1}\frac{p_1}{x_1} = \frac{\partial x_1(p,w)}{\partial p_1}\frac{p_1}{x_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w)\frac{p_1}{x_1}$$

And multiplying the last term by $\frac{w}{w}$,

$$\underbrace{\frac{\partial h_1(p,u)}{\partial p_1}\frac{p_1}{x_1}}_{\substack{\text{Substitution Price elasticity of demand} \\ \tilde{\varepsilon}_{p,Q}}} = \underbrace{\frac{\partial x_1(p,w)}{\partial p_1}\frac{p_1}{x_1}}_{\substack{\text{Price elasticity of demand} \\ \varepsilon_{p,Q}}} + \underbrace{\frac{\partial x_1(p,w)}{\partial w} x_1(p,w)\frac{p_1}{x_1}\frac{w}{w}}_{?}$$

Application of IE and SE

- Rearranging the last term, we have

$$\frac{\partial x_1(p,w)}{\partial w} x_1(p,w) \frac{p_1}{x_1} \frac{w}{w}$$

$$= \underbrace{\frac{\partial x_1(p,w)}{\partial w} \frac{w}{x_1}}_{\substack{\text{Income elasticity} \\ \text{of demand} \\ \varepsilon_{w,Q}}} \cdot \underbrace{\frac{p_1 x_1(p,w)}{w}}_{\substack{\text{Share of budget} \\ \text{spent on good 1, } \theta}}$$

- We can then rewrite the Slutsky equation in terms of elasticities as follows

$$\tilde{\varepsilon}_{p,Q} = \varepsilon_{p,Q} + \varepsilon_{w,Q} \cdot \theta$$

Application of IE and SE

- ***Example***: consider a good like housing, with $\theta = 0.4$, $\varepsilon_{w,Q} = 1.38$, and $\varepsilon_{p,Q} = -0.6$.
- Therefore,

$$\tilde{\varepsilon}_{p,Q} = \varepsilon_{p,Q} + \varepsilon_{w,Q} \cdot \theta = -0.6 + 1.38 \cdot 0.4 = -0.05$$

- If price of housing rises by 10%, and consumers do not receive a wealth compensation to maintain their welfare unchanged, consumers reduce their consumption of housing by 6%.
- However, if consumers receive a wealth compensation, the housing consumption will only fall by 0.5%.
  - Intuition: Housing is such an important share of my monthly expenses, that higher prices lead me to significantly reduce my consumption (if not compensated), but to just slightly do so (if compensated).

Application of IE and SE

- Other useful lessons from the previous expression

$$\tilde{\varepsilon}_{p,Q} = \varepsilon_{p,Q} + \varepsilon_{w,Q} \cdot \theta$$

- Price-elasticities very close $\tilde{\varepsilon}_{p,Q} \simeq \varepsilon_{p,Q}$ if
  - Share of budget spent on this particular good, θ, is very small (example: garlic).
  - The income-elasticity is really small (example: pizza).
- Advantages if $\tilde{\varepsilon}_{p,Q} \simeq \varepsilon_{p,Q}$:
  - The Walrasian and Hicksian demand are very close to each other. Hence, $CV \simeq EV \simeq CS$.

Application of IE and SE

- You can read sometimes “in this study we use the change in CS to measure welfare changes due to a price increase given that income effects are negligible”
  - What the authors are referring to is:
    - Share of budget spent on the good is relatively small and/or
    - The income-elasticity of the good is small
- Remember that our results are not only applicable to price changes, but also to changes in the sales taxes.
- For which preference relations a price change induces no income effect? Quasilinear.

Application of IE and SE

- In 1981 the US negotiated voluntary automobile export restrictions with the Japanese government.
- Clifford Winston (1987), in "Blind Intersection? Policy and the Automobile Industry", studied the effects of these export restrictions:
  - Car prices: p_{Jap} was 20% higher with restrictions than without. p_{US} was 8% higher with restrictions than without.
  - Would you use CS to gauge the effect of these higher prices on consumers' welfare? Probably not, since both θ and $\varepsilon_{w,Q}$ are relatively high.

Application of IE and SE

- Winston did not use CS. Instead, he focused on the CV. He found that CV = -$14 billion.
  - *Intuition*: The wealth compensation that domestic car owners would need after the price change (after setting the export restrictions) in order to be as well off as they were before the price change is $14 billion.
- This implies that, considering that in 1987 there were 179 million car owners in the US, the wealth compensation per car owner should have been $14 billion / 179 million ≈ $78.
- Of course, this is an underestimation, since we should divide by the number of new car owners during the period the export restriction was active (not the number of all car owners).

Application of IE and SE

- Hausman, J. and Leibtag, E. (2007) measures the welfare gain consumers obtain from the price decrease they experience after a Walmart store locates in their locality/country.
- He used CV. Why? Low-income families spend a non-negligible part of their budget in Wal-Mart.
- Result: welfare improvement of 3.75%.

Consumer as a Labor Supplier

Consumer as a Labor Supplier

- Consider the following UMP, where the consumer chooses the amount of goods, x, and leisure, L, that solve

$$\max_{x,L} \; u(x, L)$$

$$\text{s.t.} \; \sum_{i=1}^{K} p_i x_i \leq M = wz + \bar{M}$$

$$\text{and} \; T = z + L$$

where M is total wealth, coming from the z hours dedicated to work (at a wage w per hour), and the non-work income, $\bar{M}$. Total time T must be either dedicated to work (z) or leisure (L).

Consumer as a Labor Supplier

- Let us now use the Composite Commodity Theorem:
  - If the prices for all goods maintain a constant proportion with respect to the price of labor (wage), i.e., $p_1 = \alpha_1 w$, …, $p_n = \alpha_n w$, we can represent these goods all by a single (composite) commodity y, with price p.
  - Then, we have only two goods: the composite commodity y and the number of hours dedicated to work, z.

Consumer as a Labor Supplier

- Hence, the UMP can be rewritten as

$$\max_{y,z} \; v(y,z)$$
$$\text{s.t.} \quad py \leq wz + \bar{M}$$

where py represents the money spent on consumption goods, and $wz + \bar{M}$ reflects the total income originating from labor and non-labor sources.

Consumer as a Labor Supplier

- The Lagrangian of this UMP is

$$L = v(y, z) + \lambda(\bar{M} + wz - py)$$

and the FOCs (for interior optimum) are

$$\frac{\partial L}{\partial y} = v_y - \lambda p = 0 \implies \lambda = \frac{v_y}{p}$$

$$\frac{\partial L}{\partial z} = v_z + \lambda w = 0 \implies \lambda = -\frac{v_z}{w}$$

- Hence, $\frac{v_y}{p} = -\frac{v_z}{w}$.
  - That is, since v_z < 0, −v_z/w is the marginal disutility per dollar earned; at the optimum it equals v_y/p, the marginal utility per dollar spent on consumption.

Consumer as a Labor Supplier

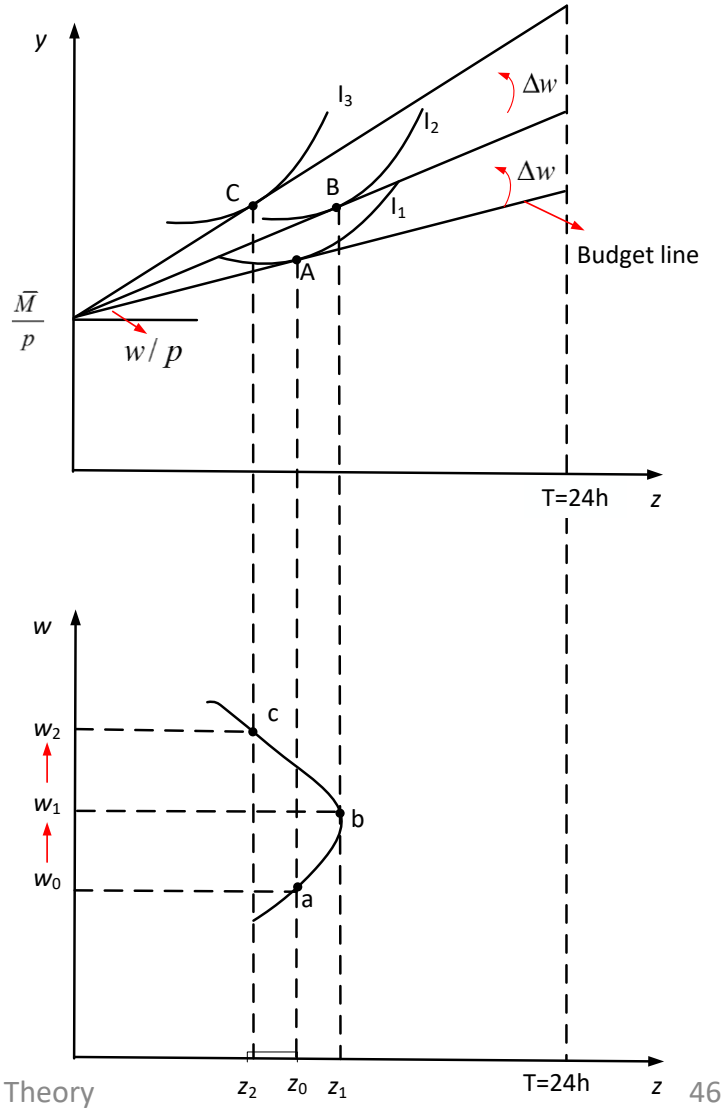
- Rearranging $\frac{v_y}{p} = -\frac{v_z}{w}$, we obtain

$$MRS_{z,y} \rightarrow -\frac{v_z}{v_y} = \frac{w}{p}$$

- Finally, using $\frac{v_y}{p} = -\frac{v_z}{w}$ and the constraint, we obtain the Walrasian demand for the composite commodity, $x_y(w, p, \bar{M})$, and the labor supply function, $x_z(w, p, \bar{M})$.

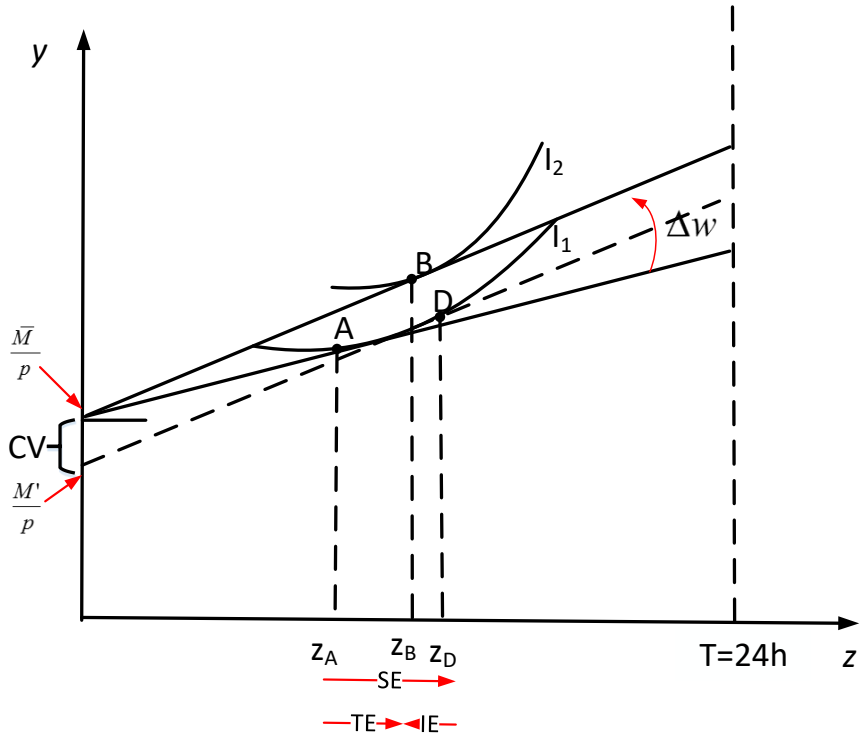
Consumer as a Labor Supplier

- Budget line: upward sloping straight line.
  - an increase in the amount of hours worked entails a larger amount of wealth.
- Indifference curve: increasing utility levels as we move northwest.
  - individual is better off when his consumption of the composite commodity increases and the number of working hours decreases.
- Labor supply: backward bending.
  - labor supply initially increases as a result of higher wages but then decreases.
  - Substitution and Income effects.



Consumer as a Labor Supplier

- SE (+):
  - $\Delta w > 0$ implies $\Delta z > 0$ (more working hours supplied).
- IE (-):
  - $\Delta w > 0$ implies $\Delta z < 0$ (fewer working hours supplied).
- If $|IE| > |SE|$, then working hours would become a Giffen good.



Consumer as a Labor Supplier

- How to relate this income and substitution effects with the Slutsky equation?
  - First, let us state the previous problems as an EMP

$$\min_{y,z} \ \bar{M} = py - wz$$

$$\text{s.t.} \quad v(y,z) = v$$

  - From this EMP we can find the optimal Hicksian demands, $h_y(w,p,v)$ and $h_z(w,p,v)$.
  - Inserting them into the objective function, we obtain the value function of this EMP (i.e., the expenditure function):

$$e(w,p,v) = ph_y(w,p,v) - wh_z(w,p,v)$$

Consumer as a Labor Supplier

- How to relate this income and substitution effects with the Slutsky equation?
  - We know that

$$x_z\left(\underbrace{w,p}_{\text{prices}}, \underbrace{e(w,p,v)}_{\text{income}}\right) = h_z(w,p,v)$$

Differentiating both sides with respect to w and using the chain rule

$$\frac{\partial x_z}{\partial w} + \frac{\partial x_z}{\partial e}\frac{\partial e}{\partial w} = \frac{\partial h_z}{\partial w} \Leftrightarrow \frac{\partial x_z}{\partial w} = \frac{\partial h_z}{\partial w} - \frac{\partial x_z}{\partial e}\frac{\partial e}{\partial w}$$

and since we know that $\frac{\partial e(w,p,v)}{\partial w} = -h_z(w,p,v)$, then

$$\frac{\partial x_z}{\partial w} = \frac{\partial h_z}{\partial w} + \frac{\partial x_z}{\partial e}h_z(w,p,v)$$

Consumer as a Labor Supplier

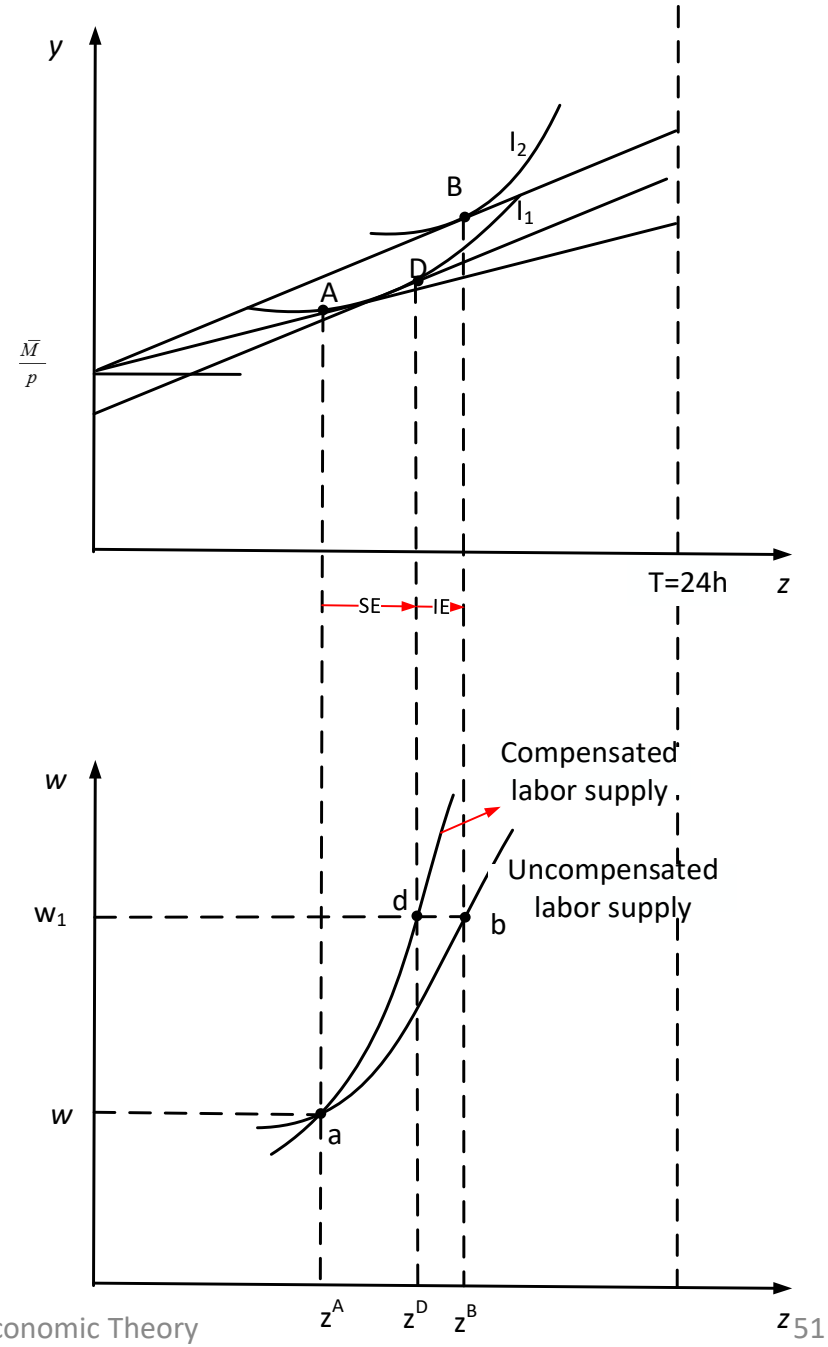
- Using the Slutsky equation (SE and IE) in the analysis of labor markets:

$$\frac{\partial x_z}{\partial w} = \frac{\partial h_z}{\partial w} + \frac{\partial x_z}{\partial e} h_z(w, p, v)$$

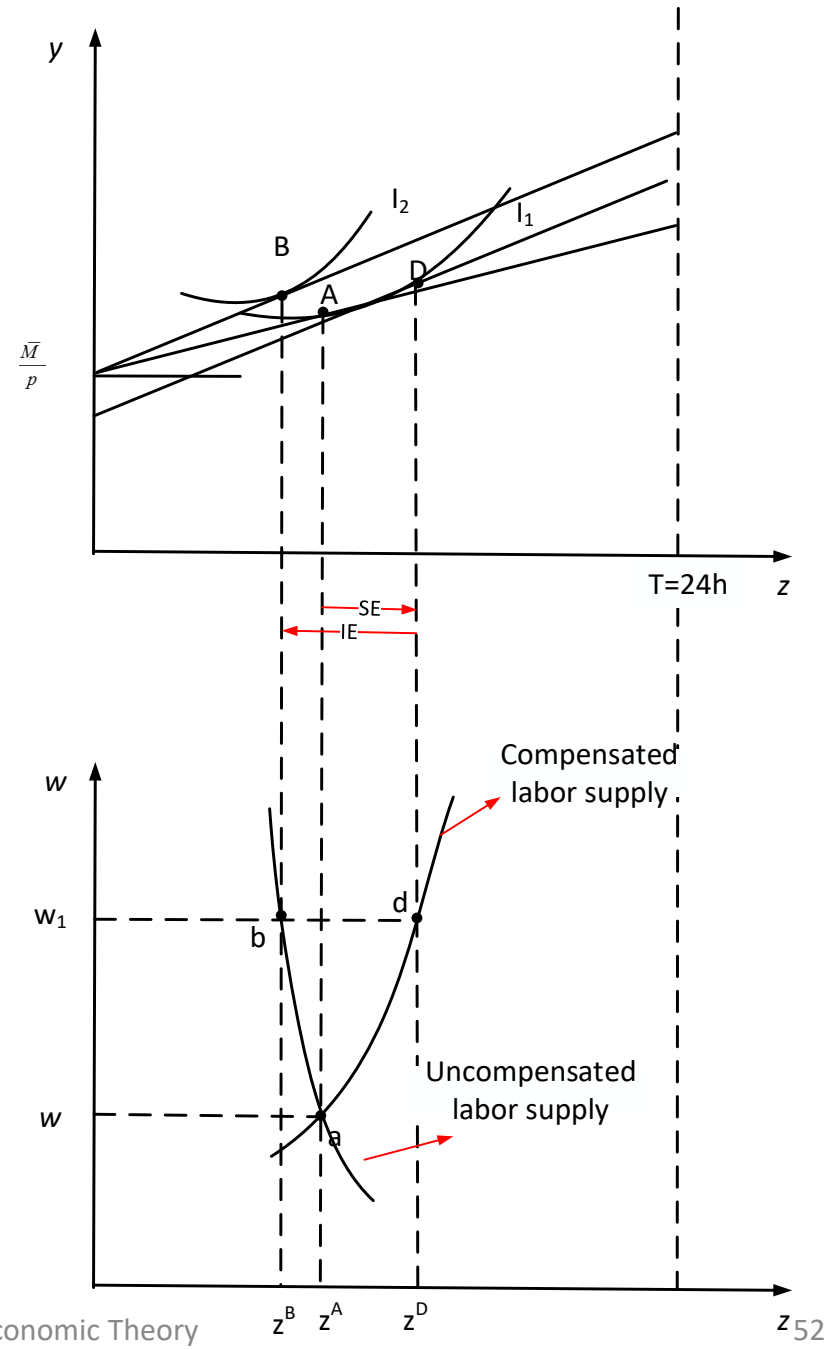
where

1) $\frac{\partial h_z}{\partial w} > 0$ is the SE effect: an increase in wages increases the worker's supply of labor, if we give him a wealth compensation.
2) $\frac{\partial x_z}{\partial e} h_z(w, p, v)$ is the IE:
   - If $\frac{\partial x_z}{\partial e} > 0$, an increase in wages makes that worker richer, and he decides to work ***more*** (this would be an upward bending supply curve);
   - If $\frac{\partial x_z}{\partial e} < 0$, an increase in wages makes that worker richer, but he decides to work ***less*** (e.g., nurses in Mass.).

- Income effect from a wage increase is positive, $IE > 0$.
  - a positively sloped labor supply curve for all wages
- The compensated supply curve is positive sloped:
  - It captures the SE due to the wage increase, but not the IE.
- The uncompensated labor supply curve, in contrast, represents both the SE and IE.

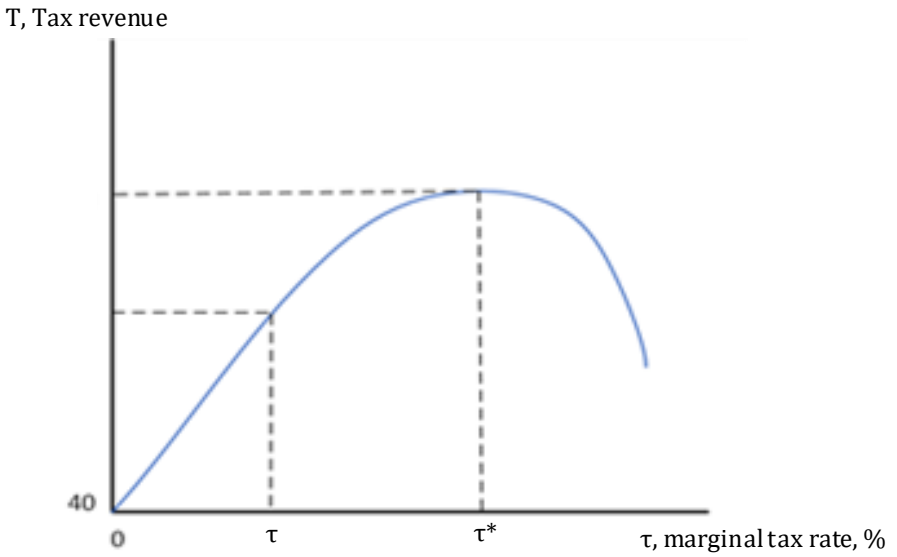


- When $IE < 0$ and $|IE| > |SE|$, implying that $TE < 0$.
- In this case, the uncompensated supply curve becomes negatively sloped.



The Laffer Curve

- An increase in the tax rate might initially *increase* tax revenue but, after a certain rate, further increments might *reduce* the tax revenues.
- Or, alternatively, a decrease in the marginal tax rate can actually increase tax revenues.
- This suggests that there is an optimal tax rate τ^* which will bring in the most tax revenue.



Consumer as a Labor Supplier

- Consider salary w per hour, and a net salary of
$$\omega = (1 - \tau)w$$
after taxes.
- Hence, $H(\omega)$ represents the number of working hours, where workers consider their net wage when deciding how many hours to work.
- Therefore, tax revenue is
$$T = \tau \cdot w \cdot H(\omega)$$

Consumer as a Labor Supplier

- Since total tax revenue is $T = \tau \cdot w \cdot H(\omega)$, the effect of marginally increasing the tax rate is

$$\frac{\partial T}{\partial \tau} = w \cdot H(\omega) + \tau \cdot w \cdot (-w)\frac{\partial H}{\partial \omega}$$

$$= \underbrace{w \cdot H(\omega)}_{\text{Positive effect}} - \underbrace{\tau \cdot w^2 \cdot \frac{\partial H}{\partial \omega}}_{\text{Negative effect}}$$

  - The *positive effect* represents that, for a given supply of working hours, an increase in the tax rate increases tax revenue.
  - The *negative effect* represents that an increase in the tax rate reduces the amount of working hours supplied and, hence, tax revenue.

Consumer as a Labor Supplier

- Therefore, under which conditions we can guarantee that $\frac{\partial T}{\partial \tau} < 0$ (so that an increase in tax rates actually decreases total tax collection, as proposed by the Laffer curve)?
  - We need

$$w \cdot H(\omega) - \tau \cdot w^2 \cdot \frac{\partial H}{\partial \omega} < 0$$

That is,

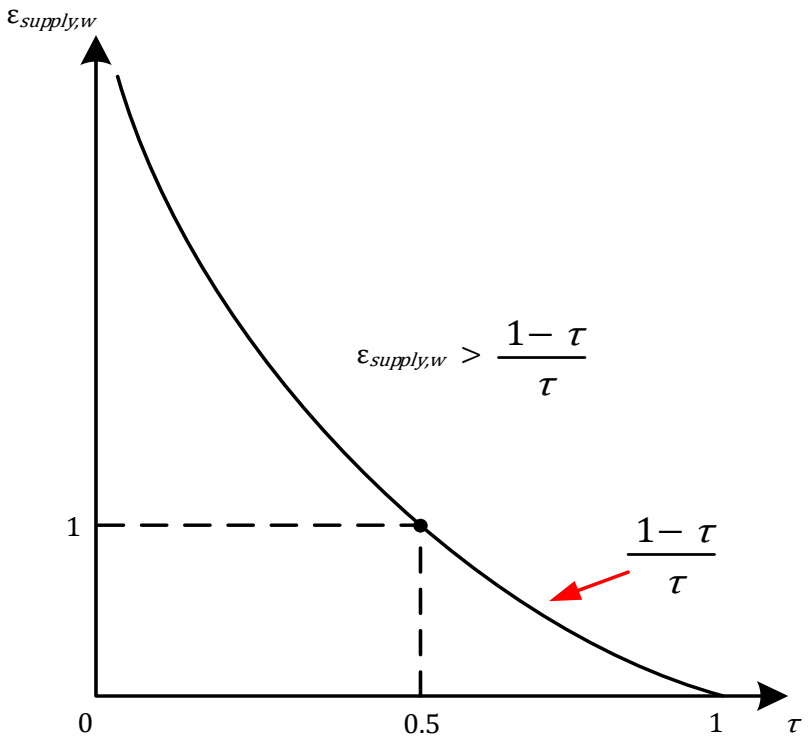
$$w \cdot H(\omega) < \tau \cdot w^2 \cdot \frac{\partial H}{\partial \omega} \quad \text{or} \quad \frac{1}{\tau} < \frac{\partial H}{\partial \omega} \frac{w}{H(\omega)}$$

Multiplying both sides by $(1-\tau)$ yields

$$\frac{1-\tau}{\tau} < \frac{\partial H}{\partial \omega} \frac{\overbrace{w(1-\tau)}^{\omega}}{H(\omega)} \quad \Rightarrow \quad \frac{1-\tau}{\tau} < \varepsilon_{\text{supply},\,\omega}$$

Consumer as a Labor Supplier

- The area above (below) cutoff $\frac{1-\tau}{\tau}$ represents combinations of the elasticity of labor supply ($\varepsilon_{\text{supply},\,\omega}$) and tax rates ($\tau$) for which a marginal increase in the tax rate yields a smaller (larger, respectively) total tax revenue.



Consumer as a Labor Supplier

- Hence, for total tax revenue to fall after an increase in the tax rate, τ, we need

$$\frac{1-\tau}{\tau} < \varepsilon_{\text{supply},\,\omega}$$

- ***Example 1***: If the marginal tax rate for the most affluent citizens is $\tau = 0.8$, then the above condition implies

$$\frac{1-0.8}{0.8} = 0.25 < \varepsilon_{\text{supply},\,\omega}$$

which is likely to be satisfied.

Consumer as a Labor Supplier

- ***Example 2***: An economy in which the maximum marginal tax rate is $\tau = 0.35$, would need

$$\frac{1 - 0.35}{0.35} = 1.85 < \varepsilon_{\text{supply},\,\omega}$$

for total tax revenue to decrease, which is very unlikely to hold for the average worker in most developed countries.

Gross/Net Complements and Gross/Net Substitutes

Demand Relationships among Goods

- So far, we were focusing on the SE and IE of varying the price of good k on the demand for good k.

- Now, we analyze the SE and IE of varying the price of good k on the demand for other good j.

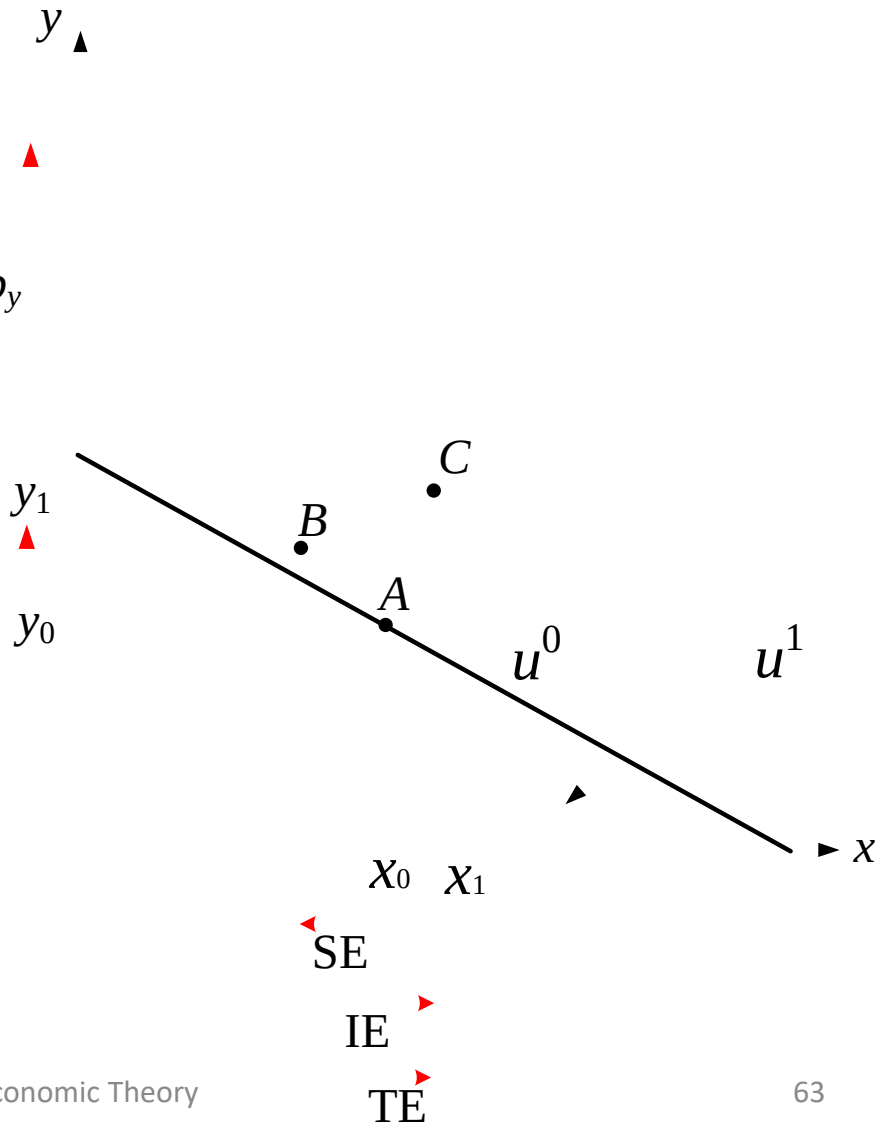
Demand Relationships among Goods

- For simplicity, let us start our analysis with the two-good case.
  - This will help us graphically illustrate the main intuitions.
- Later on we generalize our analysis to $N > 2$ goods.

Demand Relationships among Goods: The Two-Good Case

- When the price of y falls, the substitution effect may be so *small* that the consumer purchases more x and more y.
  - In this case, we call x and y ***gross complements***.

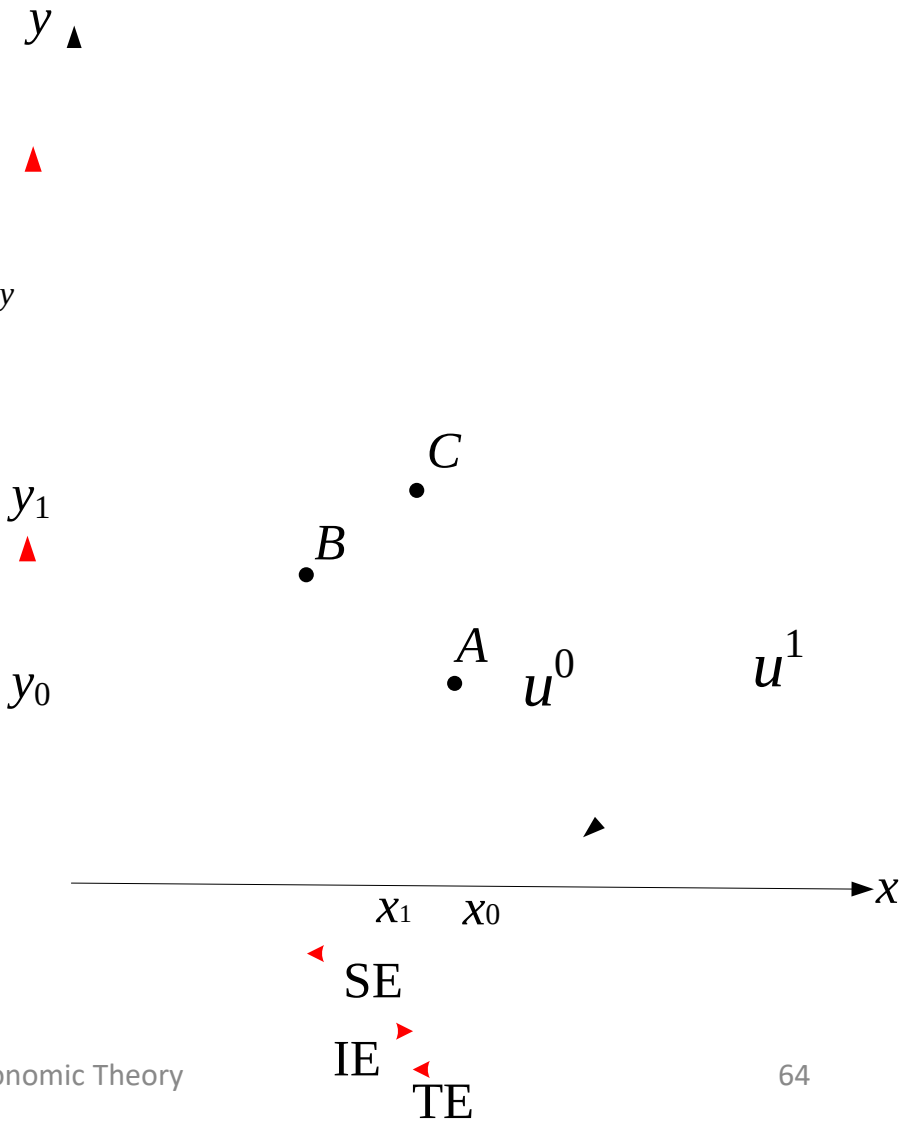
$$\frac{\partial x}{\partial p_y} < 0$$



Demand Relationships among Goods: The Two-Good Case

- When the price of y falls, the substitution effect may be so *large* that the consumer purchases less x and more y.
  - In this case, we call x and y ***gross substitutes***.

$$\frac{\partial x}{\partial p_y} > 0$$



Demand Relationships among Goods: The Two-Good Case

- A mathematical treatment
  - The change in x caused by changes in p_y can be shown by a Slutsky-type equation:

$$\frac{\partial x}{\partial p_y} = \underbrace{\frac{\partial h_x}{\partial p_y}}_{SE\ (+)} - \underbrace{y\frac{\partial x}{\partial w}}_{IE:\ (+)\text{ if } x \text{ is normal};\ (-)\text{ if } x \text{ is inferior}}$$

$$\text{Combined effect (ambiguous)}$$

$SE > 0$ is not a typo: Δp_y induces the consumer to buy more of good x, if his utility level is kept constant. Graphically, we are moving along the same indifference curve.

Demand Relationships among Goods: The Two-Good Case

- Or, in elasticity terms

$$\varepsilon_{x,p_y} = \underbrace{\tilde{\varepsilon}_{x,p_y}}_{SE\ (+)} - \underbrace{\theta_y \varepsilon_{x,w}}_{IE:\ (+)\text{ if } x \text{ is normal};\ (-)\text{ if } x \text{ is inferior}}$$

where θ_y denotes the share of income spent on good y. The combined effect of Δp_y on the observable Walrasian demand, $x(p, w)$, is ambiguous.

Demand Relationships among Goods: The Two-Good Case

- ***Example***: Let's show the SE and IE across different goods for a Cobb-Douglas utility function $u(x, y) = x^{0.5}y^{0.5}$.
  - The Walrasian demand for good x is

$$x(p, w) = \frac{1}{2}\frac{w}{p_x}$$

  - The Hicksian demand for good x is

$$h_x(p, u) = \frac{\sqrt{p_y}}{\sqrt{p_x}} \cdot u$$

Demand Relationships among Goods: The Two-Good Case

- ***Example*** (continued):
  - First, note that differentiating $x(p, w)$ with respect to p_y, we obtain

$$\frac{\partial x(p,w)}{\partial p_y} = 0$$

    i.e., variations in the price of good y do not affect consumer's Walrasian demand.
  - But,

$$\frac{\partial h_x(p,u)}{\partial p_y} = \frac{1}{2}\frac{u}{\sqrt{p_x p_y}} \neq 0$$

  - How can these two (seemingly contradictory) results arise?

Demand Relationships among Goods: The Two-Good Case

- ***Example*** (continued):
  - Answer: the SE and IE completely offset each other.
  - **Substitution Effect**: Given

$$\frac{\partial h_x(p,u)}{\partial p_y} = \frac{1}{2}\frac{u}{\sqrt{p_x p_y}},$$

    plug the indirect utility function $u = \frac{1}{2}\frac{w}{\sqrt{p_x p_y}}$ to obtain a SE of $\frac{1}{4}\frac{w}{p_x p_y}$.
  - **Income Effect**:

$$-y\frac{\partial x}{\partial w} = -\left(\frac{1}{2}\frac{w}{p_y}\right)\left(\frac{1}{2}\frac{1}{p_x}\right) = -\frac{1}{4}\frac{w}{p_x p_y}$$

Demand Relationships among Goods: The Two-Good Case

- ***Example*** (continued):
  - Therefore, the total effect is

$$\overbrace{\frac{\partial x(p,w)}{\partial p_y}}^{TE} = \overbrace{\frac{\partial h_x}{\partial p_y}}^{SE} - y\overbrace{\frac{\partial x}{\partial w}}^{IE}$$

$$= \frac{1}{4}\frac{w}{p_x p_y} - \frac{1}{4}\frac{w}{p_x p_y} = 0$$

  - Intuitively, this implies that the substitution and income effect completely offset each other.

Demand Relationships among Goods: The Two-Good Case

- Common mistake:
  - "$\frac{\partial x(p,w)}{\partial p_y} = 0$ means that good x and y cannot be substituted in consumption. That is, they must be consumed in fixed proportions. Hence, this consumer's utility function is a Leontief type."
- No! We just showed that

$$\frac{\partial x(p,w)}{\partial p_y} = 0 \implies \frac{\partial h_x}{\partial p_y} = y\frac{\partial x}{\partial w}$$

  i.e., the SE and IE completely offset each other.
- For the above statement to be true, we would need SE to be zero so that TE = IE, i.e., $\frac{\partial h_x}{\partial p_y} = 0 \implies \frac{\partial x(p,w)}{\partial p_y} = -y\frac{\partial x}{\partial w}$.

Demand Relationships among Goods: The N-Good Case

- We can, hence, generalize the Slutsky equation to the case of $N > 2$ goods as follows:

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial w}$$

  for any i and j.

- The change in the price of good j induces IE and SE on good i.

Asymmetry of the Gross Substitute and Complement

- Two goods are ***substitutes*** if one good may replace the other in use.
  - Example: tea and coffee, butter and margarine
- Two goods are ***complements*** if they are used together.
  - Example: coffee and cream, fish and chips.
- The concepts of gross substitutes and complements include both SE and IE.
  - Two goods are gross substitutes if $\frac{\partial x_i}{\partial p_j} > 0$.
  - Two goods are gross complements if $\frac{\partial x_i}{\partial p_j} < 0$.

Asymmetry of the Gross Substitute and Complement

- The definitions of gross substitutes and complements are *not* necessarily symmetric.
  - It is possible for x_1 to be a substitute for x_2 and at the same time for x_2 to be a complement of x_1.
- Let us see this potential asymmetry with an example.

Asymmetry of the Gross Substitute and Complement

- Suppose that the utility function for two goods is given by

$$U(x, y) = \ln x + y$$

- The Lagrangian of the UMP is

$$L = \ln x + y + \lambda(w - p_x x - p_y y)$$

- The first order conditions are

$$\frac{\partial L}{\partial x} = \frac{1}{x} - \lambda p_x = 0$$

$$\frac{\partial L}{\partial y} = 1 - \lambda p_y = 0$$

$$\frac{\partial L}{\partial \lambda} = w - p_x x - p_y y = 0$$

Asymmetry of the Gross Substitute and Complement

- Manipulating the first two equations, we get

$$\frac{1}{p_x x} = \frac{1}{p_y} \implies p_x x = p_y$$

- Inserting this into the budget constraint, we can find the Walrasian demand for y

$$\underbrace{p_x x}_{p_y} + p_y y = w \implies p_y y = w - p_y \implies$$

$$y = \frac{w - p_y}{p_y}$$

Asymmetry of the Gross Substitute and Complement

- An increase in p_y causes a decline in spending on y
  - Since p_x and w are unchanged, spending on x must rise $\left(x = \frac{p_y}{p_x} \Rightarrow \frac{\partial x}{\partial p_y} = \frac{1}{p_x} > 0\right)$.
    - Hence, x is a gross substitute of y.
  - But spending on y is independent of p_x $\left(\frac{\partial y}{\partial p_x} = 0\right)$.
    - Thus, y is neither a gross substitute nor complement of x.
  - This shows the asymmetry of gross substitutes and gross complements.

Asymmetry of the Gross Substitute and Complement

- Depending on how we check for gross substitutability or complementarity between two goods, there is potential to obtain different results.
- Can we use an alternative approach to check if two goods are complements or substitutes in consumption?
  - Yes. We next present such approach.

Net Substitutes and Net Complements

- The concepts of net substitutes and complements focus solely on SE.
  - Two goods are ***net (or Hicksian) substitutes*** if

$$\frac{\partial h_i}{\partial p_j} > 0$$

  - Two goods are ***net (or Hicksian) complements*** if

$$\frac{\partial h_i}{\partial p_j} < 0$$

where $h_i(p_i, p_j, u)$ is the Hicksian demand of good i.

Net Substitutes and Net Complements

- This definition looks only at the shape of the indifference curve.
- This definition is unambiguous because the definitions are perfectly symmetric

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}$$

  - This implies that every element above the main diagonal in the Slutsky matrix is symmetric with respect to the corresponding element below the main diagonal.

Net Substitutes and Net Complements

$$S(p,w) = \begin{pmatrix} \frac{\partial h_1(p,u)}{\partial p_1} & \frac{\partial h_1(p,u)}{\partial p_2} & \frac{\partial h_1(p,u)}{\partial p_3} \\ \frac{\partial h_2(p,u)}{\partial p_1} & \frac{\partial h_2(p,u)}{\partial p_2} & \frac{\partial h_2(p,u)}{\partial p_3} \\ \frac{\partial h_3(p,u)}{\partial p_1} & \frac{\partial h_3(p,u)}{\partial p_2} & \frac{\partial h_3(p,u)}{\partial p_3} \end{pmatrix}$$

Net Substitutes and Net Complements

- Proof:
  - Recall that, from Shephard's lemma, $h_k(p,u) = \frac{\partial e(p,u)}{\partial p_k}$. Hence,

$$\frac{\partial h_k(p,u)}{\partial p_j} = \frac{\partial^2 e(p,u)}{\partial p_k \partial p_j}$$

  - Using Young's theorem, we obtain

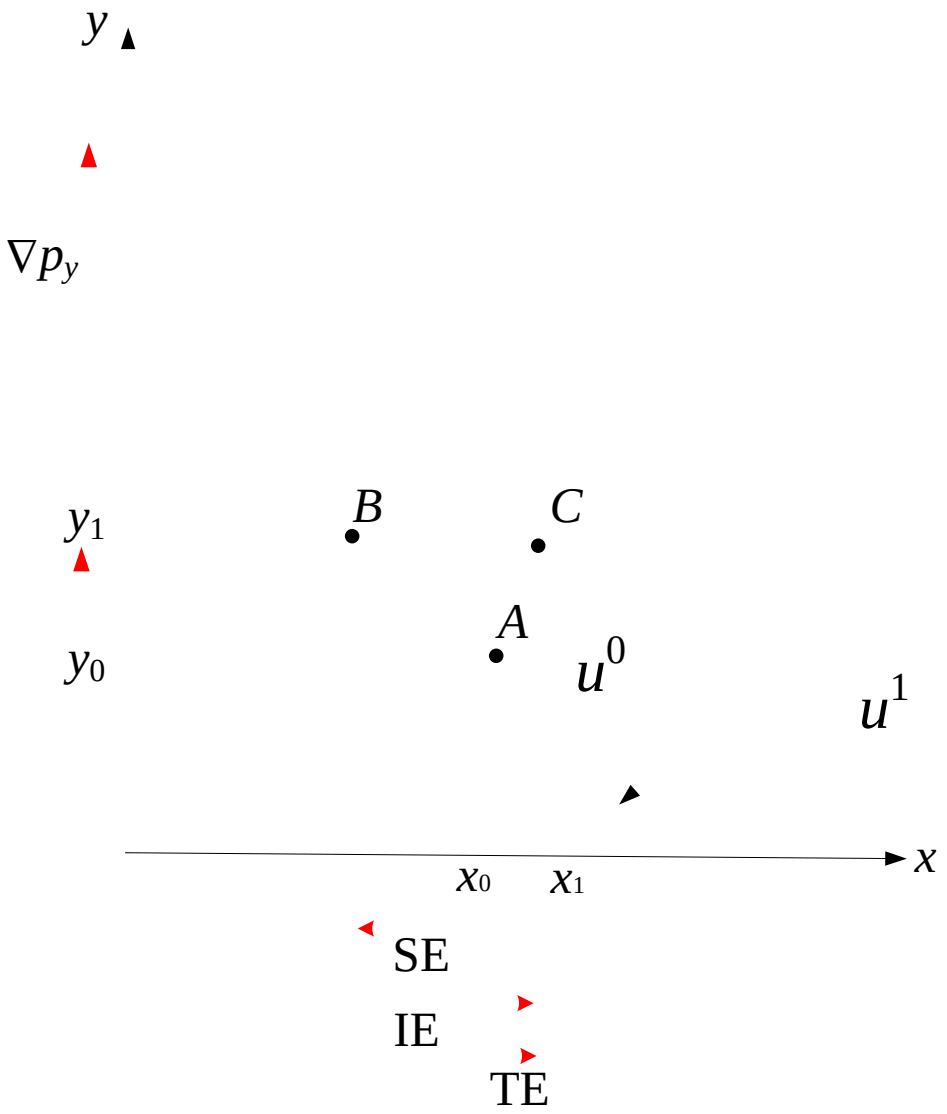
$$\frac{\partial^2 e(p,u)}{\partial p_k \partial p_j} = \frac{\partial^2 e(p,u)}{\partial p_j \partial p_k}$$

    which implies

$$\frac{\partial h_k(p,u)}{\partial p_j} = \frac{\partial h_j(p,u)}{\partial p_k}$$

Net Substitutes and Net Complements

- Even though x and y are gross complements, they are net substitutes.
- Since MRS is diminishing, the own-price SE must be negative $(SE < 0)$ so the cross-price SE must be positive $(TE > 0)$.



A Note on the Euler's Theorem

- We say that a function $f(x_1, x_2)$ is homogeneous of degree k if

$$f(tx_1, tx_2) = t^k \cdot f(x_1, x_2)$$

- Differentiating this expression with respect to x_1, we obtain

$$\frac{\partial f(tx_1, tx_2)}{\partial tx_1} \cdot \overbrace{t}^{\partial(tx_1)/\partial x_1} = t^k \cdot \frac{\partial f(x_1, x_2)}{\partial x_1}$$

or, rearranging,

$$\frac{\partial f(tx_1, tx_2)}{\partial tx_1} = t^{k-1} \cdot \frac{\partial f(x_1, x_2)}{\partial x_1}$$

A Note on the Euler's Theorem

- Last, denoting f_1 to be the partial derivative with respect to the first element, we obtain

$$f_1(tx_1, tx_2) = t^{k-1} \cdot f_1(x_1, x_2)$$

- Hence, if a function is homogeneous of degree k, its first-order derivative must be homogeneous of degree $k - 1$.

A Note on the Euler's Theorem

- Differentiating the left-hand side of the definition of homogeneity, $f(tx_1, tx_2) = t^k \cdot f(x_1, x_2)$, with respect to t yields

$$\frac{\partial f(tx_1, tx_2)}{\partial t} = f_1(tx_1, tx_2)x_1 + f_2(tx_1, tx_2)x_2$$

- Differentiating the right-hand side produces

$$\frac{\partial \left(t^k \cdot f(x_1, x_2)\right)}{\partial t} = k \cdot t^{k-1} f(x_1, x_2)$$

A Note on the Euler's Theorem

- Combining the differentiation of LHS and RHS,

$$f_1(tx_1, tx_2)x_1 + f_2(tx_1, tx_2)x_2 = k \cdot t^{k-1} f(x_1, x_2)$$

- Setting $t = 1$, we obtain

$$f_1(x_1, x_2)x_1 + f_2(x_1, x_2)x_2 = k \cdot f(x_1, x_2)$$

where k is the homogeneity order of the original function $f(x_1, x_2)$.

  - If $k = 0$, the above expression becomes 0.
  - If $k = 1$, the above expression is $f(x_1, x_2)$.

A Note on the Euler's Theorem

- ***Application***:
  - The Hicksian demand is homogeneous of degree zero in prices, that is,

$$h_i(tp_1, tp_2, \dots, tp_n, u) = h_i(p_1, p_2, \dots, p_n, u)$$

  - Hence, multiplying all prices by t does not affect the value of the Hicksian demand.
  - By Euler's theorem,

$$\frac{\partial h_i}{\partial p_1} p_1 + \frac{\partial h_i}{\partial p_2} p_2 + \cdots + \frac{\partial h_i}{\partial p_n} p_n$$
$$= 0 \cdot t^{0-1} h_i(p_1, p_2, \dots, p_n, u) = 0$$

Substitutability with Many Goods

- ***Question:*** Is net substitutability or complementarity more prevalent in real life?
- To answer this question, we can start with the compensated demand function

$$h_i(p_1, p_2, \ldots, p_n, u)$$

- Applying Euler's theorem yields

$$\frac{\partial h_i}{\partial p_1} p_1 + \frac{\partial h_i}{\partial p_2} p_2 + \cdots + \frac{\partial h_i}{\partial p_n} p_n = 0$$

- Dividing both sides by h_i, we can alternatively express the above result using compensated elasticities

$$\tilde{\varepsilon}_{i1} + \tilde{\varepsilon}_{i2} + \cdots + \tilde{\varepsilon}_{in} \equiv 0$$

Substitutability with Many Goods

- Since the negative sign of the SE implies that $\tilde{\varepsilon}_{ii} \leq 0$, then the sum of Hicksian cross-price elasticities for all other $j \neq i$ goods should satisfy

$$\sum_{j \neq i} \tilde{\varepsilon}_{ij} \geq 0$$

- Hence, "most" goods must be net substitutes.
- This is referred to as ***Hicks' second law of demand***.

Aggregate Demand

Aggregate Demand

- We now move from individual demand, $x_i(p, w_i)$, to aggregate demand,

$$\sum_{i=1}^{I} x_i(p, w_i)$$

  which denotes the total demand of a group of I consumers.

- Individual i's demand $x_i(p, w_i)$ still represents a vector of L components, describing his demand for L different goods.

Aggregate Demand

- We know individual demand depends on prices and individual's wealth.
  - When can we express aggregate demand as a function of prices and aggregate wealth?
  - In other words, when can we guarantee that aggregate demand defined as

$$x(p, w_1, w_2, \dots, w_I) = \sum_{i=1}^{I} x_i(p, w_i)$$

satisfies

$$\sum_{i=1}^{I} x_i(p, w_i) = x\left(p, \sum_{i=1}^{I} w_i\right)$$

Aggregate Demand

- This is satisfied if, for any two distributions of wealth, $(w_1, w_2, \dots, w_I)$ and $(w_1', w_2', \dots, w_I')$ such that $\sum_{i=1}^{I} w_i = \sum_{i=1}^{I} w_i'$, we have

$$\sum_{i=1}^{I} x_i(p, w_i) = \sum_{i=1}^{I} x_i(p, w_i')$$

- For such condition to be satisfied, let's start with an initial distribution $(w_1, w_2, \dots, w_I)$ and apply a differential change in wealth $(dw_1, dw_2, \dots, dw_I)$ such that the aggregate wealth is unchanged, $\sum_{i=1}^{I} dw_i = 0$.

Aggregate Demand

- If aggregate demand is just a function of aggregate wealth, then we must have that

$$\sum_{i=1}^{I} \frac{\partial x_i(p,w_i)}{\partial w_i} dw_i = 0 \text{ for every good } k$$

In words, the wealth effects of different individuals are compensated in the aggregate. That is, in the case of two individuals i and j,

$$\frac{\partial x_{ki}(p,w_i)}{\partial w_i} = \frac{\partial x_{kj}(p,w_j)}{\partial w_j}$$

for every good k.

Aggregate Demand

- This result *does not* imply that $IE_i > 0$ while $IE_j < 0$.
- In addition, it indicates that its absolute values coincide, i.e., $|IE_i| = |IE_j|$, which means that any redistribution of wealth from consumer i to j yields

$$\frac{\partial x_{ki}(p, w_i)}{\partial w_i} dw_i + \frac{\partial x_{kj}(p, w_j)}{\partial w_j} dw_j = 0$$

which can be rearranged as

$$\frac{\partial x_{ki}(p, w_i)}{\partial w_i} \underbrace{dw_i}_{-} = -\frac{\partial x_{kj}(p, w_j)}{\partial w_j} \underbrace{dw_j}_{+}$$

- Hence, $\frac{\partial x_{ki}(p,w_i)}{\partial w_i} = \frac{\partial x_{kj}(p,w_j)}{\partial w_j}$, since $|dw_i| = |dw_j|$.

Aggregate Demand

- In summary, for any
  - fixed price vector p,
  - good k, and
  - wealth level of any two individuals i and j

  the wealth effect is the same across individuals.
- In other words, the wealth effects arising from the distribution of wealth across consumers cancel out.
- This means that we can express aggregate demand as a function of aggregate wealth

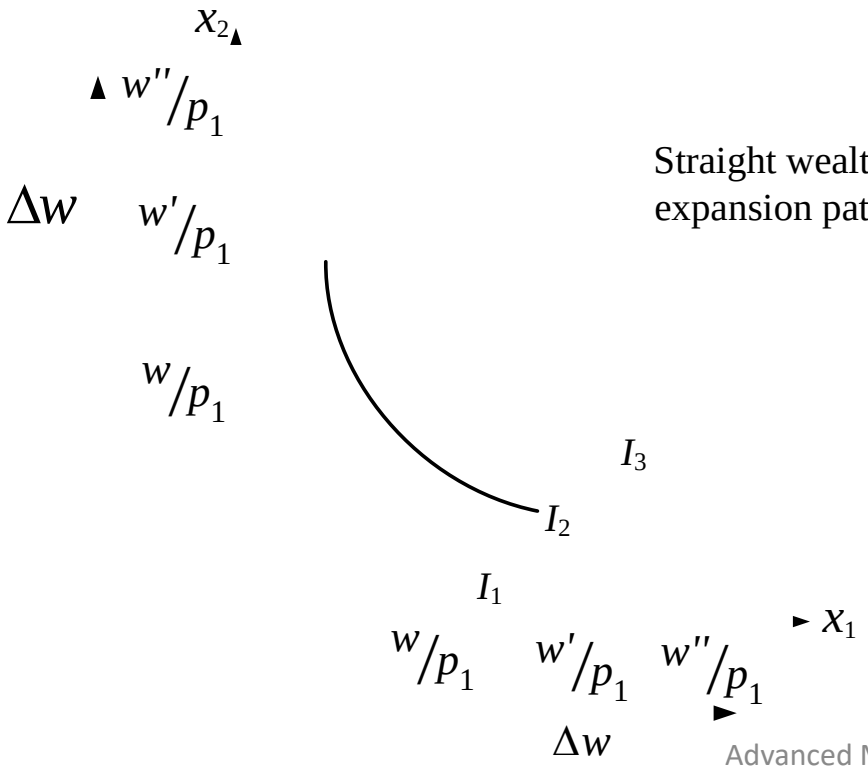
$$\sum_{i=1}^{I} x_i(p, w_i) = x\left(p, \sum_{i=1}^{I} w_i\right)$$

Aggregate Demand

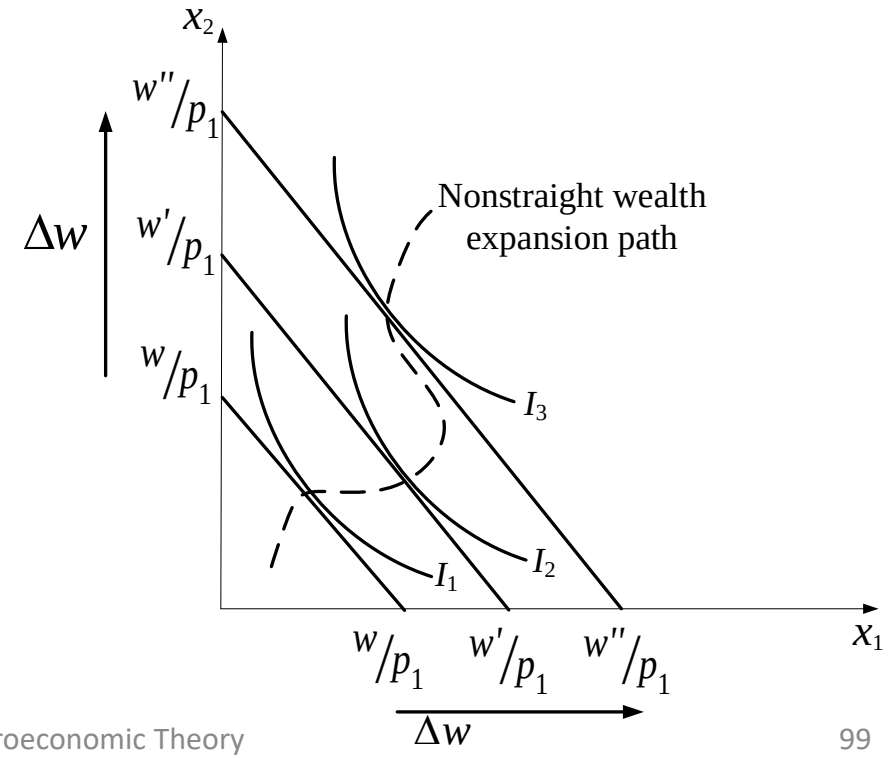
- Graphically, this condition entails that all consumers exhibit *parallel*, *straight* wealth expansion paths.
  - ***Straight***: wealth effects do not depend on the individuals' wealth level.
  - ***Parallel***: individuals' wealth effects must coincide across individuals.
    - Recall that wealth expansion paths just represent how an individual demand changes as he becomes richer.

Aggregate Demand

A given increase in wealth leads the same change in the consumption of good x_i, *regardless* of the initial wealth of the individual

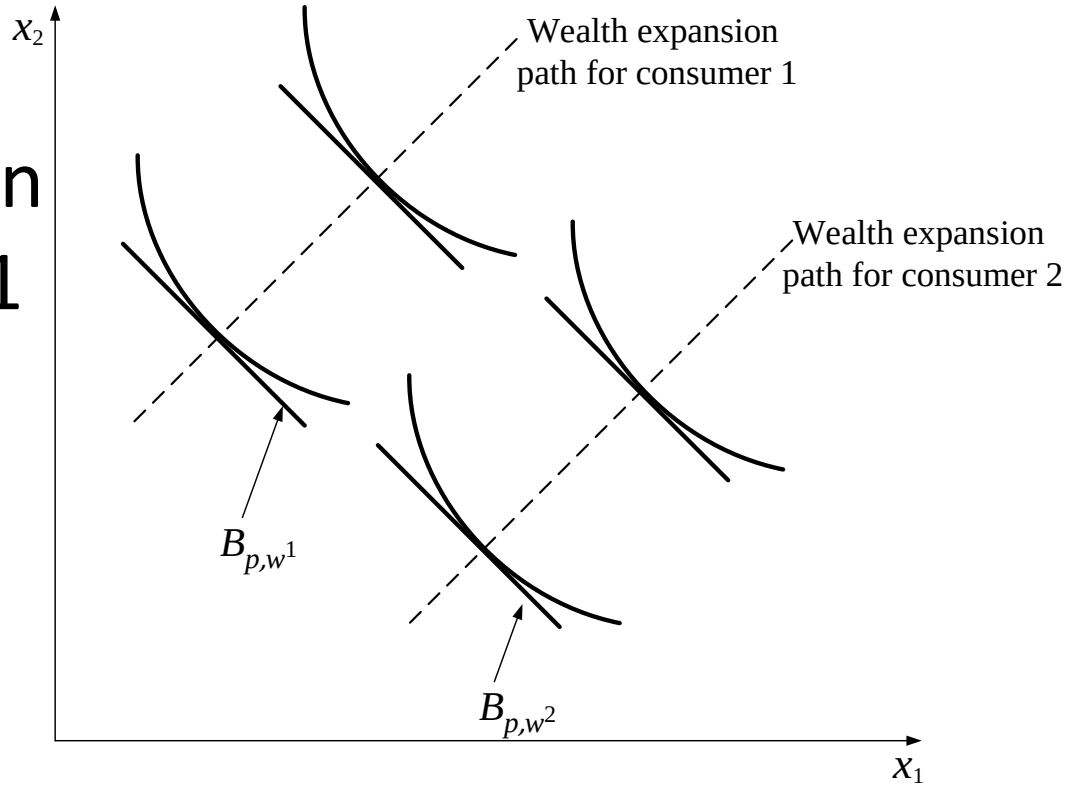


A given increase in wealth leads to changes in the consumption of good x_i that are *dependent* on the individual's wealth level



Aggregate Demand

- Individuals' wealth effects coincide.
- The wealth expansion path for consumers 1 and 2 are parallel to each other
  - both individuals' demands change similarly as they become richer.



Aggregate Demand

- Preference relations that yield *straight* wealth expansion paths:
  - Homothetic preferences
  - Quasilinear preferences
- Can we embody all these cases as special cases of a particular type of preferences?
  - Yes. We next present such cases.

Aggregate Demand: Gorman Form

- ***Gorman form.*** A necessary and sufficient condition for consumers to exhibit parallel, straight wealth expansion paths is that every consumer's indirect utility function can be expressed as:

$$v_i(p, w_i) = a_i(p) + b(p)w_i$$

  This indirect utility function is referred to as the *Gorman form*.

- Indeed, in case of quasilinear preferences

$$v_i(p, w_i) = a_i(p) + \frac{1}{p}w_i \quad \text{so that} \quad b(p) = \frac{1}{p}$$

Aggregate Demand: Gorman Form

- ***Example***:
  - Consider the Gorman form indirect utility function

$$v_i(p, w_i) = \underbrace{\gamma_i \frac{1}{\sqrt{p}}}_{a_i(p)} + \underbrace{\frac{1}{p}}_{b(p)} w_i$$

  - First of all, rearrange the above expression to yield

$$vp - \gamma_i \sqrt{p} - w_i = 0$$

  - Applying the quadratic formula, we have

$$\sqrt{p} = \frac{\gamma_i + \sqrt{4vw_i + \gamma_i^2}}{2v}$$

Aggregate Demand: Gorman Form

- ***Example***:
  - Further taking square on both sides, we find

$$p = \frac{2\gamma_i^2 + 4vw_i + 2\gamma_i\sqrt{4vw_i + \gamma_i^2}}{4v^2}$$

  - Therefore, the level sets of $v_i(p, w_i)$ become

$$p(w_i) = \frac{2vw_i + \gamma_i\left(\gamma_i + \sqrt{4vw_i + \gamma_i^2}\right)}{2v^2}$$

  - For simplicity, we set $v = 10$ and $\gamma_i = 1$

$$p(w_i) = \frac{1 + 20w_i + \sqrt{1 + 40w_i}}{200}$$

Aggregate Demand: Gorman Form

- ***Example***:
  - Further taking square on both sides, we find

$$p = \frac{2\gamma_i^2 + 4vw_i + 2\gamma_i\sqrt{4vw_i + \gamma_i^2}}{4v^2}$$

  - Therefore, the level sets of $v_i(p, w_i)$ become

$$p(w_i) = \frac{2vw_i + \gamma_i\left(\gamma_i + \sqrt{4vw_i + \gamma_i^2}\right)}{2v^2}$$

  - which is monotonically increasing in w_i since

$$\frac{\partial p(w_i)}{\partial w_i} = \frac{\gamma_i + \sqrt{4vw_i + \gamma_i^2}}{v\sqrt{4vw_i + \gamma_i^2}} > 0$$

Aggregate Demand: Gorman Form

- ***Example***:
  - Whereas, $p(w_i)$ is decreasing in v because

$$\frac{\partial p(w_i)}{\partial v} = -\frac{\left(vw_i + \gamma_i^2\right)\sqrt{4vw_i + \gamma_i^2} + 3\gamma_i vw_i}{v^3\sqrt{4vw_i + \gamma_i^2}} < 0$$

  so that a higher utility level v would dampen the level set $p(w_i)$.
  - Nevertheless, $p(w_i)$ is increasing in γ_i since

$$\frac{\partial p(w_i)}{\partial \gamma_i} = \frac{\gamma_i^2 + 2vw_i + \left(\gamma_i\sqrt{4vw_i + \gamma_i^2}\right)}{v\sqrt{4vw_i + \gamma_i^2}} > 0$$

  so when the utility function is more responsive to price (higher γ_i), the consumer is willing to pay a higher price for a given wealth w_i.

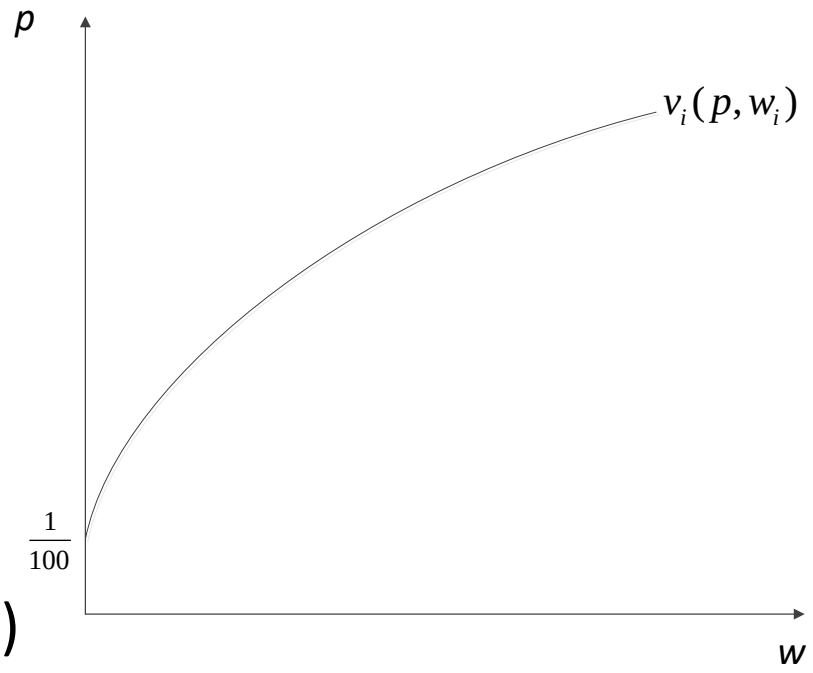
Aggregate Demand: Gorman Form

- ***Example*** (continued):
  - The vertical intercept of this function is $p(0) = \frac{1}{100}$.
  - The slope of this function is

$$\frac{\partial p(w_i)}{\partial w_i} = \frac{1}{10} + \frac{1}{10\sqrt{1+40w_i}} > 0$$

and it is decreasing in w_i (concavity)

$$\frac{\partial^2 p(w_i)}{\partial w_i^2} = -\frac{2}{(1+40w_i)^{3/2}}$$



Aggregate Demand: Gorman Form

- Let's show that, for indirect utility functions of the Gorman form, we obtain

$$\sum_{i=1}^{I} x_i(p, w_i) = x(p, \sum_{i=1}^{I} w_i)$$

- First, use Roy's identity to find the Walrasian demand associated with this indirect utility function

$$-\frac{\frac{\partial v_i(p, w_i)}{\partial p}}{\frac{\partial v_i(p, w_i)}{\partial w_i}} = x_i(p, w_i)$$

Aggregate Demand: Gorman Form

- In particular, for good j,

$$-\frac{\frac{\partial v_i(p,w_i)}{\partial p_j}}{\frac{\partial v_i(p,w_i)}{\partial w_i}} = -\frac{\frac{\partial a_i(p)}{\partial p_j}}{b(p)} - \frac{\frac{\partial b(p)}{\partial p_j}}{b(p)} w_i = x_i^j(p,w_i)$$

- In matrix notation,

$$-\frac{\nabla_p v_i(p,w_i)}{\nabla_w v_i(p,w_i)} = -\frac{\nabla_p a_i(p)}{b(p)} - \frac{\nabla_p b(p)}{b(p)} w_i = x_i(p,w_i)$$

for all goods.

Aggregate Demand: Gorman Form

- We can compactly express $x_i(p, w_i)$ as follows

$$-\frac{\nabla_p v_i(p, w_i)}{\nabla_w v_i(p, w_i)} = \alpha_i(p) + \beta(p) w_i = x_i(p, w_i)$$

where $-\frac{\nabla_p a_i(p)}{b(p)} \equiv \alpha_i(p)$ and $-\frac{\nabla_p b(p)}{b(p)} \equiv \beta(p)$.

Aggregate Demand: Gorman Form

- Hence, aggregate demand can be obtained by summing individual demands

$$\alpha_i(p) + \beta(p) w_i = x_i(p, w_i)$$

across all I consumers, which yields

$$\sum_{i=1}^{I} x_i(p, w_i) = \sum_{i=1}^{I} \alpha_i(p) + \beta(p) \sum_{i=1}^{I} w_i$$

$$= \sum_{i=1}^{I} \alpha_i(p) + \beta(p) w = x(p, \sum_{i=1}^{I} w_i)$$

where $\sum_{i=1}^{I} w_i = w$.

Advanced Microeconomic Theory

Chapter 4: Production Theory

Outline

- Production sets and production functions
- Profit maximization and cost minimization
- Cost functions
- Aggregate supply
- Efficiency (1^{st} and 2^{nd} FTWE)

Production Sets and Production Functions

Production Sets

- Let us define a production vector (or plan)

$$y = (y_1, y_2, \dots, y_L) \in \mathbb{R}^L$$

  - If, for instance, $y_2 > 0$, then the firm is producing positive units of good 2 (i.e., good 2 is an *output*).
  - If, instead, $y_2 < 0$, then the firm is producing negative units of good 2 (i.e., good 2 is an *input*).

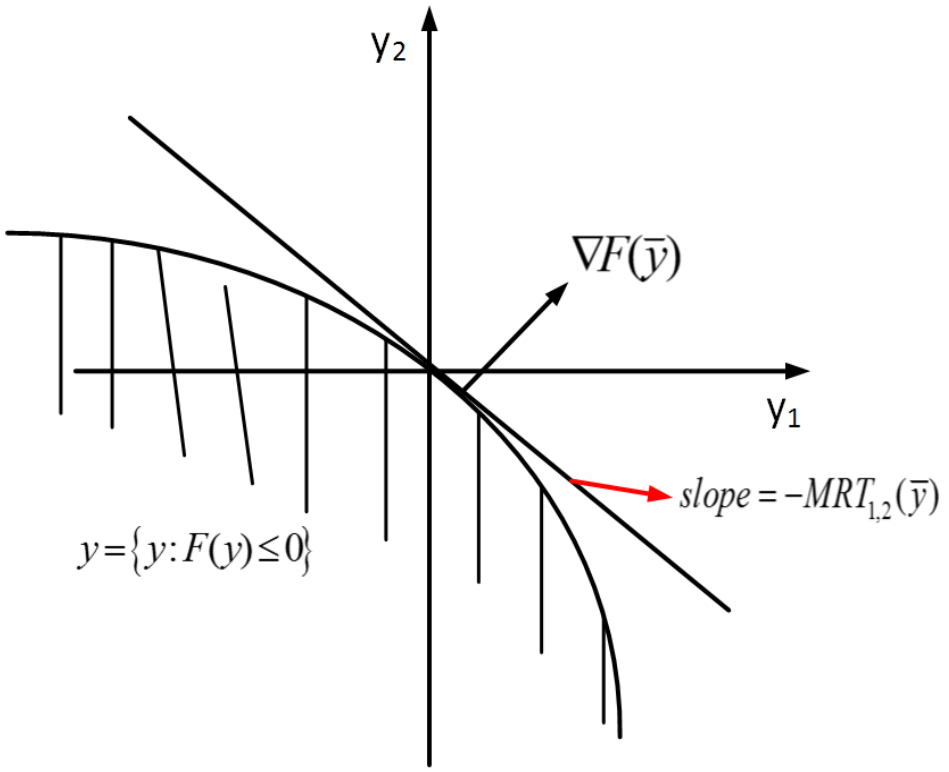
- Production plans that are technologically feasible are represented in the production set Y.

$$Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$$

  where $F(y)$ is the ***transformation function***.

Production Sets

- $F(y)$ can also be understood as a ***production function***.
- Firm uses units of y_1 as an input in order to produce units of y_2 as an output.
- Boundary of the production set is any production plan y such that $F(y) = 0$.
  - Also referred to as the ***transformation frontier***.



Production Sets

- For any production plan $\bar{y}$ on the production frontier, such that $F(\bar{y}) = 0$, we can totally differentiate $F(\bar{y})$ as follows

$$\frac{\partial F(\bar{y})}{\partial y_k} dy_k + \frac{\partial F(\bar{y})}{\partial y_l} dy_l = 0$$

solving

$$\frac{dy_l}{dy_k} = -\frac{\frac{\partial F(\bar{y})}{\partial y_k}}{\frac{\partial F(\bar{y})}{\partial y_l}}, \text{ where } \frac{\frac{\partial F(\bar{y})}{\partial y_k}}{\frac{\partial F(\bar{y})}{\partial y_l}} = MRT_{l,k}(\bar{y})$$

  - $MRT_{l,k}(\bar{y})$ measures how much the (net) output k can increase if the firm decreases the (net) output of good l by one marginal unit.

Production Sets

- What if we denote inputs and outputs with different letters?

$$q = (q_1, q_2, \dots, q_M) \geq 0 \text{ outputs}$$

$$z = (z_1, z_2, \dots, z_{L-M}) \geq 0 \text{ inputs}$$

where $L \geq M$.

- In this case, inputs are transformed into outputs by the production function, $f(z_1, z_2, \dots, z_{L-M})$, i.e., $f: \mathbb{R}^{L-M} \to \mathbb{R}^M$.

Production Sets

- ***Example***: When $M = 1$ (one single output), the production set Y can be described as

$$Y = \begin{Bmatrix} (-z_1, -z_2, \dots, -z_{L-1}, q): \\ q \leq f(z_1, z_2, \dots, z_{L-1}) \end{Bmatrix}$$

- Holding the output level fixed, $dq = 0$, totally differentiate the production function

$$\frac{\partial f(\bar{z})}{\partial z_k} dz_k + \frac{\partial f(\bar{z})}{\partial z_l} dz_l = 0$$

Production Sets

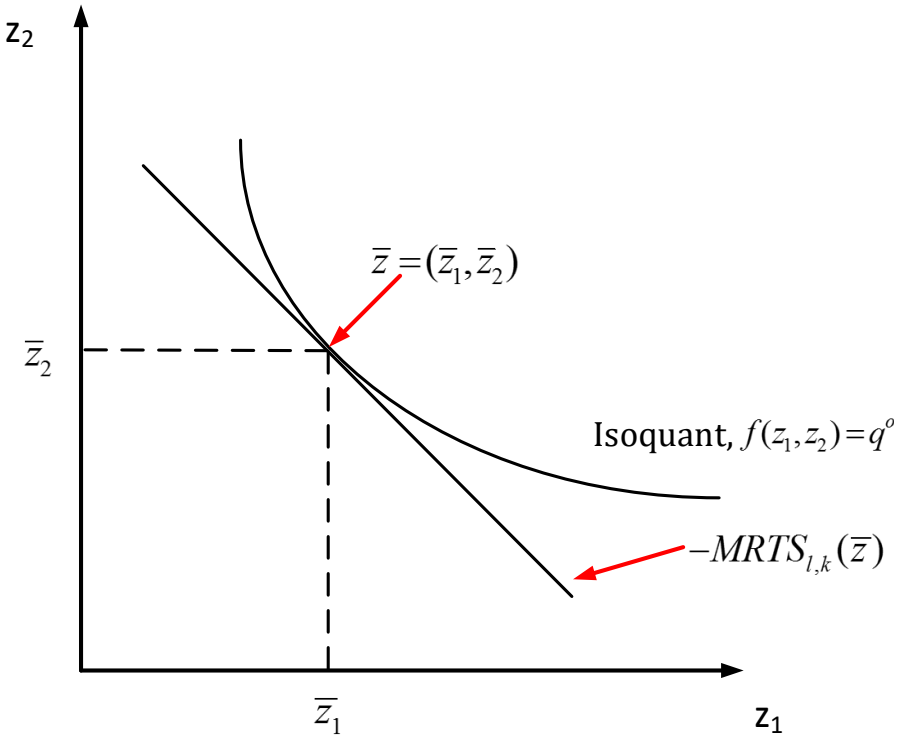
- ***Example*** (continued): and rearranging

$$\frac{dz_k}{dz_l} = -\frac{\frac{\partial f(\bar{z})}{\partial z_l}}{\frac{\partial f(\bar{z})}{\partial z_k}}, \text{ where } \frac{\frac{\partial f(\bar{z})}{\partial z_l}}{\frac{\partial f(\bar{z})}{\partial z_k}} = MRTS(\bar{z})$$

- $MRTS(\bar{z})$ measures the additional amount of input k that must be used when we marginally decrease the amount of input l, and we want to keep output level at $\bar{q} = f(\bar{z})$.
- $MRTS(\bar{z})$ in production theory is analogous to the MRS in consumer theory, where we keep the consumer's utility constant, $du = 0$.

Production Sets

- Combinations of (z_1, z_2) that produce the same total output q^0, i.e., $\{(z_1, z_2): f(z_1, z_2) = q^0\}$ is called ***isoquant***.
- The slope of the isoquant at (z_1, z_2) is $MRTS(\bar{z})$.
- Remember:
  - $MRTS$ refers to isoquants (and production function).
  - MRT refers to the transformation function.



Production Sets

- ***Example***: Find the $MRTS(z)$ for the Cobb-Douglas production function $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$, where $\alpha, \beta > 0$.
- The marginal product of input 1 is

$$\frac{\partial f(z_1, z_2)}{\partial z_1} = \alpha z_1^{\alpha-1} z_2^{\beta}$$

and that of input 2 is

$$\frac{\partial f(z_1, z_2)}{\partial z_2} = \beta z_1^{\alpha} z_2^{\beta-1}$$

Production Sets

- ***Example*** (continued): Hence, the $MRTS(z)$ is

$$MRTS(z) = \frac{\alpha z_1^{\alpha-1} z_2^{\beta}}{\beta z_1^{\alpha} z_2^{\beta-1}} = \frac{\alpha z_2^{\beta-(\beta-1)}}{\beta z_1^{\alpha-(\alpha-1)}} = \frac{\alpha z_2}{\beta z_1}$$

- For instance, for a particular vector $\bar{z} = (\bar{z}_1, \bar{z}_2) = (2,3)$, and $\alpha = \beta = \frac{1}{2}$, then

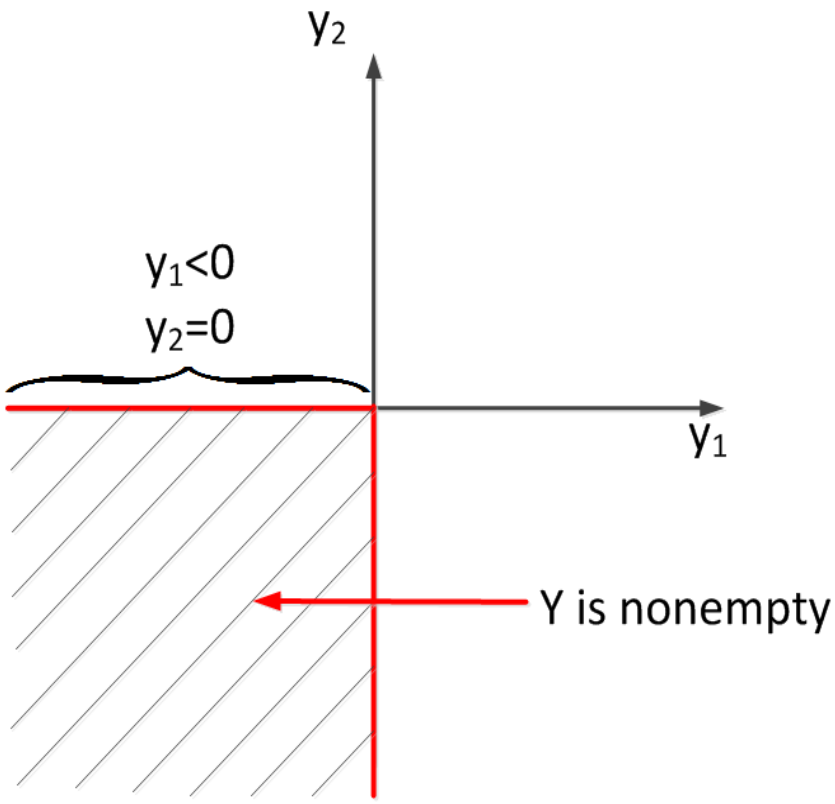
$$MRTS(\bar{z}) = \frac{3}{2} = 1.5$$

i.e., the slope of the isoquant evaluated at input vector $\bar{z} = (\bar{z}_1, \bar{z}_2) = (2,3)$ is -1.5.

Properties of Production Sets

1) ***Y is nonempty***: We have inputs and/or outputs.

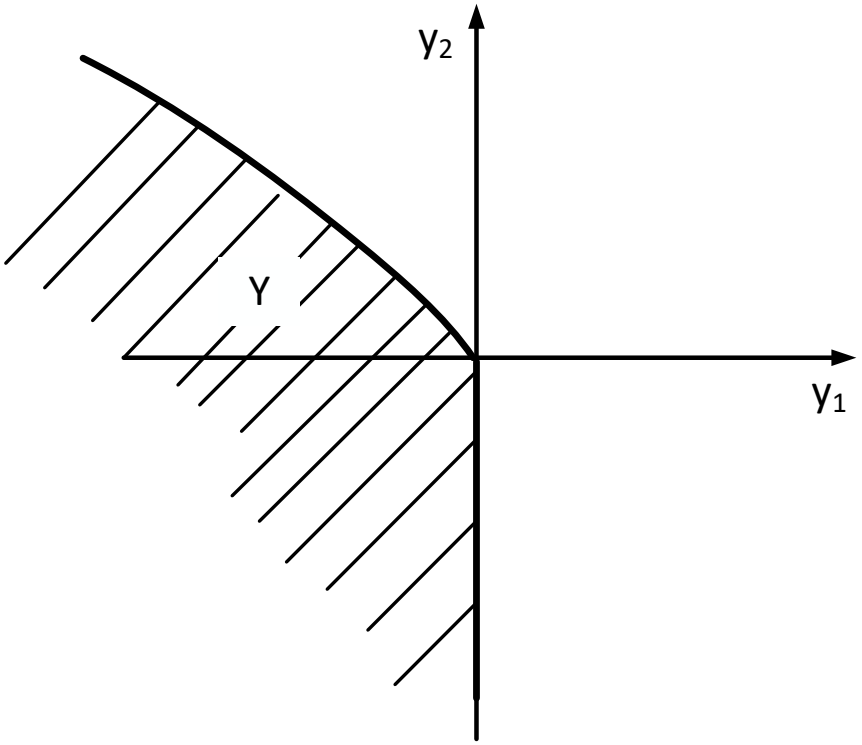
2) ***Y is closed***: The production set Y includes its boundary points.



Properties of Production Sets

3) No free lunch: No production with no resources.

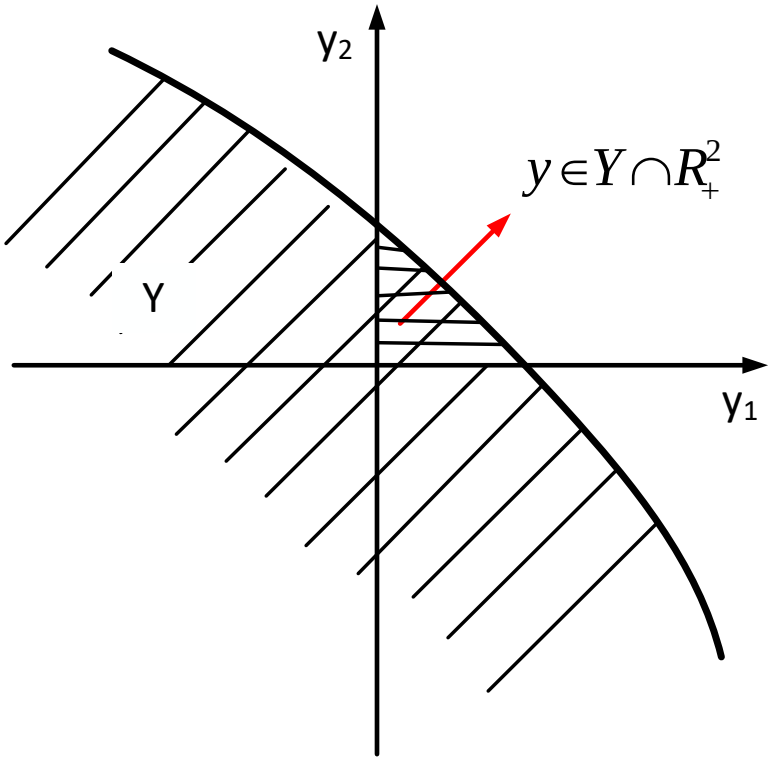
- The firm uses amounts of input y_1 in order to produce positive amounts of output y_2.



Properties of Production Sets

3) No free lunch: violation

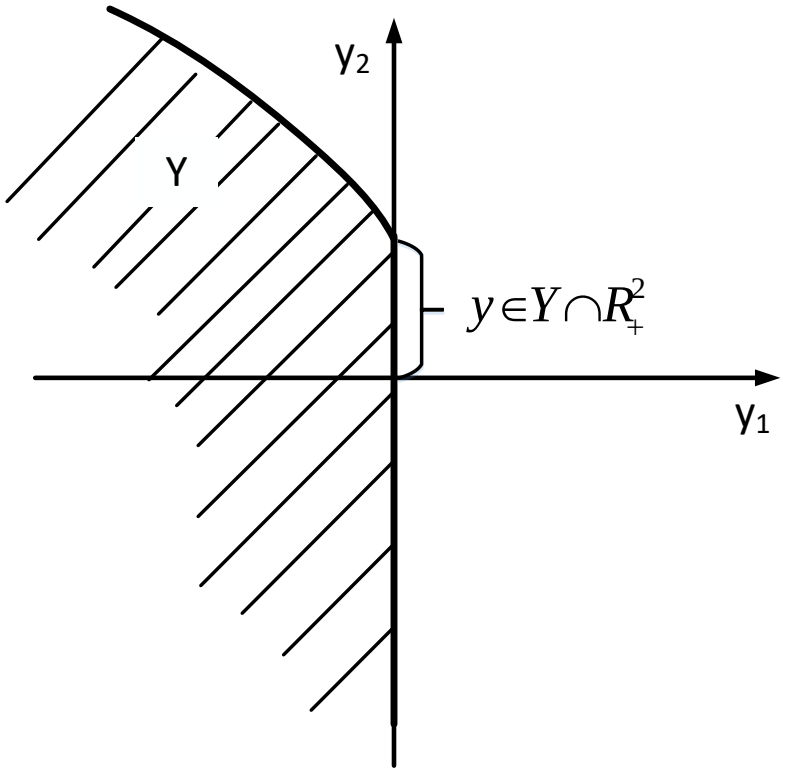
- The firm produces positive amounts of good 1 and 2 ($y_1 > 0$ and $y_2 > 0$) without the use of any inputs.



Properties of Production Sets

3) No free lunch: violation

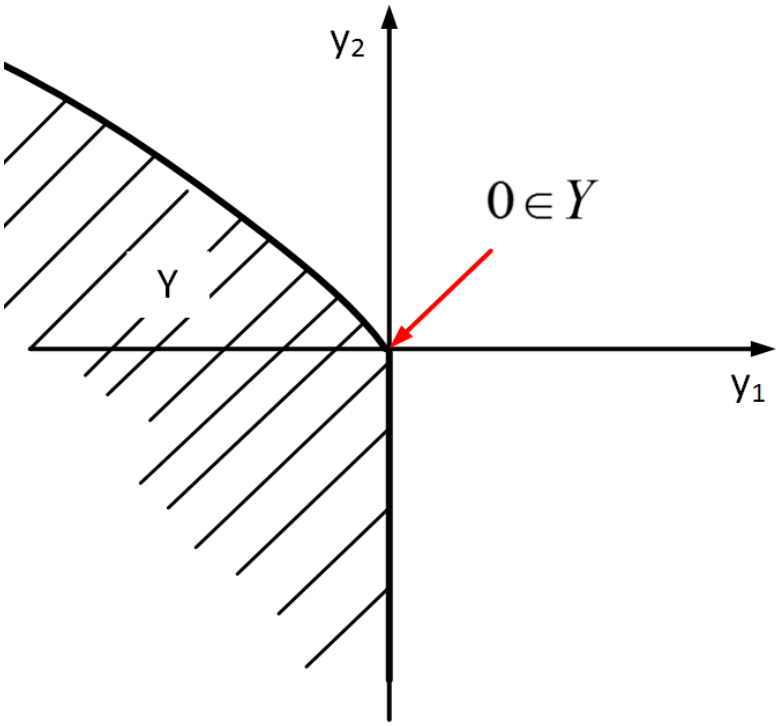
- The firm produces positive amounts of good 2 ($y_2 > 0$) with zero inputs, i.e., $y_1 = 0$.



Properties of Production Sets

4) Possibility of inaction

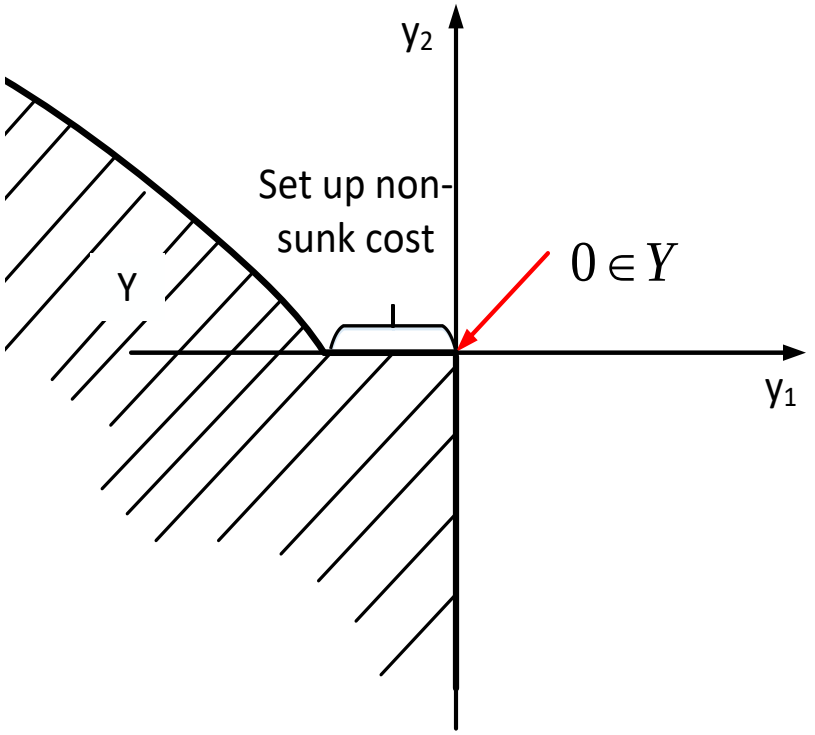
- Firm can choose to be inactive, using no inputs, and obtaining no output as a result (i.e., $0 \in Y$).



Properties of Production Sets

4) Possibility of inaction

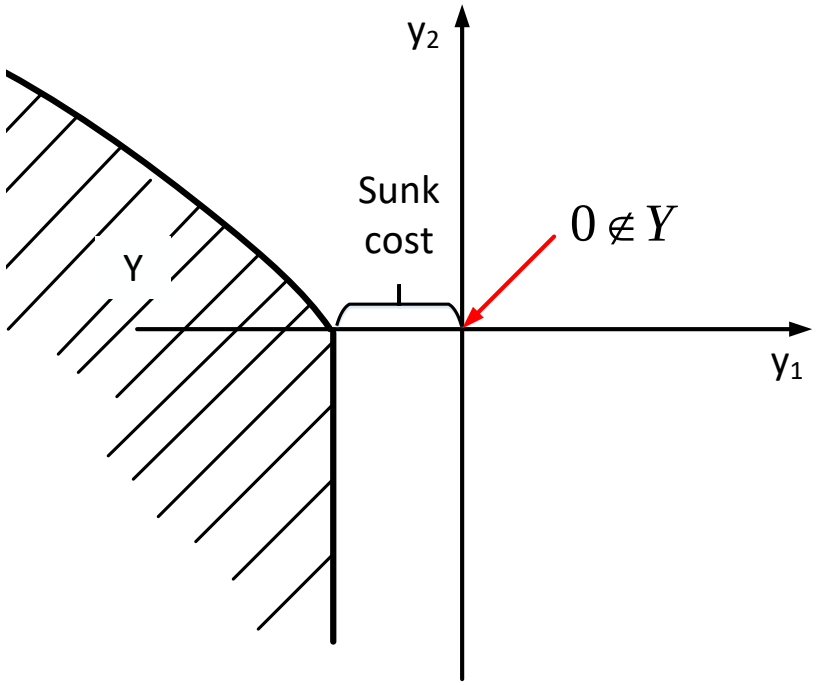
- Inaction is still possible when firms face *fixed costs* (i.e., $0 \in Y$).



Properties of Production Sets

4) Possibility of inaction

- Inaction is NOT possible when firms face *sunk costs* (i.e., $0 \notin Y$).



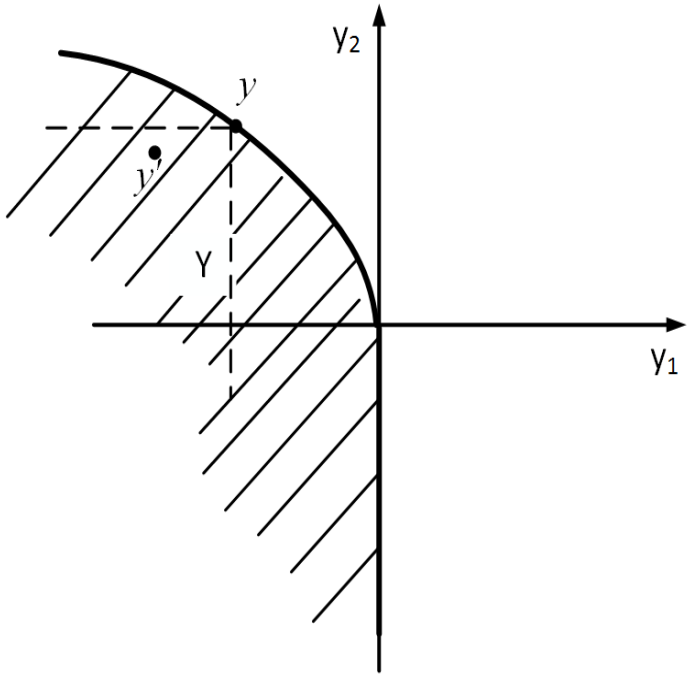
Properties of Production Sets

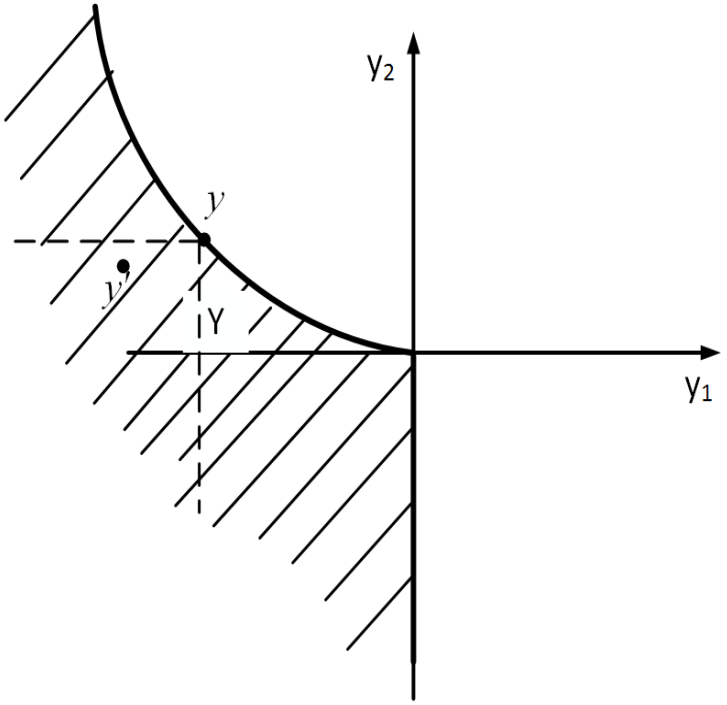
5) Free disposal: if $y \in Y$ and $y' \leq y$, then $y' \in Y$.

- y' is less efficient than y:
  - Either it produces the same amount of output with more inputs, or less output using the same inputs.
- Then, y' also belongs to the firm's production set.
- That is, the producer can use fewer inputs without the need to reduce his output:
  - The producer can dispose of (or eliminate) this additional inputs at no cost.

Properties of Production Sets

5) Free disposal (continued)

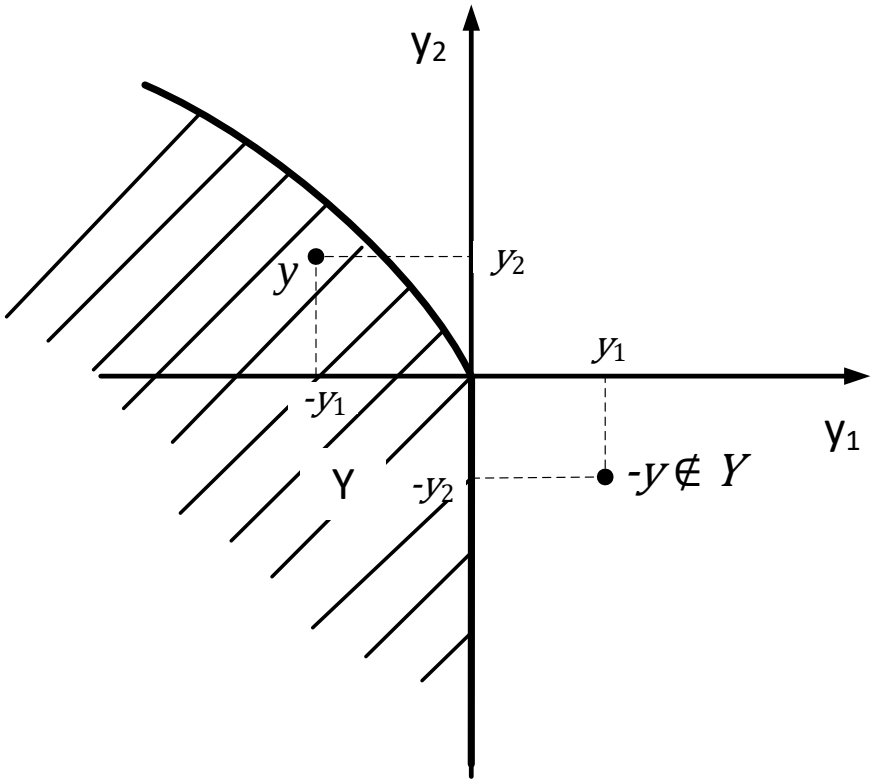




Properties of Production Sets

6) Irreversibility

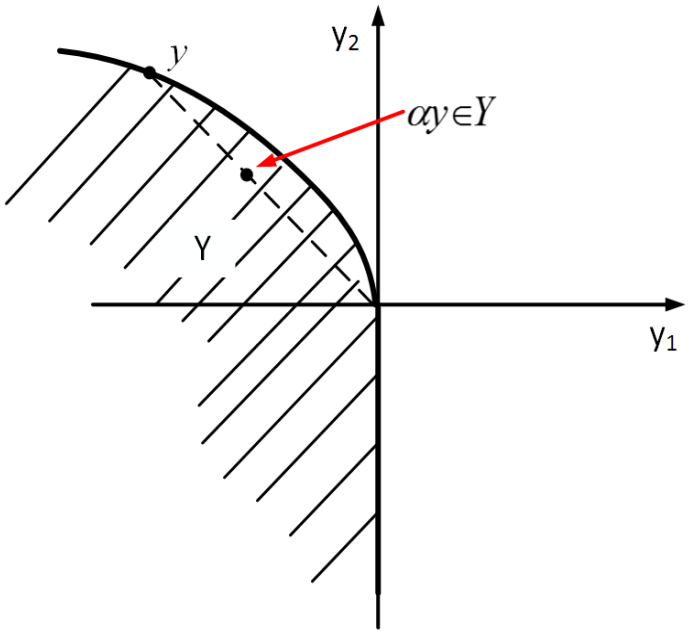
- Suppose that $y \in Y$ and $y \neq 0$. Then, $-y \notin Y$.
- "No way back"

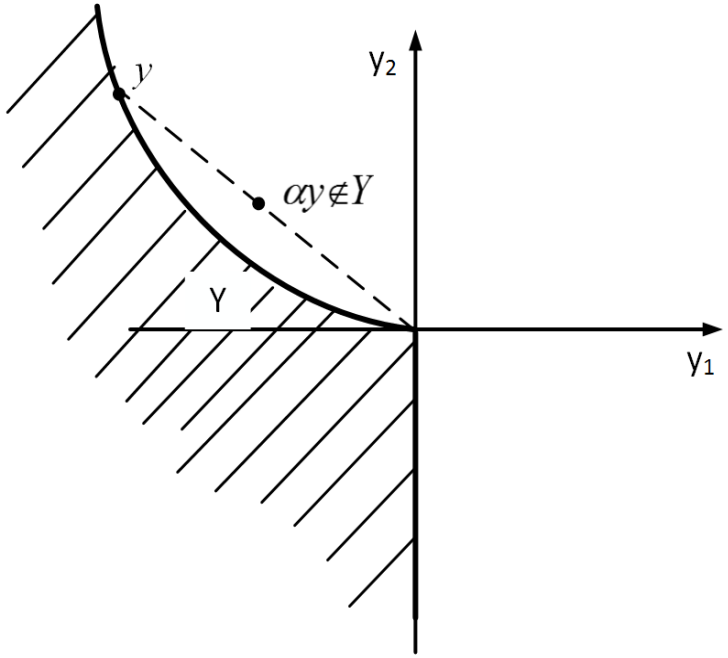


Properties of Production Sets

7) Non-increasing returns to scale: If $y \in Y$, then $\alpha y \in Y$ for any $\alpha \in [0,1]$.

- That is, any feasible vector can be scaled down.

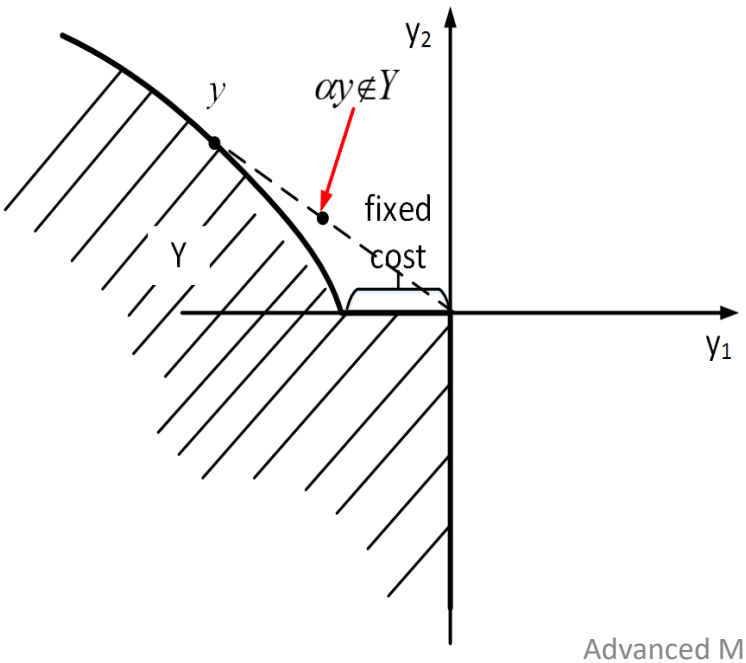


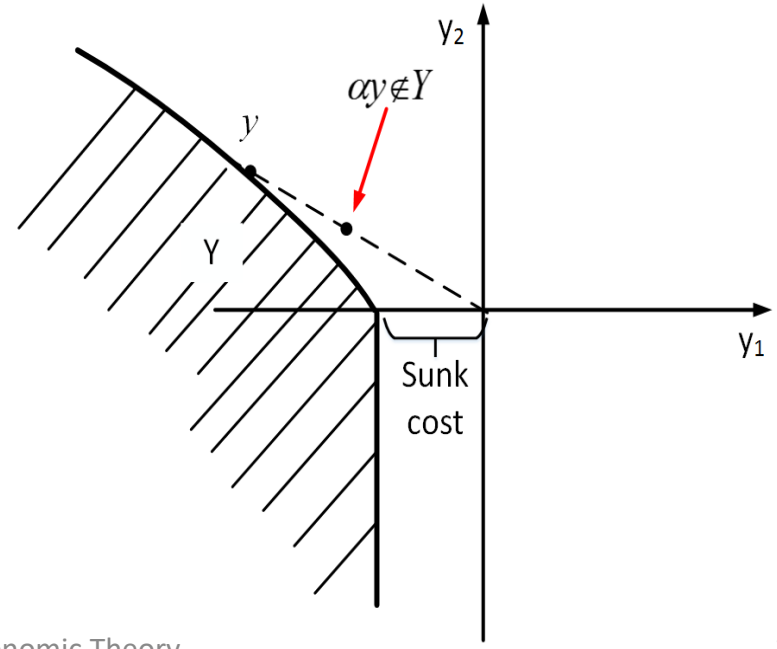


Properties of Production Sets

7) Non-increasing returns to scale

- The presence of fixed or sunk costs *violates* non-increasing returns to scale.

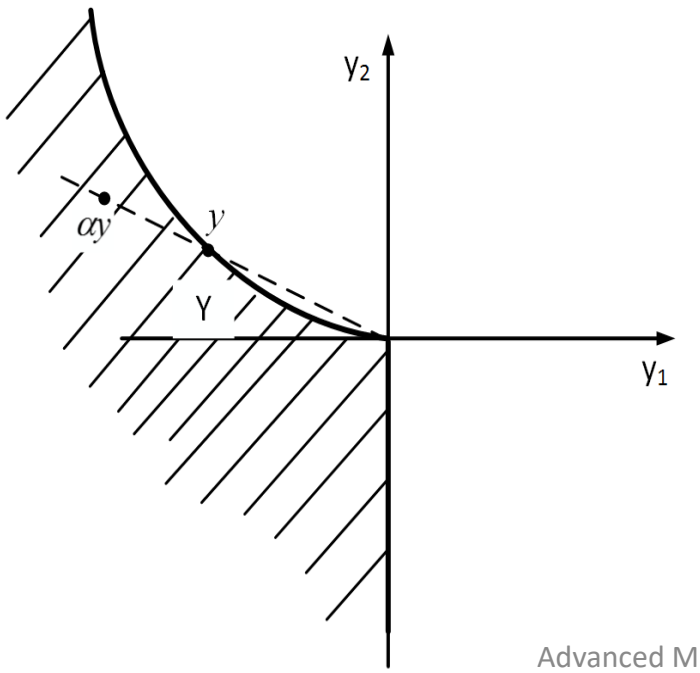


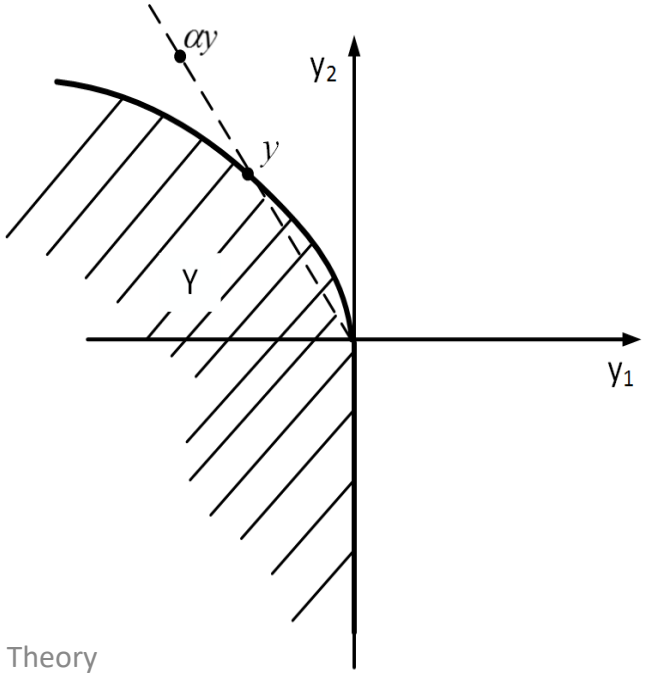


Properties of Production Sets

8) Non-decreasing returns to scale: If $y \in Y$, then $\alpha y \in Y$ for any $\alpha \geq 1$.

- That is, any feasible vector can be scaled up.

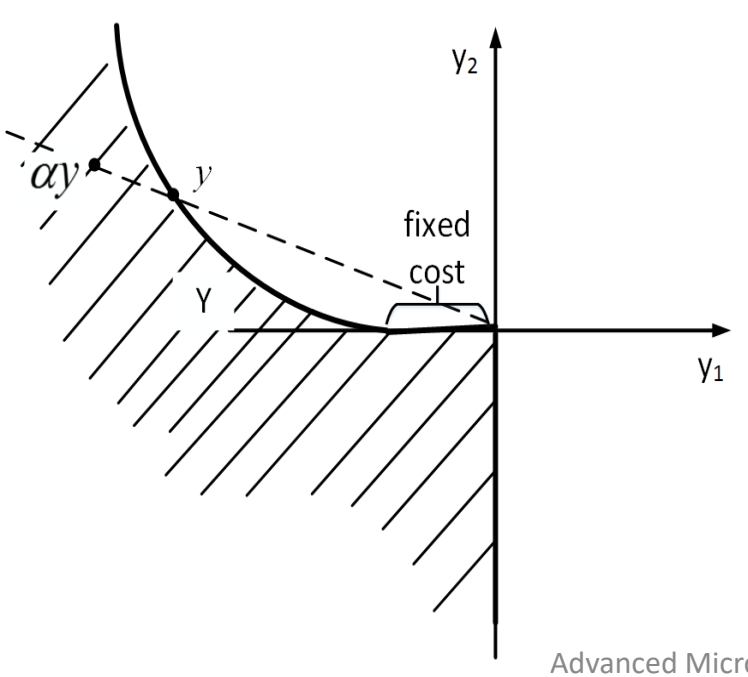


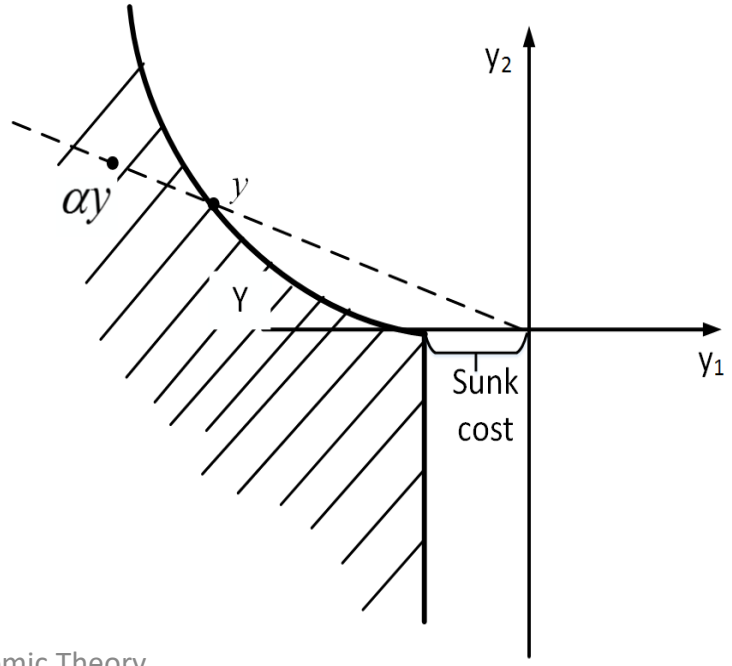


Properties of Production Sets

8) Non-decreasing returns to scale

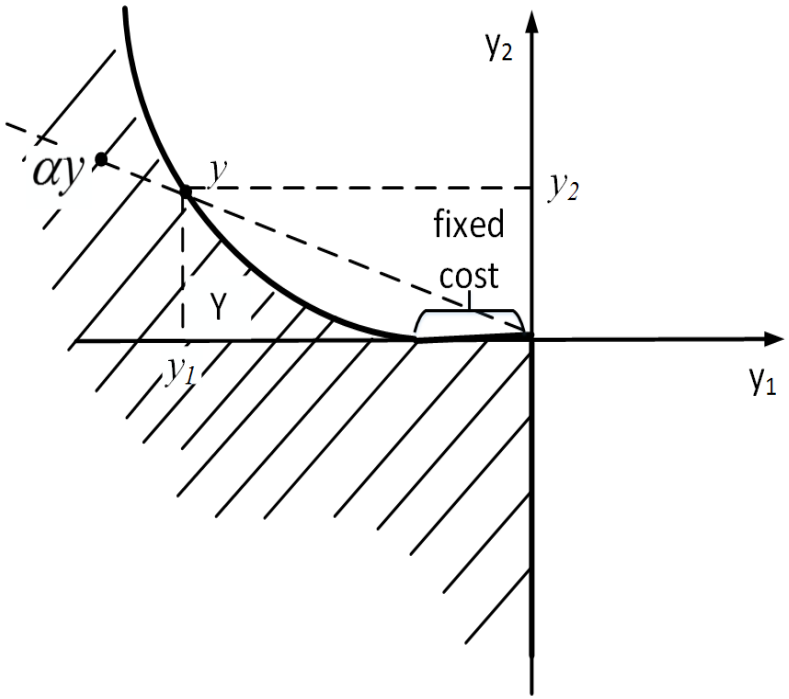
- The presence of fixed or sunk costs does NOT violate non-decreasing returns to scale.





Properties of Production Sets

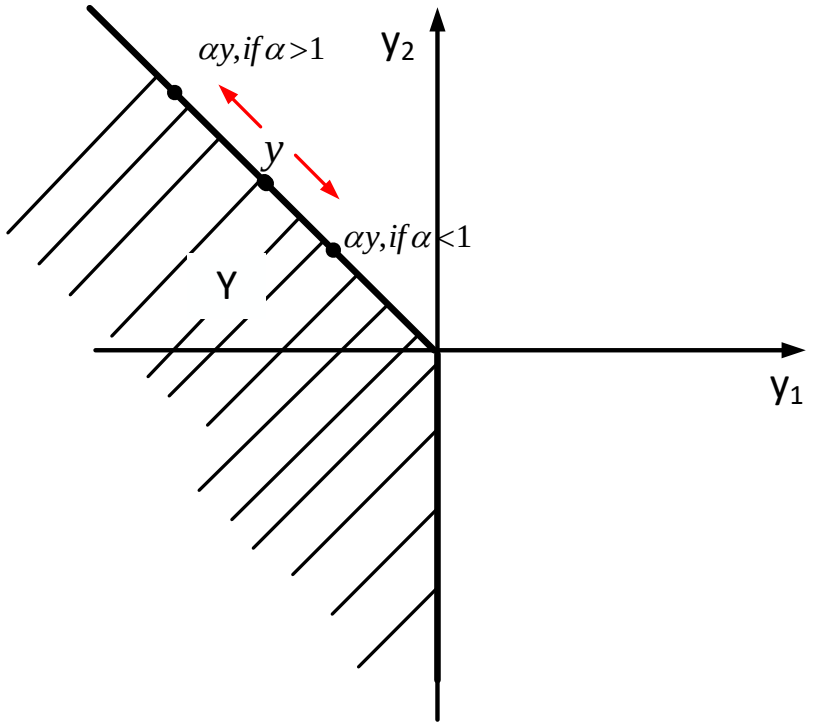
- ***Returns to scale***:
  - When scaling up/down a given production plan $y = (-y_1, y_2)$:
    - We connect y with a ray from the origin.
    - Then, the ratio $\frac{y_2}{y_1}$ must be maintained in all points along the ray.
    - Note that the angle of the ray is exactly this ratio $\frac{y_2}{y_1}$.



Properties of Production Sets

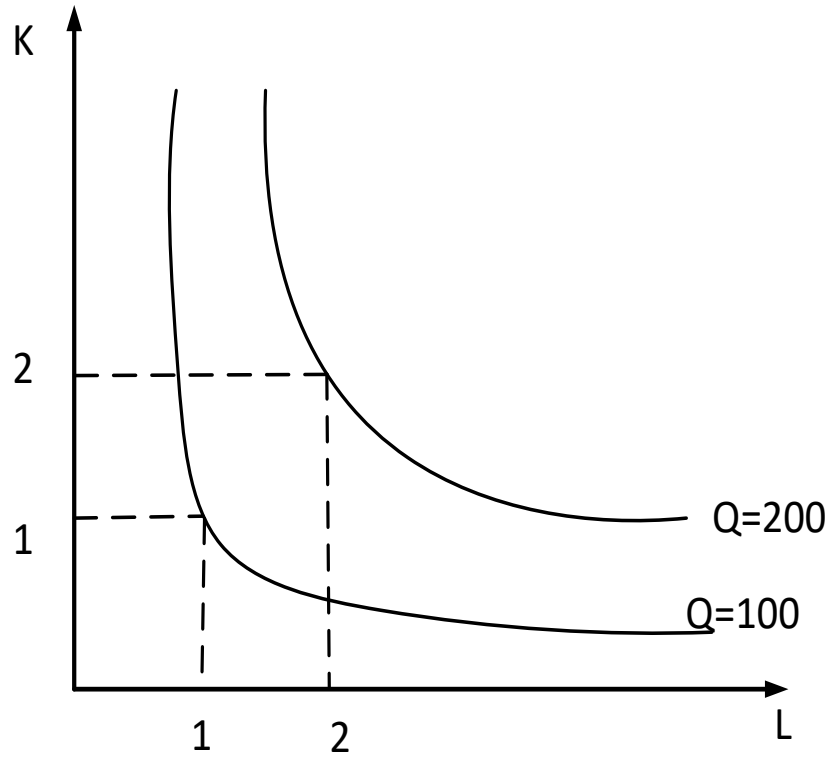
9) Constant returns to scale (CRS): If $y \in Y$, then $\alpha y \in Y$ for any $\alpha \geq 0$.

- CRS is non-increasing and non-decreasing.



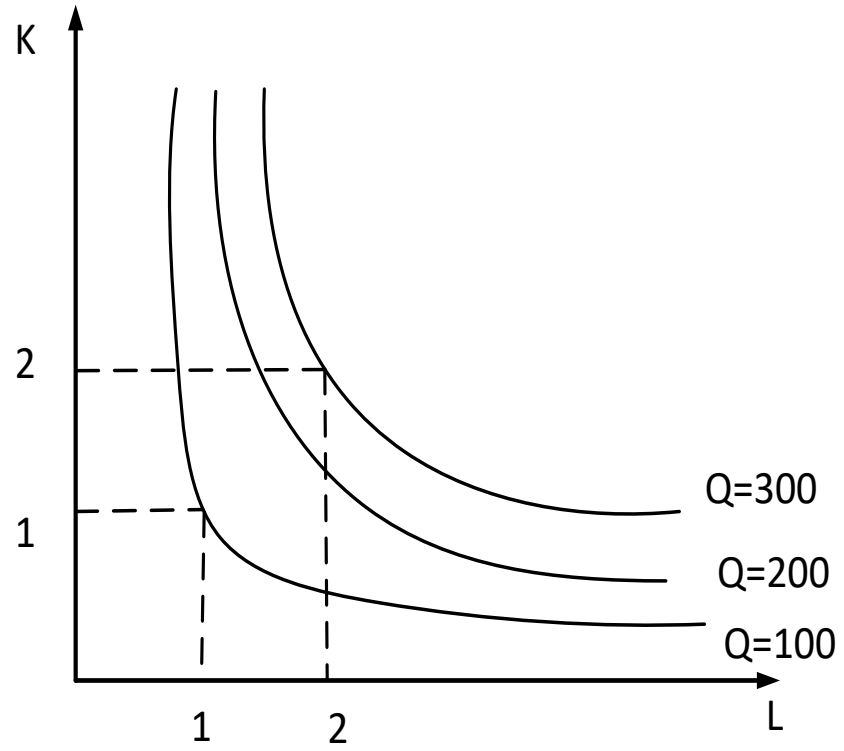
Properties of Production Sets

- Alternative graphical representation of *constant returns to scale*:
  - Doubling K and L doubles output (i.e., proportionally increase in output).



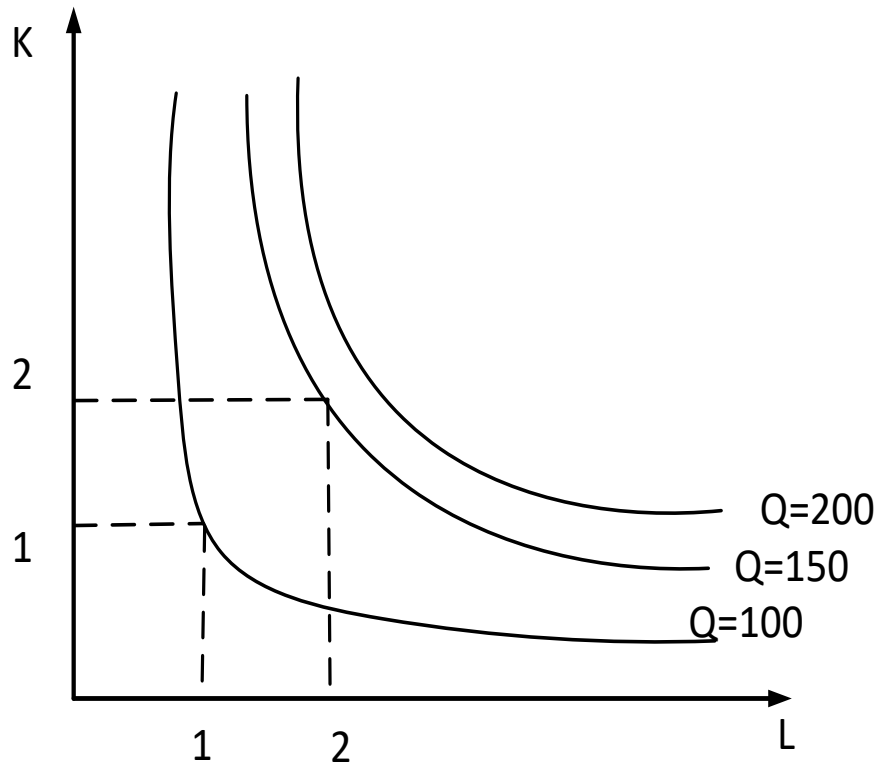
Properties of Production Sets

- Alternative graphical representation of *increasing-returns to scale*:
  - Doubling K and L increases output more than proportionally.



Properties of Production Sets

- Alternative graphical representation of *decreasing-returns to scale*:
  - Doubling K and L increases output less than proportionally.



Properties of Production Sets

Common factor	Non-convex Production Set /Increasing Returns to Scale (IRS)	Convex Production Set /Decreasing Returns to Scale (DRS)	Linear Production Set /Constant Returns to Scale (CRS)
$\alpha > 1$	✓	×	✓
$\alpha < 1$	×	✓	✓

Properties of Production Sets

- ***Example***: Let us check returns to scale in the Cobb-Douglas production function $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$. Increasing all arguments by a common factor λ, we obtain

$$f(\lambda z_1, \lambda z_2) = (\lambda z_1)^{\alpha}(\lambda z_2)^{\beta} = \lambda^{\alpha+\beta} z_1^{\alpha} z_2^{\beta}$$

  - When $\alpha + \beta = 1$, we have constant returns to scale;
  - When $\alpha + \beta > 1$, we have increasing returns to scale;
  - When $\alpha + \beta < 1$, we have decreasing returns to scale.

Properties of Production Sets

- Returns to scale in different US industries (Source: Hsieh, 1995):

	Industry	$\alpha + \beta$
Decreasing returns	Tobacco	0.51
	Food	0.91
Constant returns	Apparel and textile	1.01
	Furniture	1.02
	Electronics	1.02
Increasing returns	Paper products	1.09
	Petroleum and coal	1.18
	Primary metal	1.24

Properties of Production Sets

Homogeneity of the Production Function	Returns to Scale
$\alpha + \beta = 1$	Constant Returns
$\alpha + \beta > 1$	Increasing Returns
$\alpha + \beta < 1$	Decreasing Returns

Properties of Production Sets

- The *linear* production function exhibits CRS as increasing all inputs by a common factor t yields

$$f(tk, tl) = atk + btl = t(ak + bl)$$
$$\equiv tf(k, l)$$

- The *fixed proportion* production function $f(k, l) = \min\{ak, bl\}$ also exhibits CRS as

$$f(tk, tl) = \min\{atk, btl\} = t \cdot \min\{ak, bl\}$$
$$\equiv tf(k, l)$$

Properties of Production Sets

- Increasing/decreasing returns to scale can be incorporated into a production function $f(k,l)$ exhibiting CRS by using a transformation function $F(\cdot)$

$$F(k,l) = [f(k,l)]^{\gamma}, \text{ where } \gamma > 0$$

- Indeed, increasing all arguments by a common factor t, yields

$$F(tk,tl) = [f(tk,tl)]^{\gamma} = \left[\overbrace{t \cdot f(k,l)}^{\text{by CRS of } f(\cdot)}\right]^{\gamma}$$

$$= t^{\gamma} \cdot \underbrace{[f(k,l)]^{\gamma}}_{F(k,l)} = t^{\gamma} \cdot F(k,l)$$

Properties of Production Sets

- Hence,
  - if $\gamma > 1$, the transformed production function $F(k, l)$ exhibits *increasing* returns to scale;
  - if $\gamma = 1$, the transformed production function $F(k, l)$ exhibits *constant* returns to scale;
  - if $\gamma < 1$, the transformed production function $F(k, l)$ exhibits *decreasing* returns to scale.

Properties of Production Sets

- ***Scale elasticity***: an alternative measure of returns to scale.
  - It measures the percent increase in output due to a 1% increase in the amounts of all inputs

$$\varepsilon_{q,t} = \frac{\partial f(tk, tl)}{\partial t} \cdot \frac{t}{f(tk, tl)}$$

where t denotes the common increase in all inputs.

  - *Practice*: Show that, if a function exhibits CRS, then it has a scale elasticity of $\varepsilon_{q,t} = 1$.

Properties of Production Sets

$$\varepsilon_{q,t} = \frac{\partial F(tk,tl)}{\partial t} \cdot \frac{t}{F(tk,tl)}$$
$$= \frac{\partial [t^{\gamma} F(k,l)]}{\partial t} \cdot \frac{t}{F(tk,tl)}$$
$$= \gamma t^{\gamma-1} F(k,l) \cdot \frac{t}{t^{\gamma} F(k,l)} = \gamma$$

and the scale elasticity measures returns to scale of the production function.

– For example, when $\gamma = 1$, the production function exhibits constant returns to scale, and $\varepsilon_{q,t} = 1$.

Properties of Production Sets

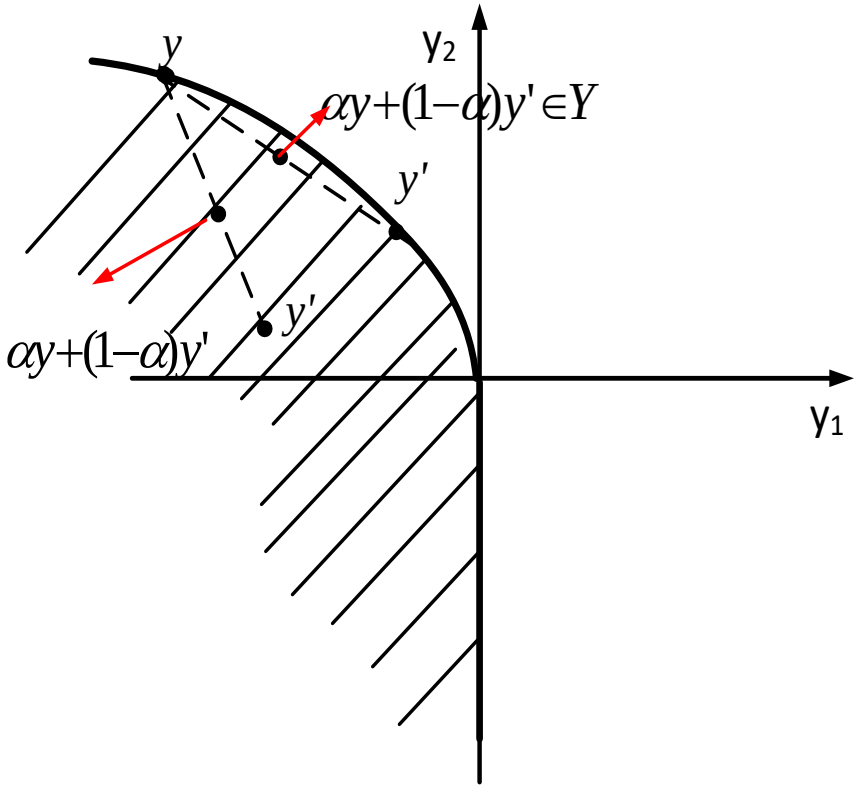
10) ***Additivity (or free entry)***: If $y \in Y$ and $y' \in Y$, then $y + y' \in Y$.

- *Interpretation*: one plant produces y, while another plant enters the market producing y'. Then, the aggregate production $y + y'$ is feasible.

Properties of Production Sets

11) Convexity: If $y, y' \in Y$ and $\alpha \in [0,1]$, then $\alpha y + (1-\alpha)y' \in Y$.

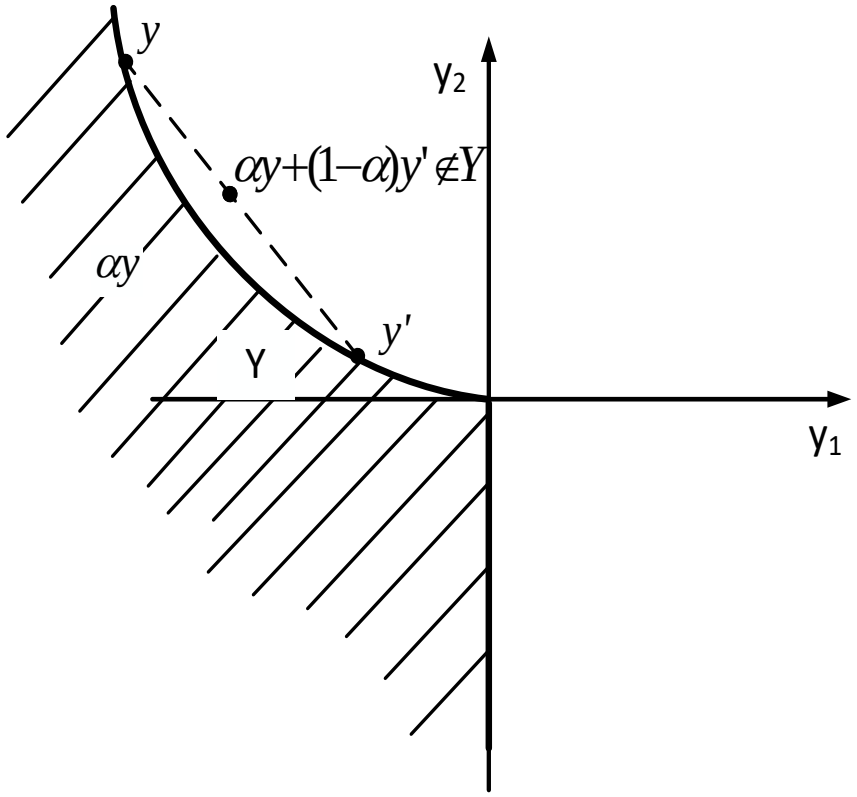
Intuition: "balanced" input-output combinations are more productive than "unbalanced" ones.



Properties of Production Sets

11) Convexity: violation

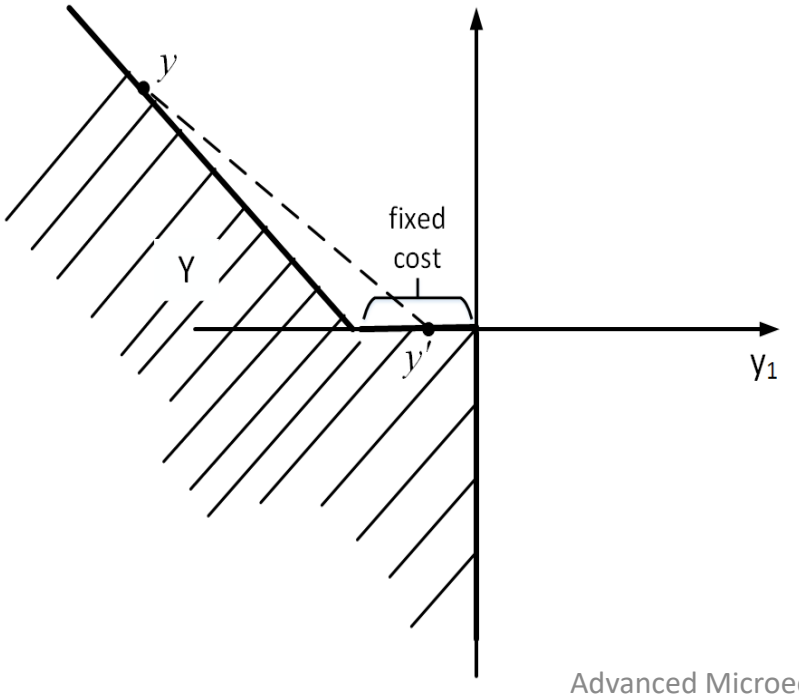
Note: The convexity of the production set maintains a close relationship with the concavity of the production function.

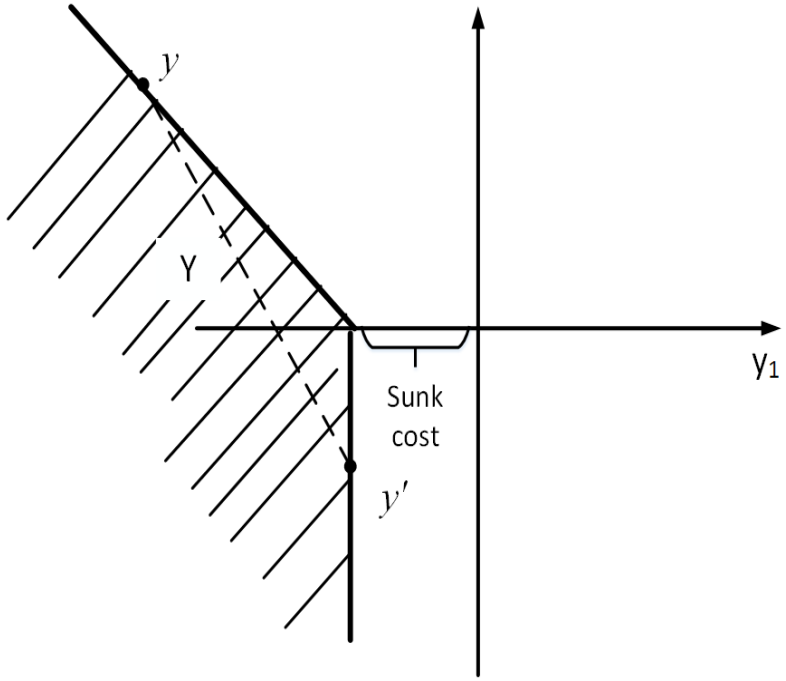


Properties of Production Sets

11) Convexity

- With fixed costs, convexity is NOT necessarily satisfied;
- With sunk costs, convexity is satisfied.





Diminishing MRTS

- The slope of the firm's isoquant is

$$MRTS_{l,k} = -\frac{dk}{dl}, \text{ where } MRTS_{l,k} = \frac{f_l}{f_k}$$

- Differentiating $MRTS_{l,k}$ with respect to labor yields

$$\frac{\partial MRTS_{l,k}}{\partial l} = \frac{f_k\left(f_{ll} + f_{lk} \cdot \frac{dk}{dl}\right) - f_l\left(f_{kl} + f_{kk} \cdot \frac{dk}{dl}\right)}{(f_k)^2}$$

Diminishing MRTS

- Using the fact that $\frac{dk}{dl} = -\frac{f_l}{f_k}$ along an isoquant and Young's theorem $f_{lk} = f_{kl}$,

$$\frac{\partial MRTS_{l,k}}{\partial l} = \frac{f_k\left(f_{ll} - f_{lk}\cdot\frac{f_l}{f_k}\right) - f_l\left(f_{lk} - f_{kk}\cdot\frac{f_l}{f_k}\right)}{(f_k)^2}$$

$$= \frac{f_k f_{ll} - f_{lk}f_l - f_l f_{lk} + f_{kk}\cdot\frac{f_l^2}{f_k}}{(f_k)^2}$$

Diminishing MRTS

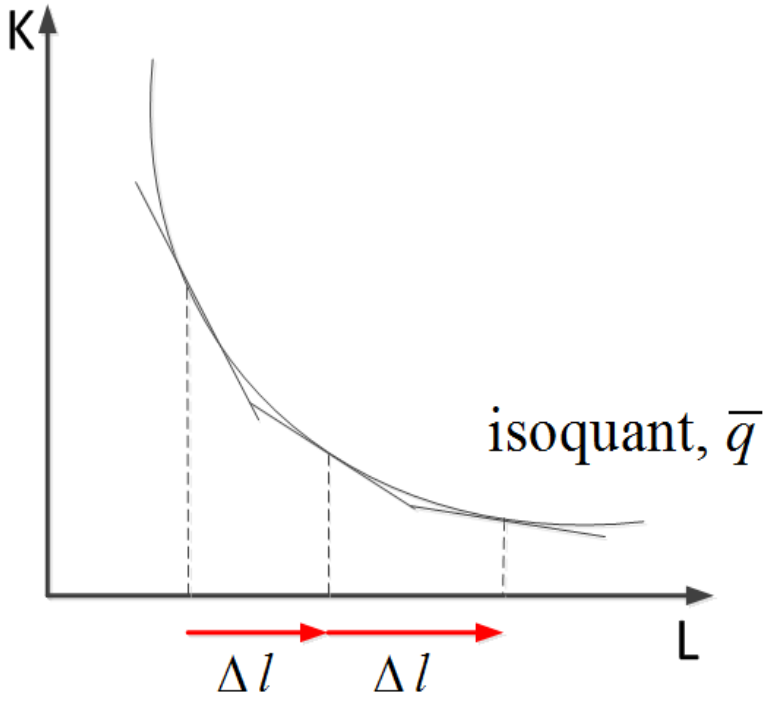
- Multiplying numerator and denominator by f_k

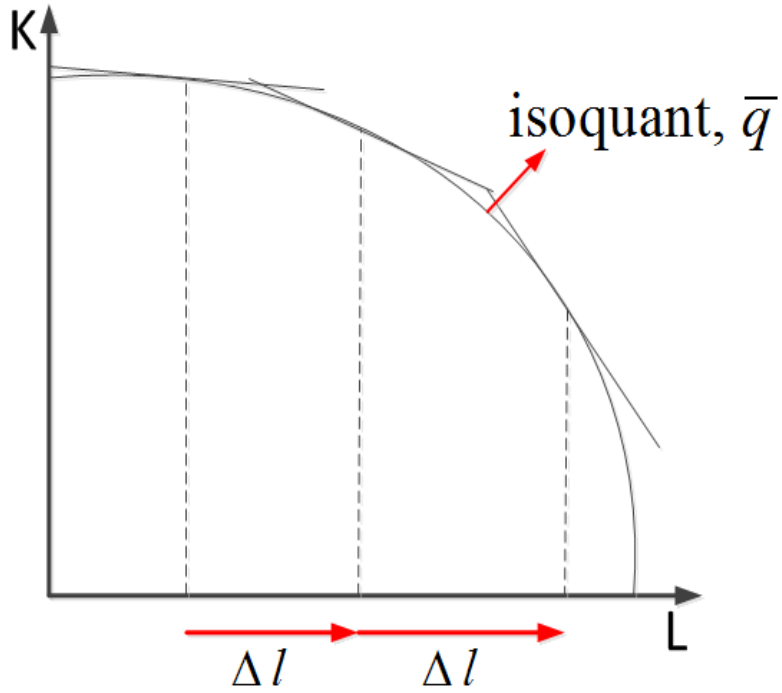
$$\frac{\partial MRTS_{l,k}}{\partial l} = \frac{\overbrace{f_k^2}^{+}\overbrace{f_{ll}}^{-} + \overbrace{f_{kk}}^{-}\overbrace{f_l^2}^{+} - \overbrace{2f_l f_k}^{+}\overbrace{f_{lk}}^{-\text{ or }+}}{(f_k)^3}$$

- Thus,
  - If $f_{lk} > 0$ (i.e., $\uparrow k \Longrightarrow \uparrow MP_l$), then $\frac{\partial MRTS_{l,k}}{\partial l} < 0$
  - If $f_{lk} < 0$, then we have

$$\left|f_k^2 f_{ll} + f_{kk} f_l^2\right| \left\{\begin{matrix}>\\<\end{matrix}\right\} \left|2 f_l f_k f_{lk}\right| \Longrightarrow \frac{\partial MRTS_{l,k}}{\partial l} \left\{\begin{matrix}<\\>\end{matrix}\right\} 0$$

Diminishing MRTS





$f_{lk} > 0$ ($\uparrow k \Longrightarrow \uparrow MP_l$), or
$f_{lk} < 0$ ($\uparrow k \Longrightarrow \downarrow MP_l$) but small $\downarrow$ in MP_l

$f_{lk} < 0$ ($\uparrow k \Longrightarrow \downarrow\downarrow MP_l$)

Diminishing MRTS

- ***Example***: Let us check if the production function $f(k,l) = 600k^2l^2 - k^3l^3$ yields convex isoquants.
  - Marginal products:

    $MP_l = f_l = 1{,}200k^2l - 3k^3l^2 > 0$ iff $kl < 400$

    $MP_k = f_k = 1{,}200kl^2 - 3k^2l^3 > 0$ iff $kl < 400$
  - Decreasing marginal productivity:

    $\frac{\partial MP_l}{\partial l} = f_{ll} = 1{,}200k^2 - 6k^3l < 0$ iff $kl > 200$

    $\frac{\partial MP_k}{\partial k} = f_{kk} = 1{,}200l^2 - 6kl^3 < 0$ iff $kl > 200$

Diminishing MRTS

- ***Example*** (continued):
  - Is $200 < kl < 400$ the sufficient condition for diminishing $MRTS$?
    - No!
    - We need $f_{lk} > 0$ too in order to guarantee diminishing $MRTS_{l,k}$.
  - Check the sign of f_{lk}:

$$f_{lk} = f_{kl} = 2{,}400kl - 9k^2l^2 > 0 \text{ iff } kl < 266$$

Diminishing MRTS

- ***Example*** (continued):
  - Alternatively, we can represent the above conditions by solving for l in the above inequalities:

$$MP_l > 0 \text{ iff } l < \frac{400}{k} \qquad \frac{\partial MP_l}{\partial l} < 0 \text{ iff } l > \frac{200}{k}$$

$$MP_k > 0 \text{ iff } l < \frac{400}{k} \qquad \frac{\partial MP_k}{\partial k} < 0 \text{ iff } l > \frac{200}{k}$$

and

$$f_{lk} > 0 \text{ iff } l < \frac{266}{k}$$

Diminishing MRTS

- ***Example*** (continued):
  - Hence, $\frac{200}{k} < l < \frac{266}{k}$ guarantees positive but diminishing marginal products and, in addition, a diminishing $MRTS$.
  - ***Figure***: $l = \frac{200}{k}$ is a curve decreasing in k, never crossing either axes. Similarly for $l = \frac{266}{k}$.

Constant Returns to Scale

- If production function $f(k,l)$ exhibits CRS, then increasing all inputs by a common factor t yields

$$f(tk,tl) = tf(k,l)$$

- Hence, $f(k,l)$ is homogenous of degree 1, thus implying that its first-order derivatives

$$f_k(k,l) \text{ and } f_l(k,l)$$

are homogenous of degree zero.

Constant Returns to Scale

- ***Proof:*** Since $f(tk, tl) = tf(k, l)$, taking the partial derivative of both sides with respect to k,

$$tf_k(tk, tl) = tf_k(k, l)$$

- Cancelling out t on both sides of the above expression, we obtain $f_k(tk, tl) = f_k(k, l)$.
- Hence, $f_k(k, l)$ is homogenous of degree zero, and the same principle applies to $f_l(k, l)$.

Constant Returns to Scale

- Therefore,

$$MP_l = \frac{\partial f(k,l)}{\partial l} = \frac{\partial f(tk,tl)}{\partial l}$$
$$= f_l(k,l) = f_l(tk,tl)$$

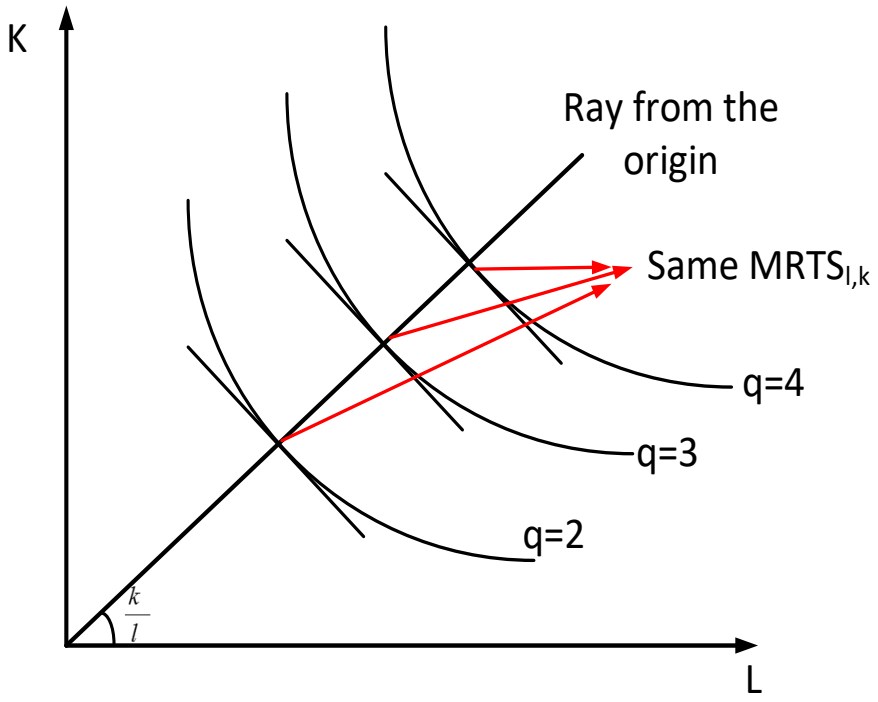
- Setting $t = \frac{1}{l}$, we obtain

$$MP_l = f_l(k,l) = f_l\left(\frac{1}{l}k, \frac{l}{l}\right) = f_l\left(\frac{k}{l}, 1\right)$$

- Hence, MP_l only depends on the ratio $\frac{k}{l}$, but not on the absolute levels of k and l that firm uses.
- A similar argument applies to MP_k.

Constant Returns to Scale

- Thus, $MRTS = \frac{MP_l}{MP_k}$ only depends on the ratio of capital to labor.
- The slope of a firm's isoquants coincides at any point along a ray from the origin.
- Firm's production function is, hence, homothetic.



Elasticity of Substitution

Elasticity of Substitution

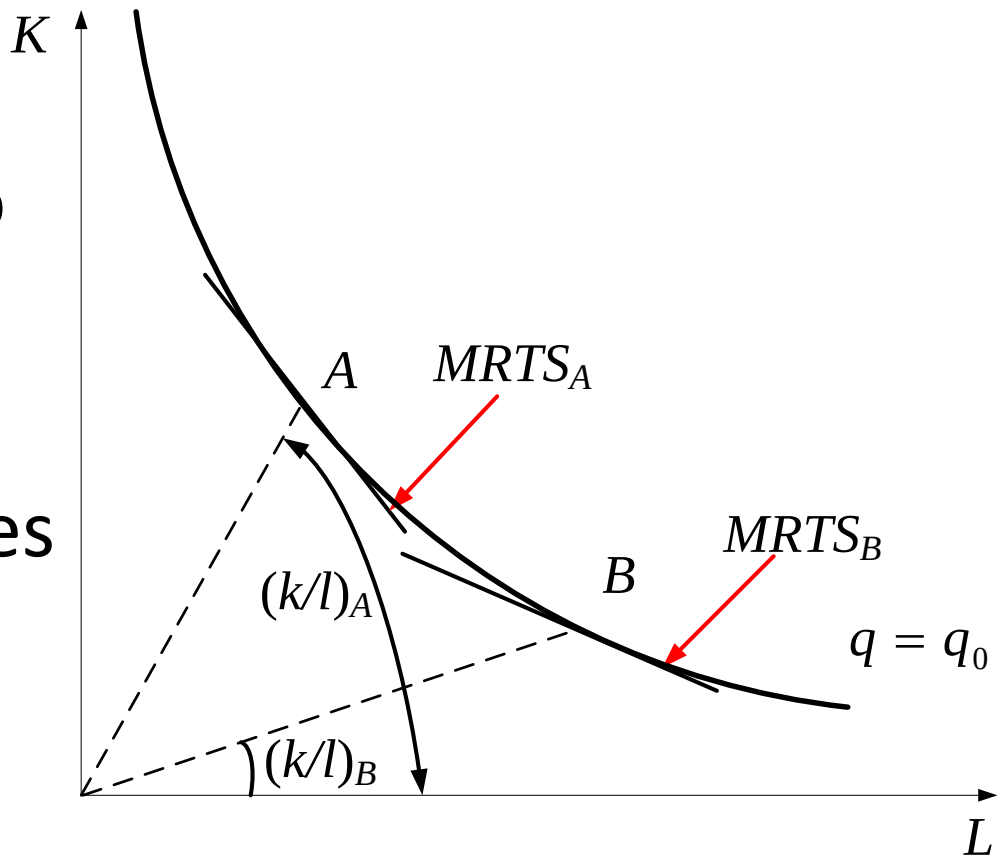
- ***Elasticity of substitution (σ)*** measures the proportionate change in the k/l ratio relative to the proportionate change in the $MRTS_{l,k}$ along an isoquant:

$$\sigma = \frac{\%\Delta(k/l)}{\%\Delta MRTS} = \frac{d(k/l)}{dMRTS} \cdot \frac{MRTS}{k/l} = \frac{\partial \ln(k/l)}{\partial \ln(MRTS)}$$

where $\sigma > 0$ since ratio k/l and $MRTS$ move in the same direction.

Elasticity of Substitution

- Both $MRTS$ and k/l will change as we move from point A to point B.
- σ is the ratio of these changes that measures the curvature of the isoquant.

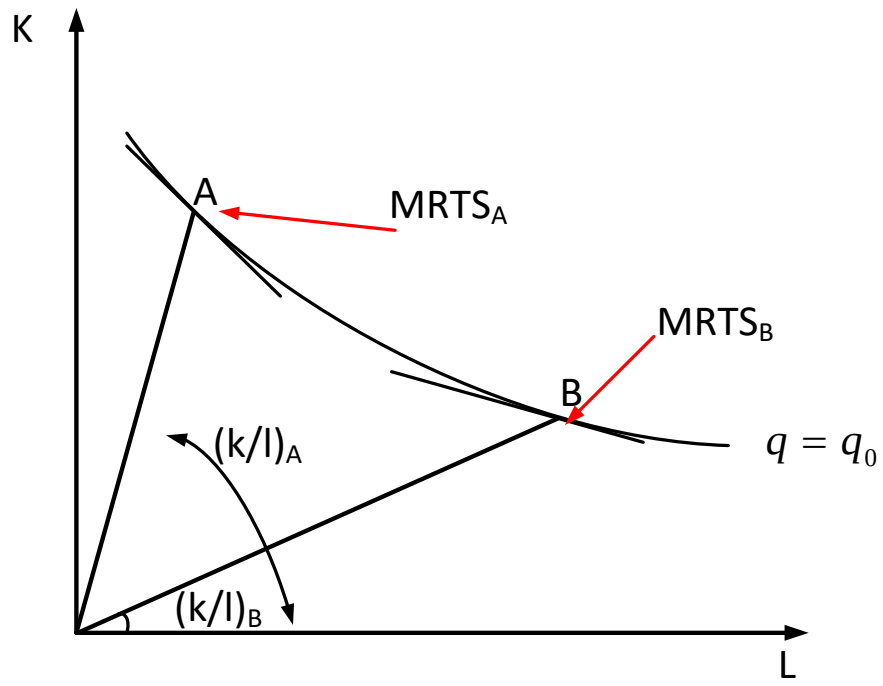


Elasticity of Substitution

- If we define the elasticity of substitution between two inputs to be proportionate change in the ratio of the two inputs to the proportionate change in $MRTS$, we need to hold:
  - output constant (so we move along the same isoquant), and
  - the levels of other inputs constant (in case we have more than two inputs). For instance, we fix the amount of other inputs, such as land.

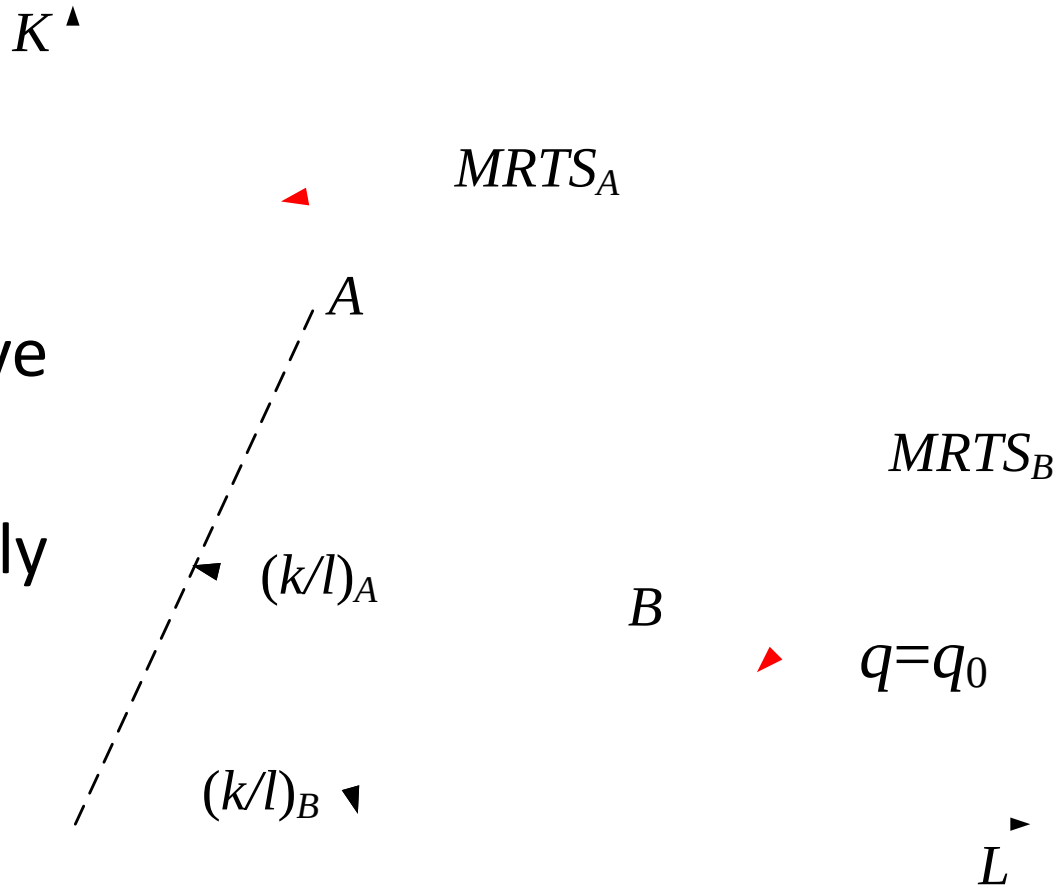
Elasticity of Substitution

- High elasticity of substitution (σ):
  - $MRTS$ does not change substantially relative to k/l.
  - Isoquant is relatively flat.



Elasticity of Substitution

- Low elasticity of substitution (σ):
  - $MRTS$ changes substantially relative to k/l.
  - Isoquant is relatively sharply curved.



Elasticity of Substitution: Linear Production Function

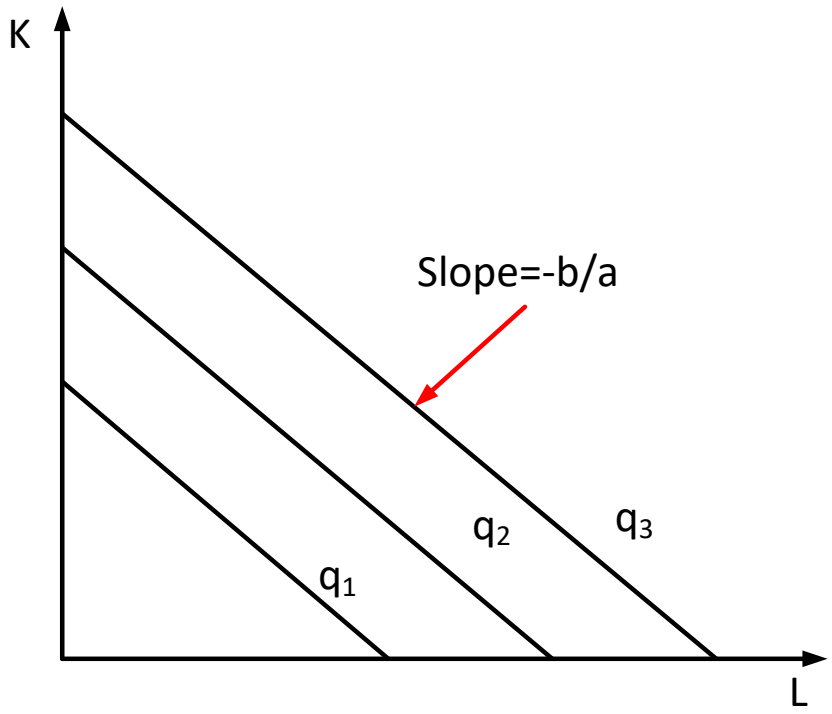
- Suppose that the production function is
$$q = f(k, l) = ak + bl$$
- This production function exhibits constant returns to scale since
$$f(tk, tl) = atk + btl = t(ak + bl)$$
$$= tf(k, l)$$
- Solving for k in q, we get $k = \frac{f(k,l)}{a} - \frac{b}{a} l$.
  - All isoquants are straight lines
  - k and l are perfect substitutes

Elasticity of Substitution: Linear Production Function

- $MRTS$ (slope of the isoquant) is constant as k/l changes.

$$\sigma = \frac{\%\Delta(k/l)}{\underbrace{\%\Delta MRTS}_{0}} = \infty$$

- This production function satisfies homotheticity.



Elasticity of Substitution: Fixed Proportions Production Function

- Suppose that the production function is

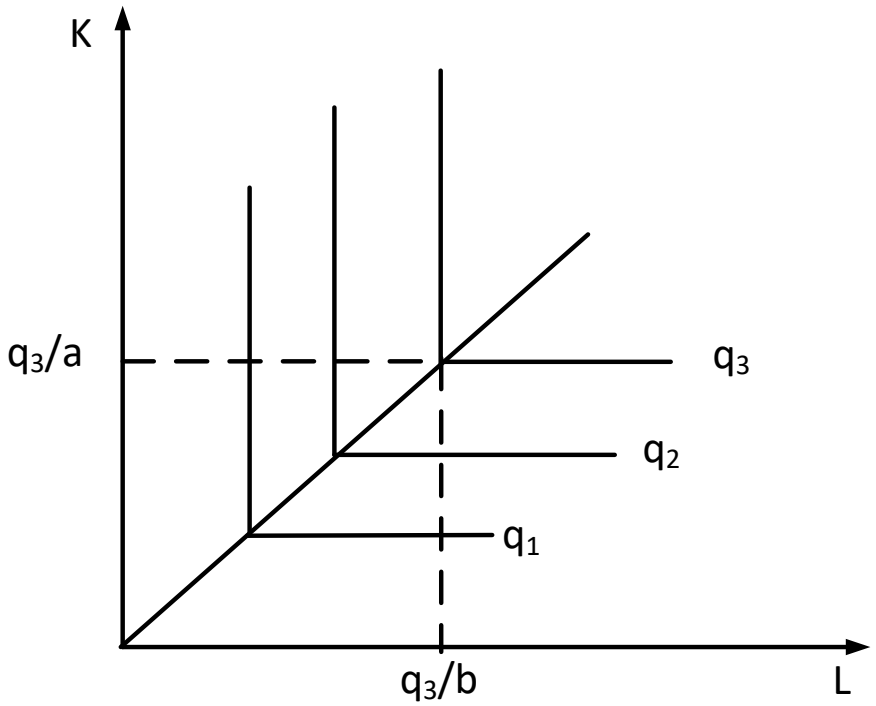
$$q = \min(ak, bl) \quad a, b > 0$$

- Capital and labor must always be used in a fixed ratio
  - No substitution between k and l
  - The firm will always operate along a ray where k/l is constant (i.e., at the kink!).
- Because k/l is constant (b/a),

$$\sigma = \frac{\%\Delta(k/l)}{\underbrace{\%\Delta MRTS}_{\infty}} = 0$$

Elasticity of Substitution: Fixed Proportions Production Function

- $MRTS = \infty$ for l before the kink of the isoquant.
- $MRTS = 0$ for l after the kink.
- This production function also satisfies homotheticity.



Elasticity of Substitution: Cobb-Douglas Production Function

- Suppose that the production function is
$$q = f(k,l) = Ak^a l^b \quad \text{where } A, a, b > 0$$
- This production function can exhibit any returns to scale since
$$f(tk,tl) = A(tk)^a(tl)^b = At^{a+b}k^a l^b$$
$$= t^{a+b} f(k,l)$$
  - If $a + b = 1 \Longrightarrow$ constant returns to scale
  - If $a + b > 1 \Longrightarrow$ increasing returns to scale
  - If $a + b < 1 \Longrightarrow$ decreasing returns to scale

Elasticity of Substitution: Cobb-Douglas Production Function

- The Cobb-Douglas production function is linear in logarithms

$$\ln(q) = \ln(A) + a\ln(k) + b\ln(l)$$

  - a is the elasticity of output with respect to k

$$\varepsilon_{q,k} = \frac{\partial \ln(q)}{\partial \ln(k)}$$

  - b is the elasticity of output with respect to l

$$\varepsilon_{q,l} = \frac{\partial \ln(q)}{\partial \ln(l)}$$

Elasticity of Substitution: Cobb-Douglas Production Function

- The elasticity of substitution (σ) for the Cobb-Douglas production function:
  - First,

$$MRTS = \frac{MP_l}{MP_k} = \frac{\frac{\partial q}{\partial l}}{\frac{\partial q}{\partial k}} = \frac{bAk^a l^{b-1}}{aAk^{a-1}l^b} = \frac{b}{a} \cdot \frac{k}{l}$$

  - Hence,

$$\ln(MRTS) = \ln\left(\frac{b}{a}\right) + \ln\left(\frac{k}{l}\right)$$

Elasticity of Substitution: Cobb-Douglas Production Function

– Solving for $\ln\left(\frac{k}{l}\right)$,

$$\ln\left(\frac{k}{l}\right) = \ln(MRTS) - \ln\left(\frac{b}{a}\right)$$

– Therefore, the elasticity of substitution between k and l is

$$\sigma = \frac{d\ln\left(\frac{k}{l}\right)}{d\ln(MRTS)} = 1$$

Elasticity of Substitution: CES Production Function

- Suppose that the production function is

$$q = f(k, l) = (k^{\rho} + l^{\rho})^{\gamma/\rho}$$

where $\rho \leq 1, \rho \neq 0, \gamma > 0$

  - $\gamma = 1 \Longrightarrow$ constant returns to scale
  - $\gamma > 1 \Longrightarrow$ increasing returns to scale
  - $\gamma < 1 \Longrightarrow$ decreasing returns to scale

- Alternative representation of the CES function

$$f(k, l) = \left(ak^{\frac{\sigma-1}{\sigma}} + bl^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

where σ is the elasticity of substitution.

Elasticity of Substitution: CES Production Function

- The elasticity of substitution (σ) for the CES production function:
  - First,

$$MRTS = \frac{MP_l}{MP_k} = \frac{\frac{\partial q}{\partial l}}{\frac{\partial q}{\partial k}} = \frac{\frac{\gamma}{\rho}(k^\rho + l^\rho)^{\frac{\gamma}{\rho}-1}(\rho l^{\rho-1})}{\frac{\gamma}{\rho}(k^\rho + l^\rho)^{\frac{\gamma}{\rho}-1}(\rho k^{\rho-1})}$$

$$= \left(\frac{l}{k}\right)^{\rho-1} = \left(\frac{k}{l}\right)^{1-\rho}$$

Elasticity of Substitution: CES Production Function

- Hence,

$$\ln(MRTS) = (1-\rho)\ln\left(\frac{k}{l}\right)$$

- Solving for $\ln\left(\frac{k}{l}\right)$,

$$\ln\left(\frac{k}{l}\right) = \frac{1}{1-\rho}\ln(MRTS)$$

- Therefore, the elasticity of substitution between k and l is

$$\sigma = \frac{d\ln\left(\frac{k}{l}\right)}{d\ln(MRTS)} = \frac{1}{1-\rho}$$

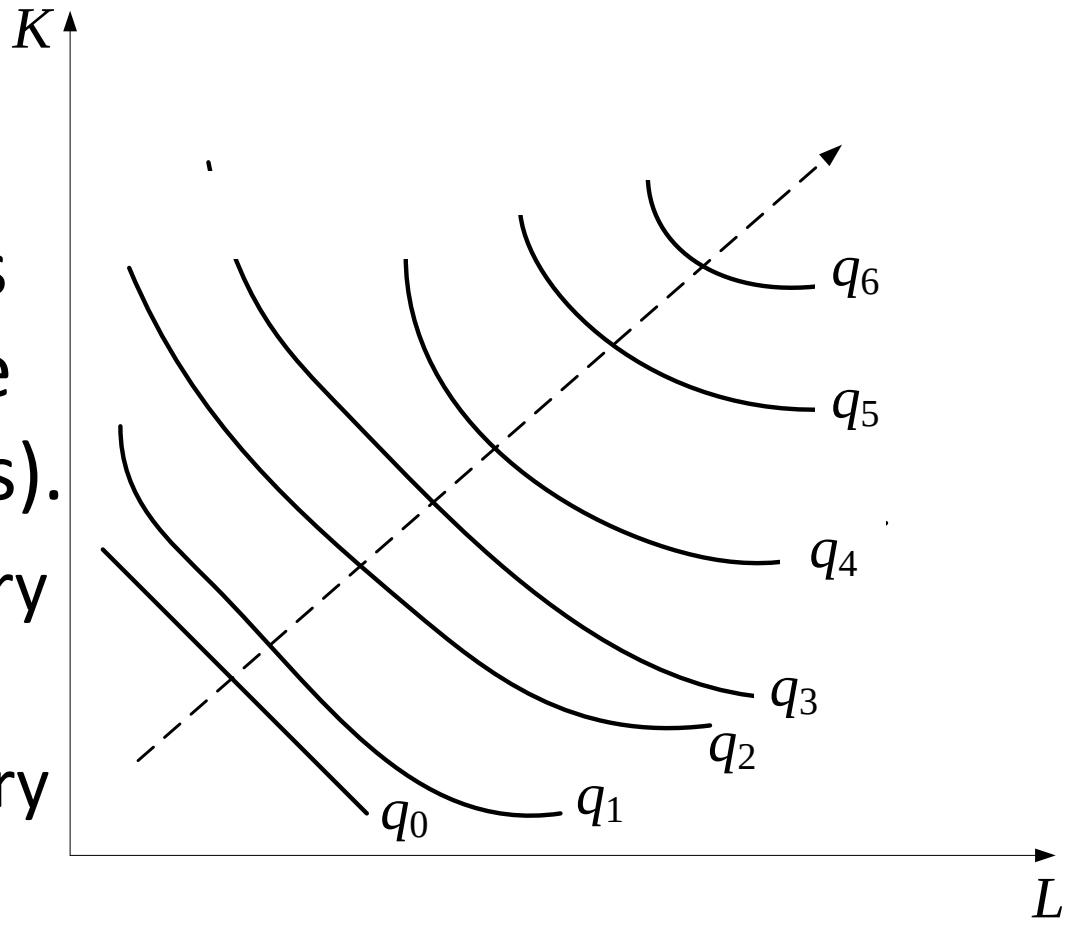
Elasticity of Substitution: CES Production Function

- Elasticity of Substitution in German Industries (Source: Kemfert, 1998):

Industry	σ
Food	0.66
Iron	0.50
Chemicals	0.37
Motor Vehicles	0.10

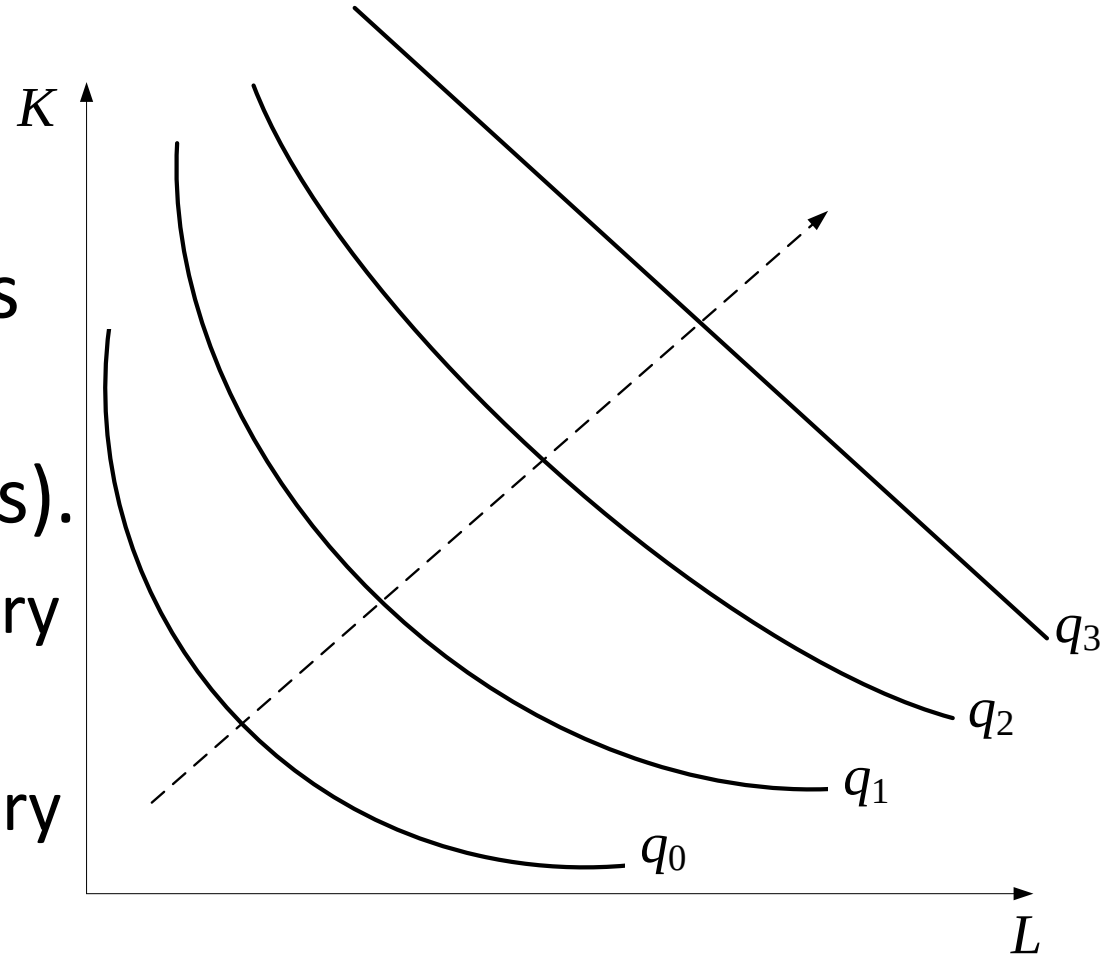
Elasticity of Substitution

- The elasticity of substitution σ between k and l is *decreasing* in scale (i.e., as q increases).
  - q_0 and q_1 have very high σ
  - q_5 and q_6 have very low σ



Elasticity of Substitution

- The elasticity of substitution σ between k and l is *increasing* in scale (i.e., as q increases).
  - q_0 and q_1 have very low σ
  - q_2 and q_3 have very high σ



Profit Maximization

Profit Maximization

- Assumptions:
  - Firms are price takers: the production plans of an individual firm do not alter price levels $p = (p_1, p_2, \dots, p_L) \gg 0$.
  - The production set satisfies: non-emptiness, closedness, and free-disposal.
- Profit maximization problem (PMP):

$$\max_{y} \quad p \cdot y$$

s.t. $y \in Y$, or alternatively, $F(y) \leq 0$

Profit Maximization

- Profit function $\pi(p)$ associates to every p the highest amount of profits (i.e., $\pi(p)$ is the value function of the PMP)

$$\pi(p) = \max_{y} \{p \cdot y : y \in Y\}$$

- And the supply correspondence $y(p)$ is the argmax of the PMP,

$$y(p) = \arg\max\{y \in Y : p \cdot y = \pi(p)\}$$

where ***positive*** component in the vector $y(p)$ is output supplied by the firm to the market, while ***negative*** components are inputs in its production process.

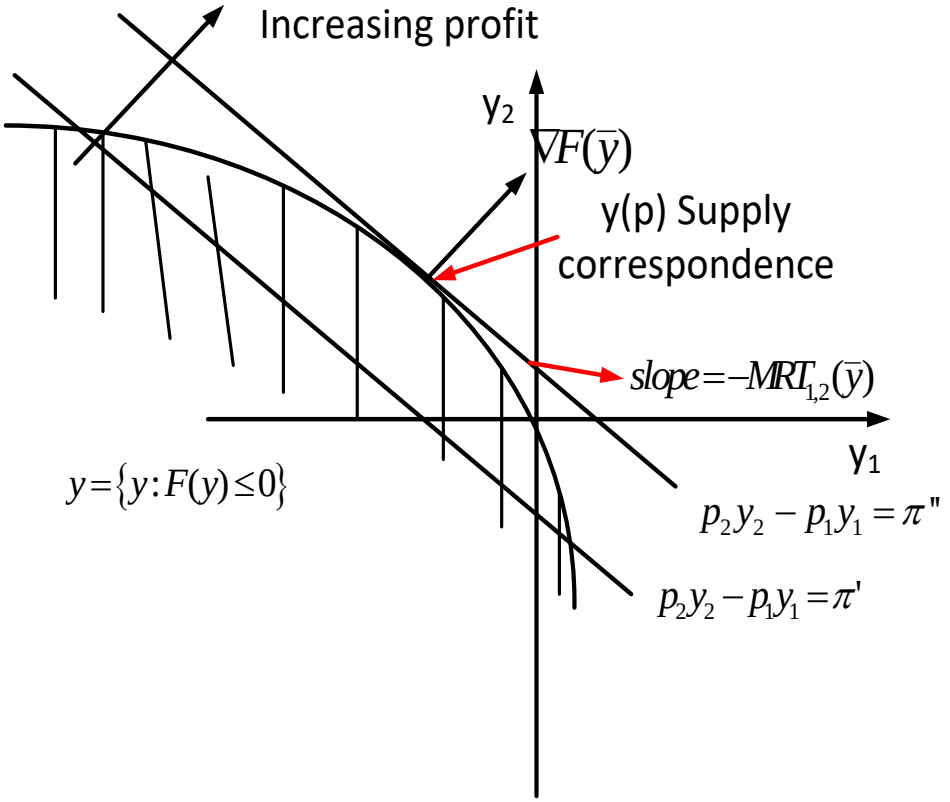
Profit Maximization

- ***Isoprofit line***: combinations of inputs and output for which the firm obtains a given level of profits.
- Note that

$$\pi^0 = p_1 y_1 + p_2 y_2$$

Solving for y_2

$$y_2 = \underbrace{\frac{\pi^0}{p_2}}_{\text{intercept}} \underbrace{-\frac{p_1}{p_2}}_{\text{slope}} y_1$$



Profit Maximization

- We can rewrite the PMP as

$$\max_{y \leq F(y)} \; p \cdot y$$

with associated Lagrangian

$$L = p \cdot y - \lambda F(y)$$

Profit Maximization

– Taking FOCs with respect to every y_k, we obtain

$$p_k - \lambda \frac{\partial F(y^*)}{\partial y_k} \leq 0$$

where $F(y^*)$ is evaluated at the optimum, i.e., $F(y^*) = F\big(y(p)\big)$.

– For interior solutions, $p_k = \lambda \frac{\partial F(y^*)}{\partial y_k}$, or in matrix notation

$$p = \lambda \nabla_y F(y^*)$$

that is, the price vector and the gradient vector are proportional.

Profit Maximization

– Solving for λ, we obtain

$$\lambda = \frac{p_k}{\frac{\partial F(y^*)}{\partial y_k}} \text{ for every good } k \Longrightarrow \frac{p_k}{\frac{\partial F(y^*)}{\partial y_k}} = \frac{p_l}{\frac{\partial F(y^*)}{\partial y_l}}$$

which can also be expressed as

$$\frac{p_k}{p_l} = \frac{\frac{\partial F(y^*)}{\partial y_k}}{\frac{\partial F(y^*)}{\partial y_l}} \quad (= MRT_{k,l}(y^*))$$

– Graphically, the slope of the transformation frontier (at the profit maximization production plan y^*), $MRT_{k,l}(y^*)$, coincides with the price ratio, $\frac{p_k}{p_l}$.

Profit Maximization

- Are there PMPs with no supply correspondence $y(p)$, i.e., there is no well defined profit-maximizing vector?
  - Yes.
- ***Example***: $q = f(z) = z$ (i.e., every unit of input z is transformed into a unit of output q)

Profit Maximization: Single Output

- Production function, $q = f(z)$, produces a single output from a vector z of inputs.

$$\max_{z \geq 0} \; pf(z) - wz$$

- The first-order conditions are

$$p \frac{\partial f(z^*)}{\partial z_k} \leq w_k \;\; \text{or} \;\; p \cdot MP_{z_k} \leq w_k$$

- For interior solutions, the market value of the marginal product obtained form using additional units of this input k, $p \frac{\partial f(z^*)}{\partial z_k}$, must coincide with the price of this input, w_k.

Profit Maximization: Single Output

- Note that for any two input, this implies

$$p = \frac{w_k}{\frac{\partial f(z^*)}{\partial z_k}} \text{ for every good } k$$

Hence,

$$\frac{w_k}{w_l} = \frac{\frac{\partial f(z^*)}{\partial z_k}}{\frac{\partial f(z^*)}{\partial z_l}} = \frac{MP_{z_k}}{MP_{z_l}} \; (= MRTS_{z_k,z_l}(z^*))$$

or

$$\frac{MP_{z_k}}{w_{z_k}} = \frac{MP_{z_l}}{w_{z_l}}$$

Intuition: Marginal productivity per dollar spent on input z_k is equal to that spent on input z_l.

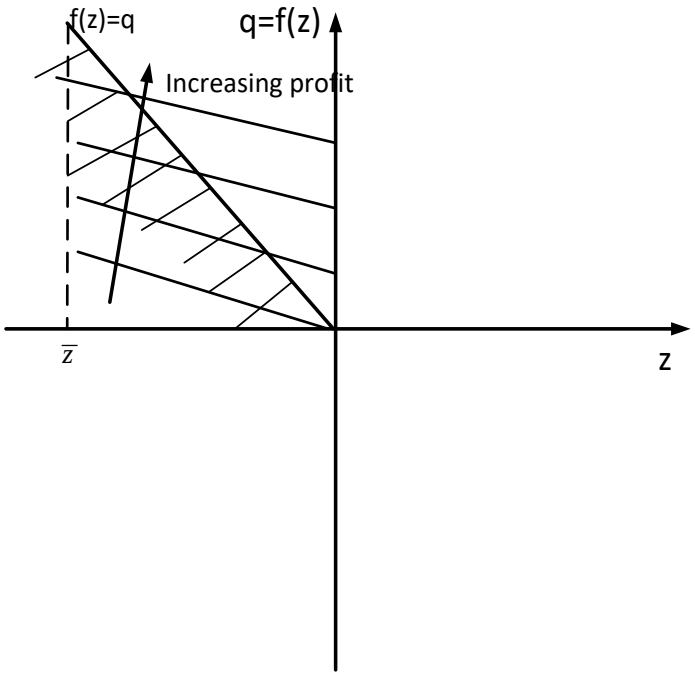
Profit Maximization

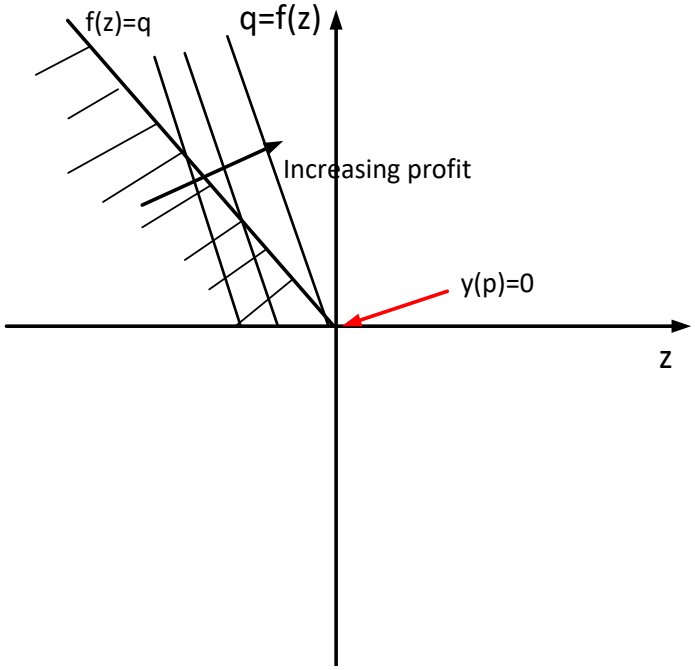
- ***Example*** : Are there PMPs with no supply correspondence $y(p)$, i.e., there is no well defined profit-maximizing vector?
  - Yes.

- If the input price p_z satisfies $p_z \geq p$, then $q = 0$ and $\pi(p) = 0$.
- If the input price p_z satisfies $p_z < p$, then $q = +\infty$ and $\pi(p) = +\infty$.
  - In this case, the supply correspondence is not well defined, since you can always increase input usage, thus increasing profits.
  - *Exception*: if input usage is constrained in the interval $[0, \bar{z}]$, then $y(p)$ is at the corner solution $y(p) = \bar{z}$, thus implying that the PMP is well defined.

Profit Maximization

- ***Example*** (continued):





If $p_z < p$, the firm can Δq and $\Delta\pi$.

If $p_z > p$, the firm chooses $q = y(p) = 0$ with $\pi(p) = 0$.

Profit Maximization: Single Output

- When are these FOCs also sufficient?
  - When the production set Y is *convex*! Let's see.
- Isocost line for the firm is

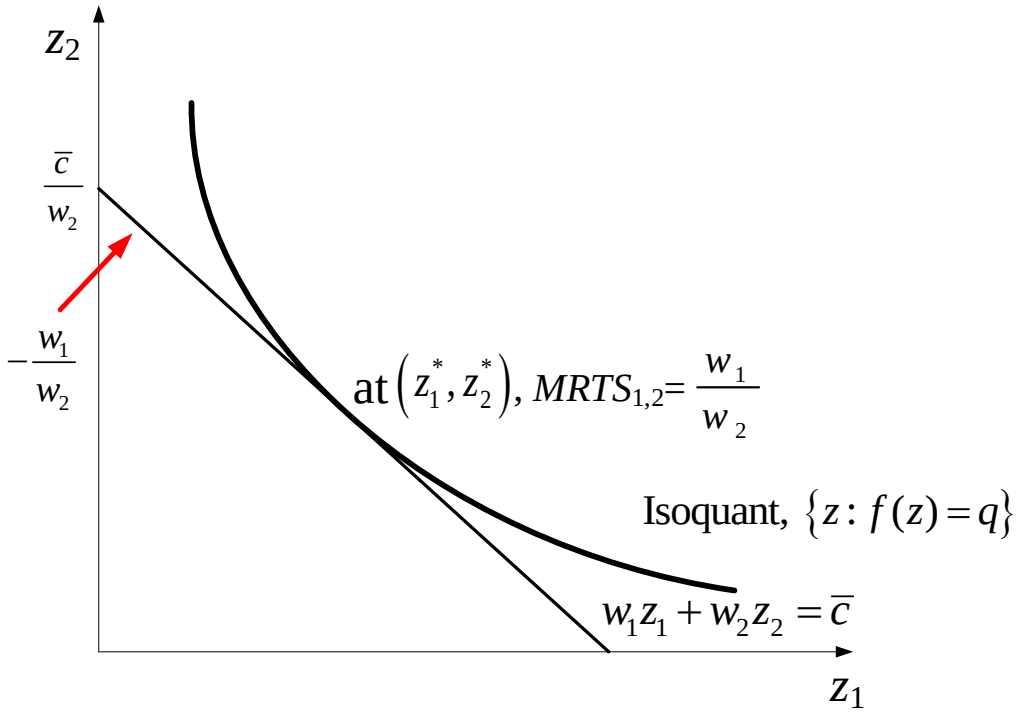
$$w_1 z_1 + w_2 z_2 = \bar{c}$$

- Solving for z_2

$$z_2 = \underbrace{\frac{\bar{c}}{w_2}}_{\text{intercept}} \underbrace{- \frac{w_1}{w_2}}_{\text{slope}} z_1$$

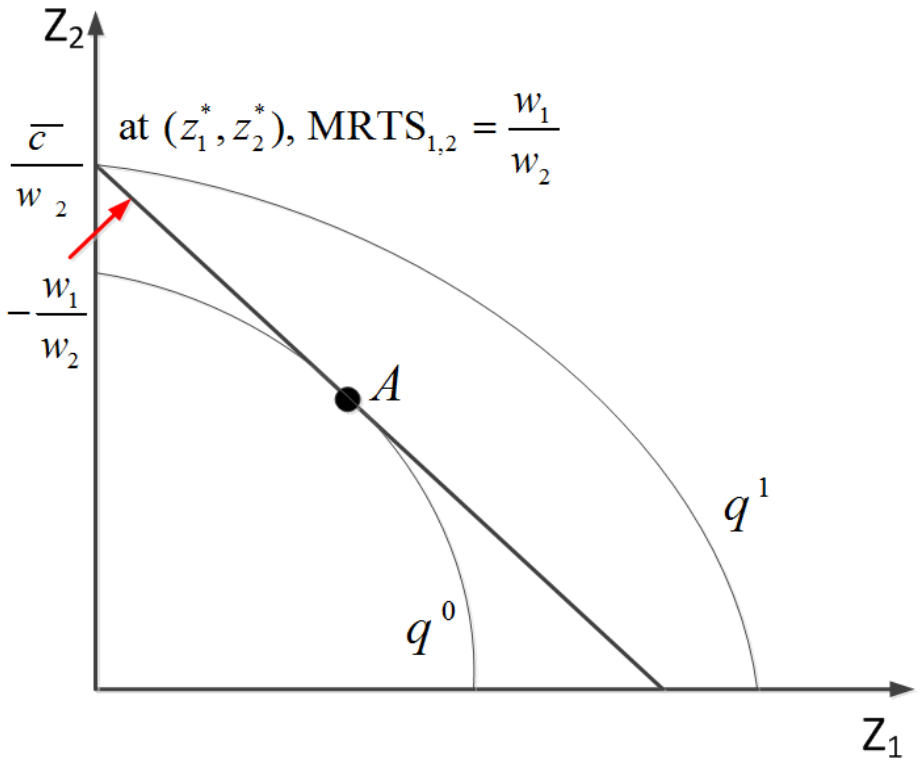
Profit Maximization: Single Output

- Convex production set
  - The FOCs (necessary) of $MRTS = \frac{w_1}{w_2}$ are also sufficient.



Profit Maximization: Single Output

- Non-convex production set
  - the FOCs are NOT sufficient for a combination of (z_1, z_2) that maximize profits.
  - the profit-maximizing vector (z_1^*, z_2^*) is at a corner solution, where the firm uses z_2 alone.



Profit Maximization: Single Output

- ***Example:*** Cobb-Douglas production function
- On your own:
  - Solve PMP (differentiating with respect to z_1 and z_2).
  - Find optimal input usage $z_1(w, q)$ and $z_2(w, q)$.
    - These are referred to as "conditional factor demand correspondences"
  - Plug them into the production function to obtain the value function, i.e., the output that arises when the firm uses its profit-maximizing input combination.

Properties of Profit Function

- Assume that the production set Y is closed and satisfies the free disposal property.

  1) Homog(1) in prices

$$\pi(\lambda p) = \lambda \pi(p)$$

  - Increasing the prices of all inputs and outputs by a common factor λ produces a proportional increase in the firm's profits.

$$\pi(p) = pq - w_1 z_1 - \cdots - w_n z_n$$

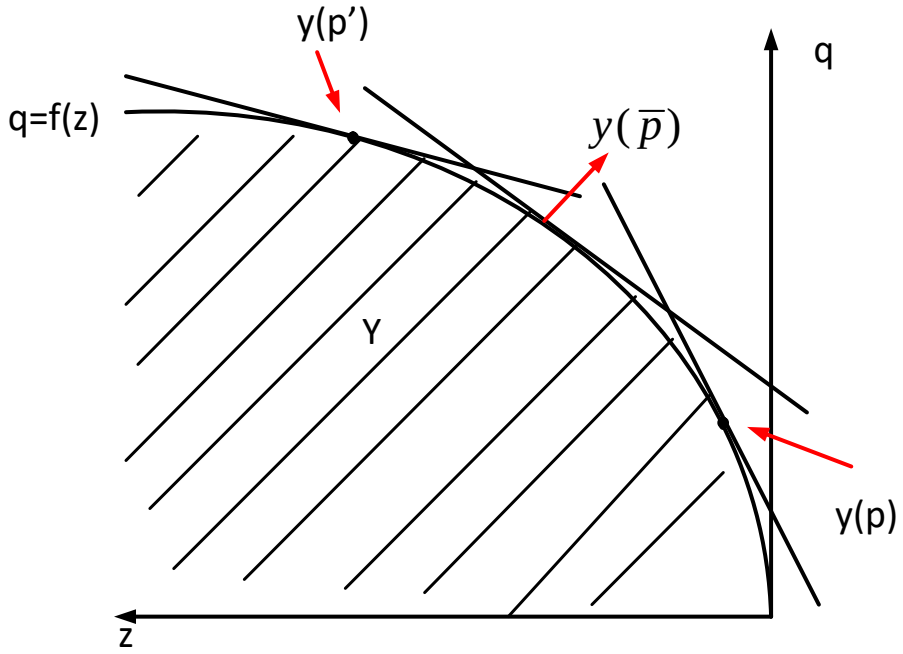
  Scaling all prices by a common factor, we obtain

$$\pi(\lambda p) = \lambda pq - \lambda w_1 z_1 - \cdots - \lambda w_n z_n$$
$$= \lambda(pq - w_1 z_1 - \cdots - w_n z_n) = \lambda \pi(p)$$

Properties of Profit Function

2) Convex in output prices

- *Intuition*: the firm obtains more profits from balanced input-output combinations, than from unbalanced combinations.



Price vector	Production plan	Profits
p	$y(p)$	$\pi(p)$
p'	$y(p')$	$\pi(p')$
$\bar{p}$	$y(\bar{p})$	$\pi(\bar{p}) \leq \alpha\pi(p) + (1-\alpha)\pi(p')$

Properties of Profit Function

3) *If the production set Y is convex, then*
$Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}$

- *Intuition*: the production set Y can be represented by this "dual" set.
- This dual set specifies that, for any given prices p, all production vectors y generate less profits $p \cdot y$, than the optimal production plan $y(p)$ in the profit function $\pi(p) = p \cdot y(p)$.

Properties of Profit Function

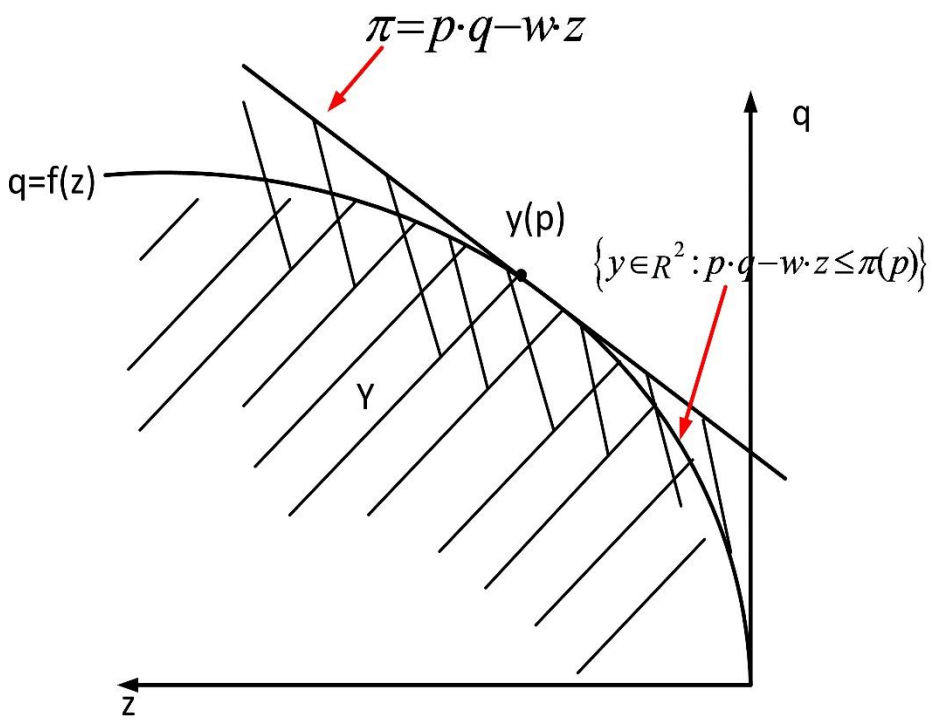
- All production plans (z, q) below the isoprofit line yield a lower profit level:

$$pq - wz \leq \pi(p)$$

- The isoprofit line $\pi(p) = pq - wz$ can be expressed as

$$q = \frac{\pi}{p} + \frac{w}{p} z$$

  - If $\frac{w}{p}$ is constant $\Longrightarrow \pi(\cdot)$ is convex.
  - What if it is not constant? Let's see next.



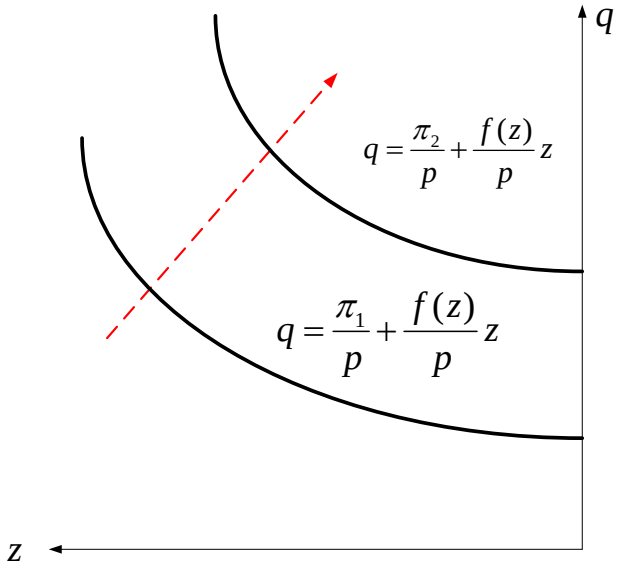
Properties of Profit Function

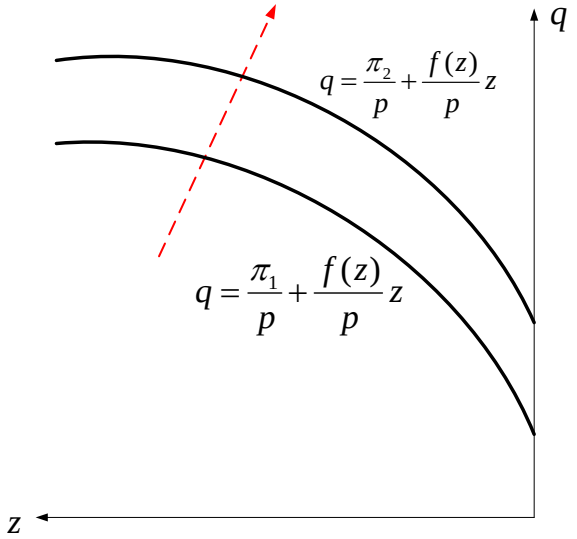
a) Input prices are a function of input usage, i.e., $w = f(z)$, where $f'(z) \neq 0$. Then, either
   - *i.* $f'(z) < 0$, and the firm gets a *price discount* per unit of input from suppliers when ordering large amounts of inputs (e.g., loans)
   - *ii.* $f'(z) > 0$, and the firm has to pay *more* per unit of input when ordering large amounts of inputs (e.g., scarce qualified labor)

b) Output prices are a function of production , i.e., $p = g(q)$, where $g'(q) \neq 0$. Then, either
   - *i.* $g'(q) < 0$, and the firm offers *price discounts* to its customers.
   - *ii.* $g'(q) > 0$, and the firm applies *price surcharges* to its customers.

Properties of Profit Function

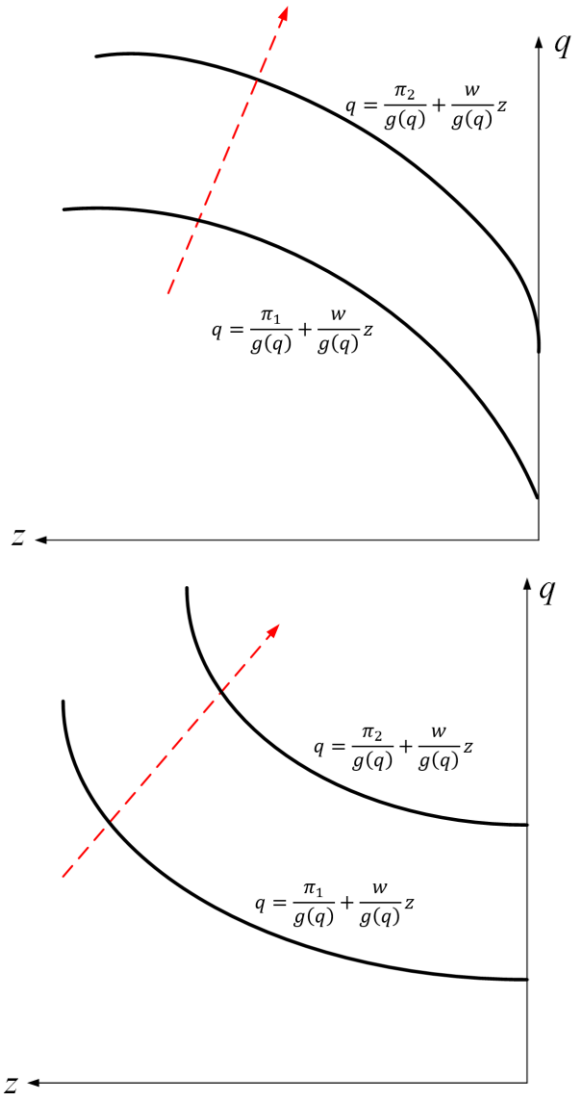
- Suppose $w = f(z)$, then the slope of the isoquant is $\frac{\partial q}{\partial z} = -\frac{\frac{\partial \pi}{\partial z}}{\frac{\partial \pi}{\partial q}} = -\frac{-f'(z)z - f(z)}{p} = \frac{f'(z)z + f(z)}{p}$.
- Further differentiating the slope with respect to z, $\frac{\partial^2 q}{\partial z^2} = \frac{f''(z)z + 2f'(z)}{p}$. Assuming negligible second-order effects where $f''(z) \simeq 0$, $\frac{\partial^2 q}{\partial z^2} = \frac{2f'(z)}{p}$.
- When $f'(z) > 0$, $\frac{\partial^2 q}{\partial z^2} > 0$, so we have strictly *convex* isoprofit curves for input price surcharges.
- When $f'(z) < 0$, $\frac{\partial^2 q}{\partial z^2} < 0$, so we have strictly *concave* isoprofit curves for input price discounts.
- When $f'(z) = 0$, $\frac{\partial^2 q}{\partial z^2} = 0$, so isoprofit curves are *linear* with a slope of w/p for fixed input prices.





Properties of Profit Function

- Suppose $p = g(q)$, then the slope of the isoquant is $\frac{\partial q}{\partial z} = -\frac{\frac{\partial \pi}{\partial z}}{\frac{\partial \pi}{\partial q}} = -\frac{-w}{g'(q)q+g(q)} = \frac{w}{g'(q)q+g(q)}$.
- Further differentiating the slope with respect to z, $\frac{\partial^2 q}{\partial z^2} = -\frac{w[g''(q)q+2g'(q)]}{[g'(q)q+g(q)]^2}$, which, when assuming negligible second-order effects where $g''(q) \simeq 0$, $\frac{\partial^2 q}{\partial z^2} = -\frac{2g'(q)w}{[g'(q)q+g(q)]^2}$.
- When $g'(q) > 0$, $\frac{\partial^2 q}{\partial z^2} < 0$, so we have strictly *concave* isoprofit curves for price surcharges.
- When $g'(q) < 0$, $\frac{\partial^2 q}{\partial z^2} > 0$, so we have strictly *convex* isoprofit curves for price discounts.
- When $g'(q) = 0$, $\frac{\partial^2 q}{\partial z^2} = 0$, so isoprofit curves are *straight* lines with a slope of w/p for fixed prices.



Remarks on Profit Function

- ***Remark 1***: the profit function is a value function, measuring firm profits *only* for the profit-maximizing vector y^*.
- ***Remark 2***: the profit function can be understood as a support function.
  - Take negative of the production set Y, i.e., $-Y$
  - Then, the support function of $-Y$ set is

$$\mu_{-Y}(p) = \min_{y} \{p \cdot (-y): y \in Y\}$$

That is, take the profits resulting form all production vectors $y \in Y$, $p \cdot y$, then take the negative of all these profits, $p \cdot (-y)$, and then choose the smallest one.

Remarks on Profit Function

- Of course, this is the same as maximizing the (positive) value of the profits resulting from all production vector $y \in Y$, $p \cdot y$.
- Therefore, the profit function, $\pi(p)$, is the support of the negative production set, $-Y$,

$$\pi(p) = \mu_{-Y}(p)$$

Remarks on Profit Function

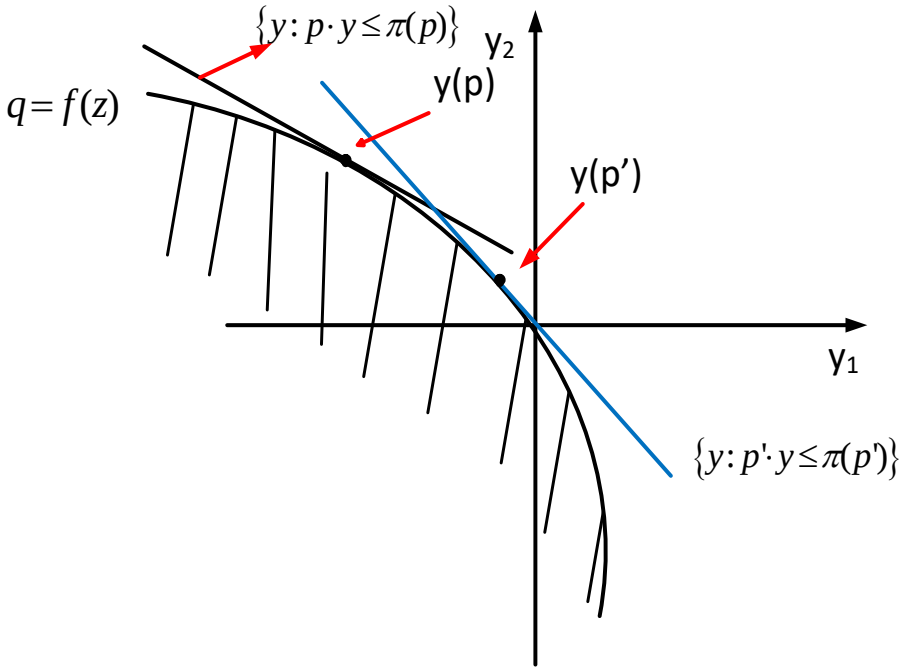
- Alternatively, the argmax of any objective function

$$y_1^* = \arg\max_y f(z)$$

coincides with the argmin of the negative of this objective function

$$y_2^* = \arg\min_y [-f(z)]$$

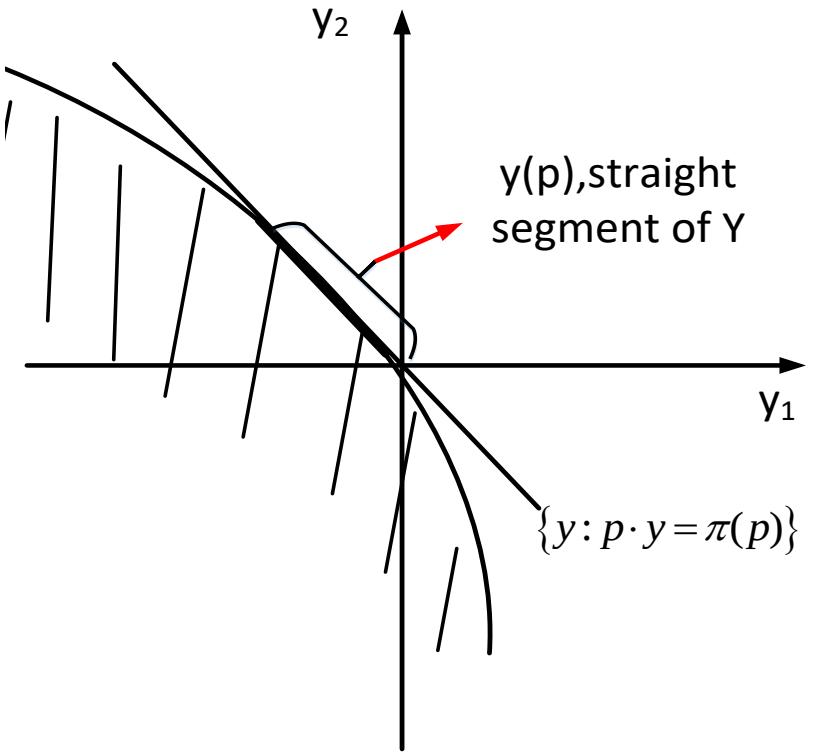
where $y_1^* = y_2^*$.



Properties of Supply Correspondence

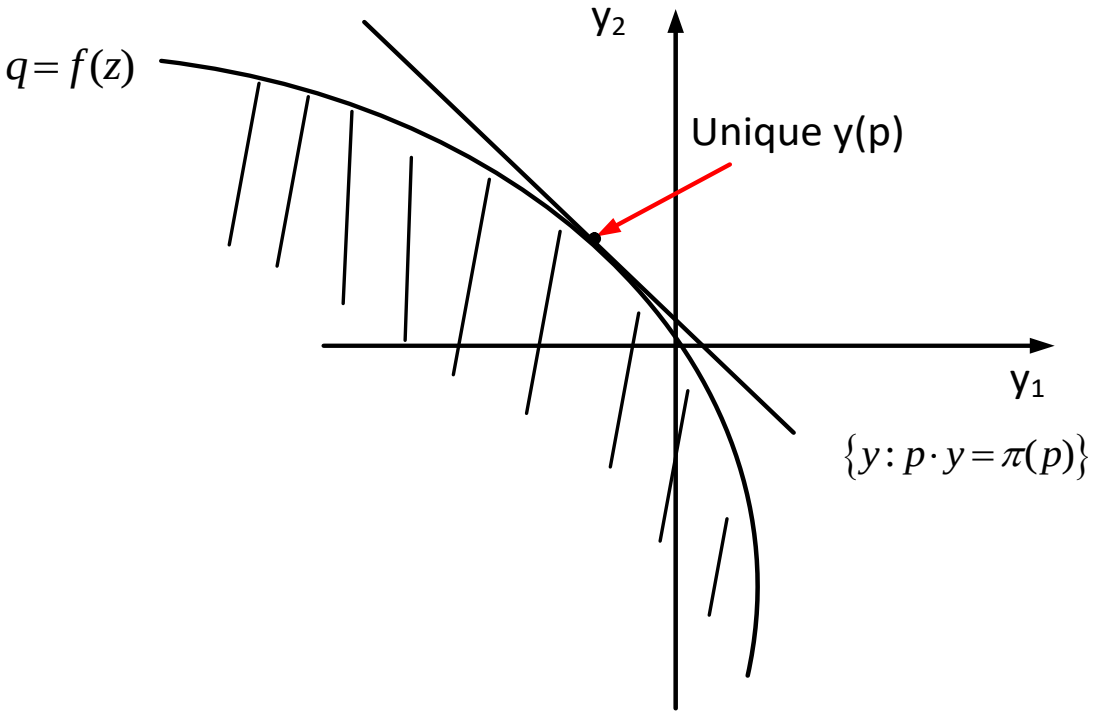
1) *If Y is weakly convex, then $y(p)$ is a convex set for all p.*

- Y has a flat surface
- $y(p)$ is NOT single valued.



Properties of Supply Correspondence

1) *(continued) If Y is strictly convex, then $y(p)$ is single-valued (if nonempty).*



Properties of Supply Correspondence

2) Hotelling's Lemma: If $y(\bar{p})$ consists of a single point, then $\pi(\cdot)$ is differentiable at $\bar{p}$. Moreover, $\nabla_p \pi(\bar{p}) = y(\bar{p})$.

– This is an application of the duality theorem from consumer theory.

- If $y(\cdot)$ is a function differentiable at $\bar{p}$, then $D_p y(\bar{p}) = D_p^2 \pi(\bar{p})$ is a symmetric and positive semidefinite matrix, with $D_p y(\bar{p})\bar{p} = 0$.
  - This is a direct consequence of the **law of supply**.

Properties of Supply Correspondence

- Since $\pi(p)$ is HD1, $p \cdot y(p) = \pi(p)$ entails that multiplying all prices by common factor $\lambda > 0$ yields $\lambda p \cdot y(\lambda p) = \pi(\lambda p) = \lambda \pi(p)$.
  - Simplifying, we obtain $p \cdot y(\lambda p) = \pi(p)$, so that $y(\lambda p) = y(p)$ for all $\lambda > 0$ and $y(p)$ is HD0.
- Differentiating both sides of $y(\lambda p) = y(p)$ with respect to λ, we find $D_\lambda y(\lambda p)p = 0$.
  - Finally, setting $\lambda = 1$ and evaluating at $\bar{p}$, yields

$$D_p y(\bar{p})\bar{p} = 0$$

Properties of Supply Correspondence

- Since $D_p y(\bar{p})\bar{p} = 0, D_p\, y(\bar{p})$ must satisfy:
  - *Own substitution effects* (main diagonal elements in $D_p y(\bar{p})$) are non-negative, i.e.,

$$\frac{\partial y_k(p)}{\partial p_k} \geq 0 \text{ for all } k$$

  - *Cross substitution effects* (off diagonal elements in $D_p y(\bar{p})$) are symmetric, i.e.,

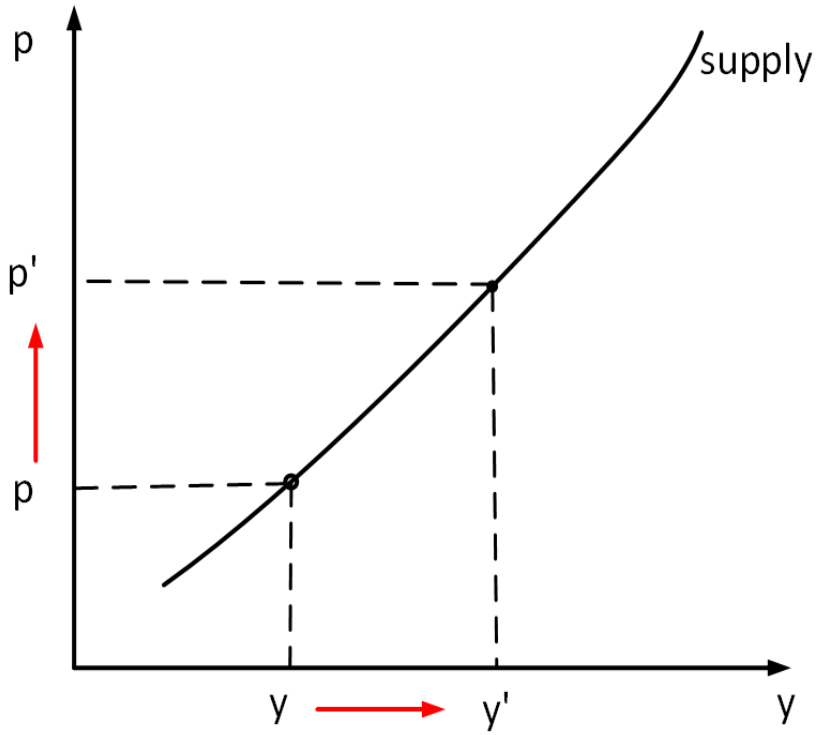
$$\frac{\partial y_l(p)}{\partial p_k} = \frac{\partial y_k(p)}{\partial p_l} \text{ for all } l \text{ and } k$$

Properties of Supply Correspondence

- $\frac{\partial y_k(p)}{\partial p_k} \geq 0$, which implies that quantities and prices move in the *same* direction,

  $(p - p')\,(y - y') \geq 0$

  – The **law of supply** holds!



Properties of Supply Correspondence

- Since there is no budget constraint, there is no wealth compensation requirement (as opposed to Demand theory).
  - This implies that there no income effects, only substitution effects.
- Alternatively, from a revealed preference argument, the law of supply can be expressed as

$$(p - p')(y - y') =$$
$$(py - py') + (p'y' - p'y) \geq 0$$

where $y \in y(p)$ and $y' \in y(p')$.

Properties of Supply Correspondence

- ***Proof***: Consider the Law of Supply,

$$(py - py') + (p'y' - p'y) \geq 0$$

  - Since $y \in y(p)$, y is the profit-maximizing bundle under the price vector p so that $py \geq py'$
  - Similarly, since $y' \in y(p')$, y' is the profit-maximizing bundle under p' so that $p'y' \geq p'y$

Cost Minimization

Cost Minimization

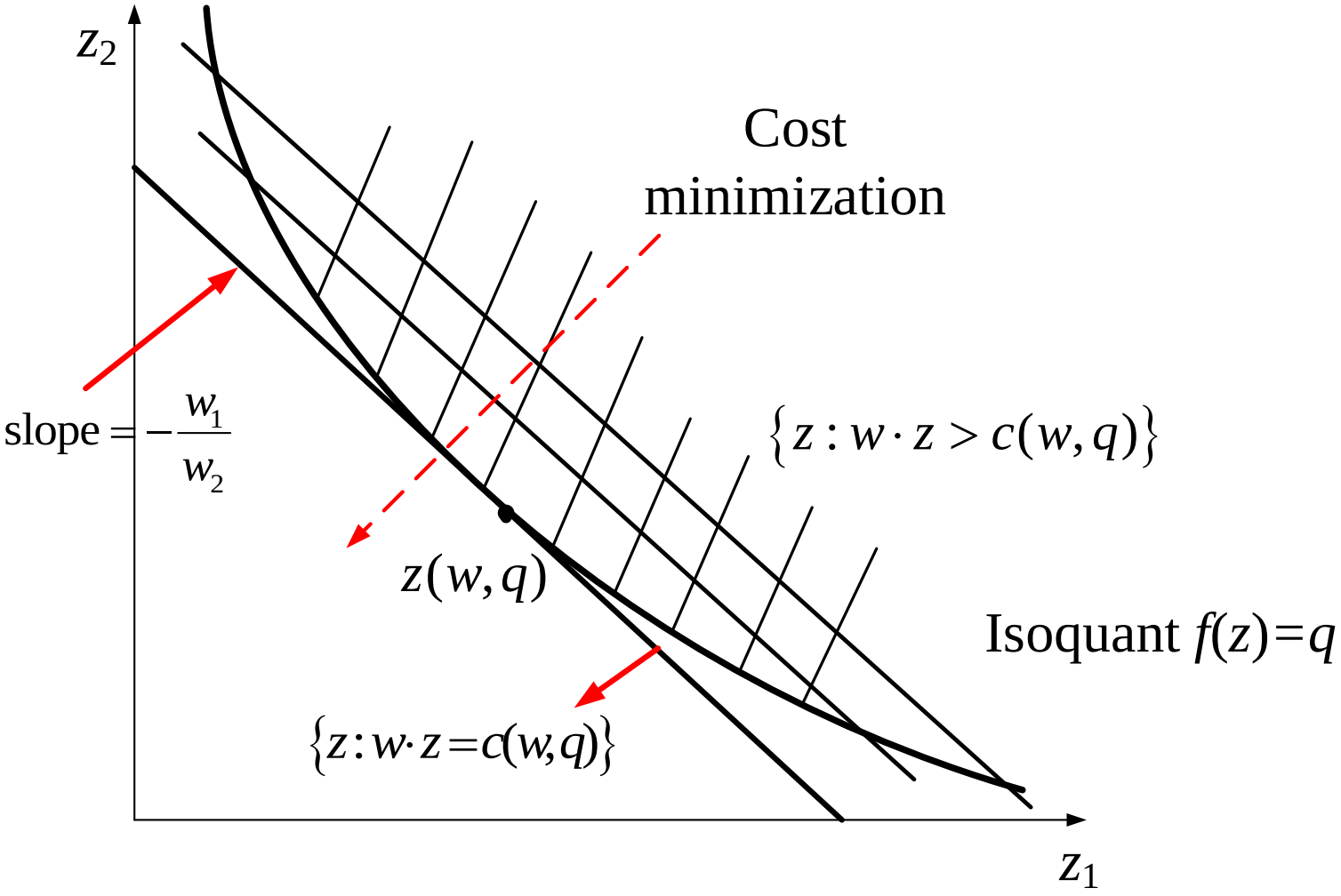
- We focus on the single output case, where
  - z is the input vector
  - $f(z)$ is the production function
  - q are the units of the (single) output
  - $w \gg 0$ is the vector of input prices
- The cost minimization problem (CMP) is

$$\min_{z \geq 0} \; w \cdot z$$
$$\text{s.t.} \;\; f(z) \geq q$$

Cost Minimization

- The optimal vector of input (or factor) choices is $z(w, q)$, and is known as the ***conditional factor demand correspondence***.
  - If single-valued, $z(w, q)$ is a function (not a correspondence)
  - Why "conditional"? Because it represents the firm's demand for inputs, conditional on reaching output level q.
- The value function of this CMP $c(w, q)$ is the ***cost function***.

Cost Minimization



Cost Minimization

- Graphically,
  - For a given isoquant $f(z) = q$, choose the isocost line associated with the lowest cost $w \cdot z$.
  - The tangency point is $z(w, q)$.
  - The isocost line associated with that combination of inputs is

    $$\{z: w \cdot z = c(w, q)\}$$

    where the cost function $c(w, q)$ represents the lowest cost of producing output level q when input prices are w.
  - Other isocost lines are associated with either:
    - output levels higher than q (with costs exceeding $c(w, q)$), or
    - output levels lower than q (with costs below $c(w, q)$).

Cost Minimization

- The Lagrangian of the CMP is

$$\mathcal{L}(z;\lambda) = wz + \lambda[q - f(z)]$$

- Differentiating with respect to z_k

$$w_k - \lambda \frac{\partial f(z^*)}{\partial z_k} \geq 0$$

$$(= 0 \text{ if interior solution, } z_k^*)$$

or in matrix notation

$$w - \lambda \nabla f(z^*) \geq 0$$

Cost Minimization

- From the above FOCs,

$$\frac{w_k}{\dfrac{\partial f(z^*)}{\partial z_k}} = \lambda \implies \frac{w_k}{w_l} = \frac{\dfrac{\partial f(z^*)}{\partial z_k}}{\dfrac{\partial f(z^*)}{\partial z_l}} \quad (= MRTS_{k,l}(z^*))$$

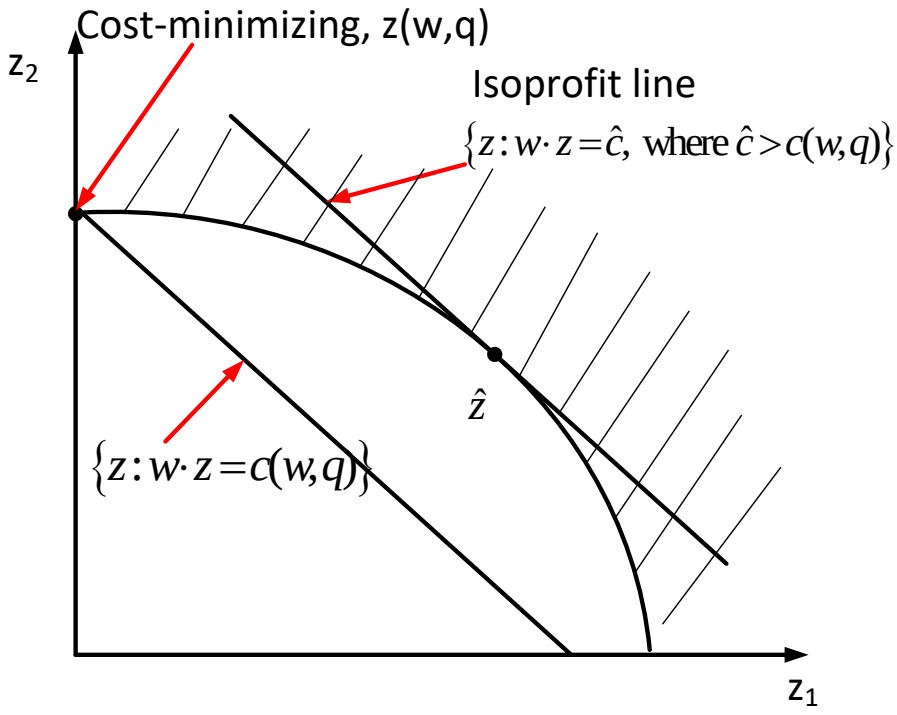
- Alternatively,

$$\frac{\dfrac{\partial f(z^*)}{\partial z_k}}{w_k} = \frac{\dfrac{\partial f(z^*)}{\partial z_l}}{w_l}$$

at the cost-minimizing input combination, the marginal product per dollar spent on input k must be equal to that of input l.

Cost Minimization

- ***Sufficiency***: If the production set is *convex*, then the FOCs are also sufficient.
- A non-convex production set:
  - The input combinations satisfying the FOCs do NOT identify a cost-minimizing input combination $z(w,q)$.
  - The cost-minimizing combination of inputs $z(w,q)$ occurs at the corner.



Cost Minimization

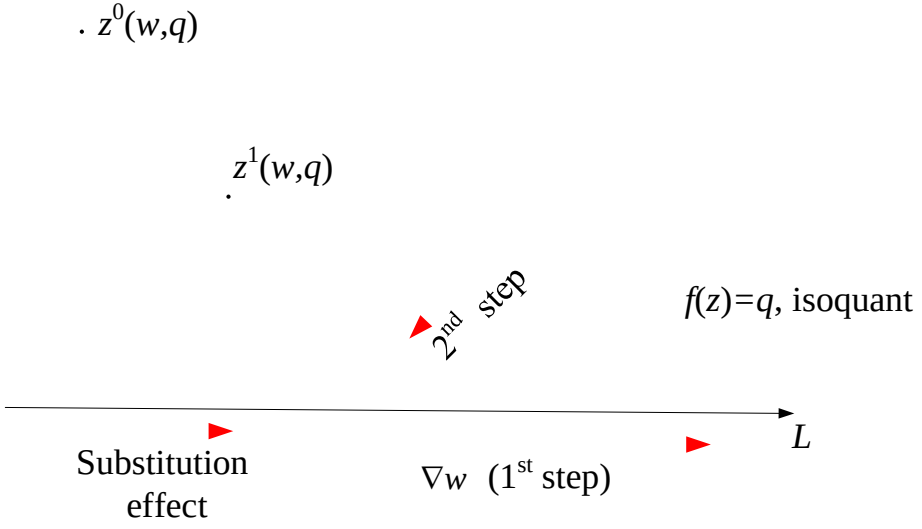
- ***Lagrange multiplier***: λ can be interpreted as the cost increase that the firm experiences when it needs to produce a higher output level q.
  - Recall that, generally, the Lagrange multiplier represents the variation in the objective function that we obtain if we relax the constraint (e.g., wealth in UMP, utility level we must reach in the EMP).
- Therefore, λ is the marginal cost of production: the marginal increase in the firm's costs from producing additional units.

Cost Minimization: SE and OE Effects

- ***Comparative statics of*** $z(w, q)$: Let us analyze the effects of an input price change. Consider two inputs, e.g., labor and capital. When the price of labor, w, falls, two effects occur:
  - *Substitution effect*: if output is held constant, there will be a tendency for the firm to substitute l for k.
  - *Output effect*: a reduction in firm's costs allows it to produce larger amounts of output (i.e., higher isoquant), which entails the use of more units of l for k.

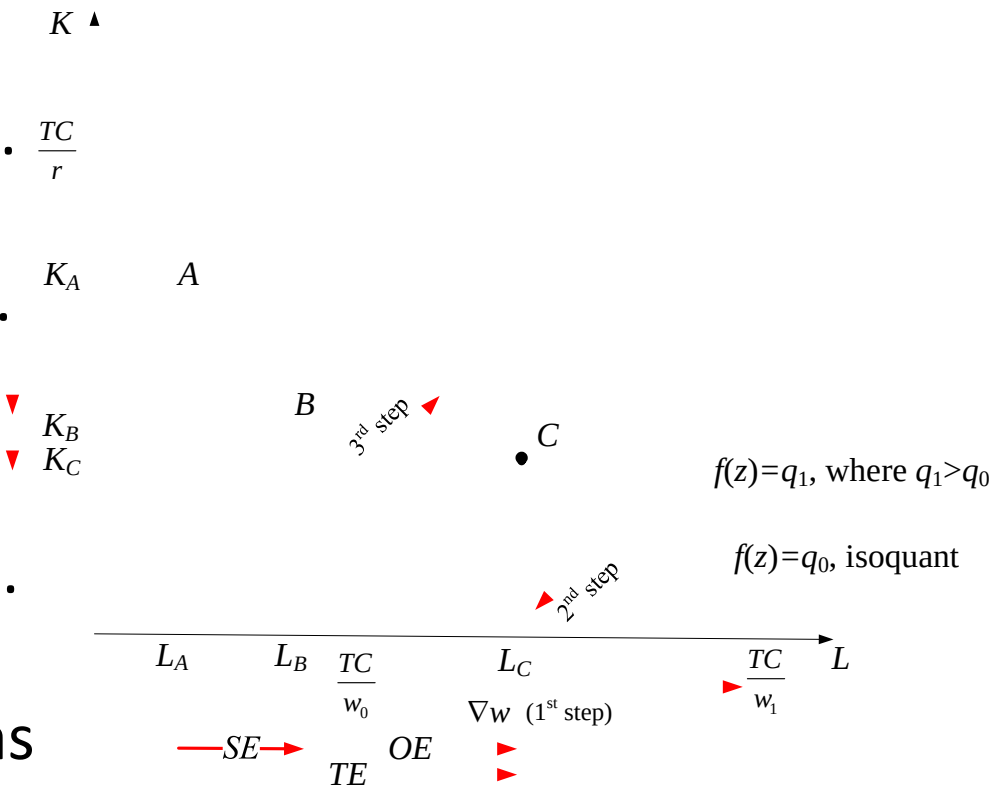
Cost Minimization: SE and OE Effects

- Substitution effect:
  - $z^0(w,q)$ solves CMP at K the initial prices.
  - $\downarrow$ in wages $\Longrightarrow$ isocost line pivots outwards.
  - To reach q, push the new isocost inwards in a parallel fashion.
  - $z^1(w,q)$ solves CMP at the new input prices (for output level q).
  - At $z^1(w,q)$, the firm uses more l and less k.



Cost Minimization: SE and OE Effects

- Substitution effect (SE):
  - increase in labor demand from L_A to L_B.
  - same output as before the input price change.
- Output effect (OE):
  - increase in labor demand from L_B to L_C.
  - output level increases, total cost is the same as before the input price change.



Cost Minimization: Own-Price Effect

- We have two concepts of demand for any input
  - the conditional demand for labor, $l^c(r, w, q)$
    - $l^c(r, w, q)$ solves the CMP
  - the unconditional demand for labor, $l(p, r, w)$
    - $l(p, r, w)$ solves the PMP
- At the profit-maximizing level of output, i.e., $q(p, r, w)$, the two must coincide

$$l(p, r, w) = l^c(r, w, q) = l^c\big(r, w, q(p, r, w)\big)$$

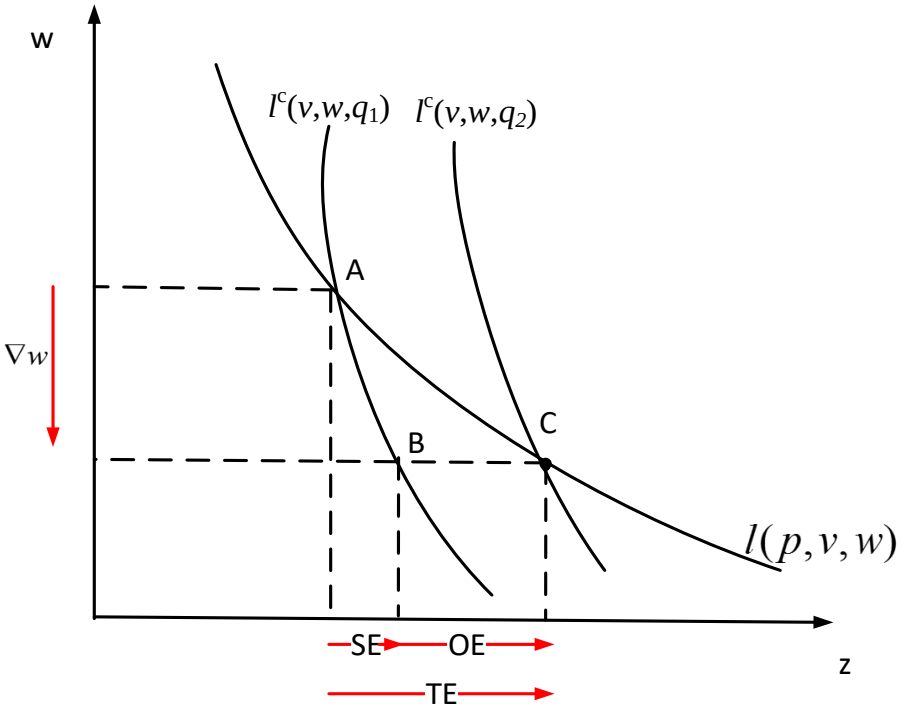
Cost Minimization: Own-Price Effect

- Differentiating with respect to w yields

$$\frac{\partial l(p,r,w)}{\partial w} = \underbrace{\underbrace{\frac{\partial l^c(r,w,q)}{\partial w}}_{SE\,(-)} + \underbrace{\overbrace{\frac{\partial l^c(r,w,q)}{\partial q}}^{(+)} \cdot \overbrace{\frac{\partial q}{\partial w}}^{(-)}}_{OE\,(-)}}_{TE\,(-)}$$

Cost Minimization: Own-Price Effect

- Since $|TE| > |SE|$, the unconditional labor demand is flatter than the conditional labor demand.
- The $(+)$ OE reinforces the $(+)$ SE, leading to a positive total effect.
  - Giffen paradox from consumer theory cannot arise in production theory.



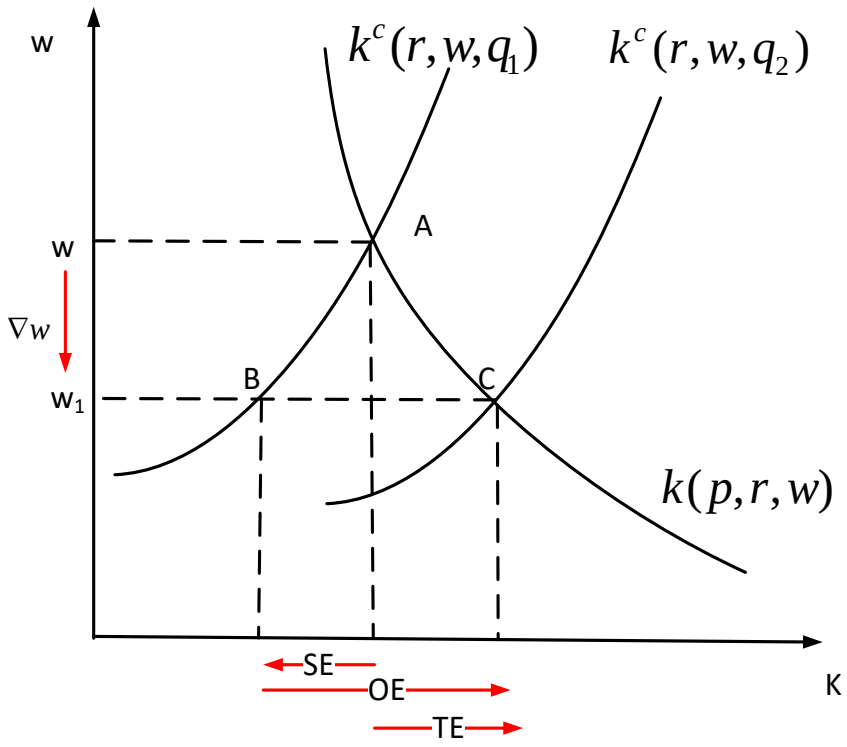
Cost Minimization: Cross-Price Effect

- No definite statement can be made about cross-price (CP) effects.
  - A fall in the wage will lead the firm to substitute away from capital.
  - The output effect will cause more capital to be demanded as the firm expands production.

$$\underbrace{\frac{\partial k(p,r,w)}{\partial w}}_{CP\ TE\ (+)\ \text{or}\ (-)} = \underbrace{\frac{\partial k^c(r,w,q)}{\partial w}}_{CP\ SE\ (+)} + \underbrace{\frac{\overbrace{\partial k^c(r,w,q)}^{(+)}}{\partial q} \cdot \overbrace{\frac{\partial q}{\partial w}}^{(-)}}_{CP\ OE\ (-)}$$

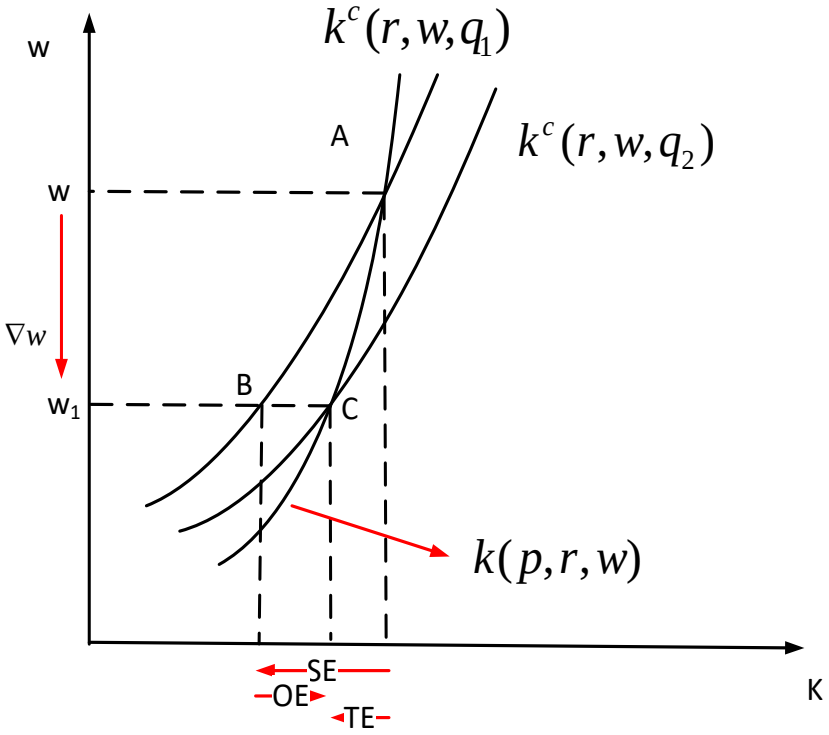
Cost Minimization: Cross-Price Effect

- The (+) cross-price OE completely offsets the (−) cross-price SE, leading to a positive cross-price TE.



Cost Minimization: Cross-Price Effect

- The (+) cross-price OE only partially offsets the (−) cross-price SE, leading to a negative cross-price TE.



Properties of Cost Function

- Assume that the production set Y is closed and satisfies the free disposal property.

  1) *$c(w, q)$ is Homog(1) in w*

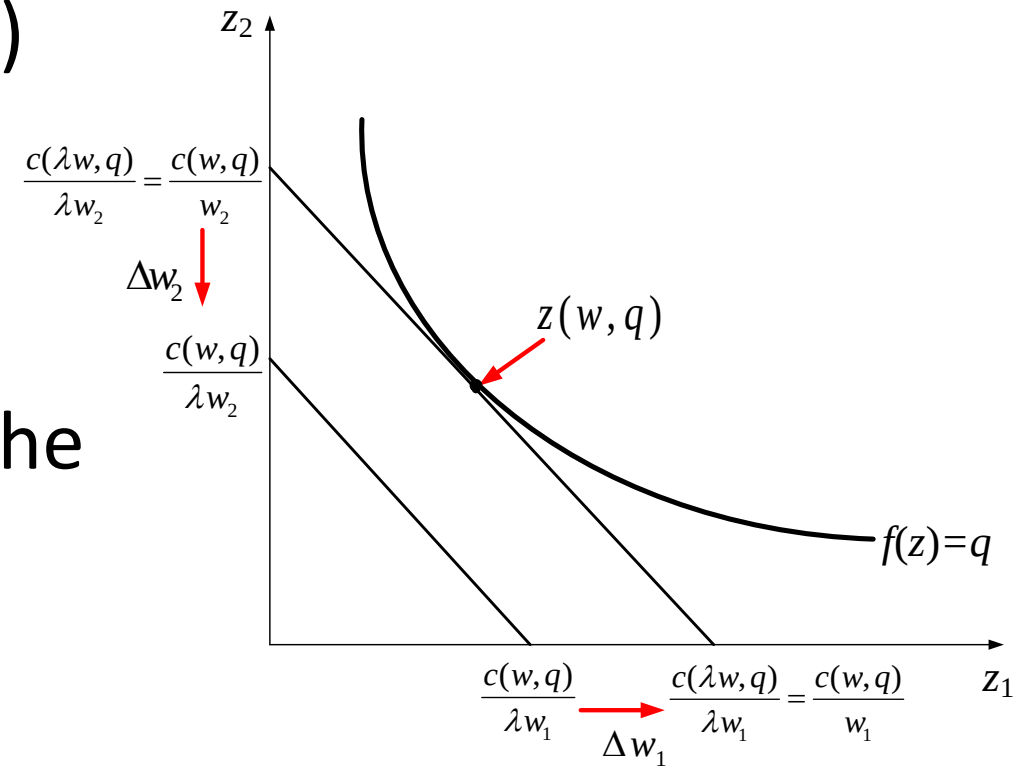
  - That is, increasing all input prices by a common factor λ yields a proportional increase in the minimal costs of production:

$$c(\lambda w, q) = \lambda c(w, q)$$

  since $c(w, q)$ represents the minimal cost of producing a given output q at input prices w.

Properties of Cost Function

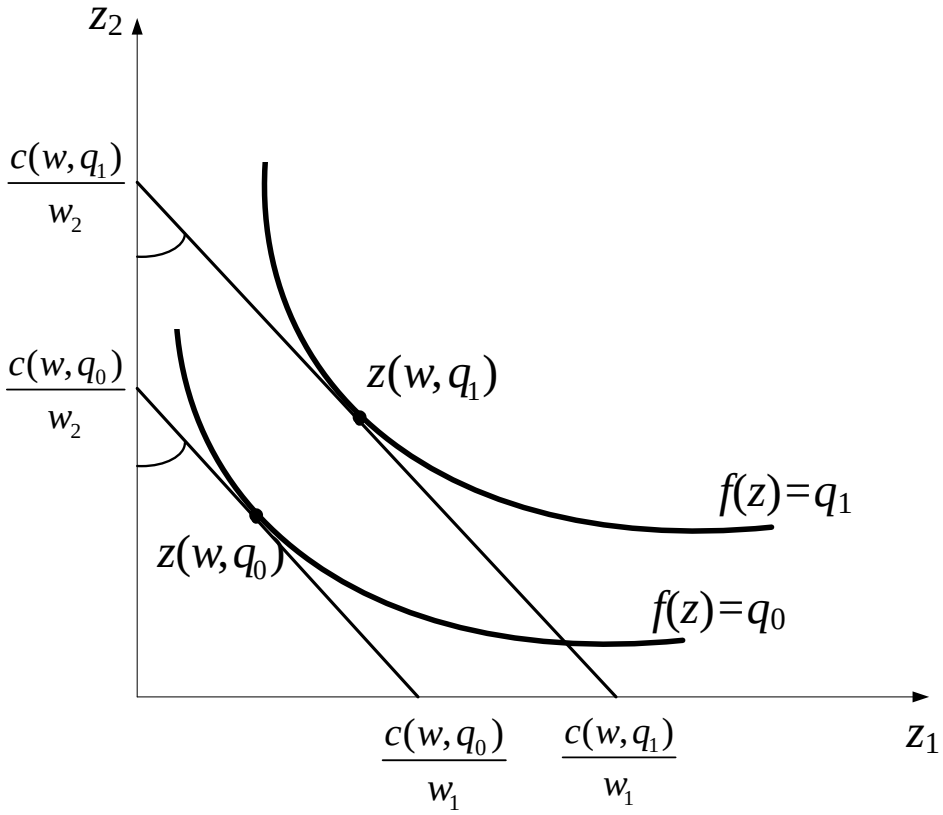
- An increase in all input prices (w_1, w_2) by the same proportion λ, produces a parallel downward shift in the firm's isocost line.



Properties of Cost Function

2) *$c(w,q)$ is non-decreasing in q*.

- Producing higher output levels implies a weakly higher minimal cost of production
- If $q_1 > q_0$, then it must be
  $c(w,q_1) > c(w,q_0)$



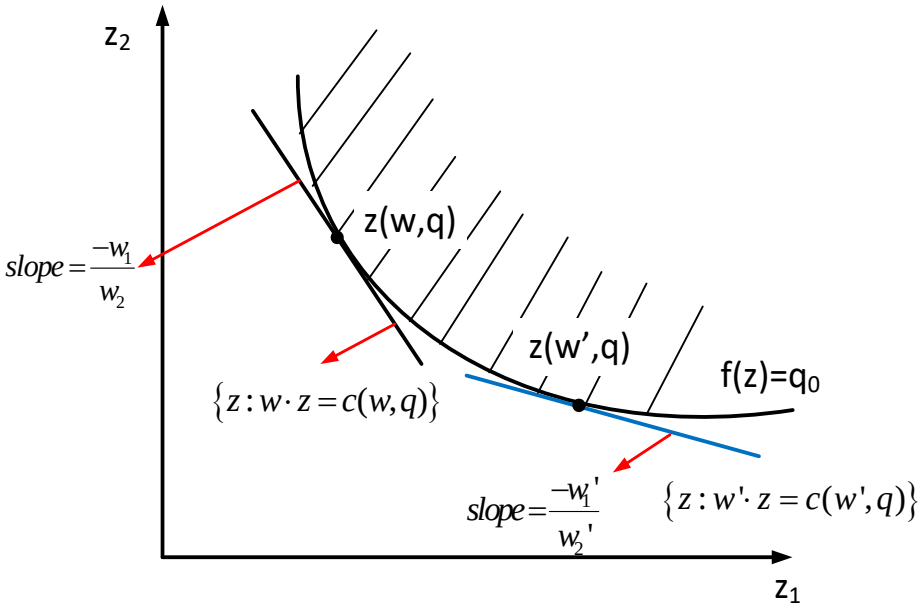
Properties of Cost Function

3) *If the set* $\{z \geq 0: f(z) \geq q\}$ *is convex for every* q*, then the production set can be described as*

$$Y = \left\{ \begin{matrix} (-z, q): w \cdot z \geq c(w, q) \\ \text{for every } \ w \gg 0 \end{matrix} \right\}$$

Properties of Cost Function

- Take $f(z) = q$.
- For input prices $w = (w_1, w_2)$, find $c(w, q)$ by solving CMP.
- For input prices $w' = (w_1', w_2')$, find $c(w', q)$ by solving CMP.
- The intersection of "more costly" input combinations $w \cdot z \geq c(w, q)$, for every input prices $w \gg 0$, describes set $f(z) \geq q$.

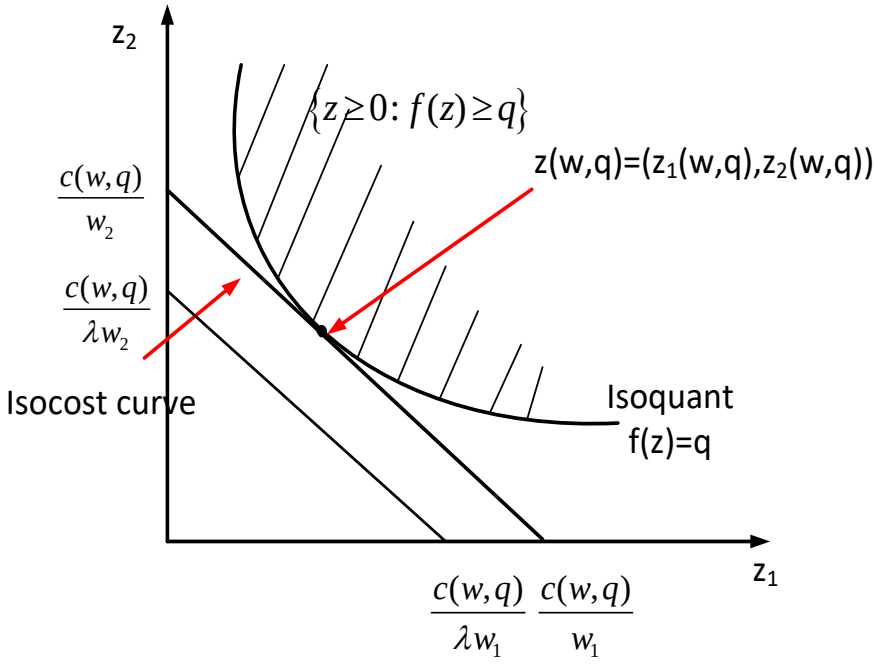


Properties of Conditional Factor Demand Correspondence

1) $z(w,q)$ *is Homog(0) in* w.

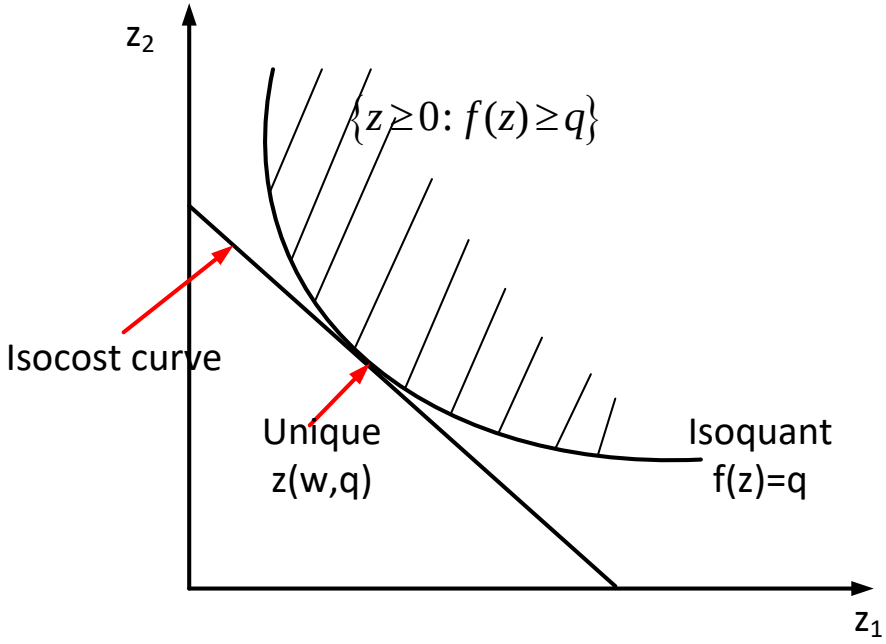
- That is, increasing input prices by the same factor λ does not alter the firm's demand for inputs at all,

$$z(\lambda w, q) = z(w, q)$$



Properties of Conditional Factor Demand Correspondence

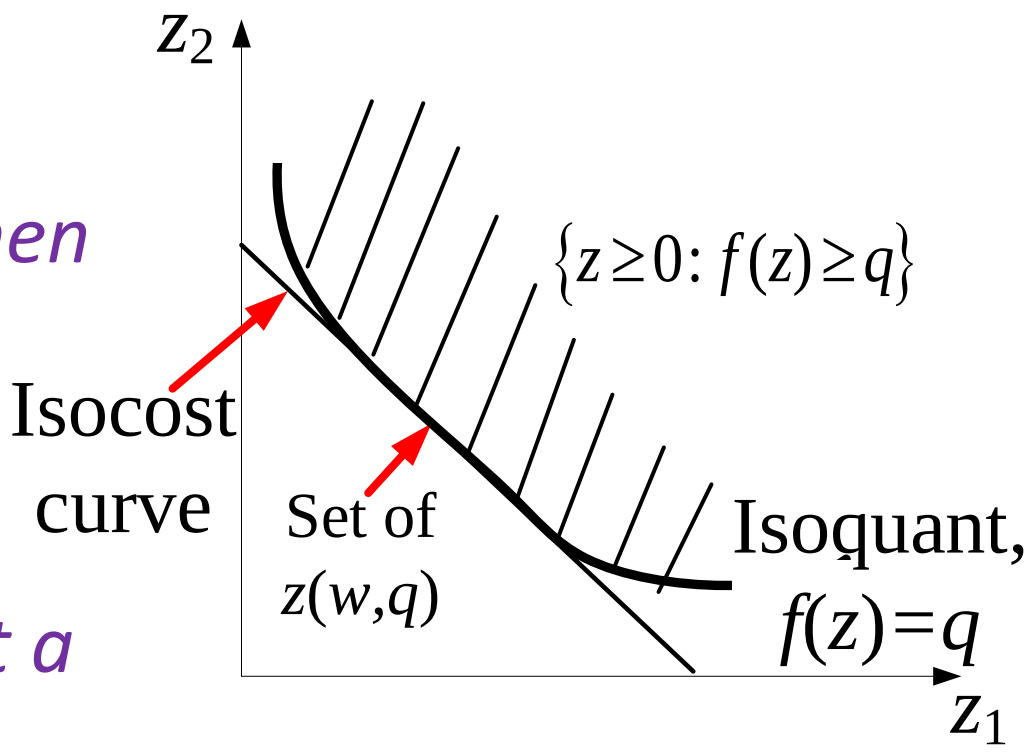
2) *If the set* $\{z \geq 0: f(z) \geq q\}$ *is strictly convex, then the firm's demand correspondence* $z(w,q)$ *is single valued.*



Properties of Conditional Factor Demand Correspondence

2) *(continued)*

If the set $\{z \geq 0: f(z) \geq q\}$ *is weakly convex, then the demand correspondence* $z(w,q)$ *is not a single-valued, but a convex set.*



Properties of Conditional Factor Demand Correspondence

3) ***Shephard's lemma***: If $z(\overline{w},q)$ consists of a single point, then $c(w,q)$ is differentiable with respect to w at $\overline{w}$, and

$$\nabla_w c(\overline{w},q) = z(\overline{w},q)$$

Properties of Conditional Factor Demand Correspondence

- Since $c(w,q)$ is HD1 and $z(w,q)$ is HD0 in w, $c(w,q) = w \cdot z(w,q)$ entails that multiplying all prices by a common factor $\lambda > 0$ yields
$$c(\lambda w, q) = \lambda w \cdot z(\lambda w, q) = \lambda w \cdot z(w,q)$$
- Differentiating both sides of $c(\lambda w, q) = \lambda w \cdot z(w,q)$ with respect to λ, we find
$$w \cdot \nabla_\lambda c(\lambda w, q) = w \cdot z(w,q)$$
- Setting $\lambda = 1$ and evaluating at $\overline{w}$, we obtain
$$\nabla_w c(\overline{w}, q) = z(\overline{w}, q)$$

Properties of Conditional Factor Demand Correspondence

4) *If $z(w,q)$ is differentiable at $\overline{w}$, then $D_w^2 c(\overline{w},q) = D_w z(\overline{w},q)$ is a symmetric and negative semidefinite matrix, with $D_w z(\overline{w},q) \cdot \overline{w} = 0$.*

- $D_w z(\overline{w},q)$ is a matrix representing how the firm's demand for every unit responds to changes in the price of such input, or in the price of the other inputs.

Properties of Conditional Factor Demand Correspondence

- Since $z(w, q)$ is HD0 in w, multiplying all prices by a common factor $\lambda > 0$ yields

$$z(\lambda w, q) = z(w, q)$$

- Differentiating both sides with respect to λ,

$$D_\lambda z(\lambda w, q) \cdot w = 0$$

- Setting $\lambda = 1$ and evaluating at $\overline{w}$, we obtain

$$D_w z(\overline{w}, q) \cdot \overline{w} = 0$$

Properties of Conditional Factor Demand Correspondence

4) *(continued)*

- Own substitution effects are non-positive,

$$\frac{\partial z_k(w,q)}{\partial w_k} \leq 0 \text{ for every input } k$$

i.e., if the price of input k increases, the firm's factor demand for this input decreases.

- Cross substitution effects are symmetric,

$$\frac{\partial z_k(w,q)}{\partial w_l} = \frac{\partial z_l(w,q)}{\partial w_k} \text{ for all inputs } k \text{ and } l$$

Properties of Production Function

1) *If $f(z)$ is Homog(1) (i.e., if $f(z)$ exhibits constant returns to scale), then $c(w,q)$ and $z(w,q)$ are Homog(1) in q.*

- Intuitively, if $f(z)$ exhibits CRS, then an increase in the output level we seek to reach induces an increase of the same proportion in the cost function and in the demand for inputs. That is,

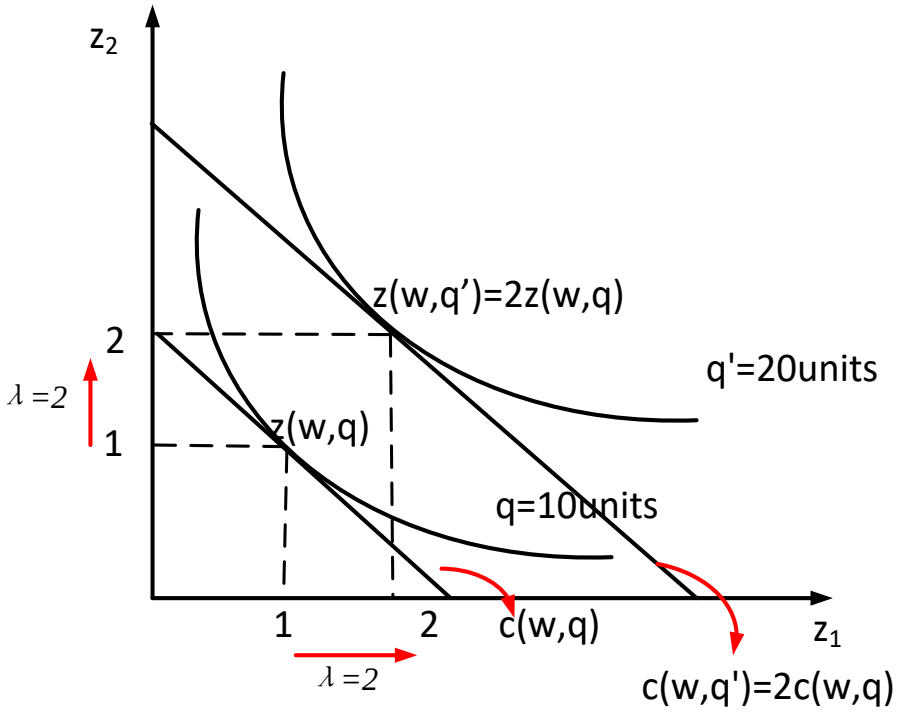
$$c(w,\lambda q) = \lambda c(w,q)$$

and

$$z(w,\lambda q) = \lambda z(w,q)$$

Properties of Production Function

- $\lambda = 2$ implies that demand for inputs doubles
  $z(w, 2q) = 2z(w, q)$
  and that the minimal costs also double
  $c(w, 2q) = 2c(w, q)$



Properties of Production Function

2) *If $f(z)$ is concave, then $c(w,q)$ is convex function of q (i.e., marginal costs are non-decreasing in q).*

- More compactly,

$$\frac{\partial^2 c(w,q)}{\partial q^2} \geq 0$$

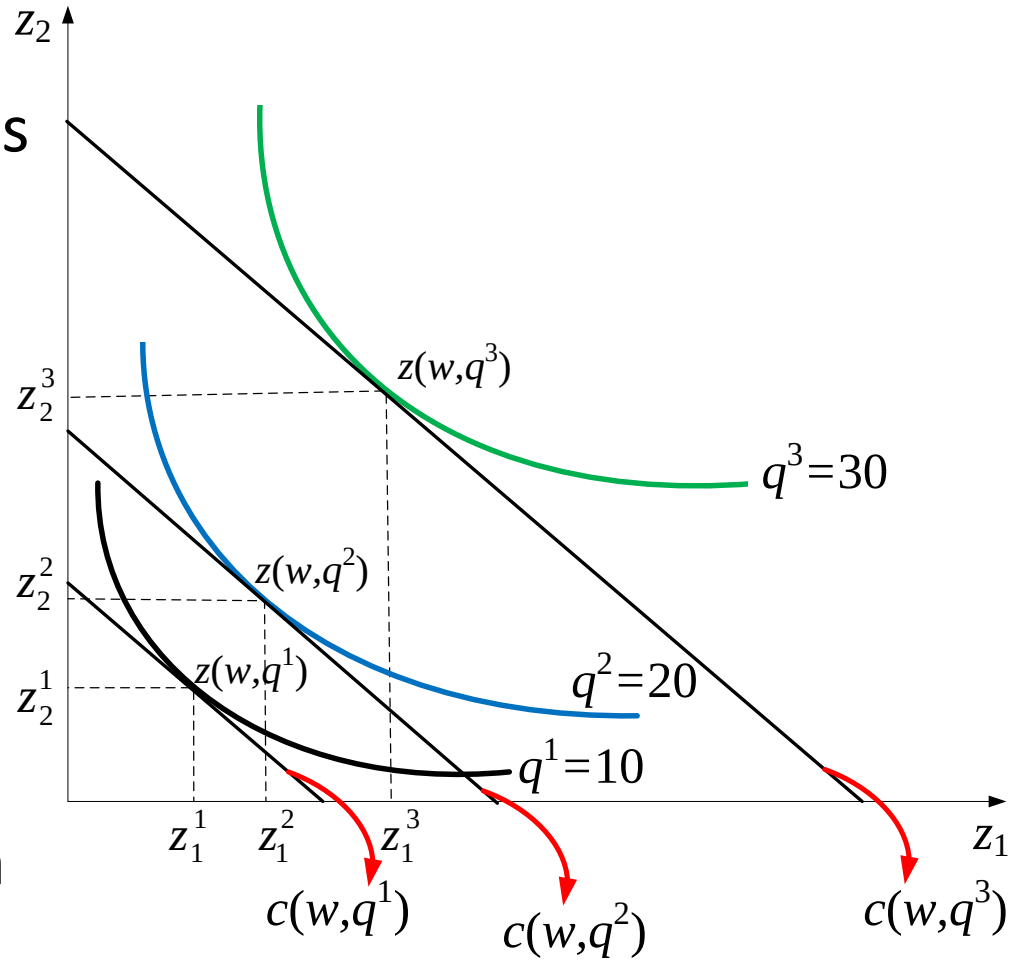
or, in other words, marginal costs $\frac{\partial c(w,q)}{\partial q}$ are weakly increasing in q.

Properties of Production Function

2) *(continued)*

- Firm uses more inputs when raising output from q^2 to q^3 than from q^1 to q^2.
- Hence,

  $c(w,q^3) - c(w,q^2) > c(w,q^2) - c(w,q^1)$
- This reflects the convexity of the cost function $c(w,q)$ with respect to q.



Alternative Representation of PMP

Alternative Representation of PMP

- Using the cost function $c(w, q)$, we write the PMP as follows

$$\max_{q \geq 0} \quad pq - c(w, q)$$

This is useful if we have information about the cost function, but not about the production function $q = f(z)$.

Alternative Representation of PMP

- Let us now solve this alternative PMP

$$\max_{q \geq 0} \quad pq - c(w, q)$$

- FOCs for q^* to be profit-maximizing are

$$p - \frac{\partial c(w, q^*)}{\partial q} \leq 0$$

and in interior solutions

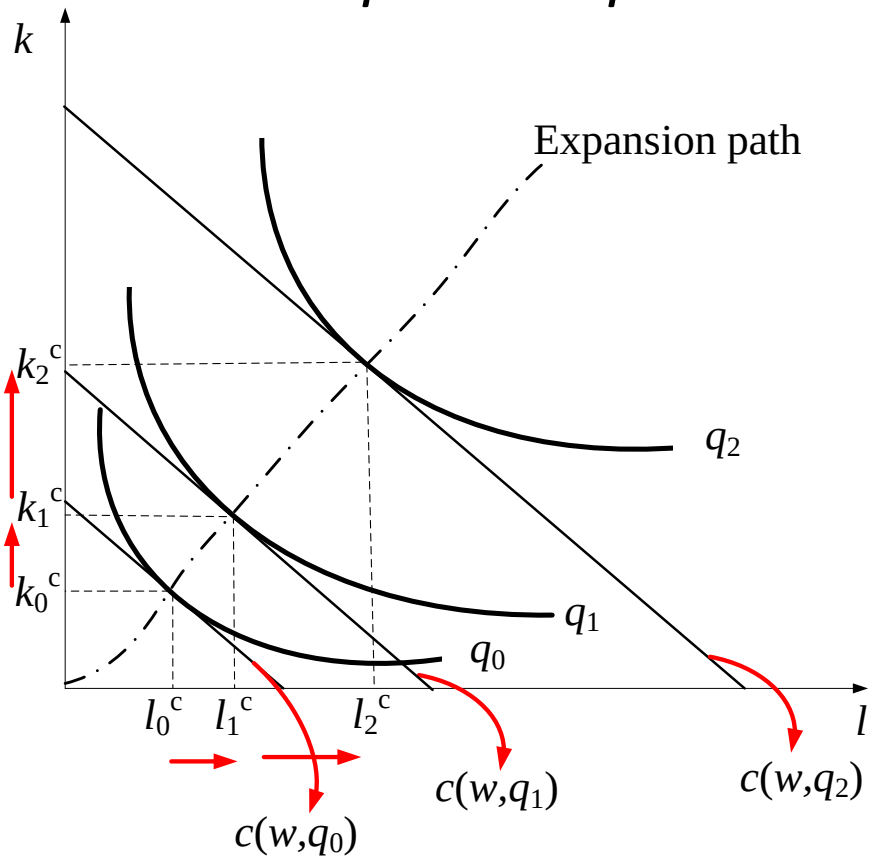
$$p - \frac{\partial c(w, q^*)}{\partial q} = 0$$

- That is, at the interior optimum q^*, price equals marginal cost, $\frac{\partial c(w,q^*)}{\partial q}$.

Firm's Expansion Path

- The ***expansion path*** is the locus of cost-minimizing tangencies. (Analogous to the *wealth expansion path* in consumer theory)
- The curve shows how inputs increase as output increases.
- Expansion path is positively sloped.
- Both k and l are *normal* goods, i.e.,

$$\frac{\partial k^c(w,q)}{\partial q} \geq 0, \frac{\partial l^c(w,q)}{\partial q} \geq 0$$



Firm's Expansion Path

- If the firm's expansion path is a ***straight line***:
  - All inputs must increase at a constant proportion as the firm increases its output.
  - The firm's production function exhibits constant returns to scale and it is, hence, homothetic.
  - If the expansion path is straight and coincides with the 45-degree line, then the firm increases all inputs by the same proportion as output increases.
- The expansion path does not have to be a straight line.
  - The use of some inputs may increase faster than others as output expands
    - Depends on the shape of the isoquants.

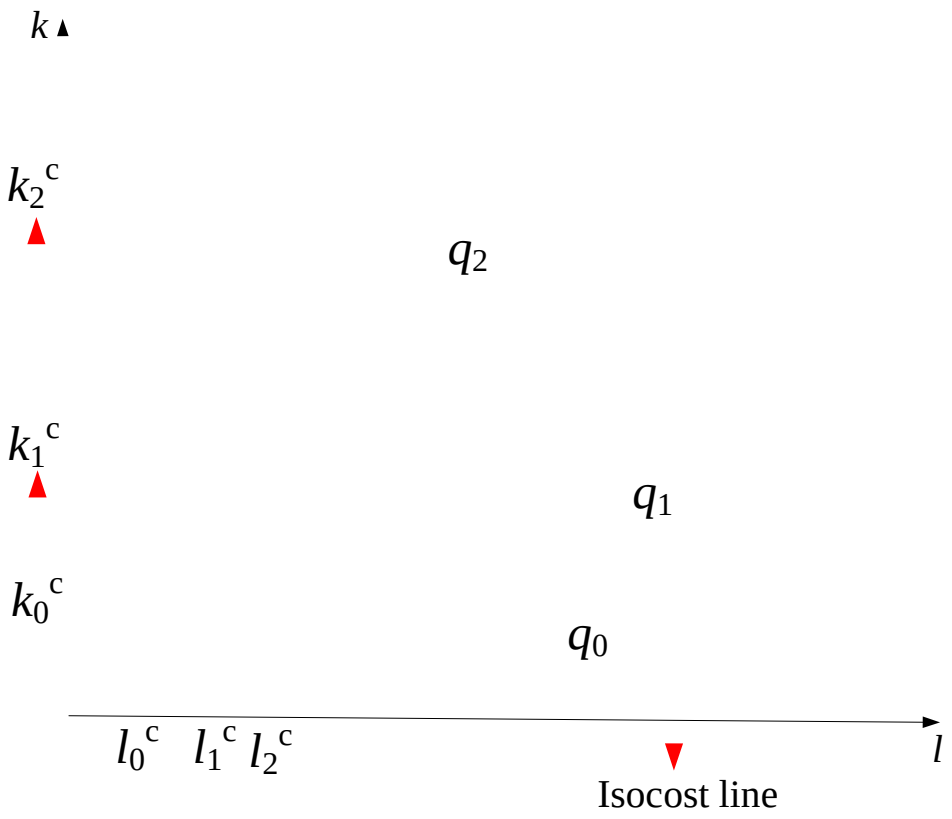
Firm's Expansion Path

- The expansion path does not have to be upward sloping.
  - If the use of an input falls as output expands, that input is an *inferior* input.
- k is *normal*

$$\frac{\partial k^c(w,q)}{\partial q} \geq 0$$

but l is inferior (at higher levels of output)

$$\frac{\partial l^c(w,q)}{\partial q} < 0$$



Firm's Expansion Path

- Are there inferior inputs out there?
  - We can identify inferior inputs if the list of inputs used by the firms is relatively disaggregated.
  - For instance, we can identify the following categories: CEOs, executives, managers, accountants, secretaries, janitors, etc.
  - These inputs do not increase at a constant rate as the firm increases output (i.e., expansion path would not be a straight line for all increases in q).
  - After reaching a certain scale, the firm might buy a powerful computer with which accounting can be done using fewer accountants.

Cost and Supply: Single Output

- Let us assume a given vector of input prices $\overline{w} \gg 0$. Then, $c(\overline{w}, q)$ can be reduced to $C(q)$. Then, average and marginal costs are

$$AC(q) = \frac{C(q)}{q} \text{ and } MC = C'(q) = \frac{\partial C(q)}{\partial q}$$

- Hence, the FOCs of the PMP can be expressed as

$p \leq C'(q)$, and in interior solutions $p = C'(q)$

i.e., all output combinations such that $p = C'(q)$ are the (optimal) supply correspondence of the firm $q(p)$.

Cost and Supply: Single Output

- We showed that the cost function $c(w, q)$ is homogenous of degree 1 in input prices, w.
  - Can we extend this property to the AC and MC? Yes!
  - For average cost function,

$$AC(tw, q) = \frac{c(tw, q)}{q} = \frac{t \cdot c(w, q)}{q}$$
$$= t \cdot AC(w, q)$$

Cost and Supply: Single Output

– For marginal cost function,

$$MC(tw,q) = \frac{\partial c(tw,q)}{\partial q} = \frac{t \cdot \partial c(w,q)}{\partial q} = t \cdot MC(w,q)$$

– Isn't this result violating Euler's theorem? No!

- The above result states that $c(w,q)$ is homog(1) in input prices, and that $MC(w,q) = \frac{\partial c(w,q)}{\partial q}$ is also homog(1) in input prices.
- Euler's theorem would say that: If $c(w,q)$ is homog(1) in input prices, then its derivative with respect to input prices, $\frac{\partial c(w,q)}{\partial w}$, must be homog(0).

Graphical Analysis of Total Cost

- With constant returns to scale, total costs are proportional to output.

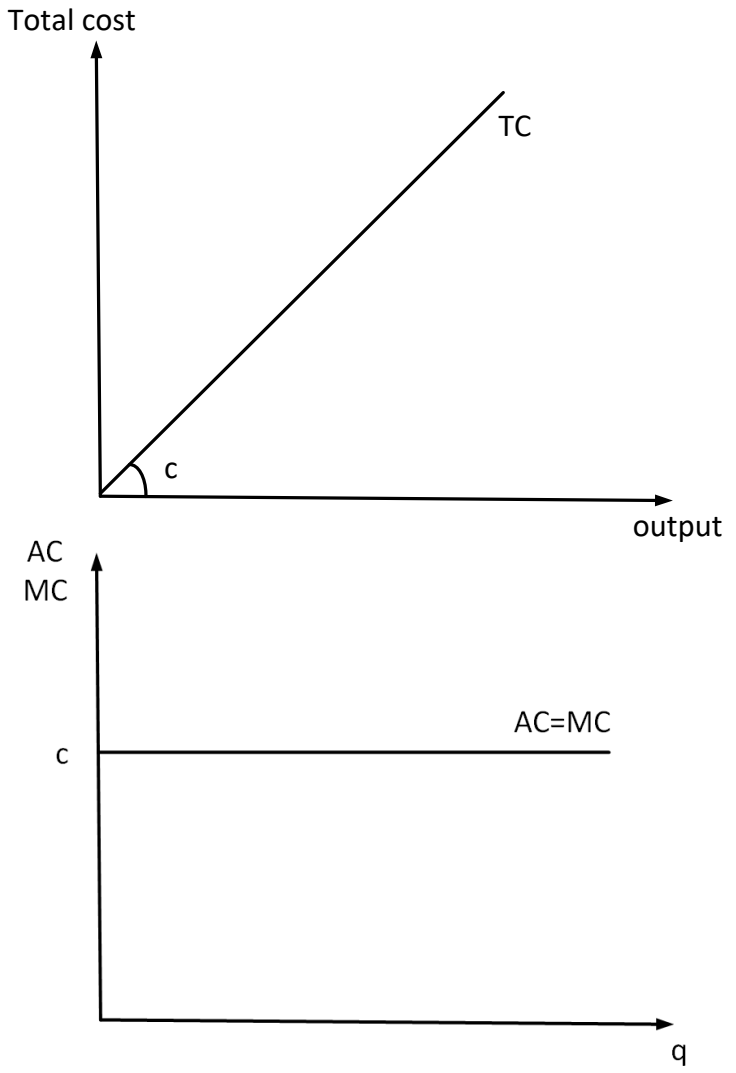
$$TC(q) = c \cdot q$$

- Hence,

$$AC(q) = \frac{TC(q)}{q} = c$$

$$MC(q) = \frac{\partial TC(q)}{\partial q} = c$$

$$\Longrightarrow AC(q) = MC(q)$$

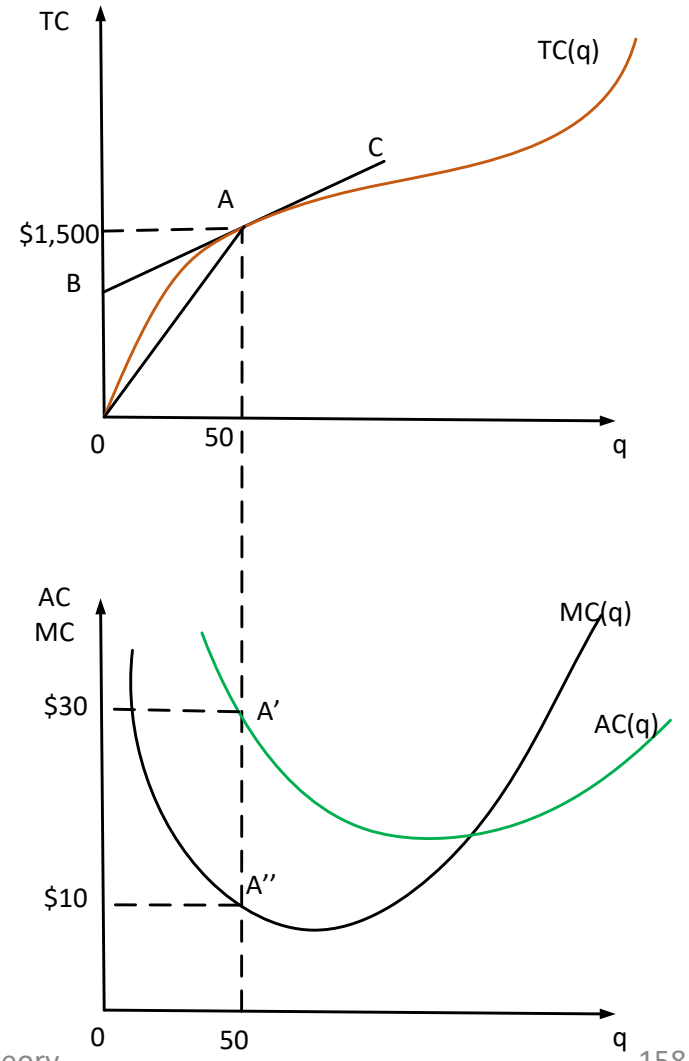


Cost and Supply: Single Output

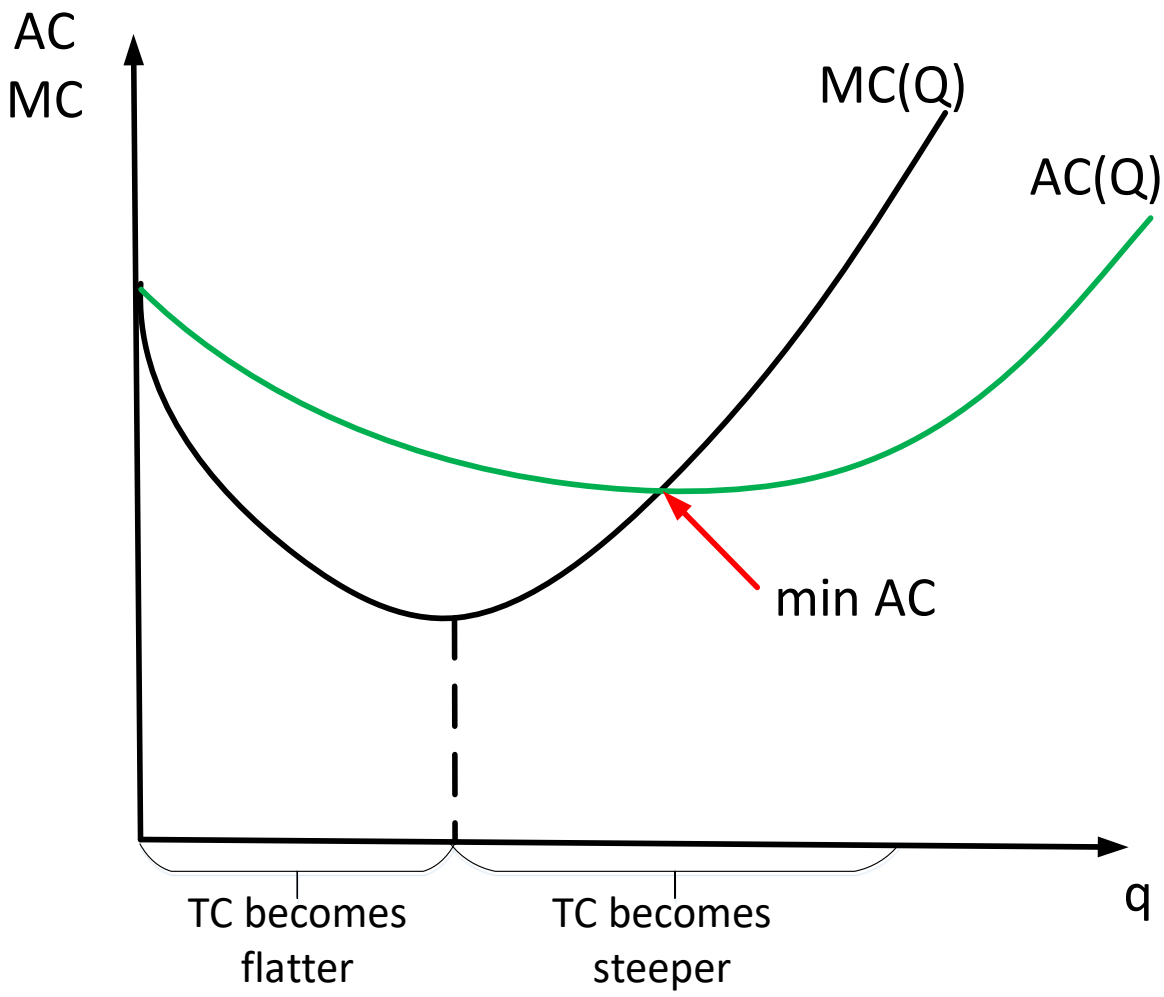
- Suppose that TC starts out as concave and then becomes convex as output increases.
  - TC no longer exhibits constant returns to scale.
  - One possible explanation for this is that there is a third factor of production that is fixed as capital and labor usage expands (e.g., entrepreneurial skills).
  - TC begins rising rapidly after *diminishing returns* set in.

Cost and Supply: Single Output

- TC initially grows very rapidly, then becomes relatively flat, and for high production levels increases rapidly again.
- MC is the slope of the TC curve.
- AC is the slope of the ray connecting any point along the TC curve.



Cost and Supply: Single Output



Cost and Supply: Single Output

- ***Remark 1***: $AC = MC$ at $q = 0$.
  - Note that we cannot compute

$$AC(0) = \frac{TC(0)}{0} = \frac{0}{0}$$

  - We can still apply L'Hôpital's rule

$$\lim_{q \to 0} AC(q) = \lim_{q \to 0} \frac{TC(q)}{q} = \lim_{q \to 0} \frac{\frac{\partial TC(q)}{\partial q}}{\frac{\partial q}{\partial q}} = \lim_{q \to 0} MC(q)$$

  - Hence, $AC = MC$ at $q = 0$, i.e., $AC(0) = MC(0)$.

Cost and Supply: Single Output

- ***Remark 2***: When MC<AC, the AC curve decreases, and when MC>AC, the AC curve increases.
  - *Intuition*: using example of grades
  - If the new exam score raises your average grade, it must be that such new grade is better than your average grade thus far.
  - If, in contrast, the new exam score lowers your average grade, it must be that such new grade is worse than your average grade thus far.

Cost and Supply: Single Output

- ***Remark 3***: AC and MC curves cross ($AC = MC$) at exactly the minimum of the AC curve.
  - Let us first find the minimum of the AC curve

$$\frac{\partial AC(q)}{\partial q} = \frac{\partial\left(\frac{TC(q)}{q}\right)}{\partial q} = \frac{q\frac{\partial TC(q)}{\partial q} - TC(q)\cdot 1}{q^2}$$
$$= \frac{q\cdot MC(q) - TC(q)}{q^2} = 0$$

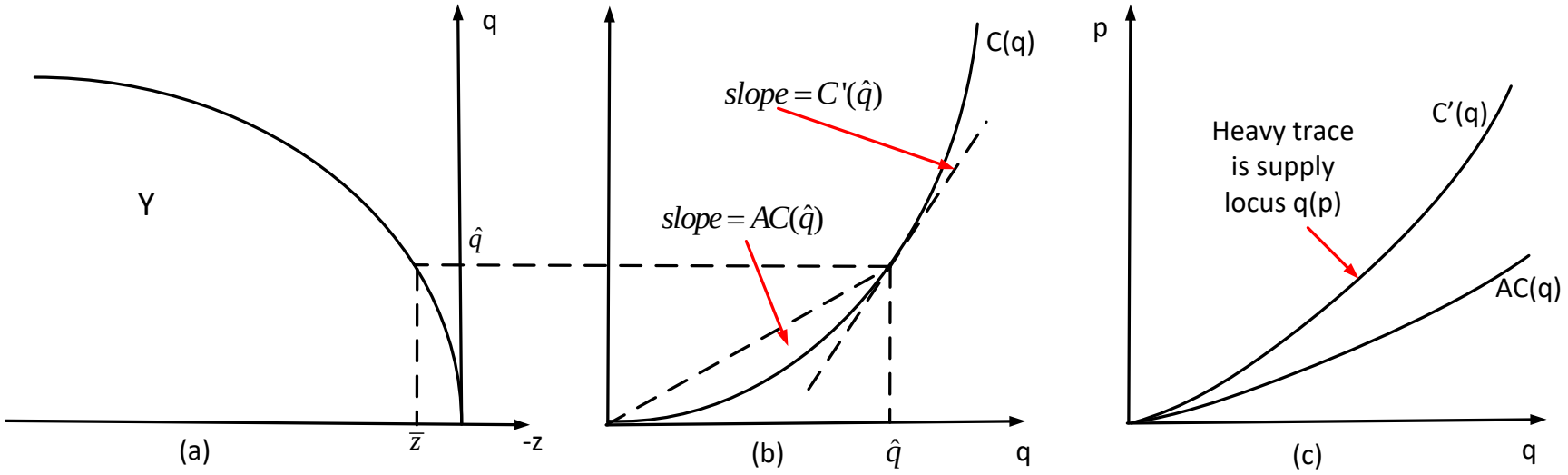
  - The output that minimizes AC must satisfy

$$q\cdot MC(q) - TC(q) = 0 \implies MC(q) = \frac{TC(q)}{q} = AC(q)$$

  - Hence, $MC = AC$ at the minimum of AC.

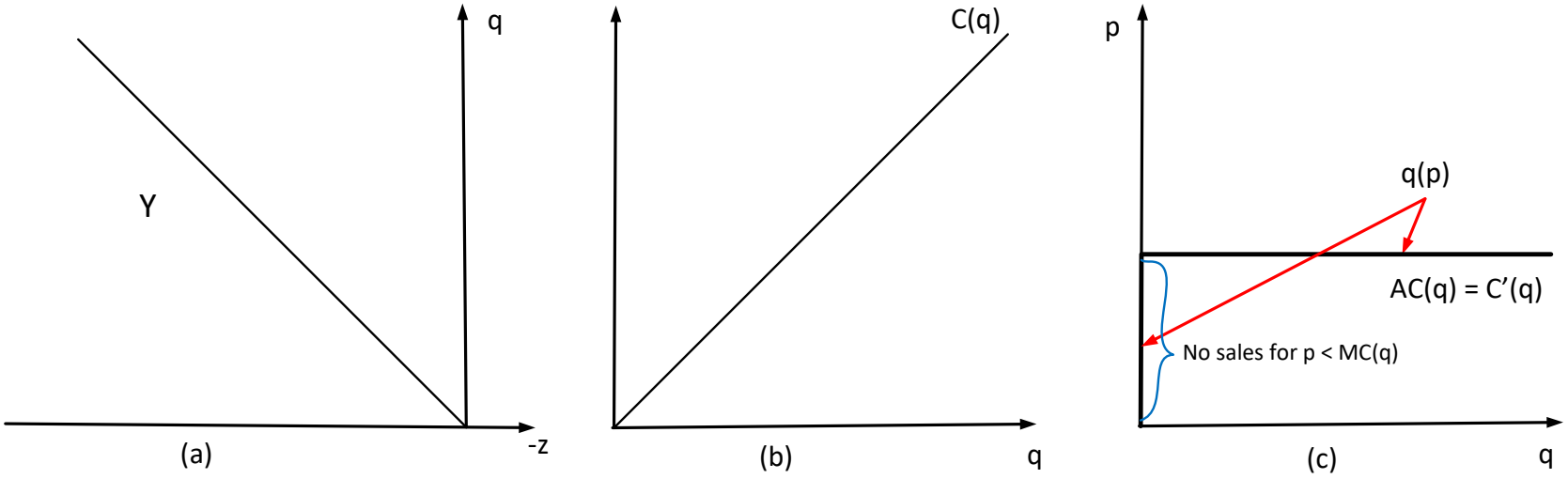
Cost and Supply: Single Output

- ***Decreasing returns to scale***:
  - an increase in the use of inputs produces a less-than-proportional increase in output.
    - production set is strictly convex
    - TC function is convex
    - MC and AC are increasing



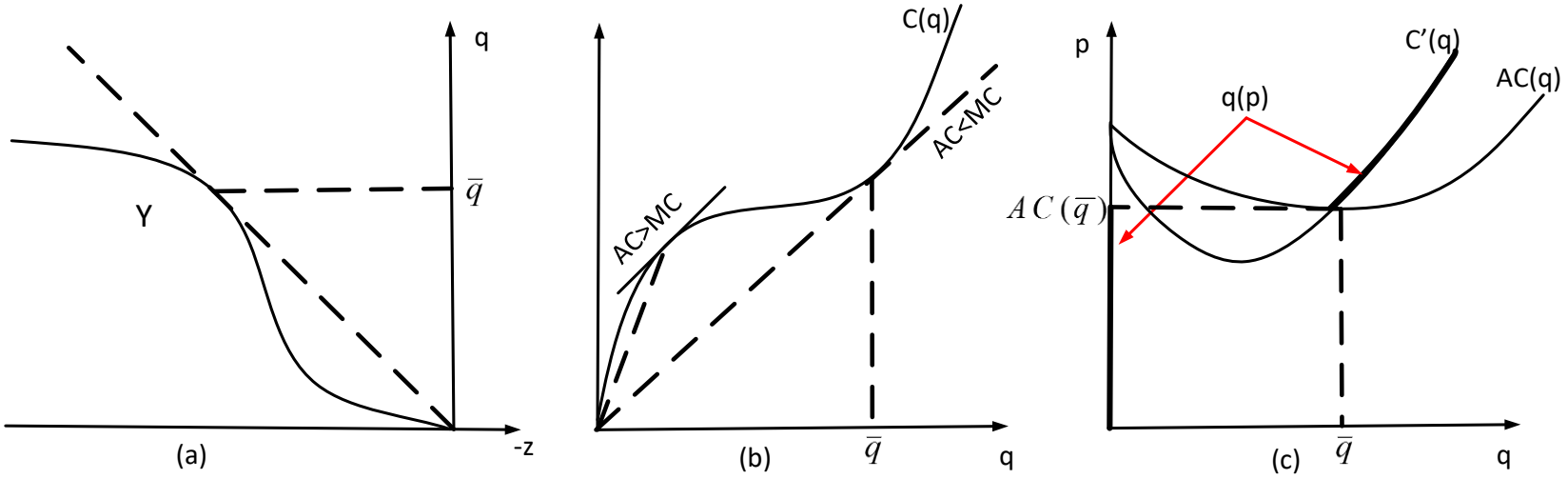
Cost and Supply: Single Output

- ***Constant returns to scale***:
  - an increase in input usage produces a proportional increase in output.
    - production set is weakly convex
    - linear TC function
    - constant AC and MC functions



Cost and Supply: Single Output

- ***Increasing returns to scale***:
  - an increase in input usage can lead to a more-than-proportional increase in output.
    - production set is non-convex
    - TC curve first increases, then becomes almost flat, and then increases rapidly again as output is increased further.



Cost and Supply: Single Output

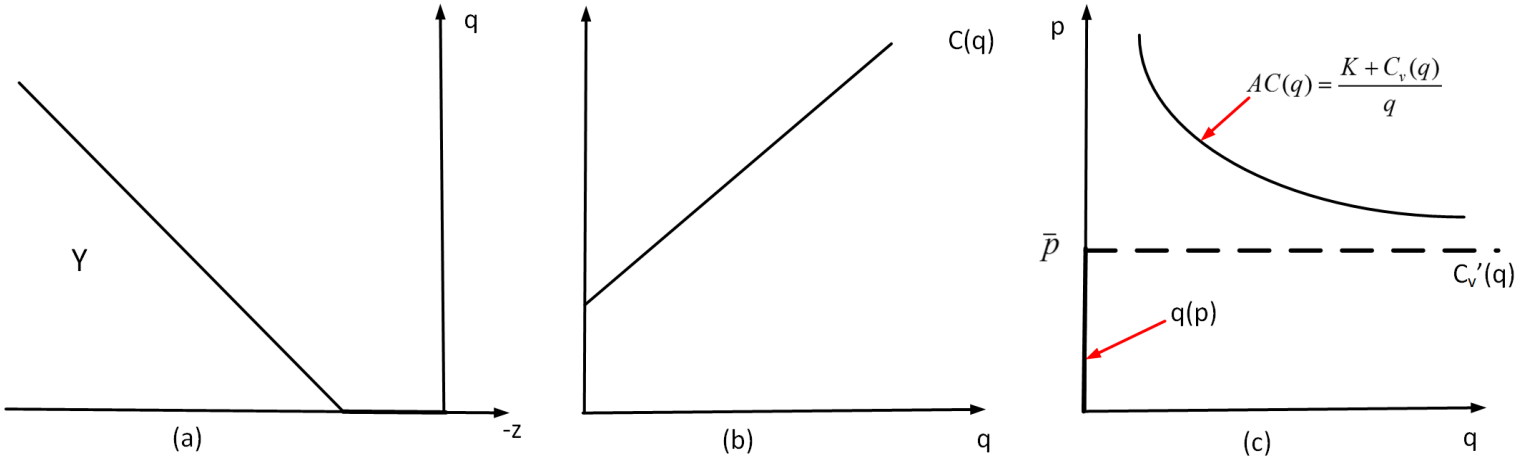
- Let us analyze the presence of *non-convexities* in the production set Y arising from:
  - Fixed set-up costs, K, that are non-sunk

$$C(q) = K + C_v(q)$$

    where $C_v(q)$ denotes variable costs
    - with strictly convex variable costs
    - with linear variable costs
  - Fixed set-up costs that are sunk
    - Cost function is convex, and hence FOCs are sufficient

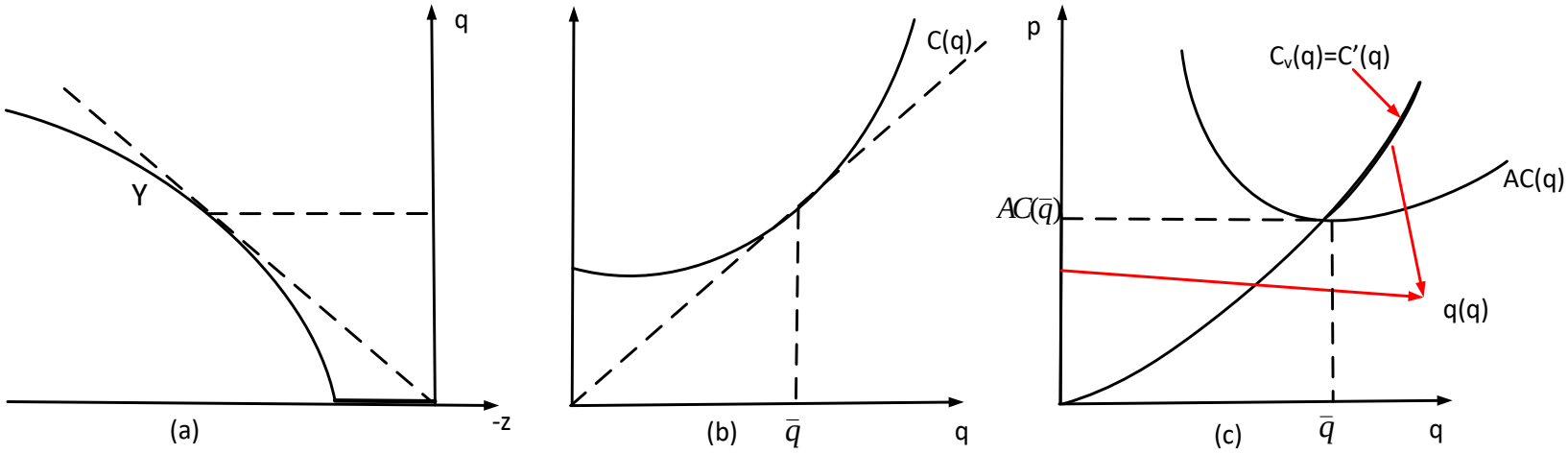
Cost and Supply: Single Output

- ***CRS technology and fixed (non-sunk) costs***:
  - If $q = 0$, then $C(q) = 0$, i.e., the firm can recover K if it shuts down its operation.
  - MC is constant: $MC = C'(q) = C_v'(q) = c$
  - AC lies above MC: $AC(q) = \frac{C(q)}{q} = \frac{K}{q} + \frac{C_v(q)}{q} = \frac{K}{q} + c$



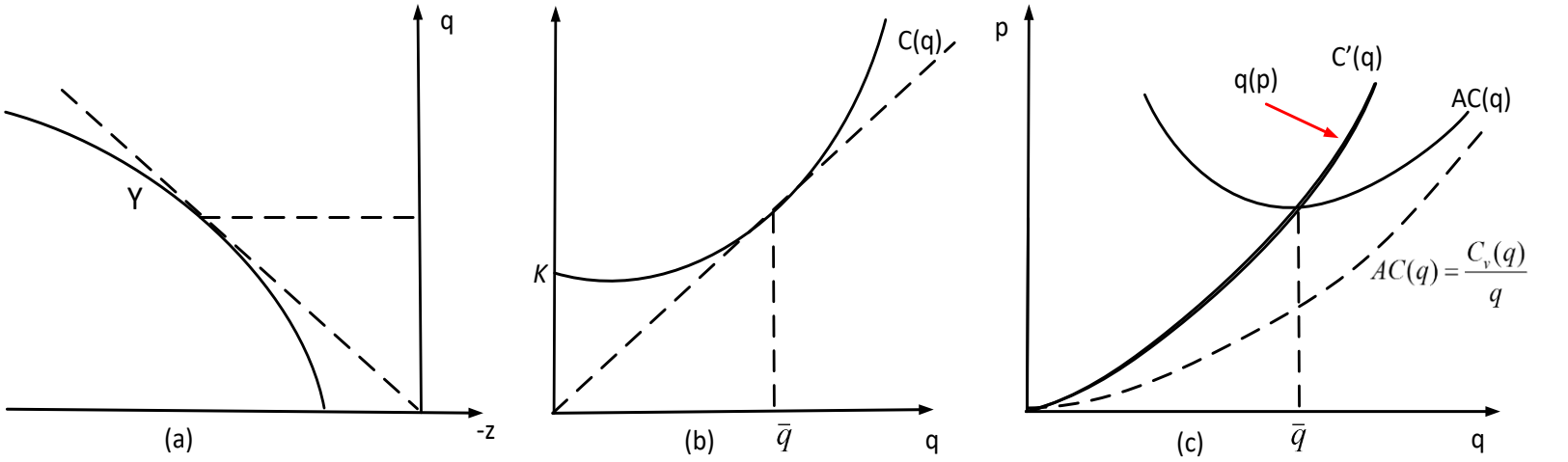
Cost and Supply: Single Output

- ***DRS technology and fixed (non-sunk) costs***:
  - MC is positive and increasing in q, and hence the slope of TC curve increases in q.
  - in the decreasing portion of the AC curve, FC is spread over larger q.
  - in the increasing portion of the AC curve, larger average VC offsets the lower average FC and, hence, total average cost increases.



Cost and Supply: Single Output

- ***DRS technology and sunk costs***:
  - TC curve originates at K, given that the firm must incur fixed sunk cost K even if it chooses $q = 0$.
  - supply locus considers the entire MC curve and not only q for which MC>AC.

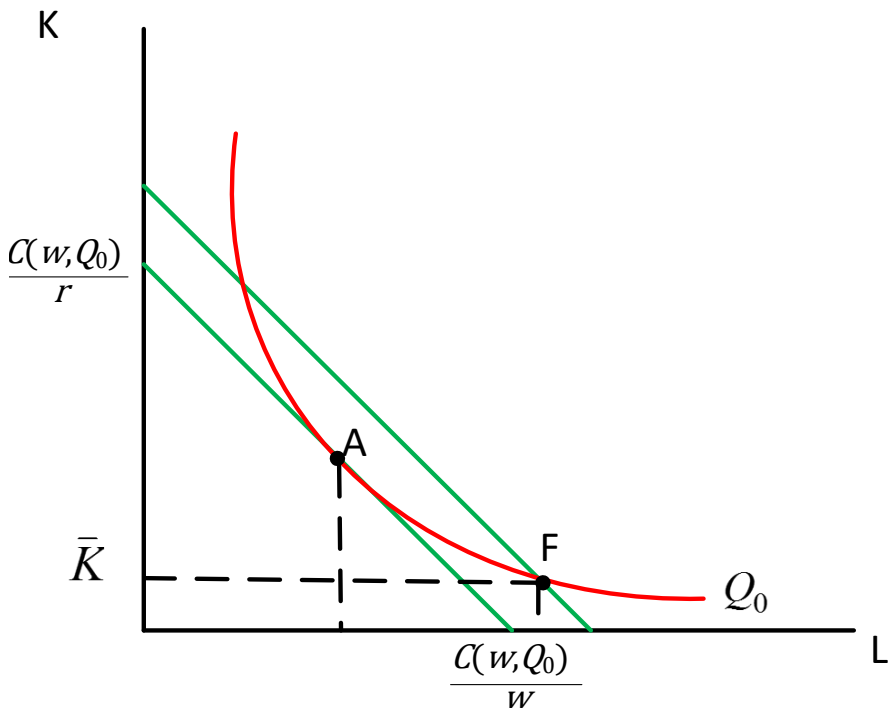


Short-Run Total Cost

- In the short run, the firm generally incurs higher costs than in the long run
  - The firm does not have the flexibility of input choice (fixed inputs).
  - To vary its output in the short-run, the firm must use non-optimal input combinations
  - The $MRTS$ will not be equal to the ratio of input prices.

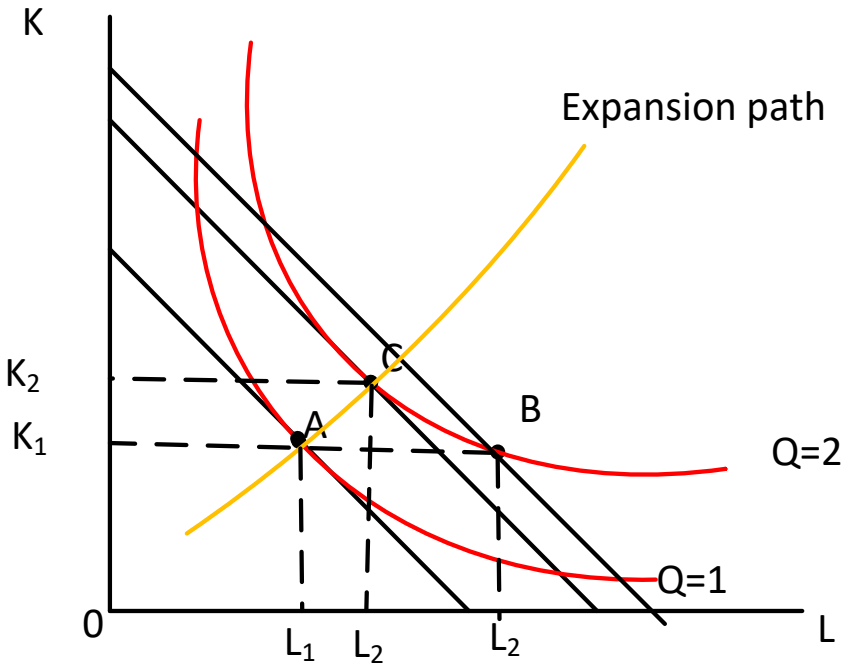
Short-Run vs Long-Run Total Cost

- In the short-run
  - capital is fixed at $\overline{K}$
  - the firm cannot equate $MRTS$ with the ratio of input prices.
- In the long-run
  - The firm can choose input vector A, which is the cost-minimizing input combination.



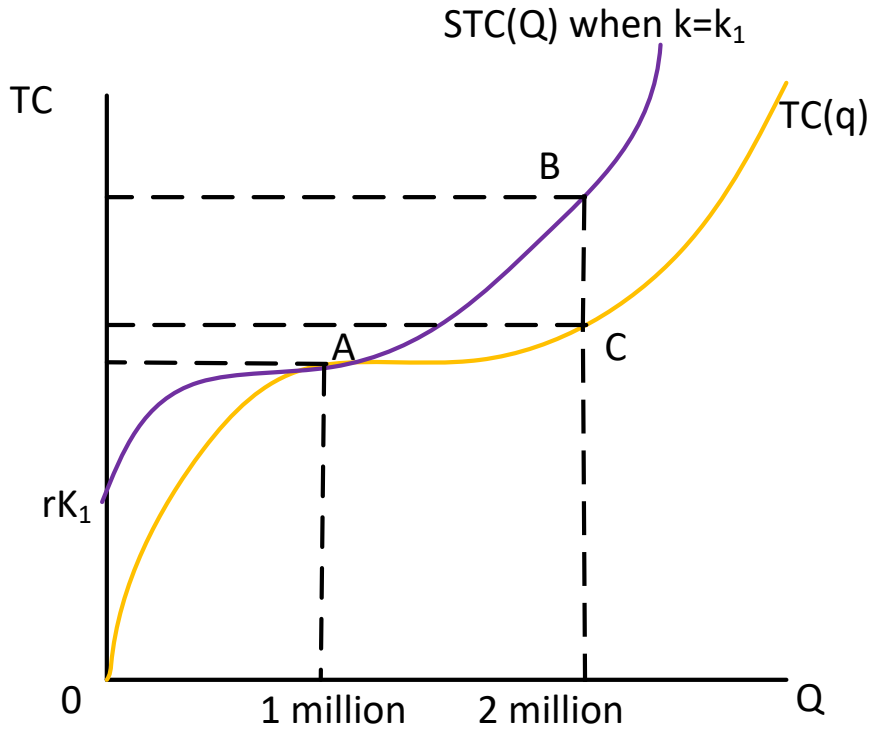
Short-Run vs Long-Run Total Cost

- $q = 1$ million units
  - Firm chooses (k_1, l_1) both in the long run and in the short run when $k = k_1$.
- $q = 2$ million units
  - Short-run (point B):
    - $k = k_1$ does not allow the firm to minimize costs.
  - Long-run (point C):
    - firm can choose cost-minimizing input combination.



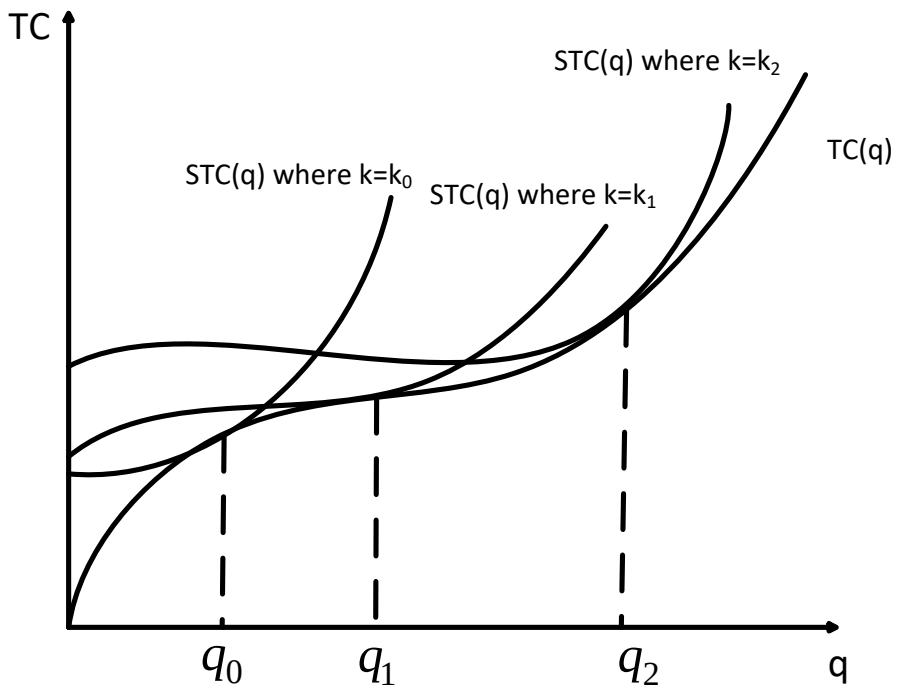
Short-Run vs Long-Run Total Cost

- The difference between long-run, $TC(q)$, and short-run, $STC(q)$, total costs when capital is fixed at $k = k_1$.



Short-Run vs Long-Run Total Cost

- The long-run total cost curve $TC(q)$ can be derived by varying the level of k.
- Short-run total cost curve $STC(q)$ lies above long-run total cost $TC(q)$.



Short-Run vs Long-Run Total Cost

- Summary:
  - In the long run, the firm can modify the values of all inputs.
  - In the short run, in contrast, the firm can only modify some inputs (e.g., labor, but not capital).

Short-Run vs Long-Run Total Cost

- ***Example***: Short- and long-run curves
  - In the long run,

$$C(q) = \bar{w}_1 z_1 + \bar{w}_2 z_2$$

    where both input 1 and 2 are variable.

  - In the short run, input 2 is fixed at $\bar{z}_2$, and thus

$$C(q|\bar{z}_2) = \bar{w}_1 z_1 + \bar{w}_2 \bar{z}_2$$

    - This implies that the only input that the firm can modify is input 1.
    - The firm chooses z_1 such that production reaches output level q, i.e., $f(z_1, \bar{z}_2) = q$.

Short-Run vs Long-Run Total Cost

- ***Example*** (continued):
  - When the demand for input 2 is at its *long-run value*, i.e., $z_2(w,q)$, then

$$C(q) = C(q|z_2(w,q)) \text{ for every } q$$

and also

$$C'(q) = C'(q|z_2(w,q)) \text{ for every } q$$

i.e., values and slopes of long- and short-run cost functions coincide.

  - Long- and short-run curves are tangent at $z_2(w,q)$.

Short-Run vs Long-Run Total Cost

- ***Example*** (continued):
  - Since

$$C(q) \leq C(q|z_2) \text{ for any given } z_2,$$

then the long-run cost curve $C(q)$ is the *lower envelope* of the short-run cost curves, $C(q|z_2)$.

Aggregation in Production

Aggregation in Production

- Let us analyze under which conditions the "law of supply" holds at the aggregate level.
- An aggregate production function maps aggregate inputs into aggregate outputs
  - In other words, it describes the maximum level of output that can be obtained if the inputs are efficiently used in the production process.

Aggregation in Production

- Consider J firms, with production sets $Y_1, Y_2, \ldots, Y_J$.
- Each Y_j is non-empty, closed, and satisfies the free disposal property.
- Assume also that every supply correspondence $y_j(p)$ is single valued, and differentiable in prices, $p \gg 0$.
- Define the aggregate supply correspondence as the sum of the individual supply correspondences

$$y(p) = \sum_{j=1}^{J} y_j(p) = \left\{ y \in \mathbb{R}^L : y = \sum_{j=1}^{J} y_j(p) \right\}$$

where $y_j \in y_j(p)$ for $j = 1,2, \ldots, J$.

Aggregation in Production

- The ***law of supply*** is satisfied at the aggregate level.
- Two ways to check it:
  1) Using the derivative of every firm's supply correspondence with respect to prices, $D_p y_j(p)$.
     - $D_p y_j(p)$ is a symmetric positive semidefinite matrix, for every firm j.
     - Since this property is preserved under addition, then $D_p y(p)$ must also define a symmetric positive semidefinite matrix.

Aggregation in Production

2) Using a revealed preference argument.
   - For every firm j,
     $$[p - p'] \cdot \left[y_j(p) - y_j(p')\right] \geq 0$$
   - Adding over j,
     $$[p - p'] \cdot [y(p) - y(p')] \geq 0$$
   - This implies that market prices and aggregate supply move in the same direction
     - the law of supply holds at the aggregate level!

Aggregation in Production

- Is there a "***representative producer***"?
  - Let Y be the aggregate production set,

$$Y = Y_1 + Y_2 + \ldots + Y_j = \left\{ y \in \mathbb{R}^L : y = \sum_{j=1}^{J} y_j \right\}$$

    for some $y_j \in Y_j$ and $j = 1,2, \ldots, J$.
  - Note that $y = \sum_{j=1}^{J} y_j$, where every y_j is just a feasible production plan of firm j, but not necessarily firm j's supply correspondence $y_j(p)$.
  - Let $\pi^*(p)$ be the profit function for the aggregate production set Y.
  - Let $y^*(p)$ be the supply correspondence for the aggregate production set Y.

Aggregation in Production

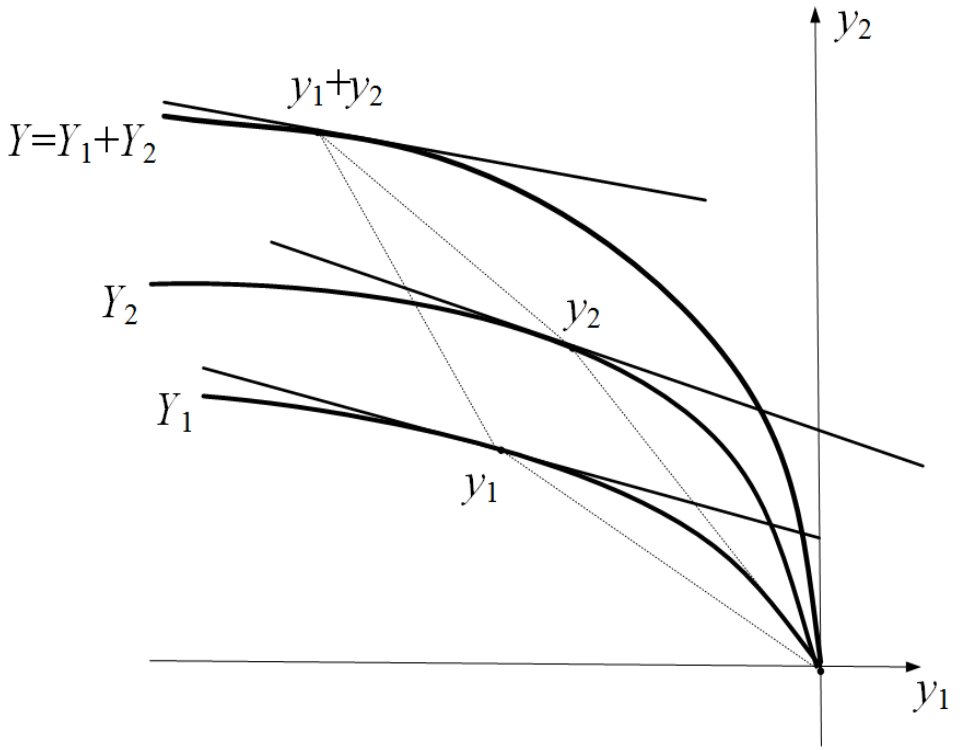
- Is there a "***representative producer***"?
  - Then, there exists a representative producer:
    - Producing an aggregate supply $y^*(p)$ that exactly coincides with the sum $\sum_{j=1}^{J} y_j\,(p)$; and
    - Obtaining aggregate profits $\pi^*(p)$ that exactly coincide with the sum $\sum_{j=1}^{J} \pi_j\,(p)$.
  - *Intuition*: The aggregate profit obtained by each firm maximizing its profits separately (taking prices as given) is the same as that which would be obtained if all firms were to coordinate their actions (i.e., y_j's) in a joint PMP.

Aggregation in Production

- Is there a "***representative producer***"?
  - It is a "decentralization" result: to find the solution of the joint PMP for given prices p, it is enough to "let each individual firm maximize its own profits" and add the solutions of their individual PMPs.
  - *Key*: price-taking assumption
    - This result does not hold if firms have market power.
    - Example: oligopoly markets where firms compete in quantities (à la Cournot).

Aggregation in Production

- Firm 1 chooses y_1 given p and Y_1.
- Firm 2 chooses y_2 given p and Y_2.
- Jointly, the two firms would be selecting $y_1 + y_2$.
- The aggregate supply correspondence $y_1 + y_2$ coincides with the supply correspondence that a single firm would select given p and $Y = y_1 + y_2$.



Efficient Production

Efficient Production

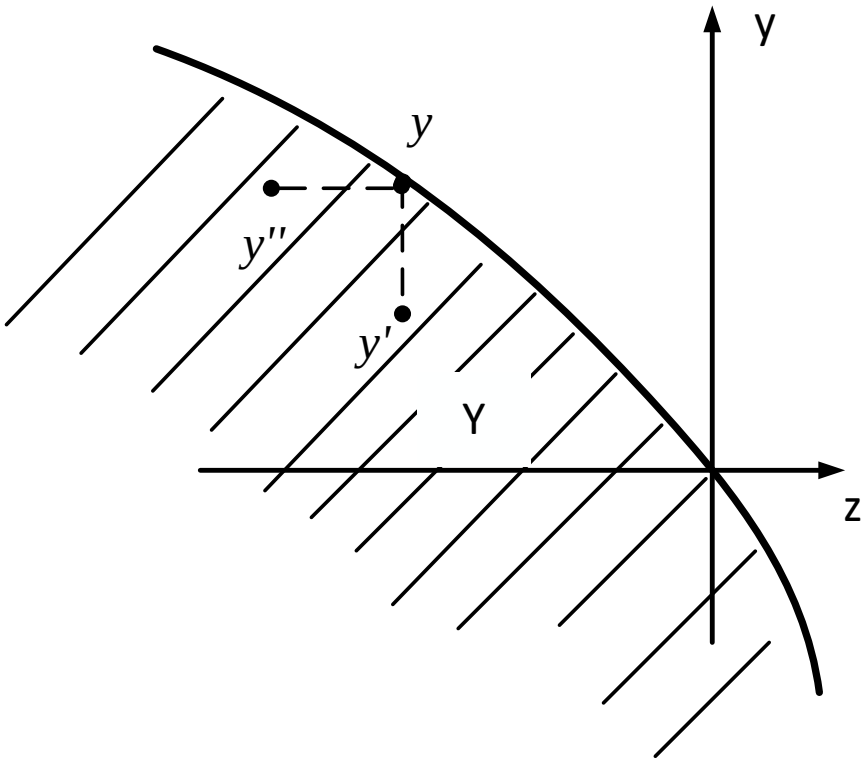
- ***Efficient production vector***: a production vector $y \in Y$ is efficient if there is no other $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.
  - That is, y is efficient if there is no other feasible production vector y' producing more output with the same amount of inputs, or alternatively, producing the same output with fewer inputs.

$$y \text{ is efficient } \Rightarrow \ y \text{ is on the boundary of } Y$$

$$y \text{ is efficient } \not\Leftarrow \ y \text{ is on the boundary of } Y$$

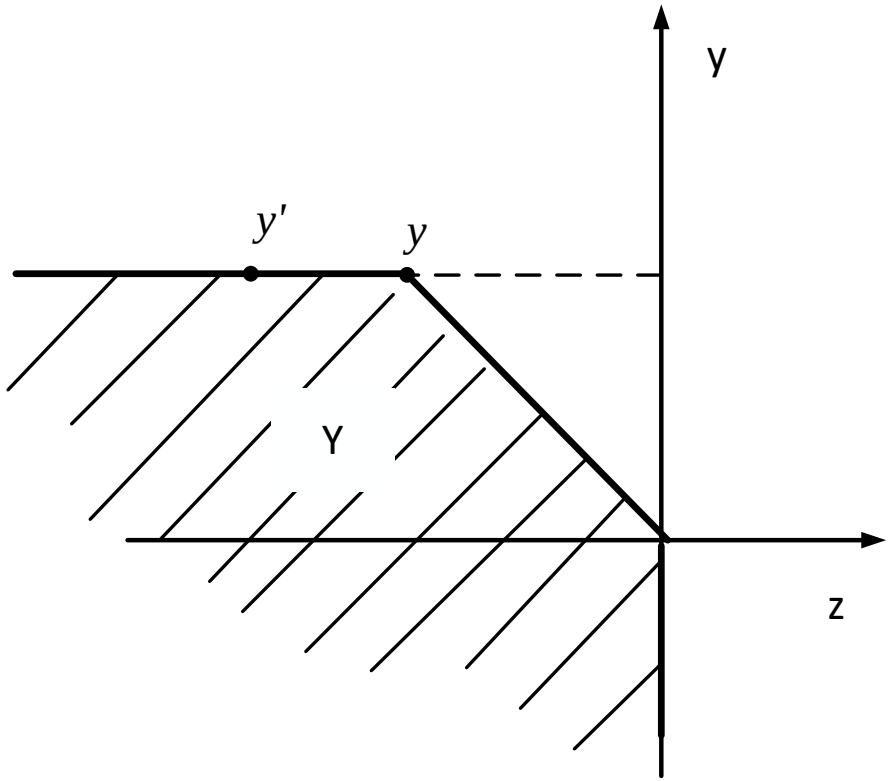
Efficient Production

- y'' produces the same output as y, but uses more inputs.
- y' uses the same inputs as y, but produces less output.
- y'' and y' are inefficient.
- y is efficient $\Rightarrow y$ lies on the frontier of the production set Y.



Efficient Production

- y is efficient
- y' is inefficient
  - it produces the same output as y, but uses more inputs.
- Hence, y' lies on the frontier of the production set $Y \nRightarrow y'$ is efficient.



Efficient Production: 1st FTWE

- ***1st Fundamental Theorem of Welfare Economics (FTWE)***: If $y \in Y$ is profit-maximizing for some price vector $p \gg 0$, then y must be efficient.

  Proof: Let us prove the 1st FTWE by contradiction. Suppose that $y \in Y$ is profit-maximizing

  $$p \cdot y \geq p \cdot y' \text{ for all } y' \in Y$$

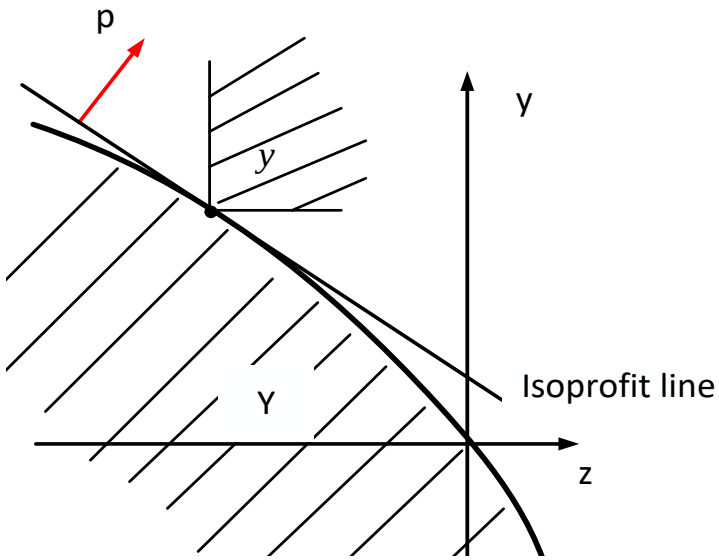
  but y is not efficient. That is, there is a $y' \in Y$ such that $y' \geq y$. If we multiply both sides of $y' \geq y$ by p, we obtain

  $$p \cdot y' \geq p \cdot y, \text{ since } p \gg 0$$

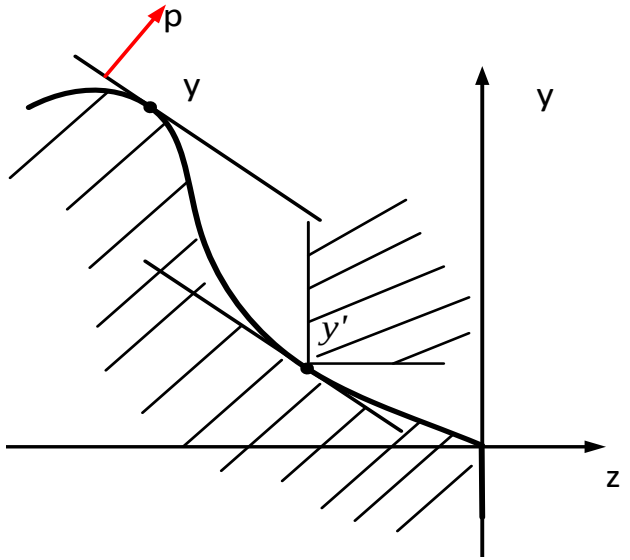
  But then, y cannot be profit-maximizing. Contradiction!

Efficient Production: 1st FTWE

- For the result in 1st FTWE, we do NOT need the production set Y to be convex.
  - y is profit-maximizing $\Rightarrow$ y lies on a tangency point



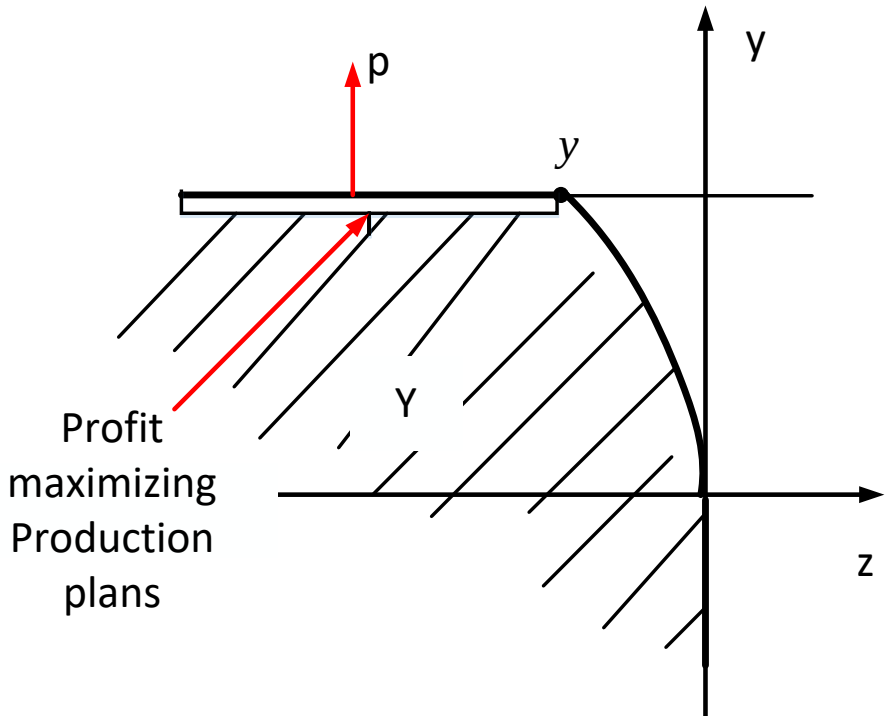
convex production set



non-convex production set

Efficient Production: 1st FTWE

- **Note**: the assumption $p \gg 0$ cannot be relaxed to $p \geq 0$.
  - Take a production set Y with an upper flat surface.
  - Any production plan in the flat segment of Y can be profit-maximizing if prices are $p = (0,1)$.
  - But only y is efficient.
  - Other profit-maximizing production plans to the left of y are NOT efficient.
  - Hence, in order to apply 1st FTWE we need $p \gg 0$.

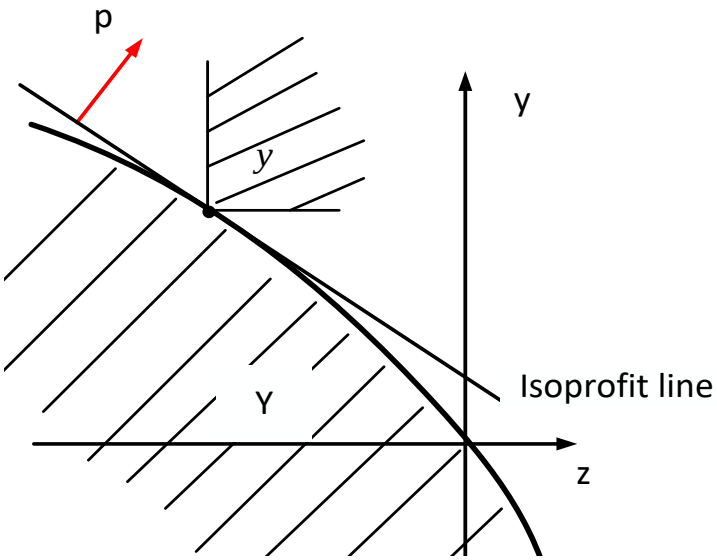


Efficient Production: 2nd FTWE

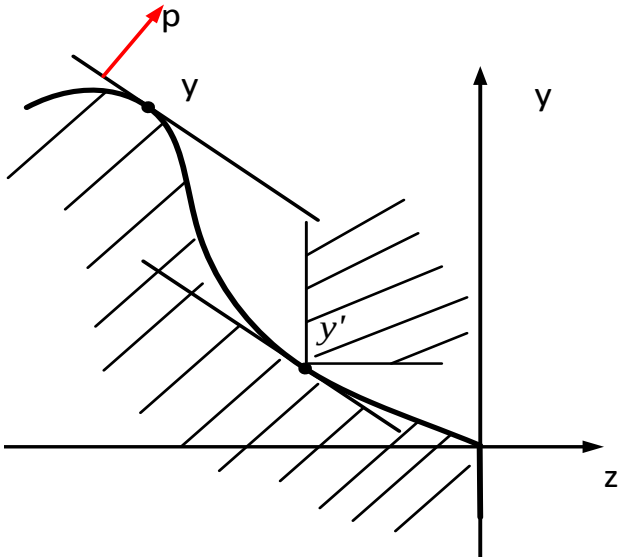
- The 2nd FTWE states the converse of the 1st FTWE:
  - If a production plan y is efficient, then it must be profit-maximizing.
- Note that, while it is true for convex production sets, it cannot be true if Y is non-convex.

Efficient Production: 2nd FTWE

- The 2nd FTWE is restricted to convex production sets.
- For non-convex production set: If y is efficient $\not\Rightarrow$ y is profit-maximizing.



convex production set



non-convex production set

Efficient Production: 2nd FTWE

- ***2nd FTWE***: If production set Y is convex, then every efficient production plan $y \in Y$ is a profit-maximizing production plan, for some non-zero price vector $p \geq 0$.

  Proof:

  1) Take an efficient production plan, such as y on the boundary of Y. Define the set of production plans that are *strictly more efficient* than y

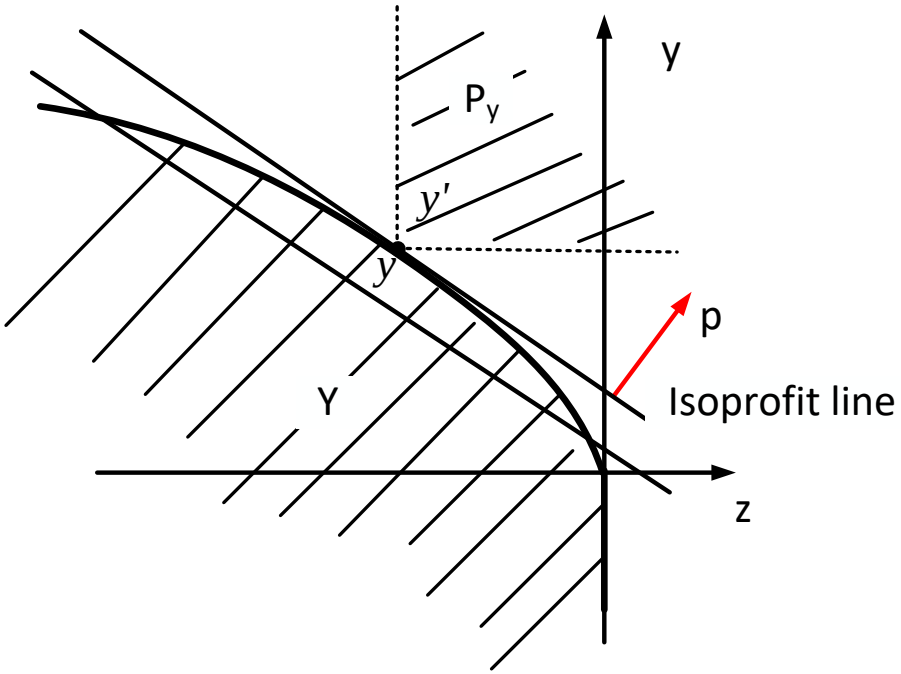
  $$P_y = \{y' \in \mathbb{R}^L : y' \gg y\}$$

  2) Note that $Y \cap P_y = \emptyset$ and P_y is a convex set.

Efficient Production: 2nd FTWE

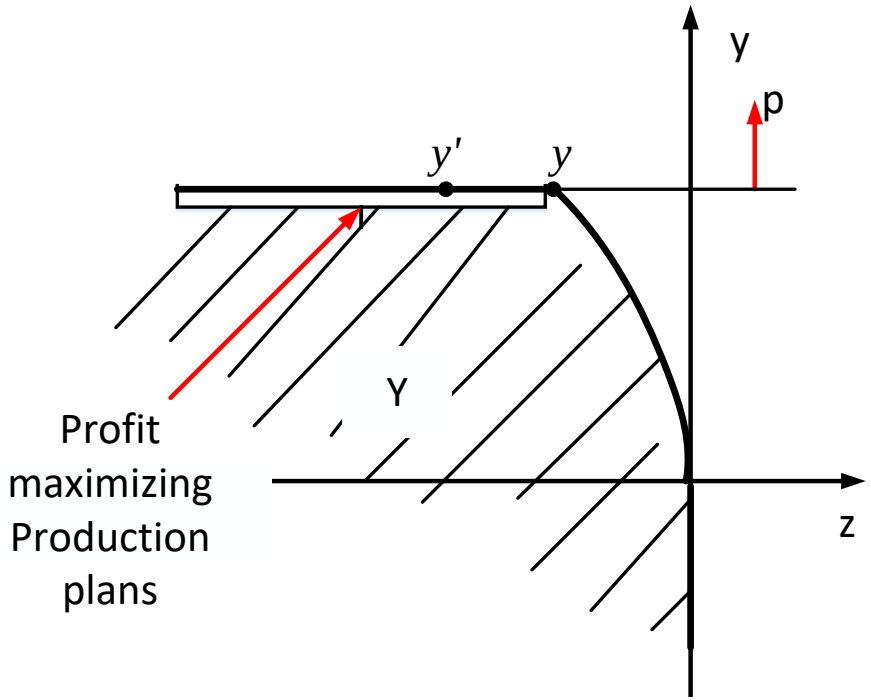
Proof (continued):

3) From the Separating Hyperplane Theorem, there is some $p \neq 0$ such that $p \cdot y' \geq p \cdot y''$, for $y' \in P_y$ and $y'' \in Y$.
4) Since y' can be made arbitrarily close to y, we can have $p \cdot y \geq p \cdot y''$ for $y'' \in Y$.
5) Hence, the efficient production plan y must be profit-maximizing.



Efficient Production: 2nd FTWE

- **Note**: we are not imposing $p \gg 0$, but $p \geq 0$.
  - We just assume that the price vector is not zero for every component, i.e., $p \neq (0,0,\ldots,0)$.
  - Hence, the slope of the isoprofit line can be zero.
  - Both y and y' are profit-maximizing, but only y is efficient.



Efficient Production: 2nd FTWE

- **Note**: the 2nd FTWE does not allow for input prices to be negative.
  - Consider the case in which the price of input l is negative, $p_l < 0$.
  - Then, we would have $p \cdot y' < p \cdot y$ for some production plan y' that is more efficient than y, i.e., $y' \gg y$, with $y'_l - y_l$ being sufficiently large.
  - This implies that the firm is essentially "paid" for using further amounts of input l.
  - For this reason, we assume $p \geq 0$.

Advanced Microeconomic Theory

Chapter 5: Choices under Uncertainty

Outline

- Simple, Compound, and Reduced Lotteries
- Independence Axiom
- Expected Utility Theory
- Money Lotteries
- Risk Aversion
- Prospect Theory and Reference-Dependent Utility
- Comparison of Payoff Distributions

Simple, Compound, and Reduced Lotteries

Simple Lotteries

- Consider a set of possible outcomes (or consequences) C.
- The set C can include
  - simple payoffs $C \in \mathbb{R}$ (positive or negative)
  - consumption bundles $C \in \mathbb{R}^L$
- Outcomes are finite (N elements in C, $n = 1,2, \dots, N$)
- Probabilities of every outcome are objectively known
  - p_1 for outcome 1, p_2 for outcome 2, etc.

Simple Lotteries

- ***Simple lottery*** is a list

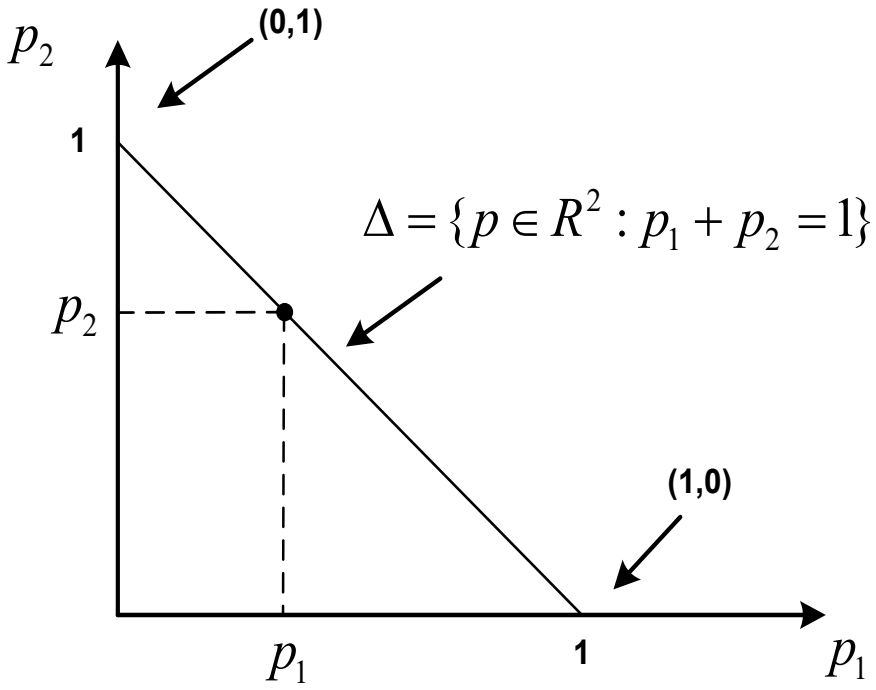
$$L = (p_1, p_2, \ldots, p_N)$$

with $p_n \geq 0$ for all n and $\sum_{n=1}^{N} p_n = 1$, where p_n is interpreted as the probability of outcome n occurring.

- In some books, lotteries are described including the outcomes too.

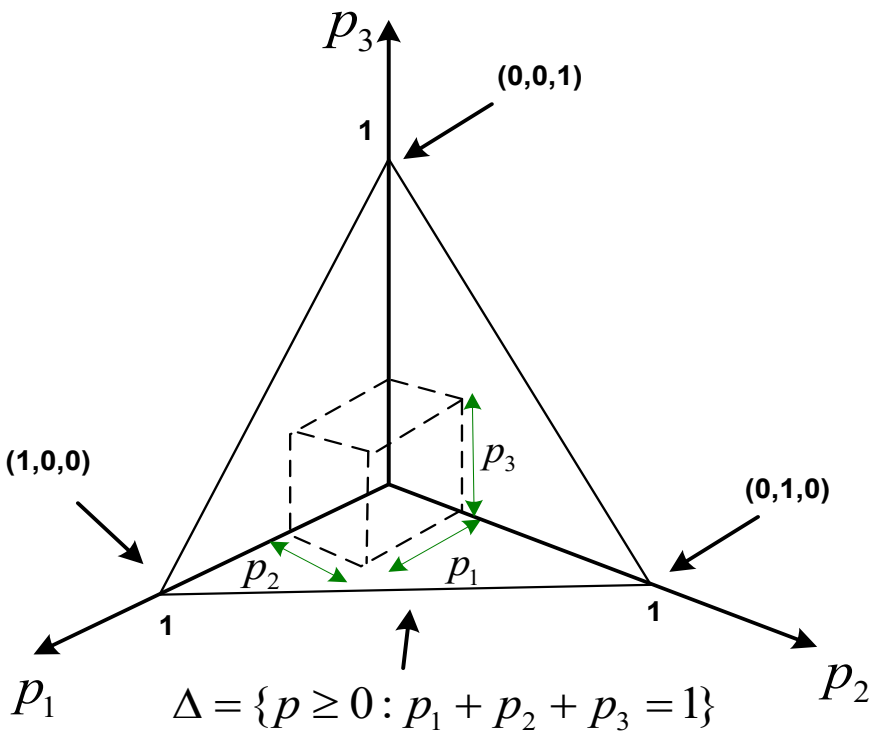
Simple Lotteries

- A simple lottery with 2 possible outcomes
- "Degenerated" probability pairs
  - at (0,1), outcome 2 happens with certainty.
  - at (1,0), outcome 1 happens with certainty.
- Strictly positive probability pairs
  - Individual faces some uncertainty, i.e., $p_1 + p_2 = 1$



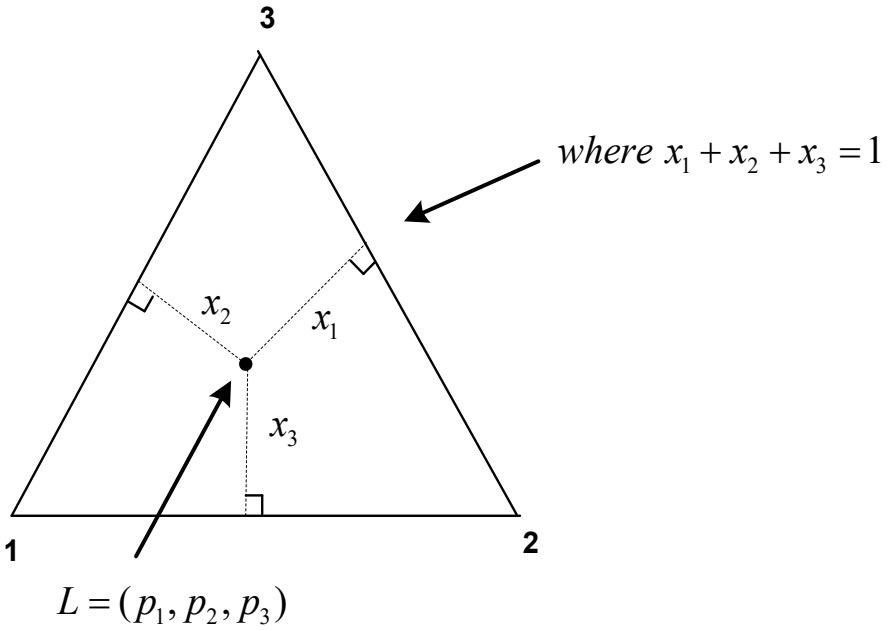
Simple Lotteries

- A simple lottery with 3 possible outcomes (i.e., 3-dim. simplex).
- Intercepts represent degenerated probabilities where one outcome is certain.
- Points strictly inside the hyperplane connecting the three intercepts denote a lottery where the individual faces uncertainty.



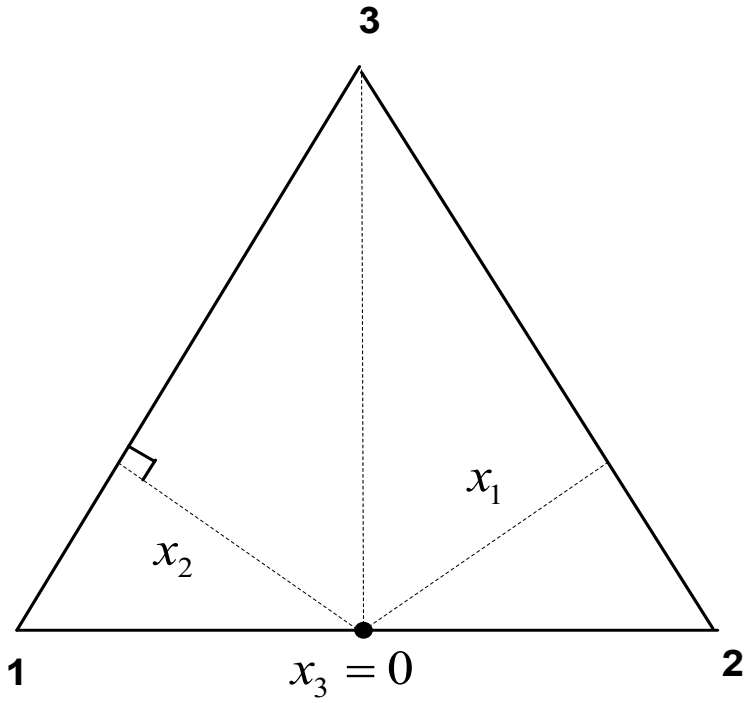
Simple Lotteries

- 2-dim. projection of the 3-dim. simplex
- Vertices represent the intercepts
- The distance from a given point to the side of the triangle measures the probability that the outcome represented at the opposite vertex occurs.



Simple Lotteries

- A lottery lies on one of the boundaries of the triangle:
  - We can only construct segments connecting the lottery to two of the outcomes.
  - The probability associated with the third outcome is zero.



Compound Lotteries

- Given simple lotteries

$$L_k = \left(p_1^k, p_2^k, \dots, p_N^k\right) \text{ for } k = 1,2, \dots, K$$

and probabilities $\alpha_k \geq 0$ with $\sum_{k=1}^{K} \alpha_k = 1$, then the ***compound lottery*** $(L_1, L_2, \dots, L_K; \alpha_1, \alpha_2, \dots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1,2, \dots, K$.

  - Think about a compound lottery as a "lottery of lotteries": first, I have probability α_1 of playing lottery 1, and if that happens, I have probability p_1^1 of outcome 1 occurring.
  - Then, the joint probability of outcome 1 is

$$p_1 = \alpha_1 \cdot p_1^1 + \alpha_2 \cdot p_1^2 + \cdots + \alpha_K \cdot p_1^K$$

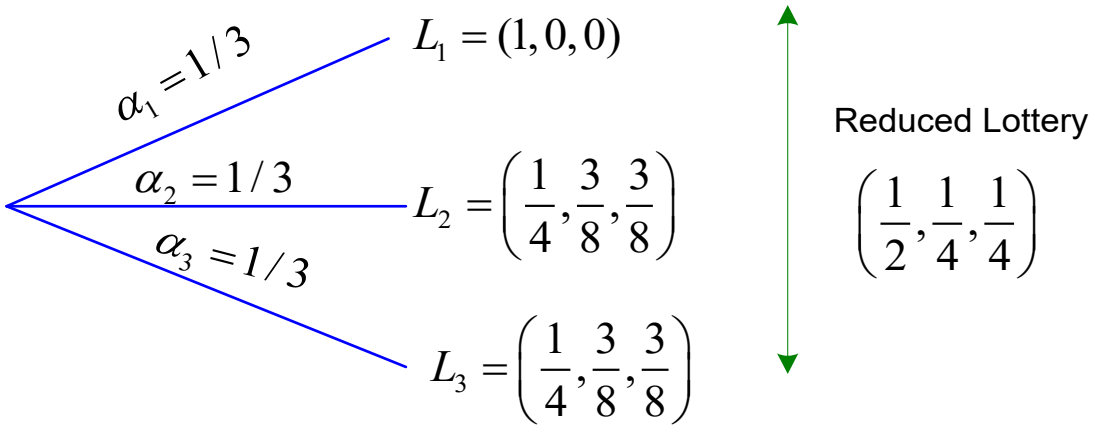
Compound and Reduced Lotteries

- Given that interpretation, the following result should come at no surprise:
  - For any compound lottery $(L_1, L_2, \dots, L_K; \alpha_1, \alpha_2, \dots, \alpha_K)$, we can calculate its corresponding ***reduced lottery*** as the simple lottery $L = (p_1, p_2, \dots, p_N)$ that generates the same ultimate probability distribution of outcomes.
- The reduced lottery L of any compound lottery can be obtained by

$$L = \alpha_1 L_1 + \alpha_2 L_2 + \cdots + \alpha_K L_K \in \Delta$$

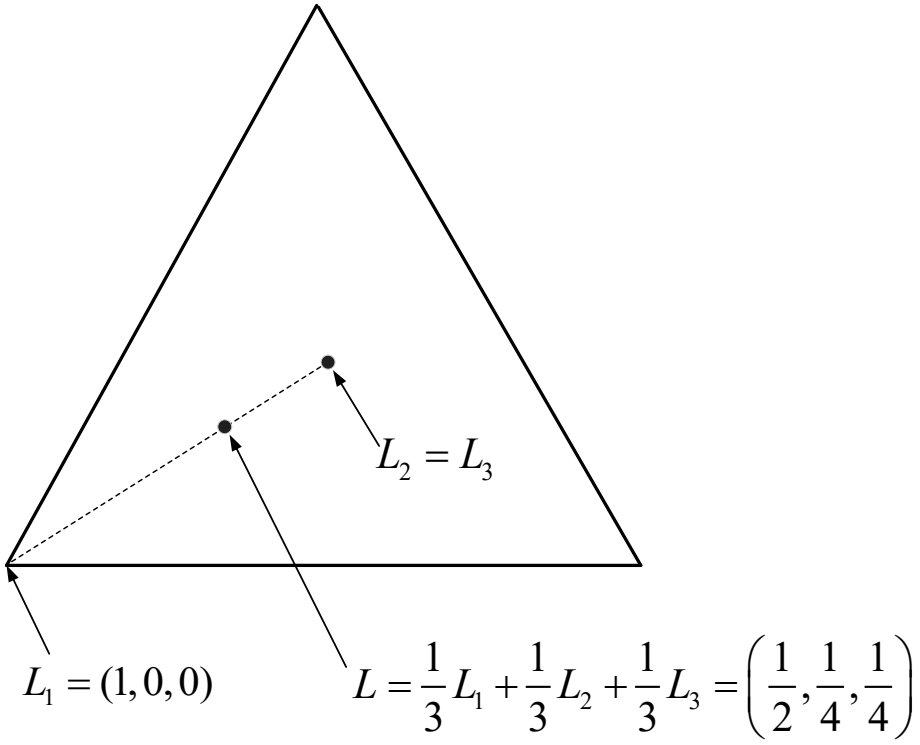
Compound and Reduced Lotteries

- ***Example 1***:
  - All three lotteries are equally likely
  - $\mathrm{P}(\text{outcome 1}) = \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{2}$
  - $\mathrm{P}(\text{outcome 2}) = \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot\frac{3}{8} + \frac{1}{3}\cdot\frac{3}{8} = \frac{1}{4}$
  - $\mathrm{P}(\text{outcome 3}) = \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot\frac{3}{8} + \frac{1}{3}\cdot\frac{3}{8} = \frac{1}{4}$



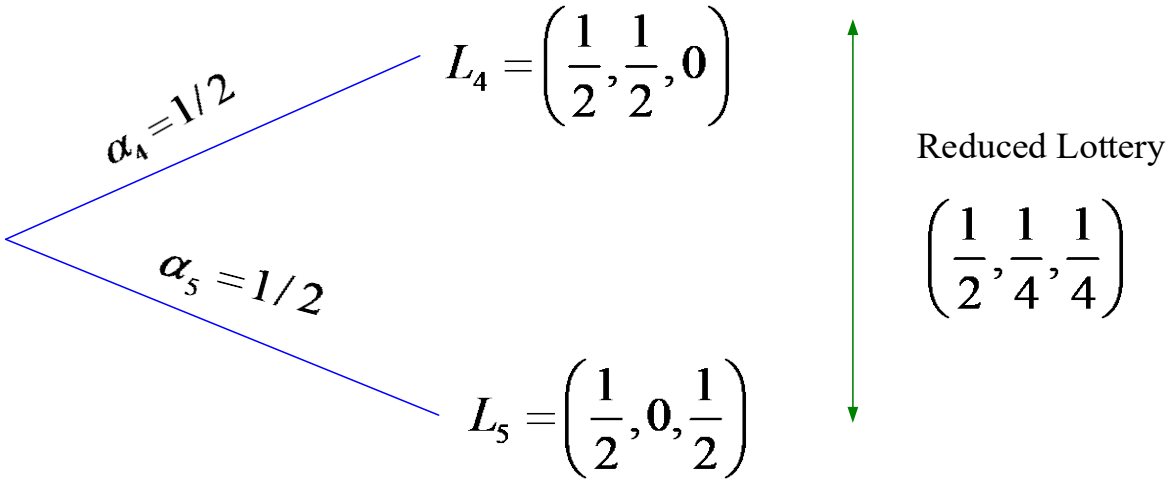
Compound and Reduced Lotteries

- ***Example 1*** (continued):
  - Probability simplex of the reduced lottery of a compound lottery
  - Reduced lottery L assigns the same probability weight to each simple lottery.



Compound and Reduced Lotteries

- ***Example 2***:
  - Both lotteries are equally likely



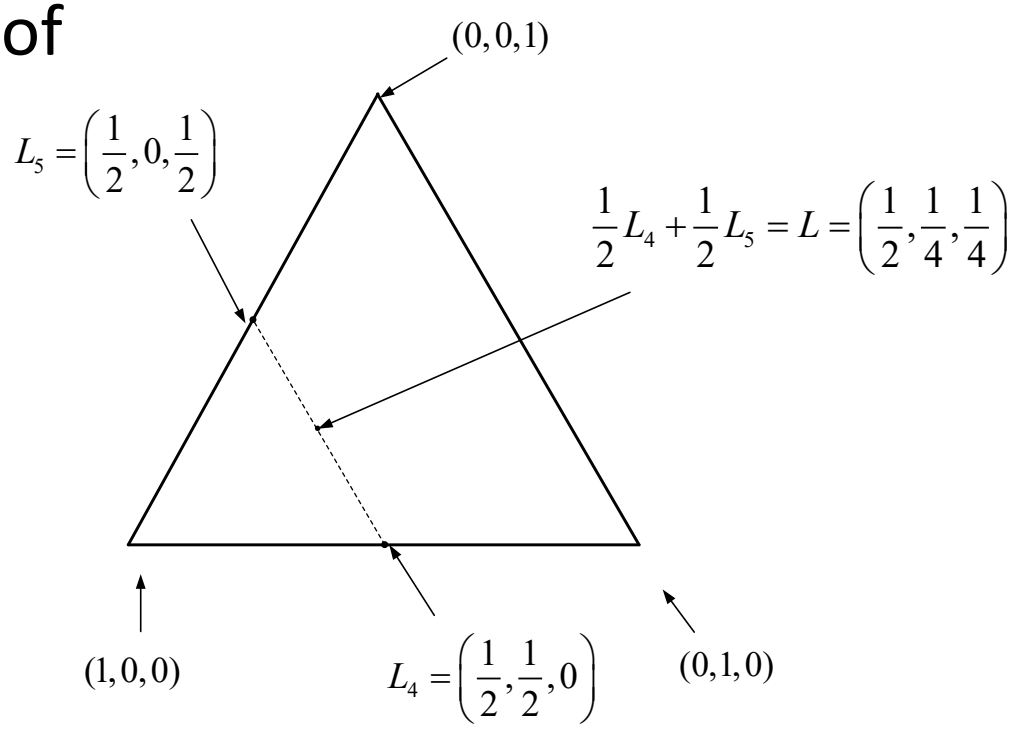
Outcome 1⟶ $\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$

Outcome 2⟶ $\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot 0=\frac{1}{4}$

Outcome 3⟶ $\frac{1}{2}\cdot 0+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$

Compound and Reduced Lotteries

- ***Example 2*** (continued):
  - Probability simplex of the reduced lottery of a compound lottery



Compound and Reduced Lotteries

- Consumer is indifferent between the two compound lotteries which induce the same reduced lottery
  - This was illustrated in the previous Examples 1 and 2 where, despite facing different compound lotteries, the consumer obtained the same reduced lottery.
- We refer to this assumption as the ***Consequentialist hypothesis***:
  - Only consequences, and the probability associated to every consequence (outcome) matters, but not the route that we follow in order to obtain a given consequence.

Preferences over Lotteries

- For a given set of outcomes C, consider the set of all simple lotteries over C, $\mathcal{L}$.
- We assume that the decision maker has a *complete* and *transitive* preference relation $\succsim$ over lotteries in $\mathcal{L}$, allowing him to compare any pair of simple lotteries L and L'.
  - ***Completeness***: For any two lotteries L and L', either $L \succsim L'$ or $L' \succsim L$, or both.
  - ***Transitivity***: For any three lotteries L, L' and L'', if $L \succsim L'$ and $L' \succsim L''$, then $L \succsim L''$.

Preferences over Lotteries

- ***Extreme preference for certainty***:
  - $L \gtrsim L'$ if and only if

$$\max_{n \in N} p_n \geq \max_{n \in N} p'_n$$

  - The decision maker is only concerned about the probability associated with the most likely outcome.

Preferences over Lotteries

- ***Example***:
  - Outcomes $C = (\$100, \$5)$, with lotteries
    - $L = (0.4, 0.6)$ and $L' = (0.55, 0.45)$.
  - The most likely outcome is
    - \$5 in L (probability 0.6), and
    - \$100 in L' (probability 0.55).
  - Since $\max_n p_n = 0.6 > 0.55 = \max_n {p'}_n$, we have that $L \succ L'$.
- ***Surprising ranking***:
  - L' pays \$100 with a higher probability (0.55 vs. 0.4), yet the decision maker prefers L.
  - He only cares about how likely the most likely outcome is, not about which outcome it is.

Preferences over Lotteries

- ***Smallest size of the support***:
  - $L \gtrsim L'$ if and only if

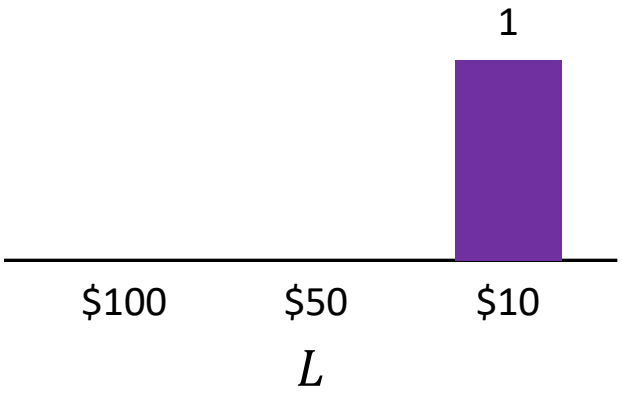
$$\text{supp}(L) \leq \text{supp}(L')$$

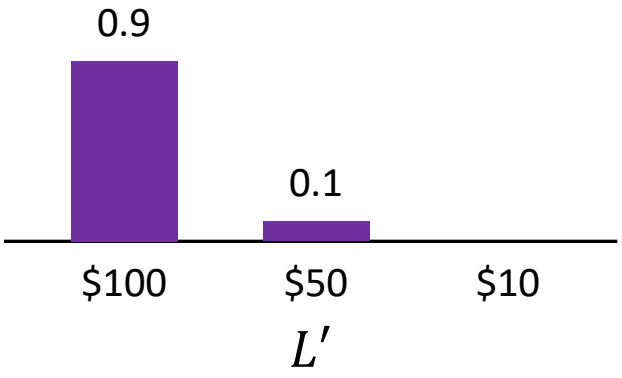
where $\text{supp}(L) = \{n \in N: p_n > 0\}$.

  - The decision maker prefers the lottery whose probability distribution is concentrated over the smallest set of possible outcomes.

Preferences over Lotteries

- ***Example***:
  - Outcomes $C = (\$100, \$50, \$10)$, with lotteries
    - $L = (0, 0, 1)$ and $L' = (0.9, 0.1, 0)$.
  - The support of L has one outcome ($10), while that of L' has two ($100 and $50), see figure. Hence, $L \succ L'$.
- ***Surprising ranking***:
  - Every outcome of L' is better than the only outcome of L, yet the decision maker prefers $10 for sure.





Preferences over Lotteries

- ***Lexicographic preferences***:
  - First, order outcomes from most preferred (outcome 1) to least preferred (outcome n).
  - Then $L \gtrsim L'$, if and only if

$$p_1 > p_1', \text{ or}$$

$$\text{If } p_1 = p_1' \text{ and } p_2 > p_2', \text{ or}$$

$$\text{If } p_1 = p_1' \text{ and } p_2 = p_2' \text{ and } p_3 > p_3', \text{ or}$$

$$\ldots$$

  - The decision maker weakly prefers lottery L to L' if outcome 1 is more likely to occur in lottery L than in lottery L'.
  - If outcome 1 is as likely to occur in both lotteries, he moves to outcome 2; and so on.

Preferences over Lotteries

- ***Example***:
  - Outcomes $C = (\$100, \$50, \$10)$, from most preferred (outcome 1) to least preferred (outcome 3), with lotteries
    - $L = (0.1, 0, 0.9)$ and $L' = (0.09, 0.91, 0)$.
  - Since $p_1 = 0.1 > 0.09 = p'_1$, we have that $L \succ L'$.
  - The decision maker never compares the probabilities of outcomes 2 and 3.
- ***Surprising ranking***:
  - L' pays at least \$50 with certainty, while L pays \$10 with probability 0.9.
  - Yet the decision maker prefers L, only because it pays \$100 with a probability 0.01 higher.

Preferences over Lotteries

- ***Example***
  - $L_1 = (p_1^1, p_2^1, p_3^1) = (1,0,0)$
  - $L_2 = (p_1^2, p_2^2, p_3^2) = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)$
  - $L_3 = (p_1^3, p_2^3, p_3^3) = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)$
  - $L_4 = (p_1^4, p_2^4, p_3^4) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$
  - $L_5 = \left(p_1^5, p_2^5, p_3^5\right) = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$

Preferences over Lotteries

- ***Under extreme preference for certainty,***
  - $\max_{n\in\{1,2,3\}} p_n^1 = 1$
  - $\max_{n\in\{1,2,3\}} p_n^2 = \max_{n\in\{1,2,3\}} p_n^3 = \frac{3}{8}$
  - $\max_{n\in\{1,2,3\}} p_n^4 = \max_{n\in\{1,2,3\}} p_n^5 = \frac{1}{2}$
  - Since $\max_{n\in\{1,2,3\}} p_n^1 > \max_{n\in\{1,2,3\}} p_n^4 = \max_{n\in\{1,2,3\}} p_n^5 > \max_{n\in\{1,2,3\}} p_n^2 = \max_{n\in\{1,2,3\}} p_n^3$, we have that

$$L_1 \succ L_4 \sim L_5 \succ L_2 \sim L_3,$$

  - because the most likely outcome happens with certainty in lottery 1, with lower probability in lotteries 4 and 5, and with the lowest probability in lotteries 2 and 3.

Preferences over Lotteries

- ***Under smallest size of the support,***
  - $\text{supp}(L_1) = 1$
  - $\text{supp}(L_2) = \text{supp}(L_3) = 3$
  - $\text{supp}(L_4) = \text{supp}(L_5) = 2$
  - Since $\text{supp}(L_1) < \text{supp}(L_4) = \text{supp}(L_5) < \text{supp}(L_2) = \text{supp}(L_3)$, we have that

$$L_1 \succ L_4 \sim L_5 \succ L_2 \sim L_3,$$

  - as lottery 1 has all probability weight concentrated in one outcome, while lotteries 4 and 5 (2 and 3) distribute probability weights between 2 (3) outcomes.

Preferences over Lotteries

- ***Under Lexicographic preferences,***
  - $p_1^1 = 1$
  - $p_1^2 = p_1^3 = \frac{1}{4}$ and $p_2^2 = p_2^3 = p_3^2 = p_3^3 = \frac{3}{8}$
  - $p_1^4 = \frac{1}{2}$ and $p_2^4 = \frac{1}{2}$
  - $p_1^5 = \frac{1}{2}$ and $p_2^5 = 0$
  - Since $p_1^1 > p_1^4 = p_1^5 > p_1^2 = p_1^3$ and $p_2^4 > p_2^5$ and $p_2^2 = p_2^3 = p_3^2 = p_3^3 = \frac{3}{8}$, we have that

$$L_1 \succ L_4 \succ L_5 \succ L_2 \sim L_3.$$

  - (Intuition in the next slide.)

Preferences over Lotteries

- ***Under Lexicographic preferences,***
  - Intuitively, outcome 1 is a sure thing in lottery 1, so it is the most preferred by the decision maker.
  - Comparing lotteries 4 and 5, they assign the same probability to outcome 1.
    - Moving to outcome 2, we see that it can happen in lottery 4 but cannot occur in lottery 5, implying that lottery 4 is preferred to 5.
  - Lotteries 2 and 3 are tied in the probability that they assign to the first, second, and third outcomes; entailing that the decision maker is indifferent between them.

Preferences over Lotteries

- ***The worst-case scenario***:
  - First, attach a number $v(z)$ to every outcome $z \in C$, that is, $v(z) \in \mathbb{R}$.
  - Then $L \gtrsim L'$ if and only if

$$\min\{v(z)\colon\, p(z) > 0\} > \min\{v(z)\colon\, p'(z) > 0\}$$

  - The decision maker prefers lottery L if the lowest utility he can get from playing lottery L is higher than the lowest utility he can obtain from playing lottery L'.

Preferences over Lotteries

- Continuity of preferences over lotteries:
  - ***Continuity 1***: For any three lotteries L, L', and L'', the sets

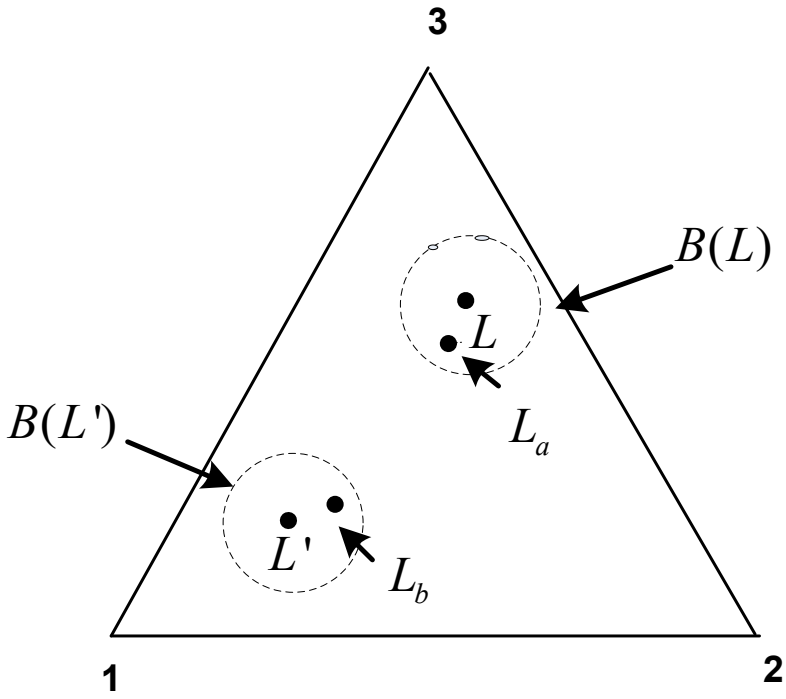
    $\{\alpha \in [0,1]: \alpha L + (1-\alpha)L' \gtrsim L''\} \subset [0,1]$ and
    $\{\alpha \in [0,1]: L'' \gtrsim \alpha L + (1-\alpha)L'\} \subset [0,1]$

    are closed.

  - ***Continuity 2***: if $L \succ L'$, then there are neighborhoods of L and L', $B(L)$ and $B(L')$, such that for all $L_a \in B(L)$ and $L_b \in B(L')$, we have $L_a \succ L_b$.

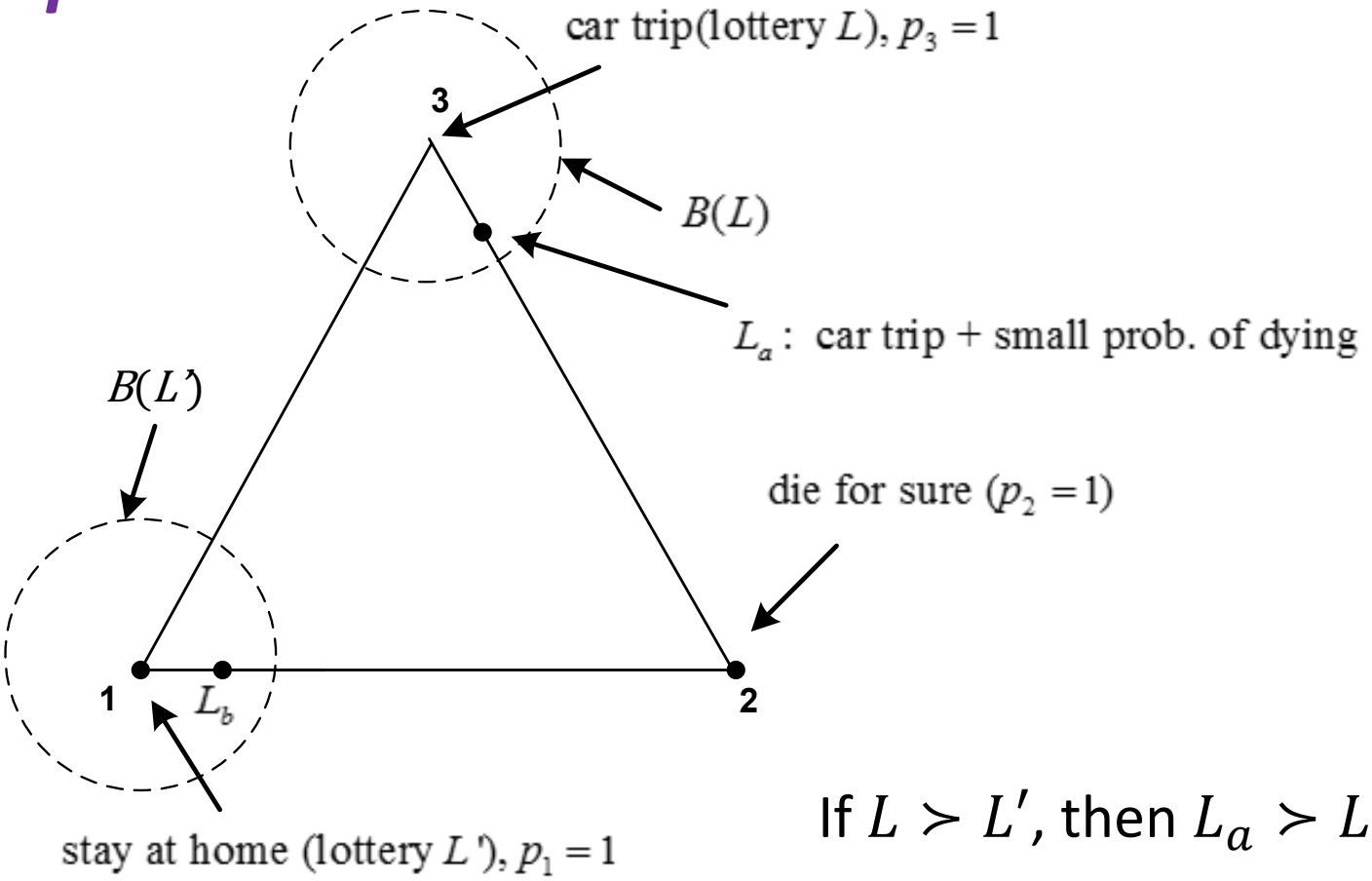
Preferences over Lotteries

- Small changes in the probability distribution of lotteries L and L' do not change the preference over the two lotteries.



Preferences over Lotteries

- ***Example***:



If $L \succ L'$, then $L_a \succ L_b$.

Preferences over Lotteries

- The continuity assumption, as in consumer theory, implies the existence of a utility function $U: \mathcal{L} \to \mathbb{R}$ such that

$$L \gtrsim L' \text{ if and only if } U(L) \geq U(L')$$

- However, we first impose an additional assumption in order to have a more structured utility function.
  - The following assumption is related with consequentialism: the *Independence axiom*.

Preferences over Lotteries

- ***Independence Axiom (IA)***: a preference relation satisfies IA if, for any three lotteries L, L', and L'', and $\alpha \in (0,1)$ we have

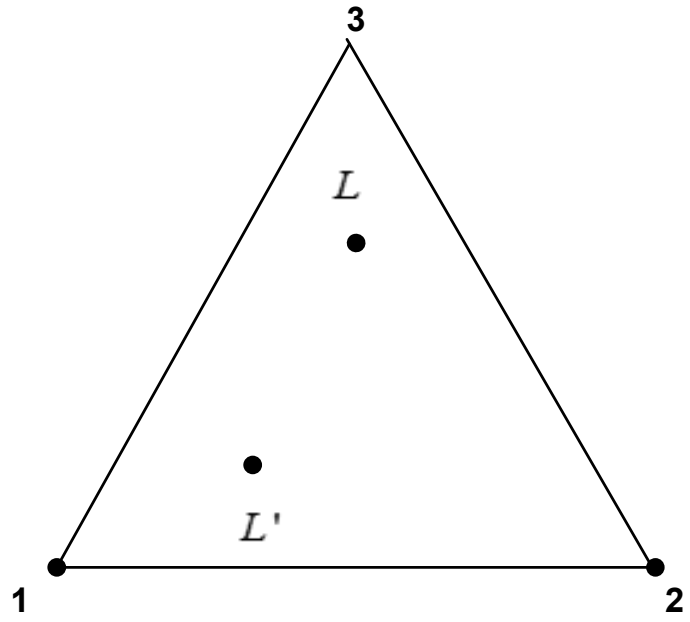
$$L \gtrsim L' \text{ if and only if}$$

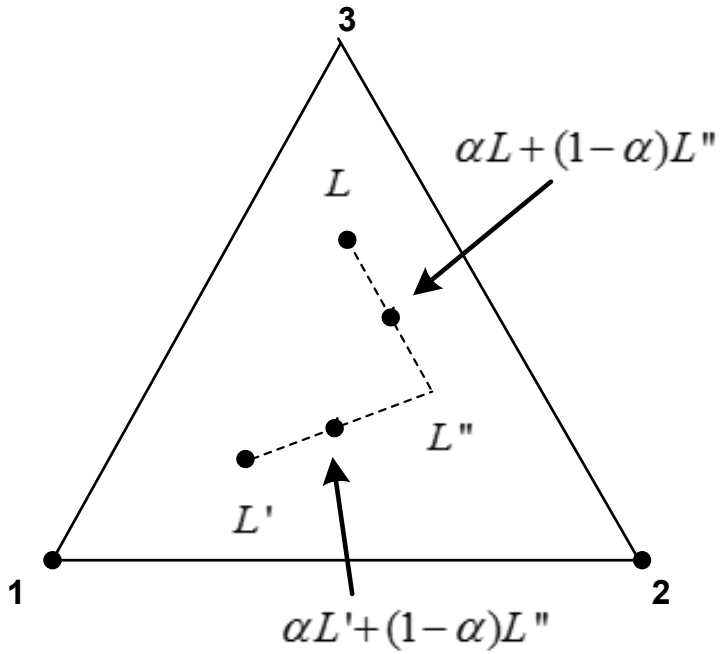
$$\alpha L + (1-\alpha)L'' \gtrsim \alpha L' + (1-\alpha)L''$$

- *Intuition*: If we mix each of two lotteries, L and L', with a third one (L''), then the preference ordering of the two resulting compound lotteries is independent of the particular third lottery .

Preferences over Lotteries

- $L \gtrsim L'$ if and only if
  $$\alpha L + (1-\alpha)L'' \gtrsim \alpha L' + (1-\alpha)L''$$





Preferences over Lotteries

- ***Example 1*** (intuition):
  - The decision maker prefers lottery L to L', $L \succsim L'$
  - Construct a compound lottery by a coin toss:
    - play lottery L if the head comes up
    - play lottery L'' if the tail comes up
  - By IA, if $L \succsim L'$, then

$$\frac{1}{2}L + \frac{1}{2}L'' \succsim \frac{1}{2}L' + \frac{1}{2}L''$$

Preferences over Lotteries

- ***Example 2*** (violations of IA):
  - Extreme preference for certainty
  - Consider two simple lotteries L and L' for which $L \sim L'$.
  - Construct two compound lotteries for which

$$\frac{1}{2}L + \frac{1}{2}L \nsim \frac{1}{2}L' + \frac{1}{2}L$$

  - If $L \sim L'$, then it must be that

$$\max\{p_1, p_2, \ldots, p_n\} = \max\{p_1', p_2', \ldots, p_n'\}$$

Preferences over Lotteries

- ***Example 2*** (violations of IA):
  - Compound lottery $\frac{1}{2}L + \frac{1}{2}L$ coincides with simple lottery L.
  - Hence, $\max\{p_1, p_2, \dots, p_n\}$ is used to evaluate lottery L.
  - But compound lottery $\frac{1}{2}L' + \frac{1}{2}L$ is a reduced lottery with associated probabilities

$$\max\left\{\frac{1}{2}p_1' + \frac{1}{2}p_1, \dots, \frac{1}{2}p_n' + \frac{1}{2}p_n\right\}$$

which might *differ* from $\max\{p_1', p_2', \dots, p_n'\}$.

Preferences over Lotteries

- ***Example 2*** (violations of IA, a numerical example):
  - Consider two simple lotteries
    $$L = (0.4, 0.5, 0.1),\ L' = (0.5, 0, 0.5)$$
  - Hence,
    $$\max\{0.4, 0.5, 0.1\} = 0.5 = \max\{0.5, 0, 0.5\}$$
    implying that $L \sim L'$.
  - However, the compound lottery $\frac{1}{2}L' + \frac{1}{2}L$ entails probabilities
    $$\left(\frac{0.4 + 0.5}{2}, \frac{0.5 + 0}{2}, \frac{0.1 + 0.5}{2}\right) = (0.45, 0.25, 0.3)$$
    implying that $\max\{0.45, 0.25, 0.3\} = 0.45$.

Preferences over Lotteries

- ***Example 2*** (violations of IA, a numerical example):
  - Therefore,

$$\max\{0.4, 0.5, 0.1\} = 0.5 > 0.45 = \max\{0.45, 0.25, 0.3\}$$

and thus $L = \frac{1}{2}L + \frac{1}{2}L \succ \frac{1}{2}L' + \frac{1}{2}L.$

  - This violates the IA, which requires

$$\frac{1}{2}L + \frac{1}{2}L \sim \frac{1}{2}L' + \frac{1}{2}L$$

Preferences over Lotteries

- ***Example 2*** (violations of IA, a numerical example):
  - The same applies to the smallest support, where $\text{supp}(L) = 3 > 2 = \text{supp}(L')$
    - since lottery L assigns positive probabilities on three outcomes while lottery L' only assigns positive probabilities on two outcomes.
  - Therefore, we obtain that $L' \succ L$, which according to the IA should yield

$$\frac{1}{2}L' + \frac{1}{2}L \succ \frac{1}{2}L + \frac{1}{2}L.$$

Preferences over Lotteries

- ***Example 2*** (violations of IA, a numerical example):
  - However, compound lottery $\frac{1}{2}L' + \frac{1}{2}L = (0.45, 0.25, 0.3)$ assigns positive probabilities on all three outcomes.
    - Thus, its support is $\text{supp}\left(\frac{1}{2}L' + \frac{1}{2}L\right) = 3$.
  - This means that both compound lotteries, $\frac{1}{2}L' + \frac{1}{2}L$ and $\frac{1}{2}L + \frac{1}{2}L$, have the same support (3), so the individual is indifferent between them:

$$\frac{1}{2}L' + \frac{1}{2}L \sim \frac{1}{2}L + \frac{1}{2}L$$

  - This is a violation of IA: we started with a strict preference in simple lotteries, $L' \succ L$, but ended with the above indifference in compound lotteries.

Preferences over Lotteries

- ***Example 3*** (violations of IA, "worst case scenario"):
  - Consider $L \succ L'$.
  - Then, the compound lottery $\frac{1}{2}L + \frac{1}{2}L$ does *not* need to be preferred to $\frac{1}{2}L' + \frac{1}{2}L$.
  - *Example*:
    - Consider the simple lotteries $L = (1,3)$ and $L' = (10,0)$, with probabilities (p_1, p_2) and (p_1', p_2'), respectively.
    - This implies

$$\min\{v(z): p(z) > 0\} = 1 \text{ for lottery } L$$
$$\min\{v(z): p'(z) > 0\} = 0 \text{ for lottery } L'$$

    - Hence, $L \succ L'$.

Preferences over Lotteries

- ***Example 3*** (violations of IA, "worst case scenario"):
  - *Example* (continued):
    - However, the compound lottery $\frac{1}{2}L + \frac{1}{2}L'$ is $\left(\frac{11}{2}, \frac{3}{2}\right)$, whose worst possible outcome is $\frac{3}{2}$, which is preferred to that of $\frac{1}{2}L + \frac{1}{2}L$, which is 1.
    - Hence, despite $L \succ L'$ over simple lotteries,

$$L = \frac{1}{2}L + \frac{1}{2}L \prec \frac{1}{2}L + \frac{1}{2}L',$$

which violates the IA.

Preferences over Lotteries

- ***Example 4*** (satisfying IA, "Lexicographic Preference"):
  - Consider lottery $L \succ L'$ where $p_k = p'_k$ for all $k < n$ but $p_n > p'_n$. Then, the Independence Axiom implies that $L \succ \alpha L' + (1-\alpha)L$ for all values of $\alpha \in (0,1)$.
  - Suppose to the contrary $L \prec \alpha L' + (1-\alpha)L$, then there exists $k \in \{1, \dots, n\}$ such that $p_k = \alpha p'_k + (1-\alpha)p_k$ for all $k < n$ but $p_n < \alpha p'_n + (1-\alpha)p_n$.
  - While the former is rearranged to $\alpha p_k = \alpha p'_k$, the latter yields $\alpha p_n < \alpha p'_n$, which simplifies to $p_n < p'_n$
    - Therefore, a contradiction results, and the Lexicographic Preference satisfies the Independence Axiom (QED).

Expected Utility Theory

Expected Utility Theory

- The utility function $U: \mathcal{L} \to \mathbb{R}$ has the ***expected utility*** (EU) form if there is an assignment of numbers $(u_1, u_2, \ldots, u_N)$ to the N possible outcomes such that, for every simple lottery $L = (p_1, p_2, \ldots, p_N) \in \mathcal{L}$ we have

$$U(L) = p_1 u_1 + \cdots + p_N u_N$$

  - A utility function with the EU form is also referred to as a **von-Neumann-Morgenstern** (vNM) expected utility function.
  - Note that this function is *linear* in the probabilities.

Expected Utility Theory

- Hence, a utility function $U: \mathcal{L} \to \mathbb{R}$ has the expected utility form if and only if it is *linear* in the probabilities, i.e.,

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k \cdot U(L_k)$$

  for any K lotteries $L_k \in \mathcal{L}$, $k = 1,2, \dots, K$, and probabilities $(\alpha_1, \alpha_2, \dots, \alpha_K) \geq 0$, where $\sum_{k=1}^{K} \alpha_k = 1$.

- *Intuition*: the utility of the expected value of the K lotteries, $U\left(\sum_{k=1}^{K} \alpha_k L_k\right)$, coincides with the expected utility of the K lotteries, $\sum_{k=1}^{K} \alpha_k U(L_k)$.

Expected Utility Theory

- Note that the utility of the expected value of playing the K lotteries is

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_n u_n \cdot \left(\sum_k \alpha_k \, p_n^k\right)$$

where $\sum_k \alpha_k \, p_n^k$ is the total joint probability of outcome n occurring.

Expected Utility Theory

- Note that the expected utility from playing the K lotteries is

$$\sum_{k=1}^{K} \alpha_k \cdot U(L_k) = \sum_k \alpha_k \cdot \left(\sum_n u_n \, p_n^k \right)$$

where $\sum_n u_n \, p_n^k$ is the expected utility from playing a given lottery k.

Expected Utility Theory

- The EU property is a *cardinal property*:
  - Not only rank matters, the particular number resulting form $U: \mathcal{L} \to \mathbb{R}$ also matters.
- Hence, the EU form is preserved only under increasing linear transformations (a.k.a. affine transformations).
  - Hence, the expected utility function $\widetilde{U}: \mathcal{L} \to \mathbb{R}$ is another vNM utility function if and only if

$$\widetilde{U}(L) = \beta U(L) + \gamma$$

for every $L \in \mathcal{L}$, where $\beta > 0$.

Expected Utility Theory: Representability

- Suppose that the preference relation $\succsim$ satisfies rationality, continuity and independence. Then, $\succsim$ admits a utility representation of the EU form.
- That is, we can assign a number u_n to every outcome $n = 1,2, \dots, N$ in such a manner that for any two lotteries

$$L = (p_1, p_2, \dots, p_N) \text{ and } L' = (p_1', p_2' \dots, p_N')$$

we have $L \succsim L'$ if and only if $U(L) \geq U(L')$, or

$$\sum_{n=1}^{N} p_n u_n \geq \sum_{n=1}^{N} p_n' u_n$$

- *Notation*: u_n is the utility that the decision maker assigns to outcome n. It is usually referred as the Bernoulli utility function.

Expected Utility Theory: Indifference Curves

- Let us next analyze the effect of the IA on indifference curves over lotteries.

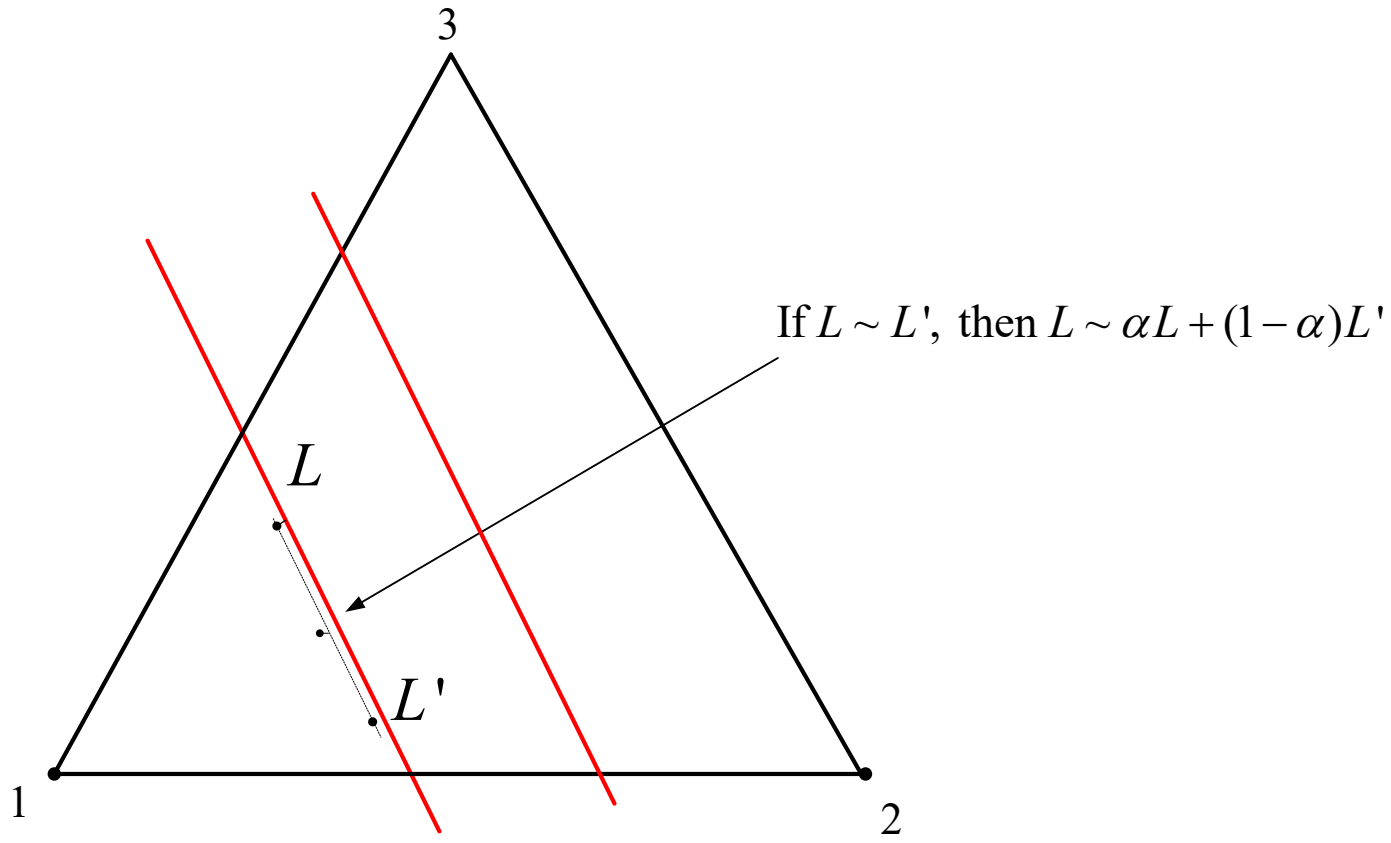
1) ***Indifference curves must be straight lines***:

   Recall that from the IA, $L \sim L'$ implies that

$$\underbrace{\alpha L + (1-\alpha)L}_{L} \sim \alpha L + (1-\alpha)L'$$

   for all $\alpha \in (0,1)$.

Expected Utility Theory: Indifference Curves



Straight indifference curves

Expected Utility Theory: Indifference Curves

- Why indifference curves must be straight?
  - We have that $L \sim L'$, but $L \prec \frac{1}{2}L + \frac{1}{2}L'$. This is equivalent to

$$\frac{1}{2}L + \frac{1}{2}L \prec \frac{1}{2}L + \frac{1}{2}L'$$

  - But from the IA we must have

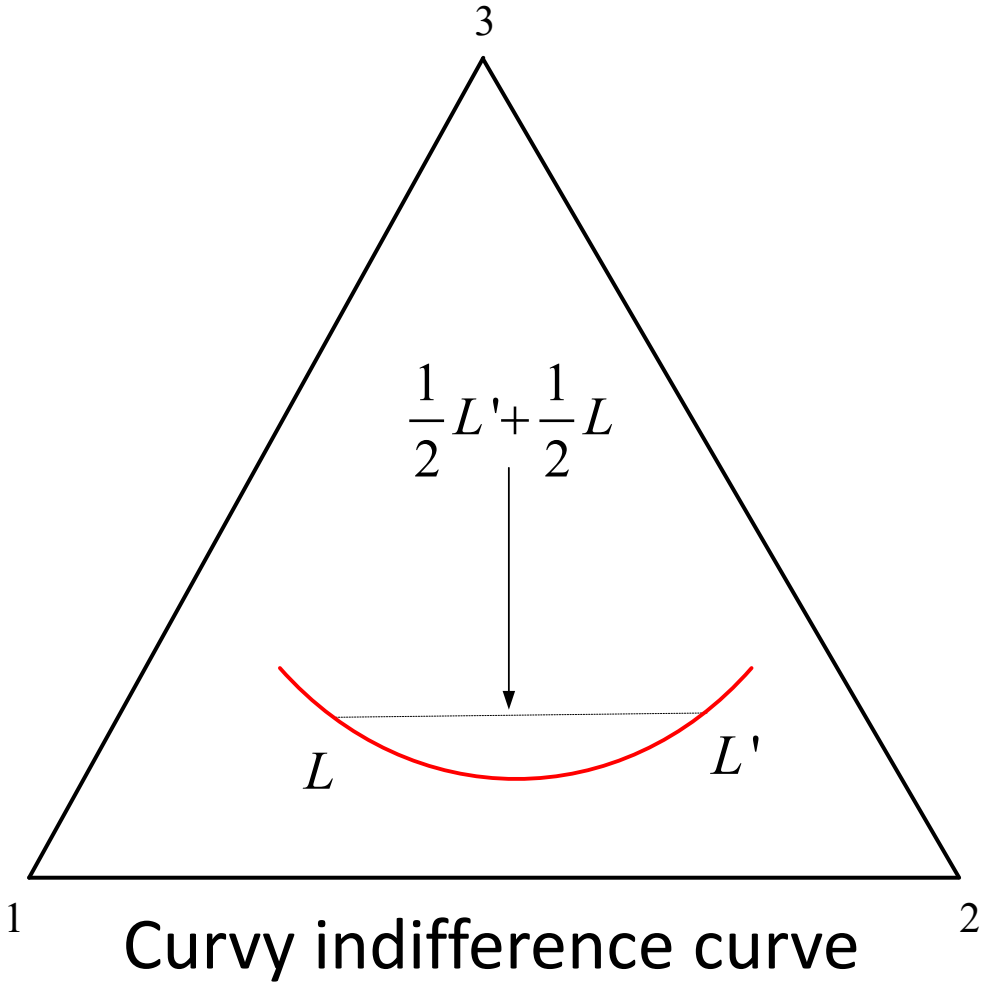
$$\frac{1}{2}L + \frac{1}{2}L \sim \frac{1}{2}L + \frac{1}{2}L'$$

  - Hence, indifference curves must be straight lines in order to satisfy the IA.

Expected Utility Theory: Indifference Curves

- Curvy indifference curves over lotteries are incompatible with the IA
  - The compound lottery $\frac{1}{2}L + \frac{1}{2}L'$ would not lie on the same indifference curve as lottery L and L'.
  - Hence, the decision maker is *not* indifferent between the compound lotteries $\frac{1}{2}L + \frac{1}{2}L$ and $\frac{1}{2}L + \frac{1}{2}L'$.

Expected Utility Theory: Indifference Curves



Curvy indifference curve

Expected Utility Theory: Indifference Curves

2) Indifference curves must be parallel lines:

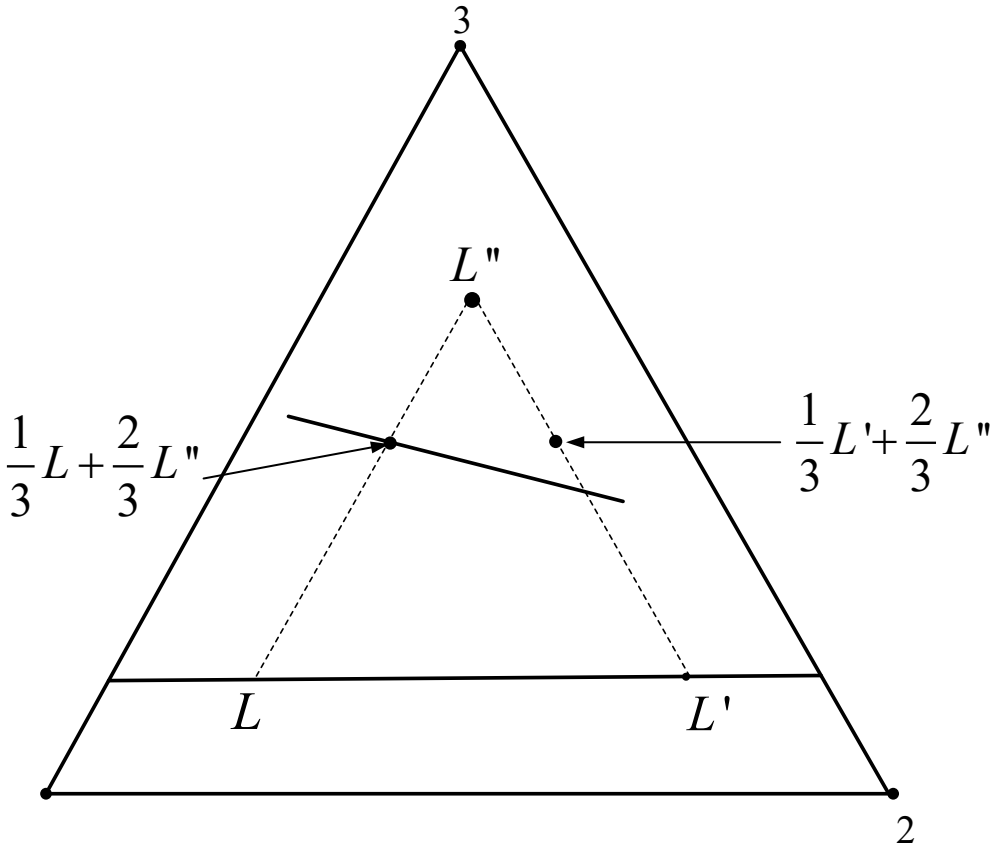
If we have that $L \sim L'$, then by the IA

$$\frac{1}{3}L + \frac{2}{3}L'' \sim \frac{1}{3}L' + \frac{2}{3}L''$$

- That is, the convex combination of L and L' with a third lottery L'' should also lie on the same indifference curve.
- This implies that the indifference curves must be parallel lines in order to satisfy the IA.

Expected Utility Theory: Indifference Curves

- Nonparallel indifference curves are incompatible with the IA.
  - If compound lotteries $\frac{1}{3}L + \frac{2}{3}L''$ and $\frac{1}{3}L' + \frac{2}{3}L''$ lie on different (nonparallel) indifference curves, then $\frac{1}{3}L + \frac{2}{3}L'' \prec \frac{1}{3}L' + \frac{2}{3}L''$ which violates the IA.



Expected Utility Theory:

Violations of the IA:

- Despite the intuitive appeal of the IA, we encounter several settings in which decision makers violate it.
- We next elaborate on these violations.

Expected Utility Theory: Violations of the IA

- ***Allais' paradox***:
  - Consider a lottery over three possible monetary outcomes:

1st prize	2nd prize	3rd prize
$2.5mln	$500,000	$0

  - First choice set:

$$L_1 = (0,1,0) \text{ and } L_1' = (\frac{10}{100}, \frac{89}{100}, \frac{1}{100})$$

  - Second choice set:

$$L_2 = (0, \frac{11}{100}, \frac{89}{100}) \text{ and } L_2' = (\frac{10}{100}, 0, \frac{90}{100})$$

Expected Utility Theory: Violations of the IA

- About 50% students surveyed expressed $L_1 \succ L_1'$ and $L_2' \succ L_2$.
- These choices violate the IA.
- To see this, consider that the decision maker's preferences over lotteries have a EU form. Hence, $L_1 \succ L_1'$ implies

$$u_5 > \frac{10}{100}u_{25} + \frac{89}{100}u_5 + \frac{1}{100}u_0$$

- By the IA, we can add $\frac{89}{100}u_0 - \frac{89}{100}u_5$ on both sides

$$u_5 + \left(\frac{89}{100}u_0 - \frac{89}{100}u_5\right) >$$

$$\frac{10}{100}u_{25} + \frac{89}{100}u_5 + \frac{1}{100}u_0 + \left(\frac{89}{100}u_0 - \frac{89}{100}u_5\right)$$

Expected Utility Theory: Violations of the IA

– Simplifying

$$\underbrace{\frac{11}{100}u_5 + \frac{89}{100}u_0}_{\text{EU of } L_2} > \underbrace{\frac{10}{100}u_{25} + \frac{90}{100}u_0}_{\text{EU of } L_2'}$$

which implies $L_2 \succ L_2'$.

– Did your own choices violate the IA?

Expected Utility Theory: Violations of the IA

- Reactions to the Allais' Paradox:
  - *Approximation to rationality*: people adapt their choices as they go.
  - *Little economic significance*: the lotteries involve probabilities that are close to zero and one.
  - *Regret theory*: the reason why $L_1 \succ L_1'$ is because I didn't want to regret a sure win of $500,000.
  - Give up the IA in favor of a weaker assumption: the *betweenness axiom*.

Expected Utility Theory: Violations of the IA

- ***Machina's paradox***:
  - Consider that

    Trip to Barcelona $\succ$ Movie about Barcelona $\succ$ Home

  - Now, consider the following two lotteries

$$L_1 = (\frac{99}{100}, \frac{1}{100}, 0) \text{ and } L_2 = (0, \frac{99}{100}, \frac{1}{100})$$

  - From the previous preferences over certain outcomes, how can we know this individual's preferences over lotteries?
    - Using the IA.

Expected Utility Theory: Violations of the IA

- From $T \succ M$ and the IA, we can construct the compound lotteries

$$\frac{99}{100}T + \frac{1}{100}M \succ \frac{99}{100}M + \frac{1}{100}M$$

- From $M \succ H$ and the IA, we have

$$\frac{99}{100}M + \frac{1}{100}M \succ \frac{99}{100}M + \frac{1}{100}H$$

- By transitivity,

$$\underbrace{\frac{99}{100}T + \frac{1}{100}M}_{L_1} \succ \underbrace{\frac{99}{100}M + \frac{1}{100}H}_{L_2}$$

- Hence, $L_1 \succ L_2$.

Expected Utility Theory: Violations of the IA

- Therefore, for preferences over lotteries to be consistent with the IA, we need $L_1 \succ L_2$.
- Many subjects in experimental settings would rather prefer L_2, thus violating the IA.
- Many people explain choosing L_2 over L_1 on grounds of the disappointment they would experience in the case of losing the trip to Barcelona, and having to watch a movie instead.
  - Similar to regret theory.

Expected Utility Theory: Violations of the IA

- ***Dutch books***:
  - In the above two anomalies, actual behavior is inconsistent with the IA.
  - Can we then rely on the IA?
  - What would happen to individuals whose behavior violates the IA?
  - They would be weeded out of the market because they would be open to the acceptance of so-called *Dutch books*, leading them to a sure loss of money.

Expected Utility Theory: Violations of the IA

- Consider that $L \succ L'$. By the IA, we should have

$$\underbrace{\alpha L + (1-\alpha)L}_{L} \succ \alpha L + (1-\alpha)L'$$

- If, instead, the IA is violated, then

$$L \prec \alpha L + (1-\alpha)L'$$

- Consider an individual with these preferences, who initially owns lottery L.
- If we offer him the compound lottery $\alpha L + (1-\alpha)L'$, for a small fee $\$x$, he would accept such a trade.

Expected Utility Theory: Violations of the IA

- After the realization stage, he owns either L or L'
  - If L', then we offer L again for \$$y$.
  - If L, then we offer $\alpha L + (1 - \alpha)L'$ for \$$y$.
- Either way, he is at the same position as he started (owning L or $\alpha L + (1 - \alpha)L'$), but having lost \$$x$ + \$$y$ in the process.
- We can repeat this process ad infinitum.
- Hence, individuals with preferences that violate the IA would be exploited by microeconomists (they would be a "money pump").

Expected Utility Theory: Violations of the IA

- Further reading:
  - "Developments in non-expected utility theory: The hunt for a descriptive theory of choice under risk" (2000) by Chris Starmer, *Journal of Economic Literature*, vol. 38(2)
  - *Choices, Values and Frames* (2000) by Nobel prize winners Daniel Kahneman and Amos Tversky, Cambridge University Press.
  - *Theory of Decision under Uncertainty* (2009) by Itzhak Gilboa, Cambridge University Press.

Theories Modifying Expected Utility Theory

1) Weighted utility theory:

- The payoff function from playing lottery L is

$$V(L) = \sum_{x \in C} w_i \cdot u(x_i)$$

where

$$w_i = \frac{g(x_i)p(x_i)}{\sum_{x \in C} g(x)p(x)} \text{ and } g: C \to \mathbb{R}$$

- The utility of outcome $x_i \in C$ is weighted according to:
  a) its probability $p(x_i)$
  b) outcome x_i itself through function $g: C \to \mathbb{R}$

Theories Modifying Expected Utility Theory

- ***Example***: Consider a lottery with two payoffs x_1 and x_2 with probabilities p and $1-p$. Then, the weighted utility is

$$V(L) = w_1 u(x_1) + w_2 u(x_2)$$
$$= \frac{g(x_1)p}{g(x_1)p + g(x_2)(1-p)} u(x_1)$$
$$+ \frac{g(x_2)(1-p)}{g(x_1)p + g(x_2)(1-p)} u(x_2)$$

If $g(x_i) = g(x_j)$ for any $x_i \neq x_j$, then

$$V(L) = pu(x_1) + (1-p)u(x_2)$$

which is a standard expected utility function.

Theories Modifying Expected Utility Theory

- The weights always add up to one,

$$\sum_i w_i = 1$$

- If $g(x_i) = g$ for all outcomes, then

$$w_i = \frac{g\, p_i}{g \sum_j p_j} = p_i$$

- If g is larger for bad outcomes, the individual inflates their weight: ***pessimism***.

Theories Modifying Expected Utility Theory

- ***Example***: \$100 with probability p, \$0 otherwise.
  - $u(x) = x$ and $g(0) = 2,\ g(100) = 1$: the bad outcome counts twice.

$$w(100) = \frac{1 \cdot p}{1 \cdot p + 2(1-p)} = \frac{p}{2-p}$$

Theories Modifying Expected Utility Theory

- Two values of p, step by step:
  - $p = 0.5$:

$$w(100) = \frac{0.5}{2 - 0.5} = \frac{1}{3},$$

$$\text{so} \quad V(L) = \frac{100}{3} \approx 33.33 < 50$$

  - $p = 0.9$:

$$w(100) = \frac{0.9}{1.1} \approx 0.818, \qquad \text{so} \quad V(L) \approx 81.82 < 90$$

- Since u is linear, $V(L)$ is also the certainty equivalent.

Theories Modifying Expected Utility Theory

- The value, $V(L) = 100\frac{p}{2-p}$, is below the expected value at every p:

p	**w(100)**	$V(L)$	**Expected value**
0.10	0.053	5.26	10
0.25	0.143	14.29	25
0.50	0.333	33.33	50
0.75	0.600	60.00	75
0.90	0.818	81.82	90

Theories Modifying Expected Utility Theory

- Why is $V(L)$ below the expected value at every $p \in (0,1)$?

$$\frac{p}{2-p} < p \quad \text{if and only if} \quad 1 < 2 - p$$

  - and $p < 1$ holds for every $p \in (0,1)$.
- The individual has linear utility, but the extra weight on the bad outcome works like risk aversion.

Theories Modifying Expected Utility Theory

- The *weighted utility theory* relies on the same axioms as *expected utility theory*, except for the IA, which is relaxed to the "weak independence axiom."
  - ***Weak independence axiom***: if we have that $L_1 \sim L_2$, we can find a pair of probabilities α and α' such that
    $$\alpha L_1 + (1-\alpha)L_3 \sim \alpha' L_2 + (1-\alpha')L_3$$
  - The IA becomes a special case if $\alpha = \alpha'$.

Theories Modifying Expected Utility Theory

- ***Weak independence, precisely***:
  - If $L_1 \sim L_2$, then for every $\alpha \in (0,1)$ there is an α' such that

$$\alpha L_1 + (1-\alpha)L_3 \sim \alpha' L_2 + (1-\alpha')L_3$$

  - This must hold for every lottery L_3.
  - The IA is the special case $\alpha' = \alpha$.

Theories Modifying Expected Utility Theory

2) Rank dependent utility theory:

- First, rank the outcomes $x_1, x_2, \ldots, x_n$ from worst (x_1) to best (x_n)
- Second, apply a probability weighting function

$$w_i = \pi(p_i + \cdots + p_n) - \pi(p_{i+1} + \cdots + p_n)$$
$$w_n = \pi(p_n)$$

  where $\pi(\cdot)$ is a non-decreasing *transformation function*, with $\pi(0) = 0$ and $\pi(1) = 1$.

- Finally, a rank-dependent utility is

$$V(L) = \sum_{x \in C} w_i \cdot u(x_i)$$

Theories Modifying Expected Utility Theory

- For a lottery with two outcomes, x_1 and x_2 where $x_2 > x_1$, the rank-dependent utility is

$$V(L) = w(p)u(x_1) + \big(1 - w(p)\big)u(x_2)$$

  where p is the probability of outcome x_1.

- This model allows for *different weight* to be attached to each outcome, as opposed to expected utility theory models in which the *same utility weight* is attached to all outcomes.
- Here, $w(p) \equiv 1 - \pi(1 - p)$ is the decision weight on the worse outcome x_1.

Theories Modifying Expected Utility Theory

- Where does the two-outcome formula come from? With $x_1 < x_2$, outcome x_2 has probability $1 - p$:

$$w_2 = \pi(1 - p)$$
$$w_1 = \pi(1) - \pi(1 - p) = 1 - \pi(1 - p)$$

- Writing $w(p) \equiv 1 - \pi(1 - p)$ gives the formula above.

Theories Modifying Expected Utility Theory

- The weights add up to one: the sum telescopes, $w_1 + w_2 + \cdots + w_n = \pi(p_1 + \cdots + p_n) = \pi(1) = 1$
- If $\pi(p) = p$, then $w_i = p_i$, and rank-dependent utility coincides with expected utility.

Theories Modifying Expected Utility Theory

- ***Example***: \$0, \$50, or \$100, each with probability $\frac{1}{3}$.
  - $u(x) = x$ and the pessimistic $\pi(p) = p^2$.
  - Probability of at least \$100: $\frac{1}{3}$. Probability of at least \$50: $\frac{2}{3}$.

Theories Modifying Expected Utility Theory

- Step 1: apply $\pi(p) = p^2$ to the probability of receiving at least each outcome:

$$\pi(\frac{1}{3}) = (\frac{1}{3})^2 = \frac{1}{9}, \qquad \pi(\frac{2}{3}) = (\frac{2}{3})^2 = \frac{4}{9}$$

- Step 2: each decision weight is the difference between consecutive values (next slide), and the weights add up to one,

$$\frac{5}{9} + \frac{3}{9} + \frac{1}{9} = 1$$

Theories Modifying Expected Utility Theory

- Decision weights:

$$w_3 = \pi(\frac{1}{3}) = \frac{1}{9}$$

$$w_2 = \pi(\frac{2}{3}) - \pi(\frac{1}{3}) = \frac{3}{9}$$

$$w_1 = 1 - \pi(\frac{2}{3}) = \frac{5}{9}$$

Theories Modifying Expected Utility Theory

- Hence

$$V(L) = \frac{5}{9}(0) + \frac{3}{9}(50) + \frac{1}{9}(100) = \frac{250}{9} \approx 27.8$$

- The worst outcome gets the largest weight, so the value is well below the expected value of 50.

Theories Modifying Expected Utility Theory

- ***Other transformation functions***:
  - The inverse-S (Prelec) function gives both extremes more weight than $\frac{1}{3}$.

$\pi(p)$	w_1 **($0)**	w_2 **($50)**	w_3 **($100)**	$V(L)$
p	0.333	0.333	0.333	50.0
p^2	0.556	0.333	0.111	27.8
$\sqrt{p}$	0.184	0.239	0.577	69.7
$e^{-(-\ln p)^{\frac{1}{2}}}$	0.471	0.178	0.351	44.0

Theories Modifying Expected Utility Theory

- Each row only needs $\pi(\frac{1}{3})$ and $\pi(\frac{2}{3})$.
- ***Optimistic***, $\pi(p) = \sqrt{p}$:

$$\pi(\frac{1}{3}) = \sqrt{\frac{1}{3}} \approx 0.577, \qquad \pi(\frac{2}{3}) = \sqrt{\frac{2}{3}} \approx 0.816$$

$$w_3 \approx 0.577, \qquad w_2 \approx 0.816 - 0.577 = 0.239$$

$$w_1 \approx 1 - 0.816 = 0.184$$

$$V(L) \approx 0.184(0) + 0.239(50) + 0.577(100) \approx 69.7$$

Theories Modifying Expected Utility Theory

- ***Prelec***, $\alpha = \frac{1}{2}$: first,

$$-\ln \frac{1}{3} = \ln 3 \approx 1.099, \qquad -\ln \frac{2}{3} = \ln 1.5 \approx 0.405$$

- Then

$$\pi(\frac{1}{3}) = e^{-\sqrt{\ln 3}} \approx e^{-1.048} \approx 0.351$$

$$\pi(\frac{2}{3}) = e^{-\sqrt{\ln 1.5}} \approx e^{-0.637} \approx 0.529$$

Theories Modifying Expected Utility Theory

- Hence

$$w_3 \approx 0.351, \qquad w_2 \approx 0.529 - 0.351 = 0.178$$

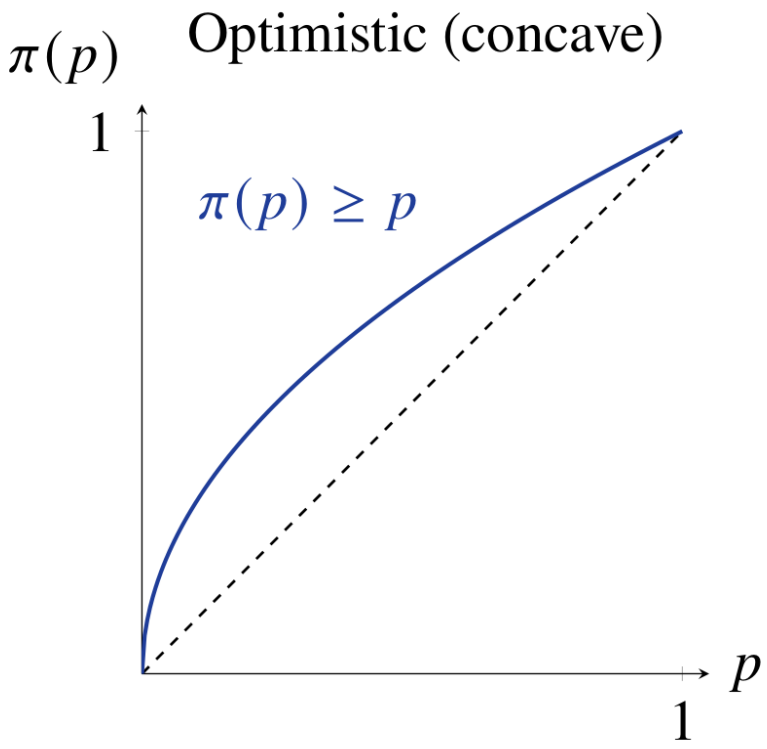
$$w_1 \approx 1 - 0.529 = 0.471$$

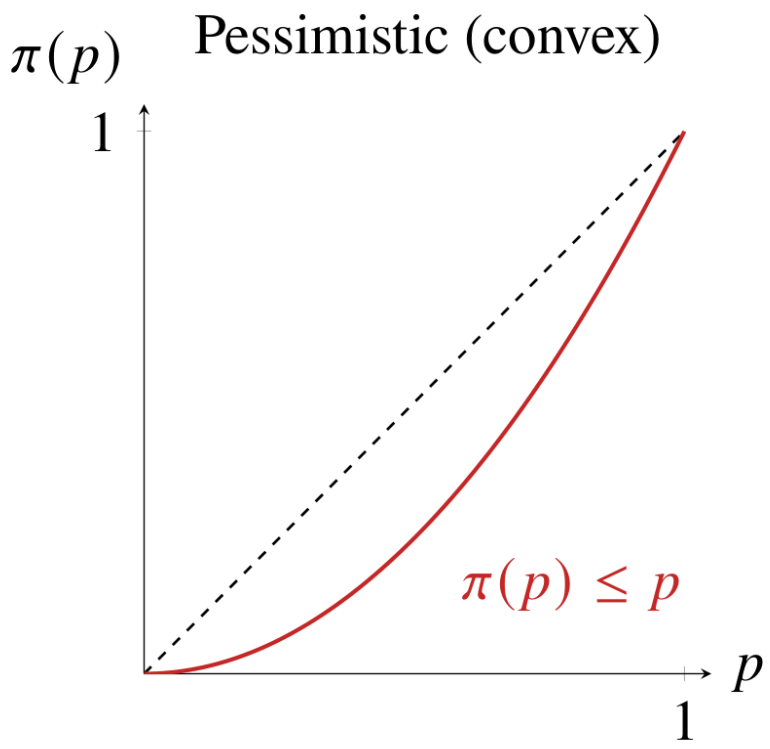
$$V(L) \approx 0.471(0) + 0.178(50) + 0.351(100) \approx 44.0$$

- Both extreme outcomes receive more weight than their probability, $\frac{1}{3}$; the middle outcome receives less.

Theories Modifying Expected Utility Theory

– Transformation function $\pi(\cdot)$



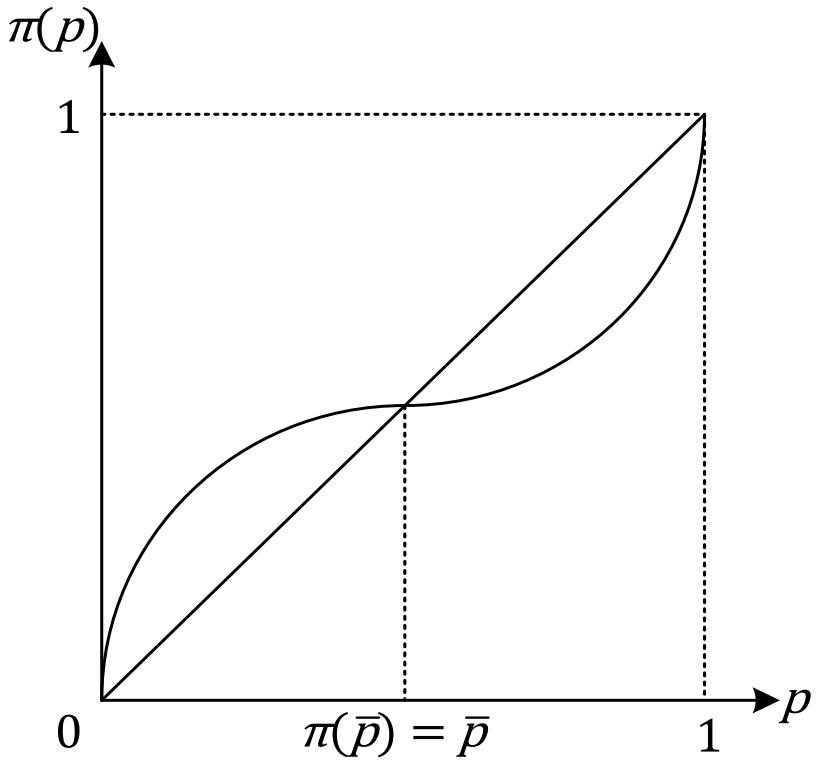


Theories Modifying Expected Utility Theory

- Since π is applied to the probability of receiving ***good*** outcomes:
  - ***Pessimism***: a convex π, with $\pi(p) \leq p$, shrinks the weight of good outcomes and shifts it to the worst ones.
  - ***Optimism***: a concave π, with $\pi(p) \geq p$, inflates the weight of good outcomes.

Theories Modifying Expected Utility Theory

- Empirical evidence suggests an inverse-S transformation function (concave, then convex).
- *Intuition*: individuals overweight small probabilities (i.e., $p < \bar{p}$), and underweight large probabilities ($p > \bar{p}$).



Theories Modifying Expected Utility Theory

- ***Prelec (1998) function***:

$$\pi(p) = e^{-(-\ln p)^{\alpha}}, \qquad \alpha \in (0,1)$$

  - It crosses the 45-degree line where $(-\ln p)^{\alpha} = -\ln p$, that is, at $\bar{p} = \frac{1}{e} \approx 0.368$.
  - Small probabilities are overweighted, and large ones underweighted.

Theories Modifying Expected Utility Theory

- The minus sign matters: since $-\ln p \geq 0$, the exponent is not positive, so $\pi(p) \in [0,1]$.
- Fixed point, step by step:
  - Take logs of $\pi(p) = p$:

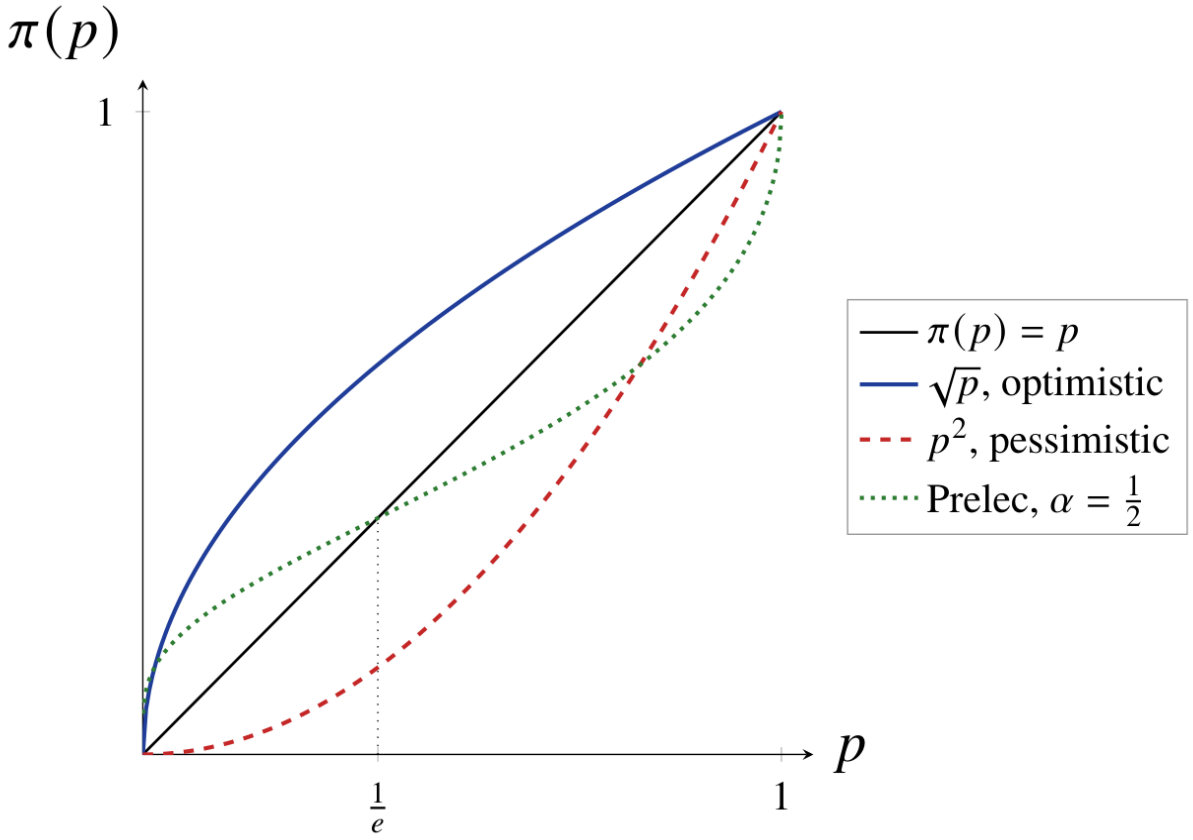
$$-(-\ln p)^{\alpha} = \ln p$$

$$z^{\alpha} = z, \qquad \text{where} \quad z \equiv -\ln p > 0$$

$$z^{\alpha-1} = 1, \qquad \text{so} \quad z = 1 \quad \text{and} \quad \bar{p} = \frac{1}{e} \approx 0.368$$

Theories Modifying Expected Utility Theory

- Three transformation functions:



Theories Modifying Expected Utility Theory

- ***Lottery tickets and insurance***: \$100 with probability p, \$0 otherwise, and $u(x) = x$, so $V(L) = 100\pi(p)$.
  - Inverse-S weighting values a long shot above its expected value, and a likely gain below it.

Theories Modifying Expected Utility Theory

- Step 1: the worse outcome, \$0, has probability $1 - p$, so the weight on \$100 is
$$\pi(1 - (1 - p)) = \pi(p)$$
- Step 2: the value of the lottery, which is also its certainty equivalent since u is linear, is
$$V(L) = [1 - \pi(p)] \cdot 0 + \pi(p) \cdot 100 = 100\pi(p)$$

Theories Modifying Expected Utility Theory

- Step 3: with Prelec ($\alpha = \frac{1}{2}$) and $p = 0.01$,

$$-\ln 0.01 \approx 4.605,$$

$$\pi(0.01) = e^{-\sqrt{4.605}} \approx e^{-2.146} \approx 0.117$$

$$V(L) \approx 100(0.117) = 11.70$$

  - More than ten times the expected value, $1.

Theories Modifying Expected Utility Theory

- Certainty equivalents (Prelec with $\alpha = \frac{1}{2}$, TK with $\beta = 0.61$):

p	**Expected value**	p^2	$\sqrt{p}$	**Prelec**	**TK**
0.01	1.00	0.01	10.00	11.70	5.53
0.05	5.00	0.25	22.36	17.71	13.16
0.10	10.00	1.00	31.62	21.93	18.63
0.368	36.79	13.53	60.65	36.79	35.37
0.50	50.00	25.00	70.71	43.49	42.06
0.90	90.00	81.00	94.87	72.28	71.17
0.99	99.00	98.01	99.50	90.46	91.16

Theories Modifying Expected Utility Theory

- With an inverse-S function:
  - A long shot ($p = 0.01$) is valued well above its expected value: more than ten times with Prelec. This helps explain the demand for lottery tickets.
  - A likely gain ($p = 0.99$) is valued below its expected value.
- Applied to losses, overweighting small probabilities helps explain the demand for insurance against unlikely losses.

Theories Modifying Expected Utility Theory

- The *rank-dependent utility theory* relies on the same axioms as *expected utility theory*, except for the IA, which is replaced by *co-monotonic independence*.

Theories Modifying Expected Utility Theory

- Shifting probability from a worse outcome to a better one never lowers $V(L)$.
  - The weights come from the probabilities of receiving ***at least*** each outcome.
- Hence rank-dependent utility respects first-order stochastic dominance, an advantage over the original version of prospect theory.

Money Lotteries

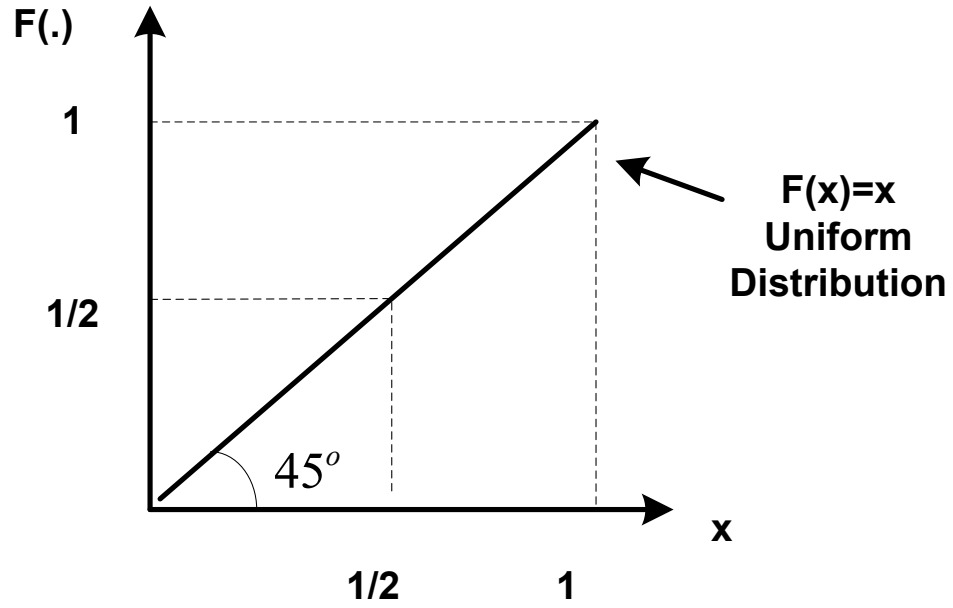
Money Lotteries

- We now restrict our attention to lotteries over *monetary* amounts, i.e., $C = \mathbb{R}$.
- Money is continuous variable, $x \in \mathbb{R}$, with cumulative distribution function (CDF)

$$F(x) = Prob\{y \leq x\} \text{ for all } y \in \mathbb{R}$$

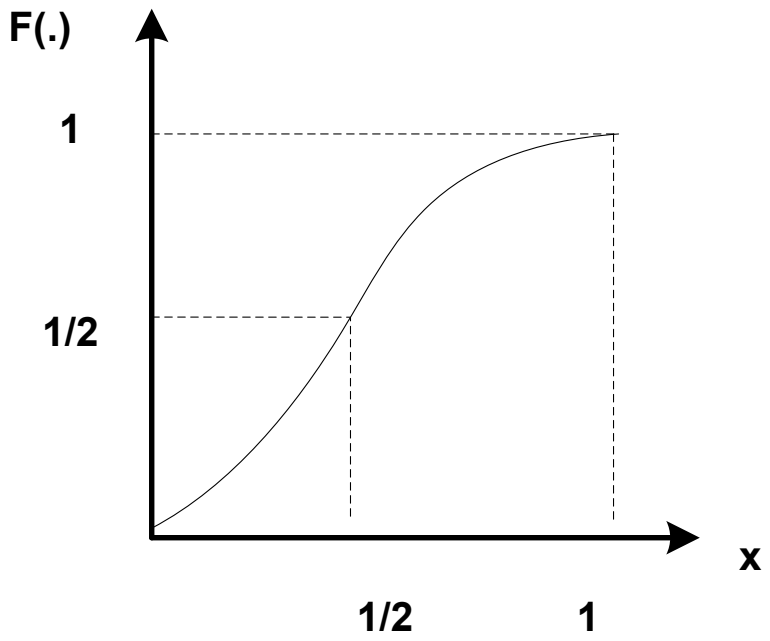
Money Lotteries

- A uniform, continuous CDF, $F(x) = x$
  - Same probability weight to every possible payoff



Money Lotteries

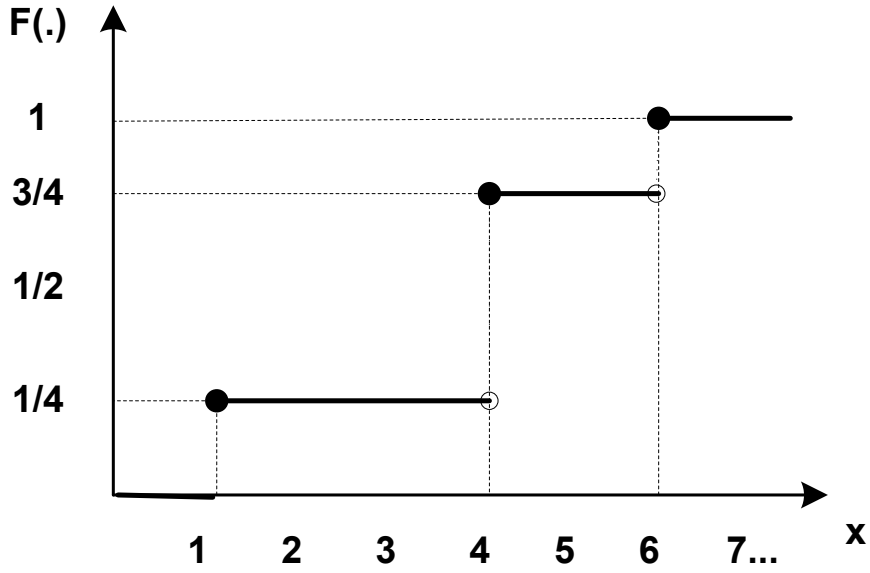
- A non-uniform, continuous CDF, $F(x)$



Money Lotteries

- A non-uniform, discrete CDF

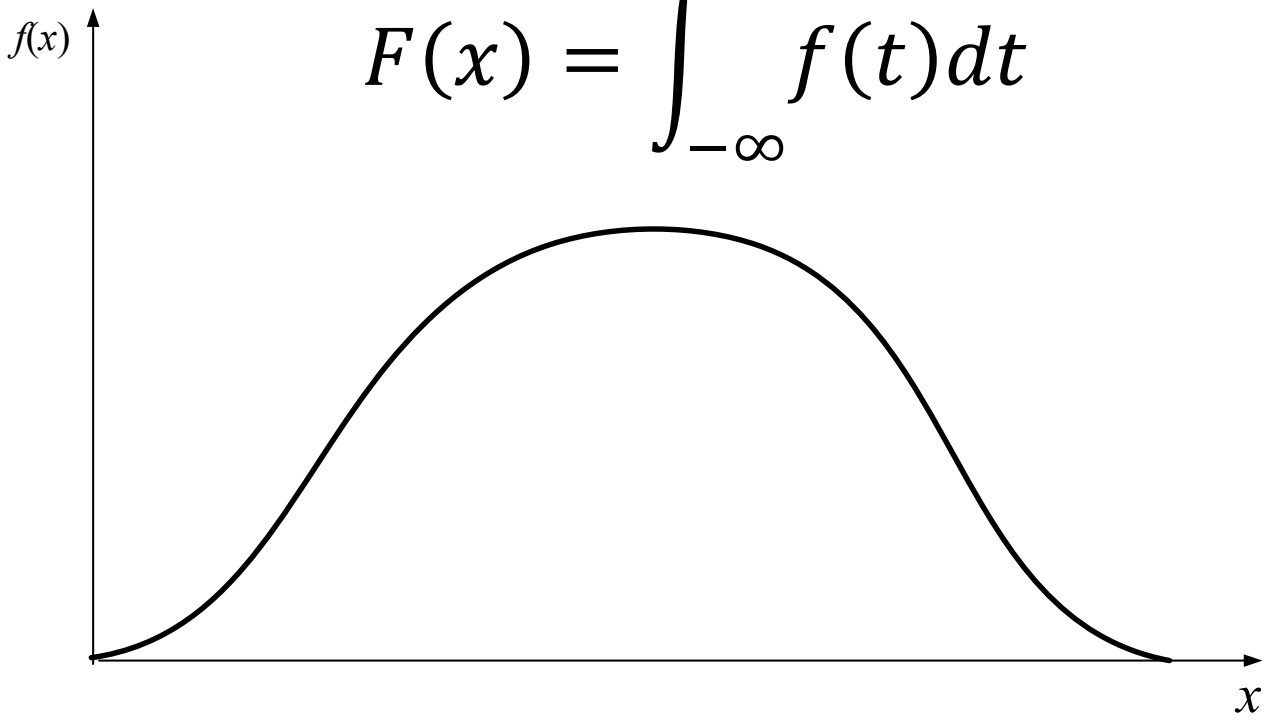
$$F(x) = \begin{cases} 0 \text{ if } x < 1 \\ \frac{1}{4} \text{ if } x \in [1,4) \\ \frac{3}{4} \text{ if } x \in [4,6) \\ 1 \text{ if } x \geq 6 \end{cases}$$



Money Lotteries

- If $f(x)$ is a density function associated with the *continuous* CDF $F(x)$, then

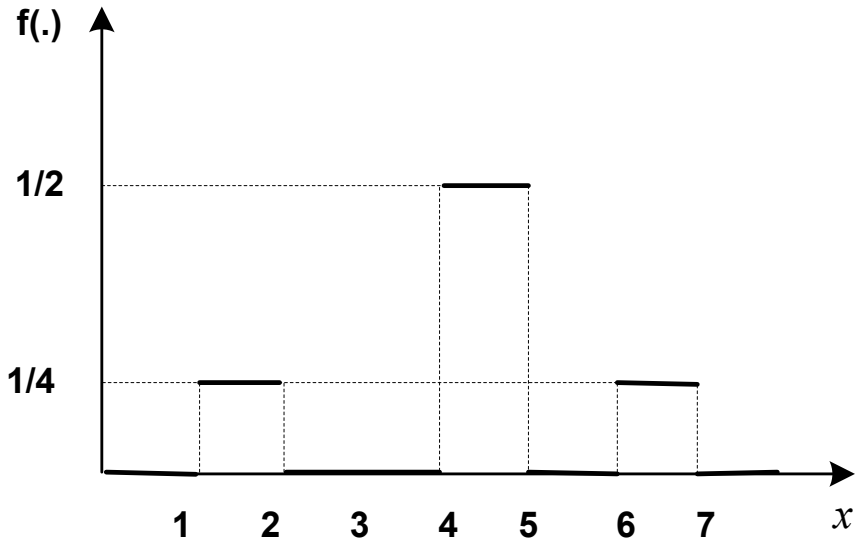
$$F(x) = \int_{-\infty}^{x} f(t)dt$$



Money Lotteries

- If $f(x)$ is a density function associated with the *discrete* CDF $F(x)$, then

$$F(x) = \sum_{t<x} f(t)$$



Money Lotteries

- We can represent *simple lotteries* by $F(x)$.
- *For compound lotteries*:
  - If the list of CDF's $F_1(x)$, $F_2(x)$, …, $F_K(x)$ represent K simple lotteries, each occurring with probability $\alpha_1, \alpha_2, \ldots, \alpha_K$, then the compound lottery can be represented as

$$F(x) = \sum_{k=1}^{K} \alpha_k F_k(x)$$

  - For simplicity, assume that CDF's are distributed over non-negative amounts of money.

Money Lotteries

– We can express EU as

$$EU(F) = \int u(x)f(x)dx \text{ or } \int u(x)dF(x)$$

where $u(x)$ is an assignment of utility value to every non-negative amount of money.

– If there is a density function $f(x)$ associated with the CDF $F(x)$, then we can use either of the expressions. If there is no, we can only use the latter.

– *Note*: we do not need to write down the limits of integration, since the integral is over the full range of possible realizations of x.

Money Lotteries

- $EU(F)$ is the mathematical expectation of the values of $u(x)$, over all possible values of x.
- $EU(F)$ is linear in the probabilities
  - In the discrete probability distribution,

$$EU(F) = p_1(u_1) + p_2(u_2) + \cdots$$

- The EU representation is sensitive not only to the *mean* of the distribution, but also to the *variance*, and *higher order moments* of the distribution of monetary payoffs.
  - Let us next analyze this property.

Money Lotteries

- ***Example***: Let us show that if $u(x) = \beta x^2 + \gamma x$, then EU is determined by the mean and the variance alone.
  - Indeed,

$$EU(x) = \int u(x)dF(x) = \int [\beta x^2 + \gamma x]dF(x)$$

$$= \beta \underbrace{\int x^2 dF(x)}_{E(x^2)} + \gamma \underbrace{\int x\, dF(x)}_{E(x)}$$

  - On the other hand, we know that

$$Var(x) = E(x^2) - \big(E(x)\big)^2 \Longrightarrow$$

$$E(x^2) = Var(x) + \big(E(x)\big)^2$$

Money Lotteries

- ***Example*** (continued):
  - Substituting $E(x^2)$ in $EU(x)$,

$$EU(x) = \underbrace{\beta Var(x) + \beta\big(E(x)\big)^2}_{\beta E(x^2)} + \gamma E(x)$$

  - Hence, the EU is determined by the mean and the variance alone.

Money Lotteries

- Recall that we refer to $u(x)$ as the Bernoulli utility function, while $EU(x)$ is the vNM function.
- We imposed few assumptions on $u(x)$:
  - Increasing in money and continuous
- We must impose an additional assumption:
  - $u(x)$ is bounded
  - Otherwise, we can end up in relatively absurd situations (*St. Petersburg-Menger paradox*).

Money Lotteries

- ***St. Petersburg-Menger paradox***:
  - Consider an unbounded Bernoulli utility function, $u(x)$. Then, we can always find an amount of money x_m such that $u(x_m) > 2^m$, for every integer m.
  - Consider a lottery in which we toss a coin repeatedly until the tail comes up. We give a monetary payoff of x_m if the tail is obtained at the m^{th} toss.
  - The probability that the tail comes up in the m^{th} toss is

$$\underbrace{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\ldots\cdot\frac{1}{2}}_{m \text{ times}} = \frac{1}{2^m}$$

Money Lotteries

- Then, the EU of this lottery is

$$EU(x) = \sum_{m=1}^{\infty} \frac{1}{2^m} u(x_m)$$

- But, because of $u(x_m) > 2^m$, we have that

$$EU(x) = \sum_{m=1}^{\infty} \frac{1}{2^m} u(x_m) \geq \sum_{m=1}^{\infty} \frac{1}{2^m} 2^m$$

$$= \sum_{m=1}^{\infty} 1 = +\infty$$

  which implies that this individual would be willing to pay infinite amounts of money to be able to play this lottery.

- Hence, we assume that the Bernoulli utility function is bounded.

Measuring Risk Preferences

Measuring Risk Preferences

- An individual exhibits ***risk aversion*** if

$$\int u(x)dF(x) \leq u\left(\int xdF(x)\right)$$

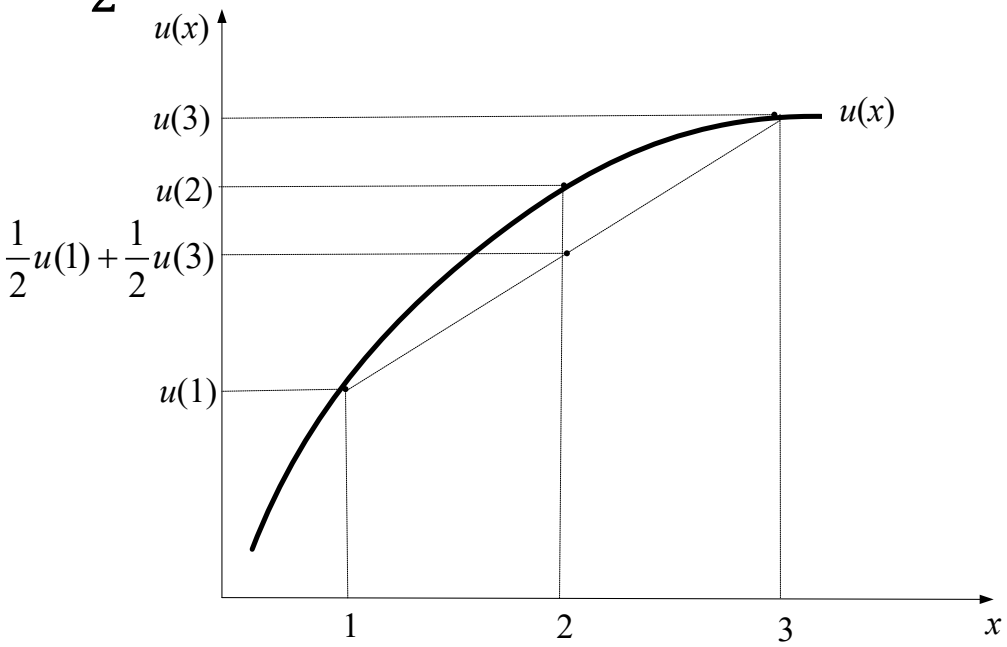
  for any lottery $F(\cdot)$
- *Intuition*:
  - The utility of receiving the expected monetary value of playing the lottery (right-hand side) is higher than…
  - The expected utility from playing the lottery (left-hand side).
- If this relationship happens with
  - *a)* $=$, we denote this individual as **risk neutral**
  - *b)* $<$, we denote him as **risk averter**
  - *c)* $>$, we denote him as **risk lover**.

Measuring Risk Preferences

- Graphical illustration:
  - Consider a lottery with two equally likely outcomes, \$1 and \$3, with associated utilities of $u(1)$ and $u(3)$, respectively.
  - *Expected value of the lottery* is $EV = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3 = 2$, with associated utility of $u(2)$.
  - *Expected utility of the lottery* is $\frac{1}{2}u(1) + \frac{1}{2}u(3)$.

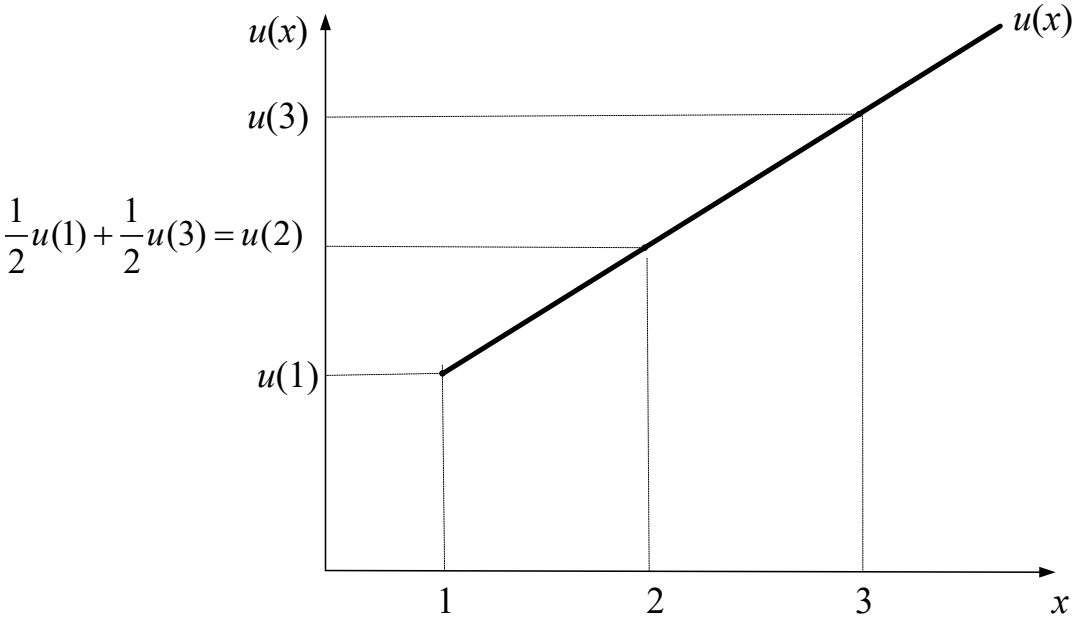
Measuring Risk Preferences

- *Risk averse individual*
  - Utility from the expected value of the lottery, $u(2)$, is **higher** than the EU from playing the lottery, $\frac{1}{2}u(1) + \frac{1}{2}u(3)$.



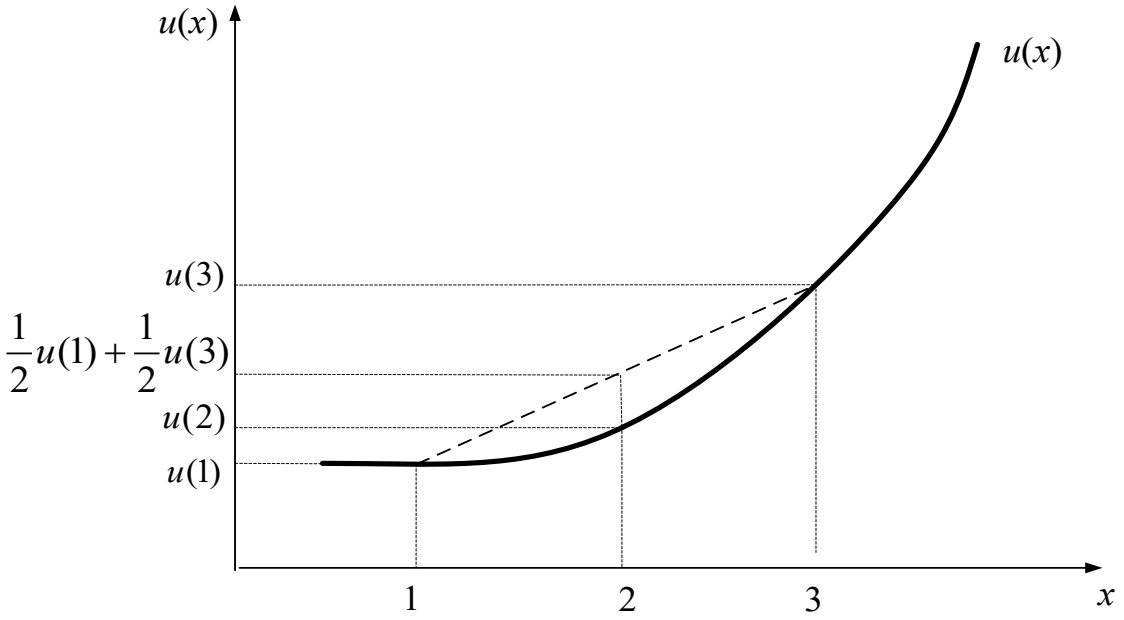
Measuring Risk Preferences

- *Risk neutral individual*
  - Utility from the expected value of the lottery, $u(2)$, **coincides** with the EU from playing the lottery, $\frac{1}{2}u(1) + \frac{1}{2}u(3)$.



Measuring Risk Preferences

- *Risk loving individual*
  - Utility from the expected value of the lottery, $u(2)$, is **lower** than the EU from playing the lottery, $\frac{1}{2}u(1) + \frac{1}{2}u(3)$.



Measuring Risk Preferences

- ***Certainty equivalent***, $c(F,u)$:
  - An alternative measure of risk aversion
  - It is the amount of money that makes the individual indifferent between playing the lottery $F(\cdot)$, and accepting a certain amount $c(F,u)$. That is,

$$u(c(F,u)) = \int u(x)dF(x) \text{ or } \sum u(x)f(x)$$

  - $c(F,u)$ is below (above) the expected value of the lottery for risk averse (lover) individuals, and exactly coincides for risk neutral individuals.

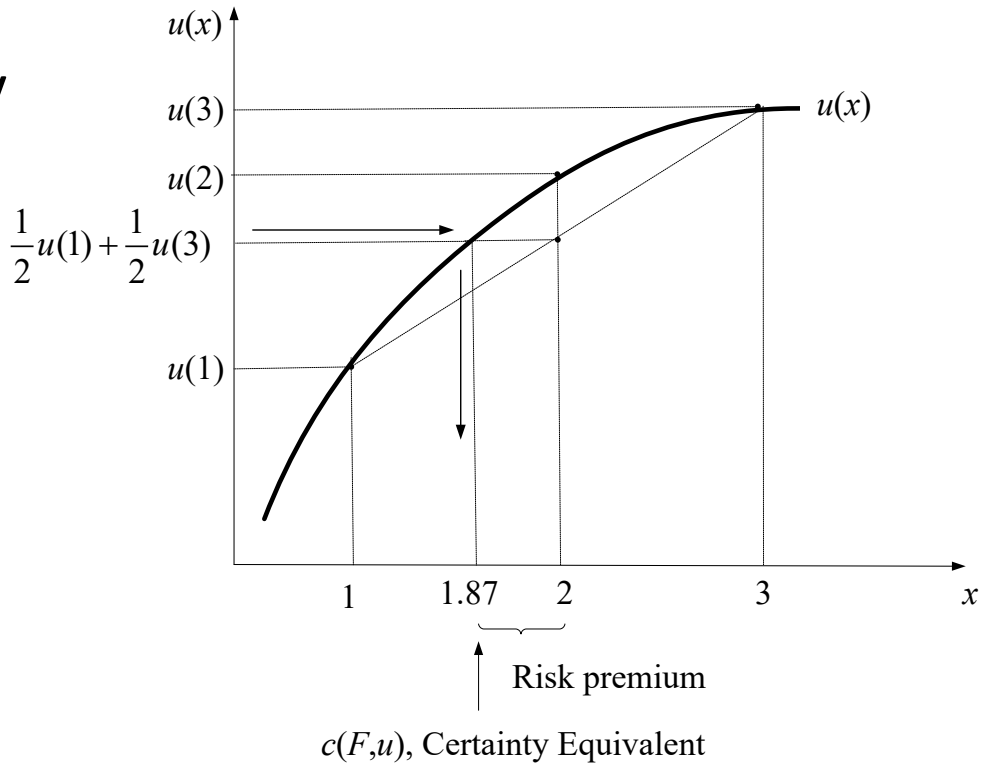
Measuring Risk Preferences

- Certainty equivalent for a risk averse individual
  - $c(F,u)$ is the amount of money (x) for which utility is equal to the EU of the lottery

$$u(c(F,u)) = \frac{1}{2}u(1) + \frac{1}{2}u(3)$$

  - ***Risk premium*** (RP): the amount that a risk-averse person would *pay* to avoid taking a risk:

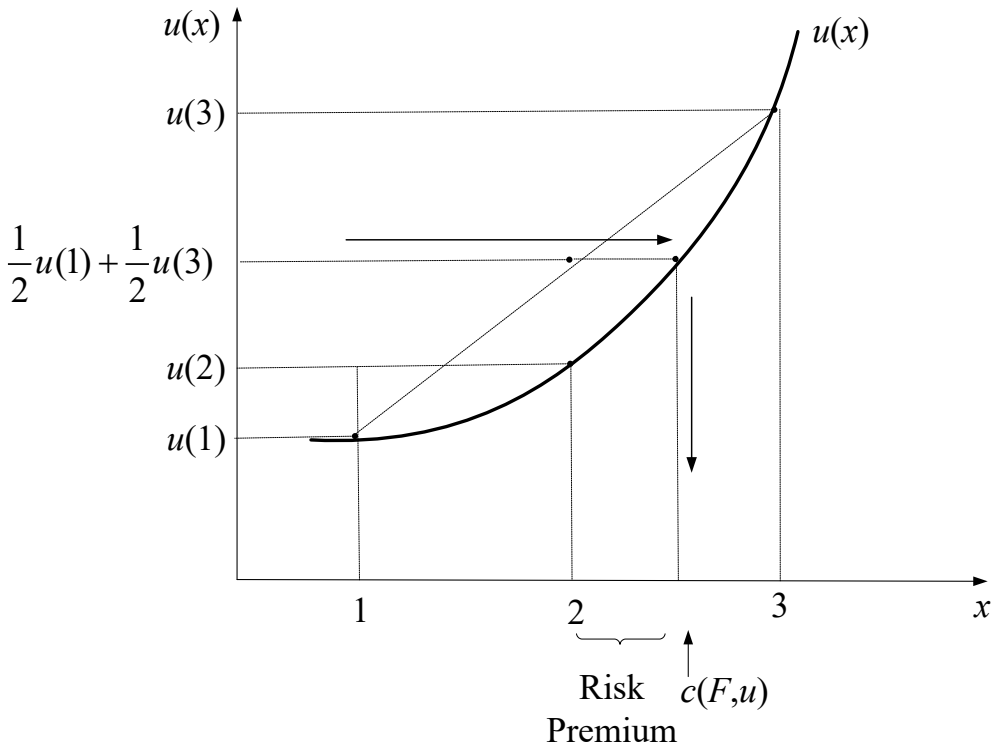
$$RP = EV - c(F,u) > 0$$



Measuring Risk Preferences

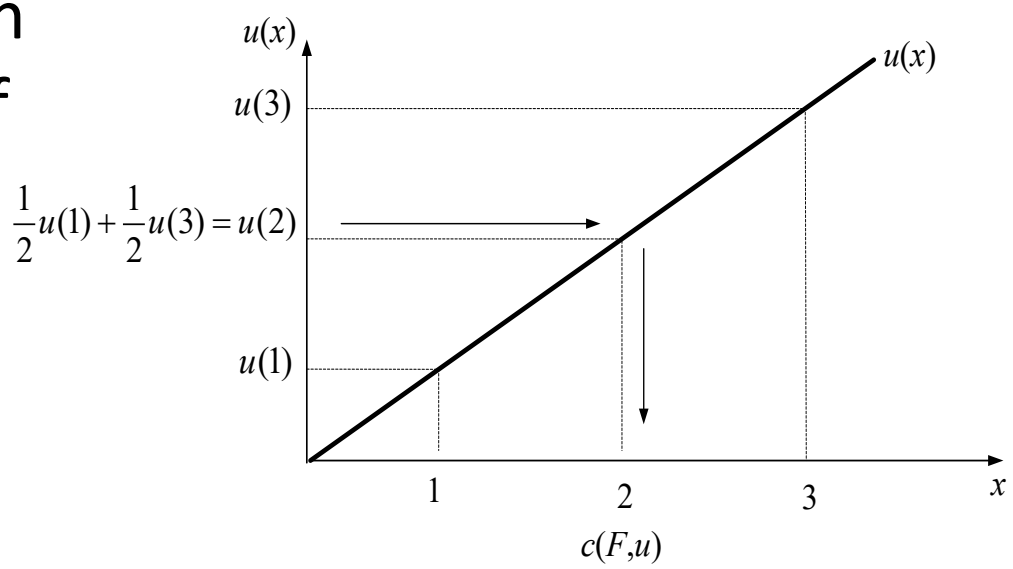
- Certainty equivalent for a risk lover
  - Individual would have to be given an amount of money *above* the expected value of the lottery in order to convince him to "stop playing" the lottery:

$$RP = EV - c(F,u) < 0$$



Measuring Risk Preferences

- Certainty equivalent for a risk neutral individual
  - The certainty equivalent $c(F,u)$ coincides with the expected value of the lottery.
  - Hence,
    $RP = EV - c(F,u) = 0$



Measuring Risk Preferences

- ***Probability premium***, $\pi(x, \varepsilon, u)$:
  - An alternative measure of risk aversion
  - It is the excess in winning probability over fair odds that makes the individual indifferent between the certainty outcome x and a gamble between the two outcomes $x + \varepsilon$ and $x - \varepsilon$:

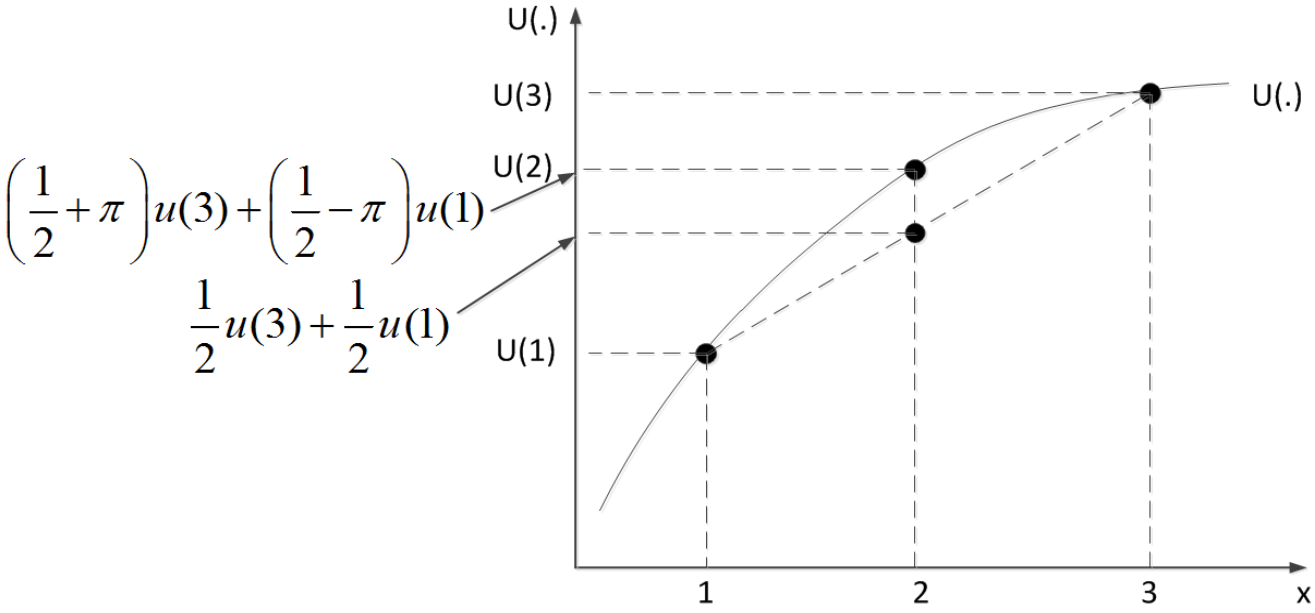
$$u(x) = \left[\frac{1}{2} + \pi(x, \varepsilon, u)\right] u(x + \varepsilon) + \left[\frac{1}{2} - \pi(x, \varepsilon, u)\right] u(x - \varepsilon)$$

  - *Intuition*: Better than fair odds must be given for the individual to accept the risk.

Measuring Risk Preferences

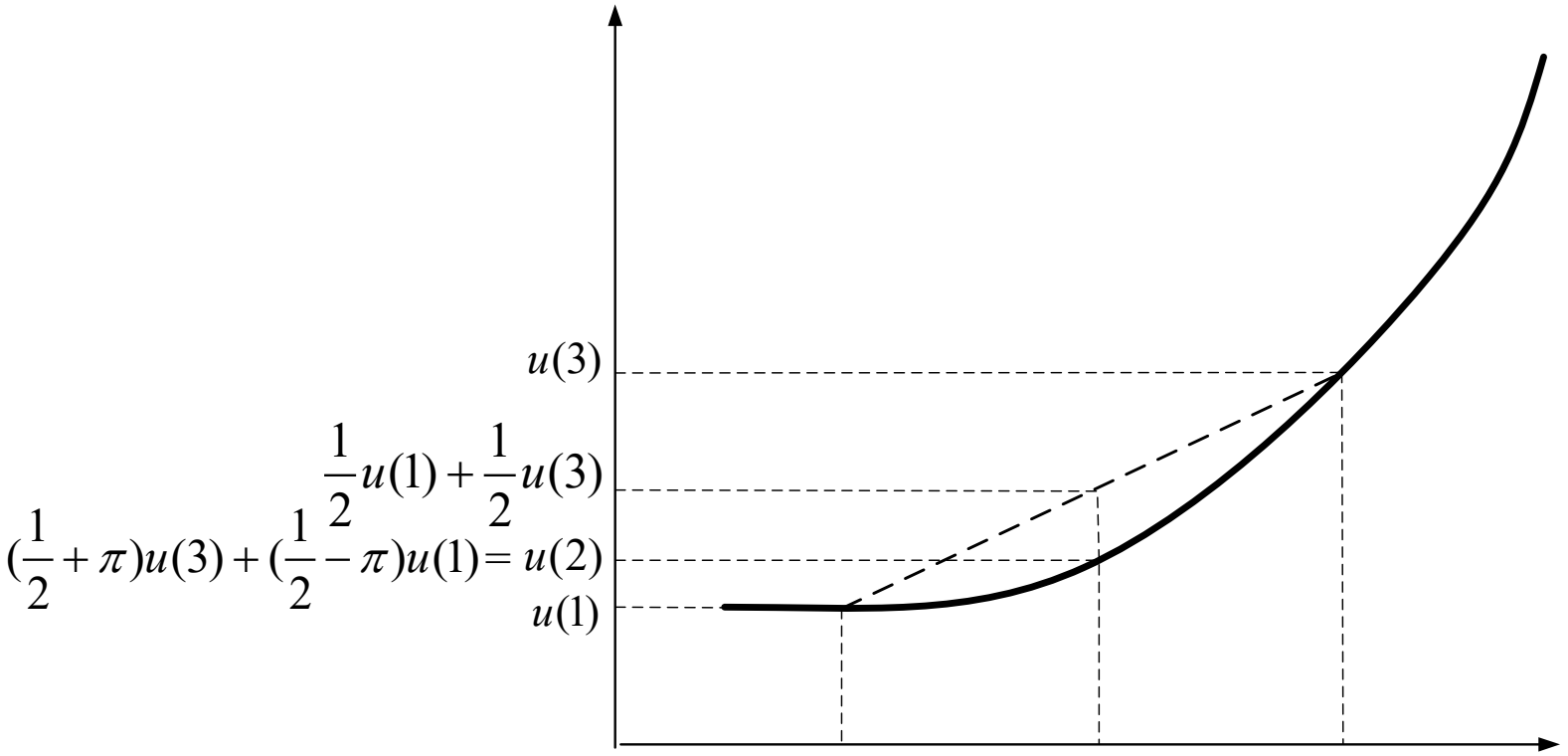
- The "extra probability" π that is needed to make the EU of the lottery coincides with the utility of the expected lottery:

$$u(2) = \left[\frac{1}{2} + \pi\right] u(3) + \left[\frac{1}{2} - \pi\right] u(1)$$



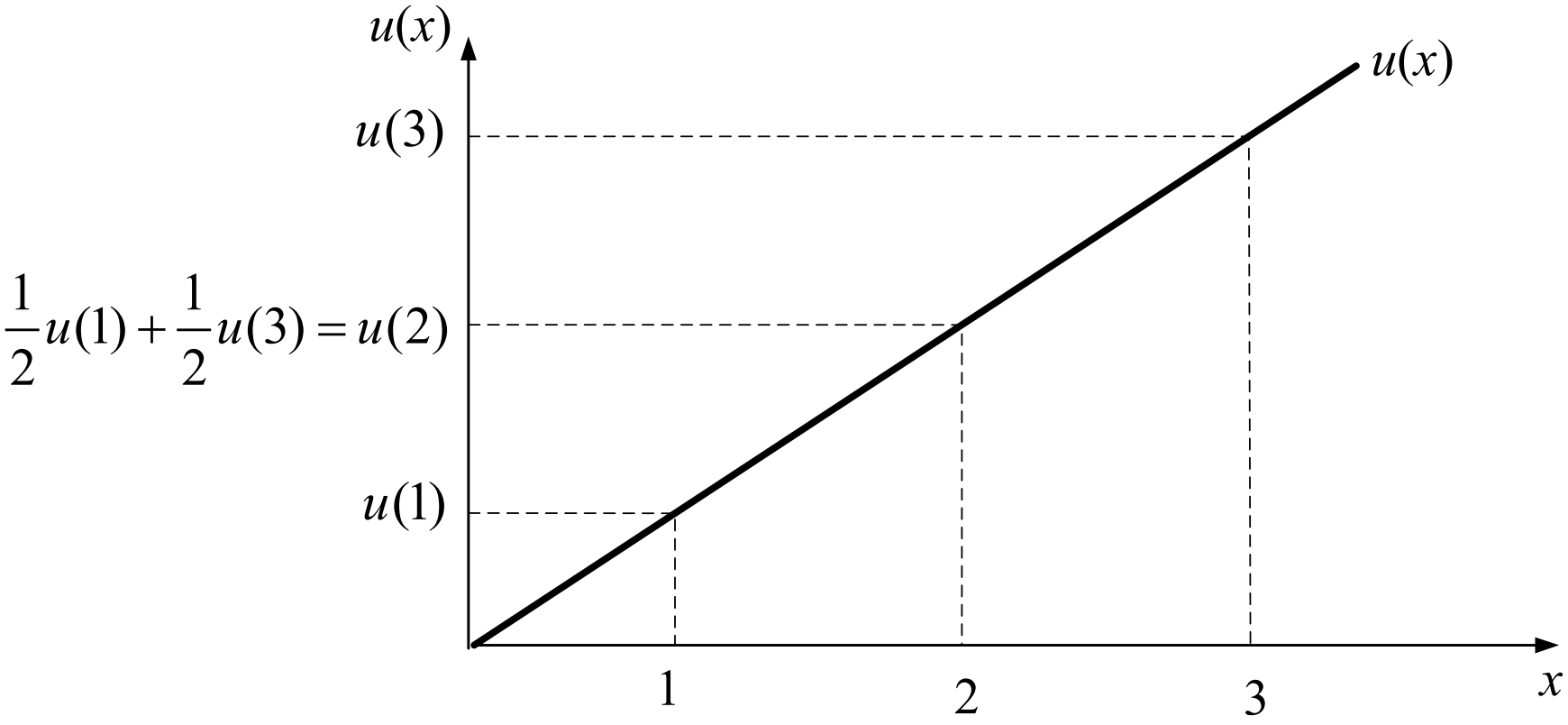
Measuring Risk Preferences

- For a risk loving individual, $\pi < 0$ so he accepts worse than fair odds (i.e., a negative probability premium) to play the lottery.



Measuring Risk Preferences

- For a risk neutral individual, $\pi = 0$ so he accepts the fair odds (i.e., zero probability premium) to play the lottery.



Measuring Risk Preferences

- The following properties are equivalent:
  1) The decision maker is risk averse.
  2) The Bernoulli utility function $u(x)$ is concave, $u''(x) \leq 0$.
  3) The certainty equivalent is lower than the expected value of the lottery, i.e., $c(F,u) \leq \int x dF(x)$.
  4) The risk premium is positive, $RP = EV - c(F,u) \geq 0$.
  5) The probability premium is positive for all x and ε, i.e., $\pi(x,\varepsilon,u) \geq 0$.

Measuring Risk Preferences

- ***Arrow-Pratt coefficient of absolute risk aversion***:

$$r_A(x) = -\frac{u''(x)}{u'(x)}$$

  - Clearly, the greater the curvature of the utility function, $u''(x)$, the larger the coefficient $r_A(x)$.
  - But, why do not we simply have $r_A(x) = u''(x)$?
    - Because it will not be invariant to positive linear transformations of the utility function, such as $v(x) = \beta u(x)$. That is, $v''(x) = \beta u''(x)$ is affected by the transformation, but the above coefficient of risk aversion is unaffected.

$$r_A(x) = -\frac{\beta u''(x)}{\beta u'(x)} = -\frac{u''(x)}{u'(x)}$$

Measuring Risk Preferences

- ***Example*** (CARA utility function).
  - Take $u(x) = -e^{-ax}$ where $a > 0$. Then

$$r_A(x) = -\frac{u''(x)}{u'(x)} = -\frac{-a^2 e^{-ax}}{ae^{-ax}} = a$$

    which is constant in wealth x.
  - The literature refers to this Bernoulli utility function as the *Constant Absolute Risk Aversion* (CARA).

Measuring Risk Preferences

- ***Risk premium of a small risk***:
  - An individual with wealth x faces a zero-mean risk $\tilde{\varepsilon}$ with variance σ^2.
  - Her risk premium RP solves

$$\mathbb{E}[u(x+\tilde{\varepsilon})] = u(x-RP)$$

Measuring Risk Preferences

- ***Pratt (1964) approximation***:
  - Taylor expansions of both sides yield

$$RP \approx \frac{1}{2} r_A(x)\sigma^2$$

  - A higher $r_A(x)$ means a larger premium to avoid the same small risk.

Measuring Risk Preferences

- ***Example***: $u(x) = \ln x$, and a fair bet that adds or subtracts \$10 with probability $\frac{1}{2}$ each.
  - Exact premium: $RP = x - \sqrt{x^2 - 100}$.
  - Approximation: $RP \approx \frac{50}{x}$.

Measuring Risk Preferences

- Log utility: the premium falls as wealth rises (DARA). CARA: it does not change.

Wealth	**Log, exact**	**Log, approximation**	**CARA,** $a = 0.02$
50	1.010	1.000	0.993
100	0.501	0.500	0.993
200	0.250	0.250	0.993
400	0.125	0.125	0.993

Measuring Risk Preferences

- If $r_A(x)$ decreases as we increase wealth x, then we say that such Bernoulli utility function satisfies *decreasing absolute risk aversion* (DARA)

$$\frac{\partial r_A(x)}{\partial x} < 0$$

- *Intuition*: wealthier people are willing to bear more risk than poorer people.
  - This is NOT due to different utility functions, but because the same utility function is evaluated at higher/lower wealth levels.

Measuring Risk Preferences

- A necessary (but not sufficient) condition for DARA is $u'''(x) > 0$, that is,

$$r_A'(x) < 0 \underset{\nLeftarrow}{\Rightarrow} u'''(x) > 0$$

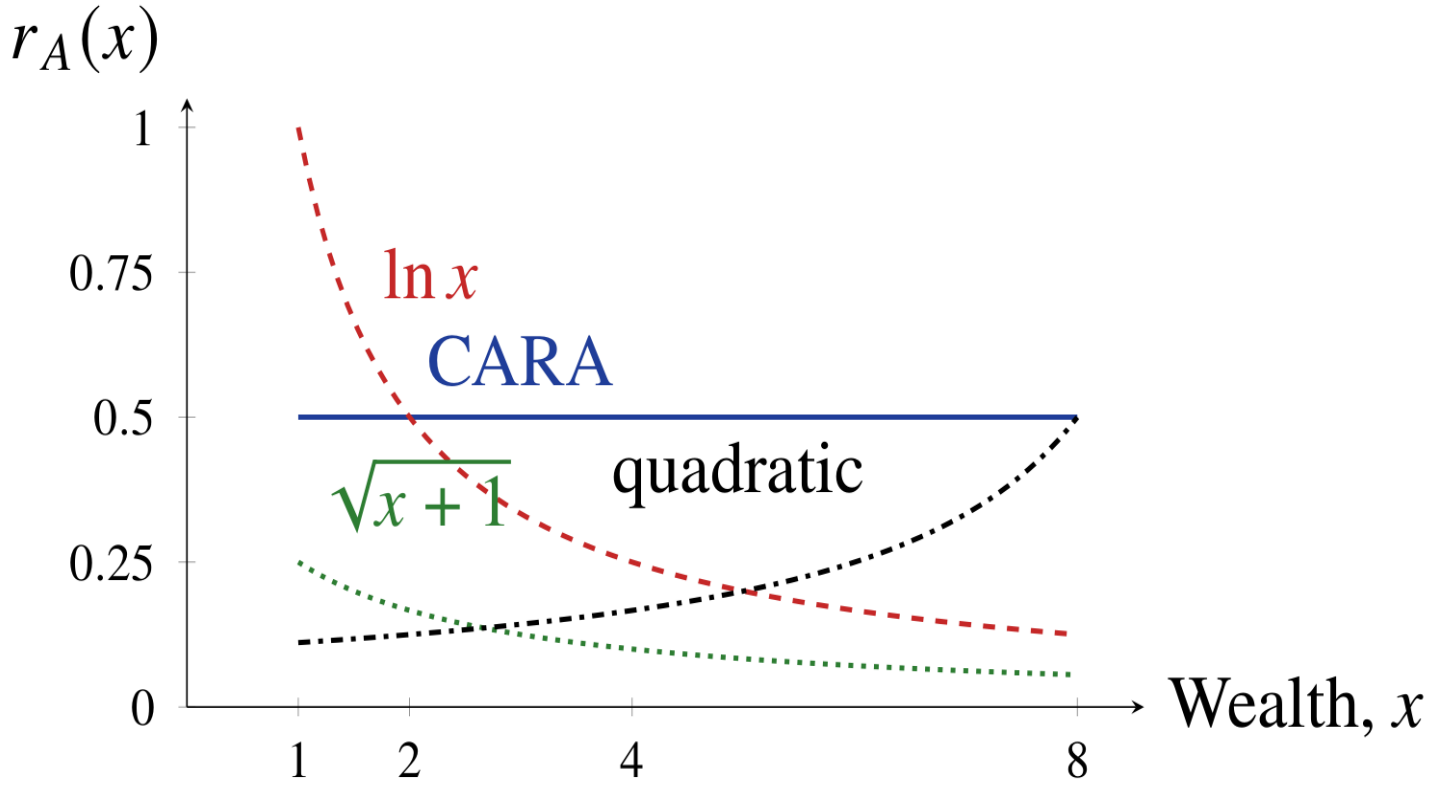
- For example, when $u(x) = -e^{-ax}$, its third-order derivative is $u'''(x) = a^3 e^{-ax} > 0$.
- However, $r_A(x) = a$ is constant, so $u'''(x) > 0$ does not guarantee DARA.

Measuring Risk Preferences

- ***Examples***:
  - CARA, $u(x) = -e^{-ax}$: $r_A(x) = a$, constant.
  - Log, $u(x) = \ln x$: $r_A(x) = \frac{1}{x}$, DARA.
  - $u(x) = (x + \alpha)^{\beta}$: $r_A(x) = \frac{1-\beta}{x+\alpha}$, DARA.
  - Quadratic, $u(x) = \alpha + \beta x - \frac{c}{2}x^2$: $r_A(x) = \frac{c}{\beta - cx}$, IARA, since $u'''(x) = 0$.

Measuring Risk Preferences

- Absolute risk aversion as wealth increases:



Measuring Risk Preferences

- ***Arrow-Pratt coefficient of relative risk aversion***:

$$r_R(x) = -x \cdot \frac{u''(x)}{u'(x)} \quad \text{or} \quad r_R(x) = x \cdot r_A(x)$$

  - $r_R(x) = -\partial \log u'(x) / \partial \log x$ is the wealth elasticity of marginal utility, which measures the percentage change in marginal utility given a one percent increase in wealth.
  - We can show that

$$\frac{\partial r_R(x)}{\partial x} = \underbrace{r_A(x)}_{+} + x \cdot \frac{\partial r_A(x)}{\partial x}$$

  - Therefore,

$$\frac{\partial r_R(x)}{\partial x} < 0 \quad \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \quad \frac{\partial r_A(x)}{\partial x} < 0$$

Measuring Risk Preferences

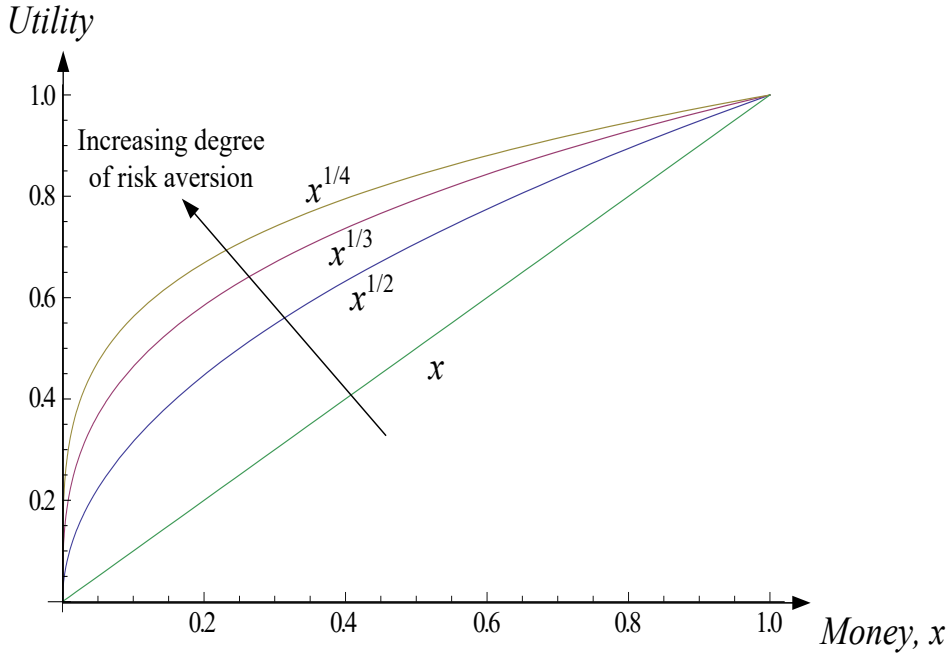
- ***Example***:
  - Take $u(x) = x^b$. Then

$$r_R(x) = -x \cdot \frac{b(b-1)x^{b-2}}{bx^{b-1}} = 1 - b$$

    for all x.
  - The literature refers to this Bernoulli utility function as the *Constant Relative Risk Aversion* (CRRA).

Measuring Risk Preferences

- ***Example*** (continued):
  - Consider a CRRA utility function $u(x) = x^b$ for $b = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.
  - $r_R(x)$ increases when b decreases from 1 to $\frac{1}{2}, \frac{1}{3}$, and $\frac{1}{4}$, making the utility function more concave.



Measuring Risk Preferences

- ***Relative risk premium***:
  - If wealth becomes $x(1+\tilde{\varepsilon})$, the risk premium as a share of wealth is

$$\frac{RP}{x} \approx \frac{1}{2} r_R(x)\sigma^2$$

Measuring Risk Preferences

- ***Example***: CRRA, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, so $r_R(x) = \gamma$.
  - With $x = 100$ and a ±20 percent bet ($\sigma^2 = 0.04$), $RP \approx 2\gamma$.

γ	**RP, exact**	**RP, approximation**
0.5	1.01	1.00
1	2.02	2.00
2	4.00	4.00
4	7.56	8.00

Measuring Risk Preferences

- ***Decreasing relative risk aversion (DRRA) implies DARA***:

$$r_R'(x) = r_A(x) + x r_A'(x)$$

  - Since $r_A(x) > 0$, a negative $r_R'(x)$ requires a negative $r_A'(x)$.

Measuring Risk Preferences

- ***The converse fails***: $u(x) = (x + \alpha)^{\beta}$ with $\alpha > 0$.
  - $r_A(x) = \frac{1-\beta}{x+\alpha}$ decreases in x (DARA).
  - $r_R(x) = \frac{(1-\beta)x}{x+\alpha}$ increases in x (increasing relative risk aversion).

Measuring Risk Preferences

- A utility function $u_A(\cdot)$ exhibits stronger risk aversion than another utility function $u_B(\cdot)$ if, there is a constant $\lambda > 0$,

$$\frac{u_A''(x_1)}{u_B''(x_1)} \geq \lambda \geq \frac{u_A'(x_2)}{u_B'(x_2)}$$

- In addition, if $x_1 = x_2$, the above condition can be re-written as

$$-\frac{u_A''(x_1)}{u_A'(x_1)} \geq -\frac{u_B''(x_1)}{u_B'(x_1)}$$

- Then, $u_A(\cdot)$ also exhibits more absolute risk aversion than $u_B(\cdot)$.

Measuring Risk Preferences

- ***The converse does not hold***: higher absolute risk aversion does not imply stronger risk aversion.
  - Take $u_i(x) = -e^{-a_i x}$ with $a_A = 2 > a_B = 1$.
  - Then u_A has higher absolute risk aversion at every wealth level.

Measuring Risk Preferences

- At $x_1 = 1$ and $x_2 = 0$,

$$\frac{u_A''(1)}{u_B''(1)} = 4e^{-1} \approx 1.47, \qquad \frac{u_A'(0)}{u_B'(0)} = 2$$

- No constant λ satisfies $1.47 \geq \lambda \geq 2$.

Measuring Risk Preferences

- For two utility functions u_1 and u_2, where u_2 is a concave transformation of u_1, the following properties are equivalent:
  1) There exists an increasing concave function $\varphi(\cdot)$ such that $u_2(x) = \varphi\big(u_1(x)\big)$ for any x. That is, $u_2(\cdot)$ is more concave than $u_1(\cdot)$.
  2) $r_A(x, u_2) \geq r_A(x, u_1)$ for any x.
  3) $c(F, u_2) \leq c(F, u_1)$ for any lottery $F(\cdot)$.
  4) $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for any x and ε.

Measuring Risk Preferences

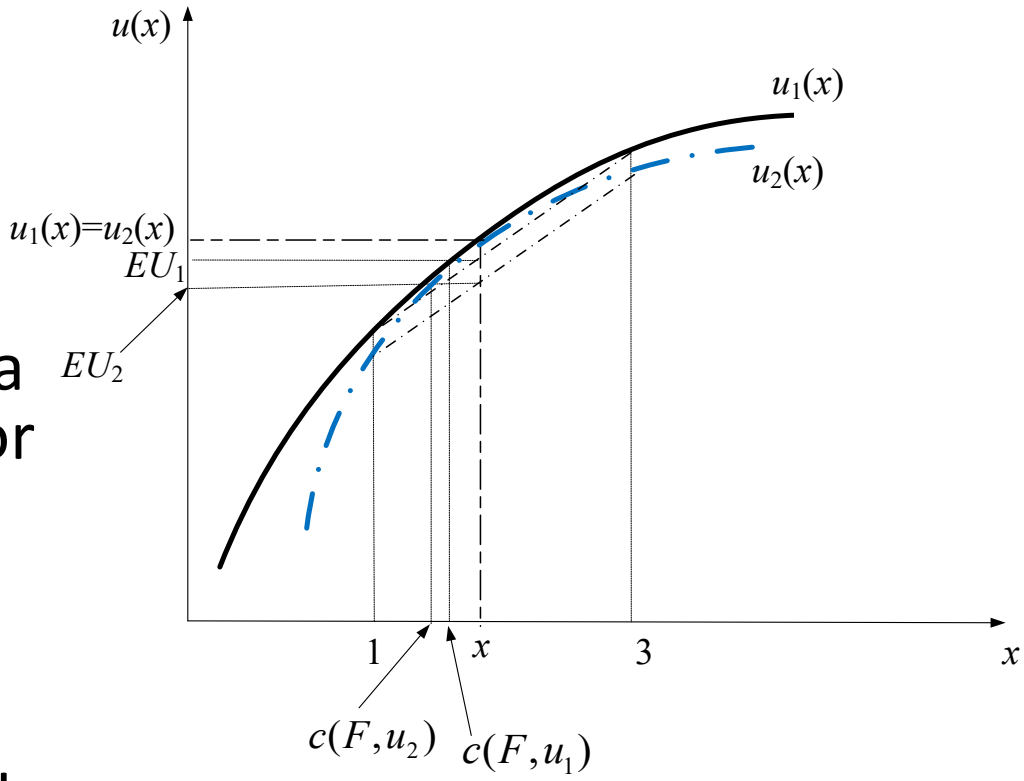
5) Whenever $u_2(\cdot)$ finds a lottery $F(\cdot)$ at least as good as a riskless outcome $\bar{x}$, then $u_1(\cdot)$ also finds such a lottery $F(\cdot)$ at least as good as $\bar{x}$. That is

$$EU_2 = \int u_2(x)dF(x) \geq u_2(\bar{x}) \Longrightarrow$$

$$EU_1 = \int u_1(x)dF(x) \geq u_1(\bar{x})$$

Measuring Risk Preferences

- Different degrees of risk aversion
- $u_1(\cdot)$ and $u_2(\cdot)$ are evaluated at the same wealth level x.
- The same lottery yields a larger expected utility for the individual with *less risk averse* preferences, $EU_1 > EU_2$.
- $c(F, u_2) < c(F, u_1)$, reflecting that individual 2 is more risk averse.



Measuring Risk Preferences

- ***Example***: $u_1(x) = \sqrt{x}$ and $u_2(x) = \ln x$.
  - $u_2 = \varphi(u_1)$, where $\varphi(t) = 2 \ln t$ is increasing and concave.
  - Hence $r_A(x, u_1) = \frac{1}{2x} < \frac{1}{x} = r_A(x, u_2)$.
  - Lottery F: \$1 or \$3 with probability $\frac{1}{2}$ each.

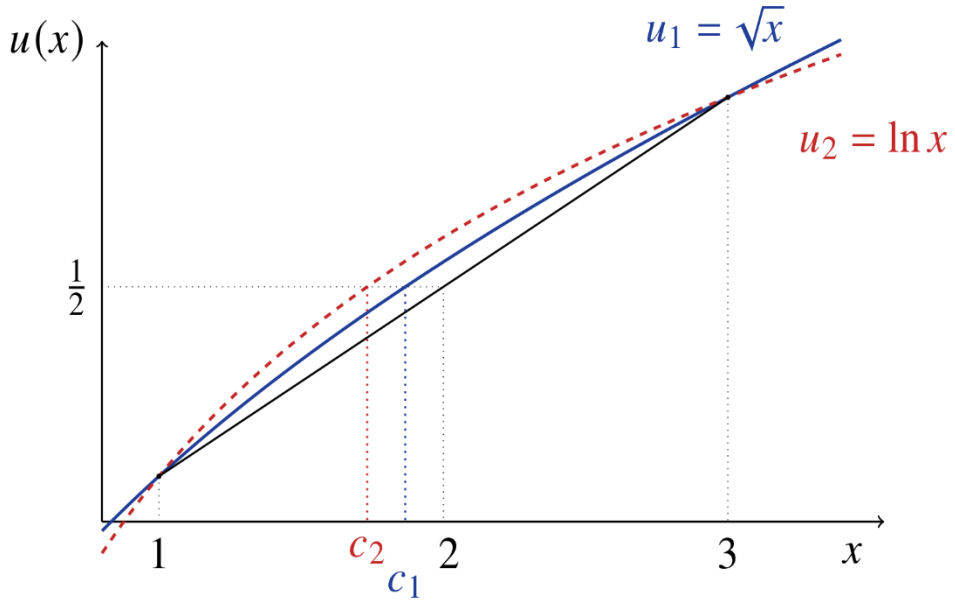
Measuring Risk Preferences

- Comparing the two individuals:

Individual	$r_A(2)$	**Certainty equivalent**	**Risk premium**	**Probability premium**
$1, \sqrt{x}$	0.25	1.866	0.134	0.066
$2, \ln x$	0.50	1.732	0.268	0.131

Measuring Risk Preferences

- $c(F, u_2) = \sqrt{3} \approx 1.73$ is lower than $c(F, u_1) \approx 1.87$, since u_2 is more concave.
  - Both utilities are normalized so that $u(1) = 0$ and $u(3) = 1$.



Measuring Risk Preferences

Prudence

- Introduced by Kimball (1990).
- It measures an individual's tendency to prepare for future risks.
- An individual is considered to be "prudent" if the third derivative of his utility function is positive, i.e., $u'''(x) > 0$.
- The formula for ***absolute*** *prudence* is:

$$K_A(x) = -\frac{u'''(x)}{u''(x)}$$

- The more prudent the individual becomes, the more precautionary saving will this individual prepare for.

Measuring Risk Preferences

- ***Prudence and precautionary saving***
  - With wealth w, she saves s today; tomorrow she faces a zero-mean risk $\tilde{\varepsilon}$.

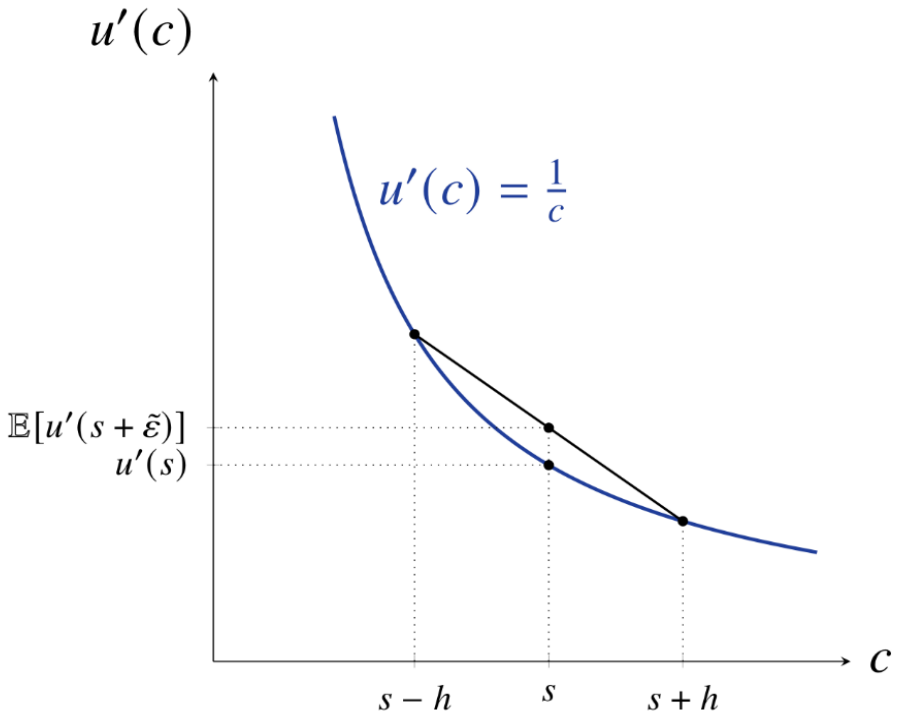
$$\max_s u(w-s) + \mathbb{E}[u(s+\tilde{\varepsilon})]$$

  - First-order condition:

$$u'(w-s) = \mathbb{E}[u'(s+\tilde{\varepsilon})]$$

Measuring Risk Preferences

- Without risk, $s = \frac{w}{2}$.
- If $u'''(x) > 0$, then u' is convex, so $\mathbb{E}[u'(s + \tilde{\varepsilon})] > u'(s)$: she saves more.



Measuring Risk Preferences

- ***Precautionary premium*** ψ:

$$\mathbb{E}[u'(x+\tilde{\varepsilon})] = u'(x-\psi)$$

- Taylor expansions yield

$$\psi \approx \frac{1}{2} K_A(x)\sigma^2$$

- A more prudent individual (higher $K_A(x)$) saves more.

Measuring Risk Preferences

- ***Example***: $w = 10$, and $\tilde{\varepsilon}$ equals 2 or −2 with probability $\frac{1}{2}$ each.
  - Without risk, $s = 5$. Precautionary saving rises with $K_A(5)$:

Utility function	$K_A(5)$	**Savings with risk**	**Precautionary saving**
Quadratic	0	5.000	0.000
$\ln c$	0.40	5.372	0.372
$-e^{-\frac{c}{2}}$ (CARA)	0.50	5.434	0.434
$-\frac{1}{c}$ (CRRA, γ = 2)	0.60	5.507	0.507

Measuring Risk Preferences

Prudence

- The formula for ***relative*** *prudence* is given by

$$K_R(x) = -x\frac{u'''(x)}{u''(x)}$$

- It measures the elasticity of concavity of the marginal utility function, i.e., how more bent the utility function becomes given one percent increase in wealth, where

$$K_R(x) = x \cdot K_A(x) = -\frac{d\log|u''(x)|}{d\log x}$$

Measuring Risk Preferences

- The Arrow-Pratt coefficient of absolute risk aversion, $r_A(x)$, and the coefficient of relative prudence, $K_R(x)$, are closely related.
- Differentiating $r_A(x)$ yields

$$r_A'(x) = -\frac{u''' \cdot u' - (u'' \cdot u'')}{(u')^2}$$

$$= -\frac{u'''}{u'} + \left(\frac{u''}{u'}\right)^2$$

Dividing both sides by $r_A(x) = -u''/u'$ yields

$$\frac{r_A'(x)}{r_A(x)} = -\frac{u'''}{u'}\left(-\frac{u'}{u''}\right) + \left(\frac{u''}{u'}\right)^2\left(-\frac{u'}{u''}\right)$$

$$= \frac{u'''}{u''} - \frac{u''}{u'}$$

$$= -K_A(x) + r_A(x)$$

Measuring Risk Preferences

- Hence, solving for $K_A(x)$, we obtain

$$K_A(x) = r_A(x) - \frac{r_A'(x)}{r_A(x)}$$

- Further multiplying both sides by x,

$$x \cdot K_A(x) = x \cdot r_A(x) - x \cdot \frac{r_A'(x)}{r_A(x)}$$

- Since $K_R(x) = x \cdot K_A(x)$ and $r_R(x) = x \cdot r_A(x)$,

$$K_R(x) = r_R(x) - x \cdot \frac{r_A'(x)}{r_A(x)}$$

Measuring Risk Preferences

- ***Example*** (CARA utility function).
  - Take $u(x) = -e^{-ax}$ where $a > 0$.
  - Then, the relative prudence is

$$K_R(x) = x \cdot a - x\frac{0}{a} = x \cdot a = r_R(x)$$

which coincides with relative risk aversion.

Measuring Risk Preferences

- ***Example***:
  - Take now $u(x) = x^b$.
  - Then $r_A(x) = \frac{1-b}{x}$ and $r_R(x) = 1 - b$, yielding a relative prudence of

$$K_R(x) = (1-b) - x \cdot \frac{-(1-b)/x^2}{(1-b)/x}$$
$$= 1 - b + 1$$
$$= 2 - b$$

  - implying that, in this case, $K_R(x) > r_R(x)$.

Measuring Risk Preferences

Cautiousness

- Cautiousness measures the individual's tendency to hedge against the downside risk of an investment.
- The formula for cautiousness is given by

$$C(x) = \frac{K_R(x)}{r_R(x)}$$

  and a ratio $C(x) > 1$ implies that $K_R(x) > r_R(x)$.
- *Examples*:
  - The CARA utility function yields $C(x) = 1$.
  - The CRRA utility function yields $C(x) > 1$.

Measuring Risk Preferences

Cautiousness

- For DARA utility function, $r_A'(x) < 0$, implying that

$$K_R(x) = r_R(x) - x \cdot \underbrace{\frac{r_A'(x)}{r_A(x)}}_{<0} > r_R(x)$$

so that $K_R(x) > r_R(x)$ yields $C(x) > 1$.

Measuring Risk Preferences

Temperance

- Temperance measures the individual's tendency to reduce the total exposure to risks.
- The formula for temperance is

$$T(x) = -\frac{u''''(x)}{u'''(x)}$$

- An individual is deemed as "temperate" when the fourth derivative of his utility function is negative, i.e., $u''''(x) < 0$.

Measuring Risk Preferences

- ***Summary***:

$u(x)$	$r_A(x)$	$r_R(x)$	$K_A(x)$	$K_R(x)$	$C(x)$	$T(x)$
$-e^{-ax}$	a	ax	a	ax	1	a
$\ln x$	$\frac{1}{x}$	1	$\frac{2}{x}$	2	2	$\frac{3}{x}$
$\frac{x^{1-\gamma}}{1-\gamma}$	$\frac{\gamma}{x}$	γ	$\frac{\gamma+1}{x}$	$\gamma+1$	$\frac{\gamma+1}{\gamma}$	$\frac{\gamma+2}{x}$
$\alpha+\beta x-\frac{c}{2}x^2$	$\frac{c}{\beta-cx}$	$\frac{cx}{\beta-cx}$	0	0	0	not defined

Prospect Theory and Reference-Dependent Utility

Prospect Theory

- ***Prospect theory***: a decision maker's total value from a list of possible outcomes $x = (x_1, x_2, \ldots, x_n)$ with associated probabilities $p = (p_1, p_2, \ldots, p_n)$ is

$$v(x, p) = \sum_{i=1}^{n} w(p_i) \cdot v(x_i)$$

  where
  - $w(p_i)$ is a "probability weighting function"
  - $v(x_i)$ is the "value function" the individual obtains from outcome x_i

Prospect Theory

- **Three main differences relative to standard expected utility theory**:
- First, $w(p_i) \neq p_i$:
  - if $w(p_i) > p_i$, individuals *overestimate* the likelihood of outcome x_i
  - if $w(p_i) < p_i$, individuals *underestimate* the likelihood of outcome x_i
  - if $w(p_i) = p_i$, the model coincides with standard expected utility theory.

Prospect Theory

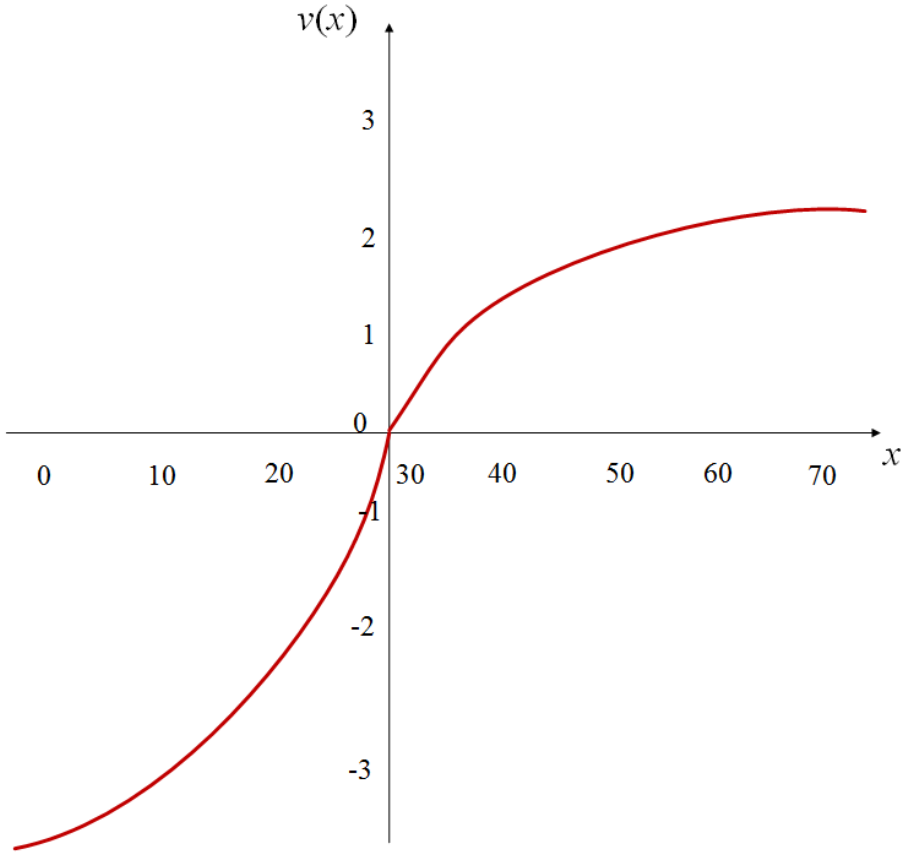
- Second, every payoff x_i is evaluated relative to a "reference point" x_0, with the value function $v(x_i)$, which is
  - Increasing and concave, $v''(x_i) < 0$, for all $x_i > x_0$,
    - That is, the individual is risk averse for gains.
  - Decreasing and convex, $v''(x_i) > 0$, for all $x_i < x_0$
    - That is, the individual is risk lover for losses.
  - *Extremes*:
    - if $x_0 = 0$, the individual is risk averse for all payoffs;
    - if $x_0 = +\infty$, he is risk lover for all payoffs.

Prospect Theory

- Third, value function $v(x_i)$ has a kink at the reference point x_0.
  - The curve becomes steeper for losses (to the left of x_0) than for gains (to the right of x_0).
    - Loss aversion:
    - A given loss of $\$a$ produces a larger disutility than a gain of the same amount.

Prospect Theory

- Value function in prospect theory



Prospect Theory

- ***Example***:
  - Consider as in Tversky and Kahneman (1992)

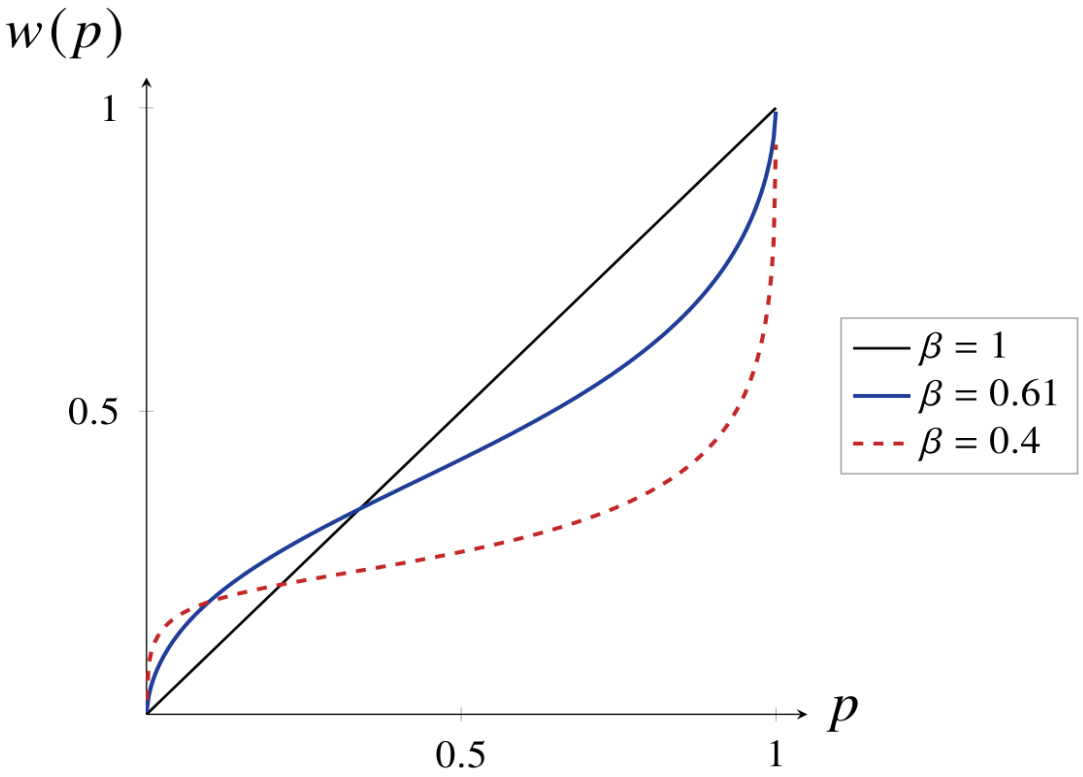
$$w(p) = \frac{p^{\beta}}{\left[p^{\beta}+(1-p)^{\beta}\right]^{\frac{1}{\beta}}} \text{ and } v(x) = x^{\alpha}$$

where $0 < \alpha < 1$ and $0 < \beta < 1$.

- Note that this implies probability weighting, but does not consider a value function with loss aversion relative to a reference point.

Prospect Theory

- ***Example*** (continued):
  - Depicting the probability weighting function



Prospect Theory

- ***Properties of the Tversky-Kahneman function***:
  - It crosses the 45-degree line below one half: $p \approx 0.34$ for $\beta = 0.61$, and $p \approx 0.21$ for $\beta = 0.4$.
  - It is increasing only if $\beta \approx 0.28$ or higher.
  - Estimates: $\beta = 0.61$ for gains and $\beta = 0.69$ for losses.

Prospect Theory

- Why does $w(\frac{1}{2}) < \frac{1}{2}$? At $p = \frac{1}{2}$, the denominator is

$$[(\frac{1}{2})^{\beta}+(\frac{1}{2})^{\beta}]^{\frac{1}{\beta}}= [2(\frac{1}{2})^{\beta}]^{\frac{1}{\beta}}= 2^{\frac{1}{\beta}} \cdot \frac{1}{2}$$

- so

$$w(\frac{1}{2}) = \frac{(\frac{1}{2})^{\beta}}{2^{\frac{1}{\beta}} \cdot \frac{1}{2}} = 2^{1-\beta-\frac{1}{\beta}}$$

Prospect Theory

- Since

$$\beta + \frac{1}{\beta} > 2 \text{ for } \beta \neq 1, \qquad \text{so } 1 - \beta - \frac{1}{\beta} < -1$$

- the exponent is below −1, and $w(\frac{1}{2}) < \frac{1}{2}$.
- ***Example***:

$$\beta = 0.61: \quad 1 - 0.61 - 1.639 = -1.249,$$

$$\text{so } w(\frac{1}{2}) = 2^{-1.249} \approx 0.421$$

Prospect Theory

- Values of the weighting function:

p	**0.01**	**0.10**	**0.25**	**0.50**	**0.75**	**0.90**	**0.99**
$w(p), \beta = 0.61$	0.055	0.186	0.291	0.421	0.568	0.712	0.912
$w(p), \beta = 0.4$	0.111	0.186	0.221	0.268	0.343	0.447	0.695

Prospect Theory

- ***Example***:
  - A common value function is

$$v(x_i) = {x_i}^{\alpha} \quad \text{if } x_i \geq x_0, \text{ and}$$

$$= -\lambda(-x_i)^{\alpha} \quad \text{if } x_i < x_0$$

where $0 < \alpha \leq 1$, and $\lambda \geq 1$ represents loss aversion.

- If $\lambda = 1$ the individual does not exhibit loss aversion.

Prospect Theory

- ***Example***:
  - Average estimates $\lambda = 2.25$ and $\alpha = 0.88$
  - Common simplifications, assume $\alpha = \beta = 1$ (which implies no probability weighting, and linear value functions), to estimate λ.
  - Tversky and Kahneman (1992) estimate $\alpha = 0.88, \quad \lambda = 2.25, \quad \beta = 0.61 \text{ (gains)}, 0.69 \text{ (losses)}.$

Prospect Theory

- With $\alpha = 0.88$ and $\lambda = 2.25$:

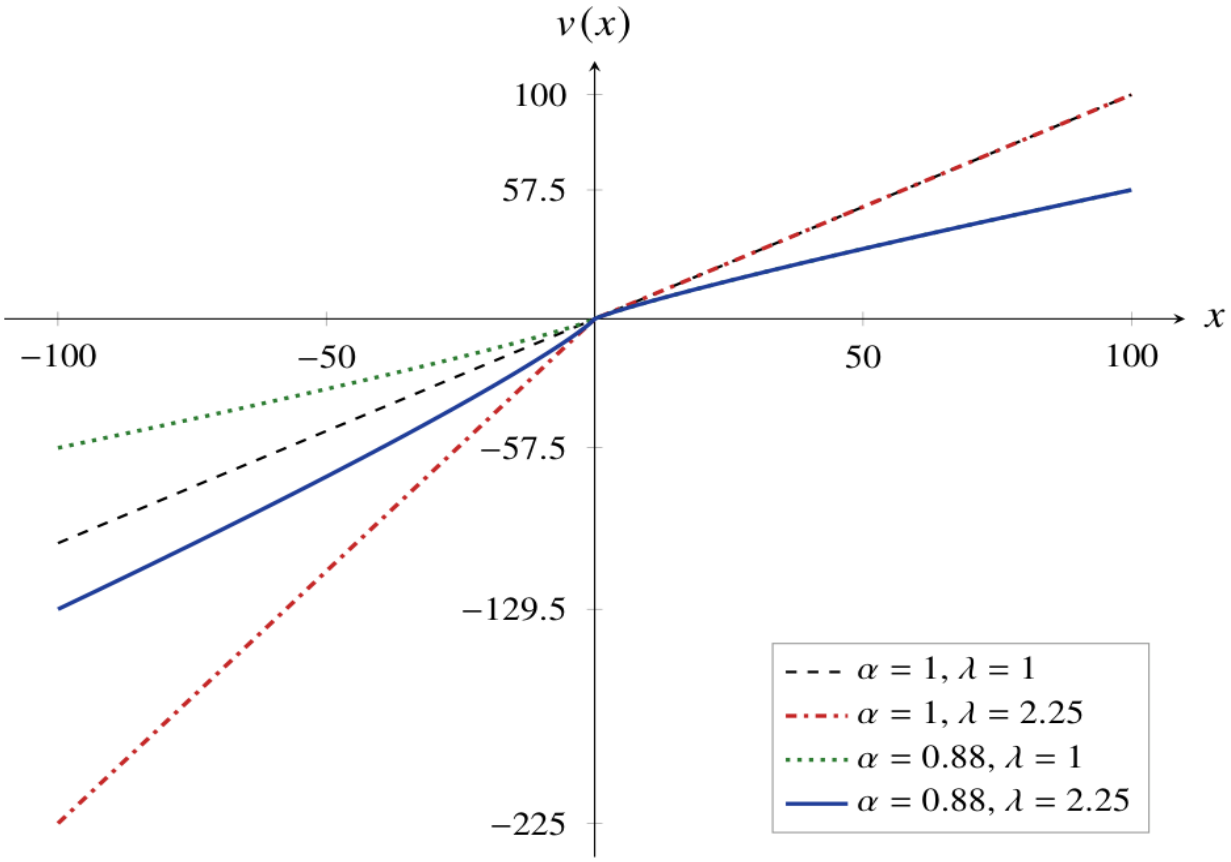
$$v(100) = 100^{0.88} \approx 57.5$$

$$v(-100) = -2.25(100)^{0.88} \approx -129.5$$

- A loss of \$100 hurts about 2.25 times as much as a gain of \$100 helps.

Prospect Theory

- Value functions for four pairs of parameters:



Prospect Theory

- On the gains side, curves with the same α coincide: λ only affects losses.
- A larger λ makes the loss side steeper:

$$\alpha = 1: \quad v(-100) = -2.25(100) = -225$$

$$\alpha = 0.88: \quad v(-100) = -2.25(100)^{0.88} \approx -129.5$$

- A smaller α bends the curve: concave over gains and convex over losses.

Prospect Theory

- ***Loss aversion in a mixed gamble***: win $G or lose $100, each with probability $\frac{1}{2}$.
  - She accepts if $w(\frac{1}{2})G^{\alpha} - w(\frac{1}{2})\lambda(100)^{\alpha} \geq 0$, that is, $G \geq 100\lambda^{\frac{1}{\alpha}}$.
  - With $\lambda = 2.25$, she rejects a fair coin flip that wins $200 or loses $100.

Prospect Theory

- Step 1: both outcomes have probability one half, so both get the weight $w(\frac{1}{2})$, and the individual accepts if

$$w(\frac{1}{2})G^{\alpha} - w(\frac{1}{2})\lambda(100)^{\alpha} \geq 0$$

- Step 2: dividing by $w(\frac{1}{2}) > 0$,

$$G^{\alpha} \geq \lambda(100)^{\alpha}$$

- Step 3: raising both sides to the power $1/\alpha$,

$$G \geq 100\lambda^{\frac{1}{\alpha}}$$

Prospect Theory

- With $\alpha = 1$, the gain must be λ times the loss: $G \geq 100\lambda$.
- With $\alpha = 0.88$ and $\lambda = 2.25$,

  $2.25^{\frac{1}{0.88}} = 2.25^{1.136} \approx 2.513, \qquad \text{so} \quad G \geq 251.3$

  – The concavity of v over gains raises the required gain further, from 225 to 251.3.

Prospect Theory

- Minimum acceptable gain:

λ	$G, \alpha = 1$	$G, \alpha = 0.88$
1	100	100.0
1.5	150	158.5
2	200	219.8
2.25	225	251.3
2.5	250	283.3
3	300	348.5

Prospect Theory

- An individual with $\lambda = 2.25$ rejects a fair coin flip that wins \$200 or loses \$100, although its expected value is \$50.
- Loss aversion, not the curvature of v, drives the rejection:
  - With $\lambda = 1$, the minimum gain is \$100 for any α.

Prospect Theory

- ***Prospect theory and the Allais paradox***

$$L_1 = (0, 1, 0), \qquad L_1' = (0.10, 0.89, 0.01)$$

$$L_2 = (0, 0.11, 0.89), \qquad L_2' = (0.10, 0, 0.90)$$

  - With $v(x) = x^{\alpha}$ and $v(0) = 0$.

Prospect Theory

- Values of the four lotteries:

$$V(L_1) = v(0.5)$$
$$V(L_1') = w(0.10)v(2.5) + w(0.89)v(0.5)$$
$$V(L_2) = w(0.11)v(0.5)$$
$$V(L_2') = w(0.10)v(2.5)$$

Prospect Theory

- L_1 is certain, so its probability is not distorted, $w(1) = 1$.
- Then $L_1 \succ L_1'$ requires
$$v(0.5) > w(0.10)v(2.5) + w(0.89)v(0.5)$$
- Subtracting $w(0.89)v(0.5)$ from both sides,
$$[1 - w(0.89)]v(0.5) > w(0.10)v(2.5)$$

Prospect Theory

- Since $v(2.5) > v(0.5)$, the condition can only hold if

$$1 - w(0.89) > w(0.10), \quad \text{that is,}$$
$$w(0.89) + w(0.10) < 1$$

  - Otherwise, the left side would be at most $w(0.10)v(0.5)$, which is smaller than the right side.
- This property is ***subcertainty***: the weights of the two positive outcomes in $L_1{}'$ add up to less than one.

Prospect Theory

- $L_1 \succ L_1'$ requires subcertainty, $w(0.89) + w(0.10) < 1$.
  - This condition is necessary but not sufficient.

- For $L_2' \succ L_2$, we follow the same steps (next slides).

Prospect Theory

- ***Second choice***, step by step.
- Step 1: neither lottery is certain, so both probabilities are distorted:

$$V(L_2) = w(0.11)v(0.5)$$
$$V(L_2') = w(0.10)v(2.5)$$

- Step 2: $L_2' \succ L_2$ requires

$$w(0.10)v(2.5) > w(0.11)v(0.5)$$

Prospect Theory

- Step 3: dividing both sides by $w(0.10)v(0.5) > 0$,

$$\frac{v(2.5)}{v(0.5)} > \frac{w(0.11)}{w(0.10)}$$

  - Left side: how much more \$2.5 million is worth than \$0.5 million.
  - Right side: how much more weight probability 0.11 gets than probability 0.10.

Prospect Theory

- Step 4: with $v(x) = x^{\alpha}$, the left side is

$$\frac{2.5^{\alpha}}{0.5^{\alpha}} = \left(\frac{2.5}{0.5}\right)^{\alpha} = 5^{\alpha}$$

$$5^{1} = 5, \qquad 5^{0.25} \approx 1.50$$

- The right side is close to one, since 0.11 and 0.10 are close:

$$\beta = 1: \quad \frac{0.11}{0.10} = 1.1$$

$$\beta = 0.61: \quad \frac{0.195}{0.186} \approx 1.05$$

Prospect Theory

- Hence the condition fails only if v is almost flat:

$$5^{\alpha} > 1.05 \quad \text{if and only if} \quad \alpha > \frac{\ln 1.05}{\ln 5} \approx 0.029$$

- With the Tversky-Kahneman estimates ($\alpha = 0.88$),

$$5^{0.88} \approx 4.12 > 1.05,$$

$$\text{so} \quad V(L_2') \approx 0.417 > V(L_2) \approx 0.106$$

Prospect Theory

- ***Tversky-Kahneman estimates***: $\alpha = 0.88$ and $\beta = 0.61$.
- Step 1: values,

$$v(0.5) = 0.5^{0.88} \approx 0.543,$$
$$v(2.5) = 2.5^{0.88} \approx 2.240$$

- Step 2: weights, from the Tversky-Kahneman function,

$$w(0.10) \approx 0.186, \qquad w(0.11) \approx 0.195,$$
$$w(0.89) \approx 0.699$$

Prospect Theory

- Step 3: the four lotteries,

$$V(L_1') \approx 0.186(2.240) + 0.699(0.543)$$
$$\approx 0.417 + 0.380 = 0.797$$
$$V(L_1) = v(0.5) \approx 0.543 < 0.797$$

$$V(L_2) \approx 0.195(0.543) \approx 0.106 <$$
$$V(L_2') \approx 0.186(2.240) \approx 0.417$$

- Hence $L_1' \succ L_1$, although subcertainty holds:

$$w(0.89) + w(0.10) \approx 0.885 < 1$$

Prospect Theory

- ***The answer depends on the parameters***:

α	β	$V(L_1)$	$V(L_1')$	$V(L_2)$	$V(L_2')$	$L_1 \succ L_1'$?	$L_2' \succ L_2$?
1	1	0.500	0.695	0.055	0.250	No	Yes
0.88	0.61	0.543	0.797	0.106	0.417	No	Yes
0.5	0.61	0.707	0.789	0.138	0.295	No	Yes
0.5	0.4	0.707	0.602	0.134	0.294	Yes	Yes
0.25	0.5	0.841	0.736	0.172	0.249	Yes	Yes
0.25	0.4	0.841	0.600	0.159	0.233	Yes	Yes

Prospect Theory

- With the Tversky-Kahneman estimates (second row), $L_1' \succ L_1$.
- The Allais pattern requires strong curvature in both v and w.

Prospect Theory

- Why strong curvature?
  - A small α makes the extra \$2 million in ${L_1}'$ worth little.
  - A small β shrinks $w(0.89)$, the weight on \$0.5 million in ${L_1}'$.
- Computing the weights with the cumulative (rank-dependent) version of prospect theory gives the same Yes/No answers in every row.

Prospect Theory

- ***Two cutoffs on α***: solve each Allais choice for α, given β, with $v(x) = x^{\alpha}$.
- First choice, $L_1 \succ L_1'$: subtract $w(0.89)v(0.5)$ from both sides,

$$[1 - w(0.89)]0.5^{\alpha} > w(0.10)2.5^{\alpha}$$

- then divide by $w(0.10)0.5^{\alpha} > 0$,

$$\frac{1 - w(0.89)}{w(0.10)} > \left(\frac{2.5}{0.5}\right)^{\alpha} = 5^{\alpha}$$

Prospect Theory

- Taking logs, and dividing by ln 5 > 0,

$$\alpha < \alpha_A(\beta) \equiv \frac{1}{\ln 5} \ln \frac{1 - w(0.89)}{w(0.10)}$$

  - This cutoff is positive only under subcertainty, $1 - w(0.89) > w(0.10)$.

- Example: with $\beta = 0.61$,

$$\alpha_A(0.61) \approx \frac{0.480}{1.609} \approx 0.298$$

Prospect Theory

- Second choice, $L_2' \succ L_2$: divide by $w(0.10)0.5^{\alpha} > 0$,

$$w(0.10)2.5^{\alpha} > w(0.11)0.5^{\alpha}, \qquad \text{so} \quad 5^{\alpha} > \frac{w(0.11)}{w(0.10)}$$

- Taking logs,

$$\alpha > \alpha_B(\beta) \equiv \frac{1}{\ln 5} \ln \frac{w(0.11)}{w(0.10)}$$

  - Since $w(0.11)$ and $w(0.10)$ are close, $\alpha_B(\beta)$ is small: the second choice fails only if v is almost flat.

Prospect Theory

- Both Allais choices hold if and only if

$$\alpha_B(\beta) < \alpha < \alpha_A(\beta)$$

β	**0.3**	**0.4**	**0.5**	**0.61**	**0.7**	**0.8**	**1**
$\alpha_B(\beta)$	0.002	0.011	0.020	0.029	0.036	0.044	0.059
$\alpha_A(\beta)$	1.040	0.690	0.468	0.298	0.198	0.118	0.059

Prospect Theory

- Can neither choice hold? It would require α above $\alpha_A(\beta)$ and below $\alpha_B(\beta)$. But
$$\alpha_A(\beta) \geq \alpha_B(\beta)$$
  - if and only if
$$w(0.11) + w(0.89) \leq 1$$
- and, for the Tversky-Kahneman function with $\beta < 1$,
$$w(p) + w(1-p) = [p^\beta + (1-p)^\beta]^{1-\frac{1}{\beta}} < 1$$
  - The term in brackets exceeds $p + (1-p) = 1$, and the exponent is negative.

Prospect Theory

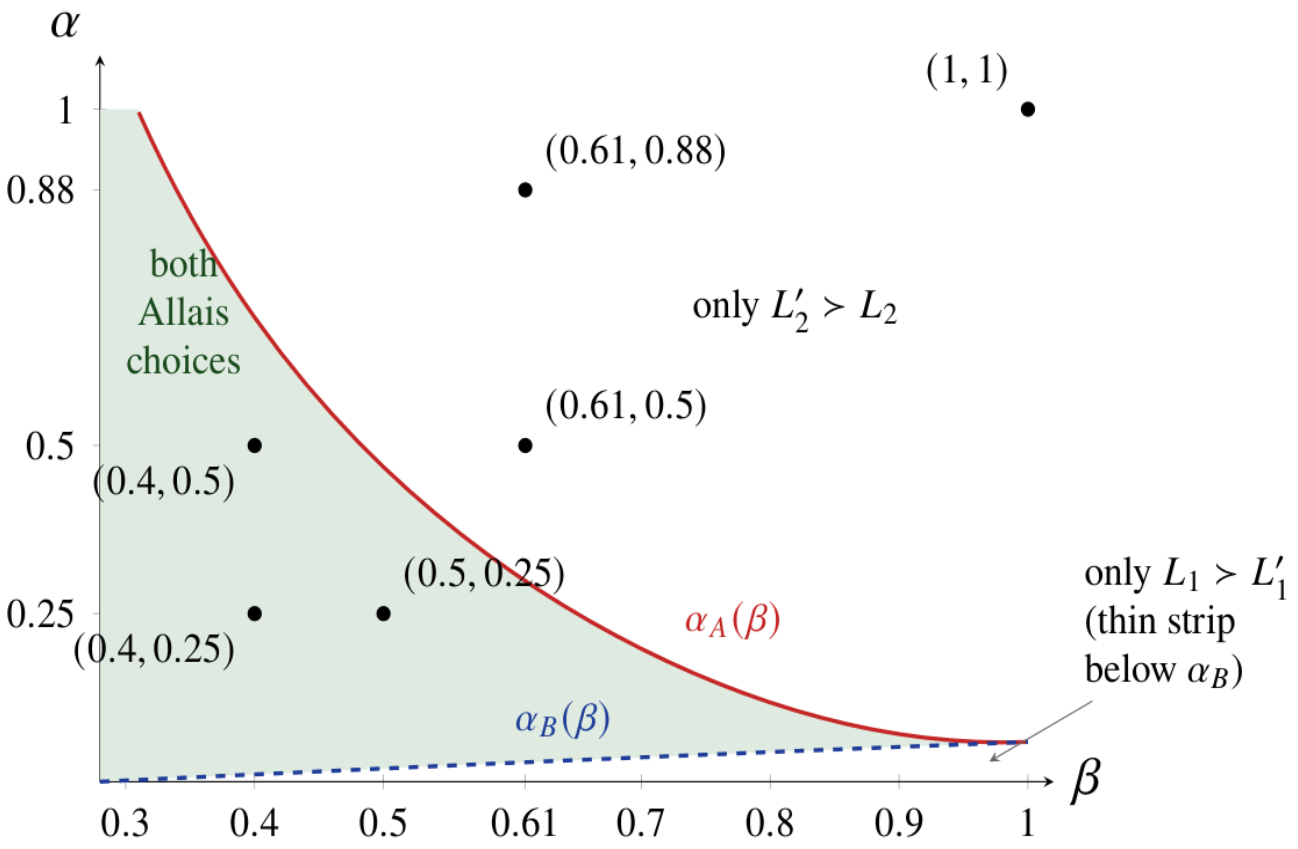
- Hence $\alpha_A(\beta) > \alpha_B(\beta)$ for every $\beta < 1$: the region where neither choice holds is empty.
- With $\beta = 1$ (no weighting), both cutoffs coincide:

$$\alpha_A(1) = \alpha_B(1) = \frac{\ln 1.1}{\ln 5} \approx 0.059$$

  - Exactly one choice holds, as expected utility predicts: an individual who chooses L_1 also chooses L_2.
- Probability weighting opens a gap between the cutoffs; the Allais pattern lives in that gap.

Prospect Theory

- The three bottom rows of the table lie between the two cutoffs.



Prospect Theory

- The rows $(\beta, \alpha) = (0.4, 0.5), (0.5, 0.25), \text{and } (0.4, 0.25)$ lie in the shaded region.
  - For instance, $0.5 < 0.690 \approx \alpha_A(0.4)$.
- The other three rows lie above $\alpha_A(\beta)$: with the Tversky-Kahneman estimates, $0.88 > 0.298 \approx \alpha_A(0.61)$.
- With $\alpha = 0.88$, the first Allais choice requires $\beta < 0.34$, approximately.

Prospect Theory

- ***Endowment effect***: a mug of value $M = \$6$, with $\alpha = 1$.
  - Buying: $M - \lambda P \geq 0$, so $WTP = \frac{M}{\lambda}$.
  - Selling: $P - \lambda M \geq 0$, so $WTA = \lambda M$.
  - Hence $\frac{WTA}{WTP} = \lambda^2$.

Prospect Theory

- ***Buying***: a subject without the mug gains the mug, M, and loses the price, P, a loss weighted by λ.

$$M - \lambda P \geq 0 \quad \text{if and only if} \quad P \leq \frac{M}{\lambda}$$

- ***Selling***: a subject who owns the mug gains P and loses the mug.

$$P - \lambda M \geq 0 \quad \text{if and only if} \quad P \geq \lambda M$$

Prospect Theory

- In each case, the item given up is coded as a loss.

$$\frac{WTA}{WTP} = \frac{\lambda M}{\frac{M}{\lambda}} = \lambda^2$$

- ***Example***: $M = \$6$ and $\lambda = 2.25$:

$$\text{WTP} = \frac{6}{2.25} \approx 2.67, \qquad \text{WTA} = 2.25(6) = 13.50$$

$$\frac{13.50}{2.67} \approx 2.25^2 \approx 5.06$$

  - With $\lambda = 1$, WTP = WTA = \$6.

Prospect Theory

- The gap grows with the square of loss aversion:

λ	**WTP**	**WTA**	$\frac{WTA}{WTP}$
1	6.00	6.00	1.00
1.5	4.00	9.00	2.25
2	3.00	12.00	4.00
2.25	2.67	13.50	5.06
2.5	2.40	15.00	6.25
3	2.00	18.00	9.00

Prospect Theory

- Further reading:
  - Nicholas Barberis (2013) “Thirty Years of Prospect Theory in Economics: A Review and Assessment,” *Journal of Economic Perspectives*, 27(1), pp. 173-96.
  - R. Duncan Luce and Peter C. Fishburn (1991) “Rank and sign-dependent linear utility models for binary gambles.” *Journal of Economic Theory*, 53, pp. 75–100.
  - Daniel Kahneman and Amos Tversky (1992) “Advances in prospect theory: Cumulative representation of uncertainty” *Journal of Risk and Uncertainty*, 5(4), pp. 297–323.
  - Peter Wakker and Amos Tversky (1993) “An axiomatization of cumulative prospect theory.” *Journal of Risk and Uncertainty*, 7, pp. 147–176.

Reference-Dependent Utility

- Individual preferences are affected by *reference points*. Thus, gains and losses can be evaluated differently.
- Consider a consumption vector $x \in \mathbb{R}^n$ which is evaluated against a n-dimensional reference vector $r \in \mathbb{R}^n$. Utility function is

$$u(x|r) = m(x) + n(x|r)$$

where $n(x_k|r_k) = \mu(m_k(x_k) - m_k(r_k))$ measures the gain/loss of consuming x_k units of good k relative to its reference amount r_k.

Reference-Dependent Utility

- For lotteries with cumulative distribution function $F(x)$,

$$U(F|r) = \int u(x|r)dF(x)$$

- For lotteries over the set of reference points

$$U(F|G) = \int \int u(x|r)dG(r)dF(x)$$

Reference-Dependent Utility

- Kőszegi and Rabin (2006) treat the reference point as the individual's expectations.
- They close the model by requiring that expectations be consistent with behavior: a ***personal equilibrium***.
  - If the individual expects lottery F and then chooses F, the reference distribution satisfies $F = G$.

Reference-Dependent Utility

- ***Example***: one good, $u(x|r) = x + \mu(x - r)$.
  - $\mu(z) = \eta z$ if $z \geq 0$, and $\mu(z) = \eta\lambda z$ if $z < 0$.
  - $\eta = 1, \lambda = 2.25$.

	$r = 0$	$r = 10$
$x = 0$	0	−22.5
$x = 10$	20	10

Reference-Dependent Utility

- Each cell adds consumption utility and gain-loss utility:

$$u(0|0) = 0 + \mu(0) = 0$$
$$u(0|10) = 0 + \mu(-10) = -(1)(2.25)(10) = -22.5$$
$$u(10|0) = 10 + \mu(10) = 10 + 10 = 20$$
$$u(10|10) = 10 + \mu(0) = 10$$

  - The loss in the second line is multiplied by $\lambda = 2.25$.

Reference-Dependent Utility

- Receiving 10 is worth 20 if she expected 0, but only 10 if she expected 10.
- Receiving 0 when she expected 10 costs 22.5.

Reference-Dependent Utility

- ***Uncertain outcome, correctly expected***: the individual receives 10 with probability q, and 0 otherwise.
- She expects this lottery, so her reference point is drawn from the same lottery, $F = G$:
  - her reference point is 10 with probability q, and 0 otherwise.

Reference-Dependent Utility

- Goal: compute her expected utility, $U(F|F)$, which adds
  - expected consumption utility, and
  - expected gain-loss utility.
- We compute it in four steps (next slides).

Reference-Dependent Utility

- The four cases.
- Step 1: consumption and the reference point are drawn independently from the same lottery, so there are four cases:

Case	Probability	Gain-loss term
$x = 10, r = 10$	q^2	0
$x = 0, r = 0$	$(1-q)^2$	0
$x = 10, r = 0$	$q(1-q)$	$\eta(10)$ (gain)
$x = 0, r = 10$	$q(1-q)$	$\eta\lambda(-10)$ (loss)

Reference-Dependent Utility

- Step 2: average consumption utility over the cases.
  - The individual consumes 10 with probability q, so consumption utility is $10q$.
- Gains and losses only arise when consumption and the reference point differ: a gain with probability $q(1-q)$ and a loss with the same probability.

Reference-Dependent Utility

- Step 3: add up the gain and the loss terms.
  - Factoring out $q(1-q)$:

$$10\eta q(1-q) - 10\eta\lambda q(1-q)$$
$$= -10\eta(\lambda - 1)q(1-q)$$

- Step 4: with $q = \frac{1}{2}$, consumption utility is $10q$ = 5, and

$$-10(1.25)\frac{1}{4} = -3.125,$$
$$\text{so} \quad U(F|F) = 5 - 3.125 = 1.875$$

Reference-Dependent Utility

- Negative under uncertainty, and largest in absolute value at one half:

q	$10q$	**Gain-loss**	$U(F\|F)$
0	0.0	0.000	0.000
0.25	2.5	−2.344	0.156
0.50	5.0	−3.125	1.875
0.75	7.5	−2.344	5.156
1	10.0	0.000	10.000

Reference-Dependent Utility

- A certain outcome generates no gains or losses.
- ***Example***: \$2 for sure, correctly expected, gives

$$u(2|2) = 2 + \mu(0) = 2 > 1.875$$

  - The individual prefers \$2 for sure to the 50-50 lottery over \$10 and \$0, although the lottery's expected value is \$5.

Reference-Dependent Utility

- Further reading:
  - "Reference-Dependent Consumption Plans" (2009) by Koszegi and Rabin, *American Economic Review*, vol. 99(3).
  - "Rational Choice with Status Quo Bias" (2005) by Masatlioglu and Ok, *Journal of Economic Theory,* vol. 121(1).
  - "On the complexity of rationalizing behavior" (2007) Apesteguia and Ballester, *Economics Working Papers* 1048.

Comparison of Payoff Distributions

Comparison of Payoff Distributions

- So far we compared utility functions, but not the distribution of payoffs.
- Two main ideas:
  1) $F(\cdot)$ yields unambiguously *higher returns* than $G(\cdot)$. We will explore this idea in the definition of first order stochastic dominance (FOSD);
  2) $F(\cdot)$ is unambiguously *less risky* than $G(\cdot)$. We will explore this idea in the definition of second order stochastic dominance (SOSD).

Comparison of Payoff Distributions

- ***FOSD***: $F(\cdot)$ FOSD $G(\cdot)$ if, for every non-decreasing function u: $\mathbb{R} \to \mathbb{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

- The distribution of monetary payoffs $F(\cdot)$ FOSD the distribution of monetary payoffs $G(\cdot)$ if and only if

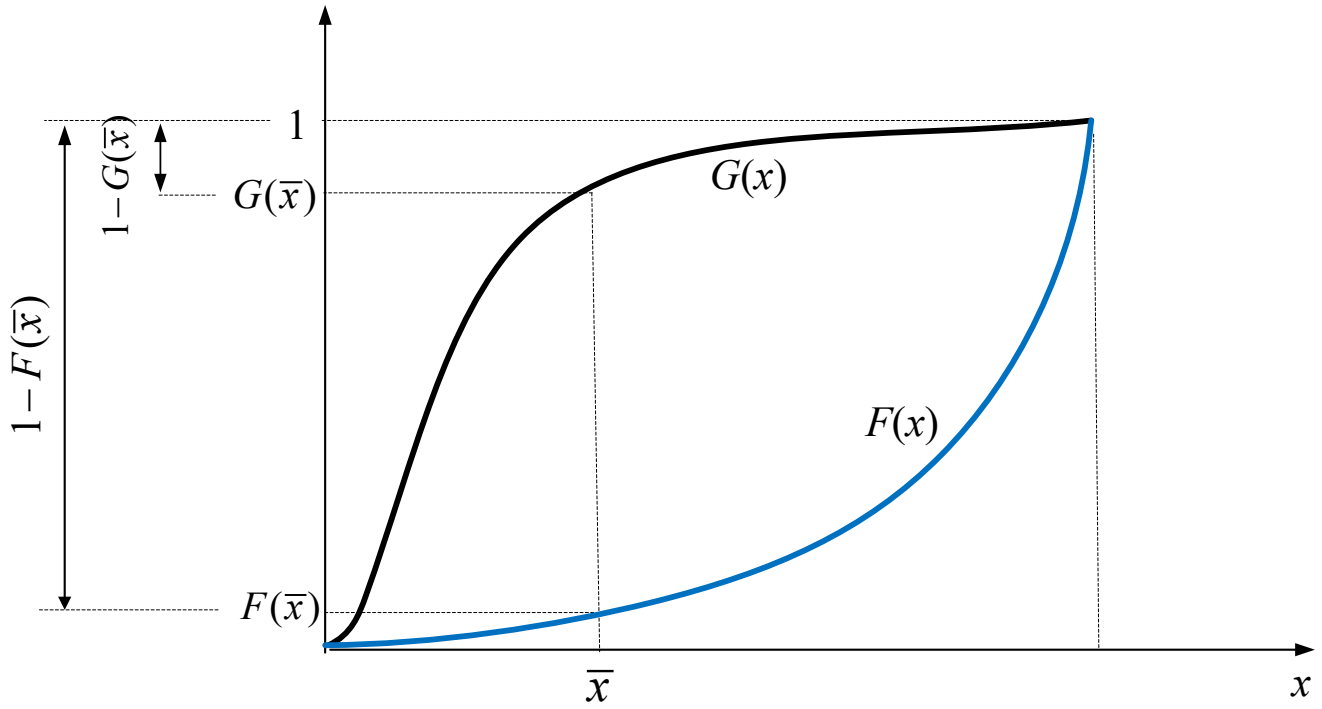
$$F(x) \leq G(x) \text{ or } 1 - F(x) \geq 1 - G(x)$$

  for every x.

- *Intuition*: For every amount of money x, the probability of getting at least x is higher under $F(\cdot)$ than under $G(\cdot)$.

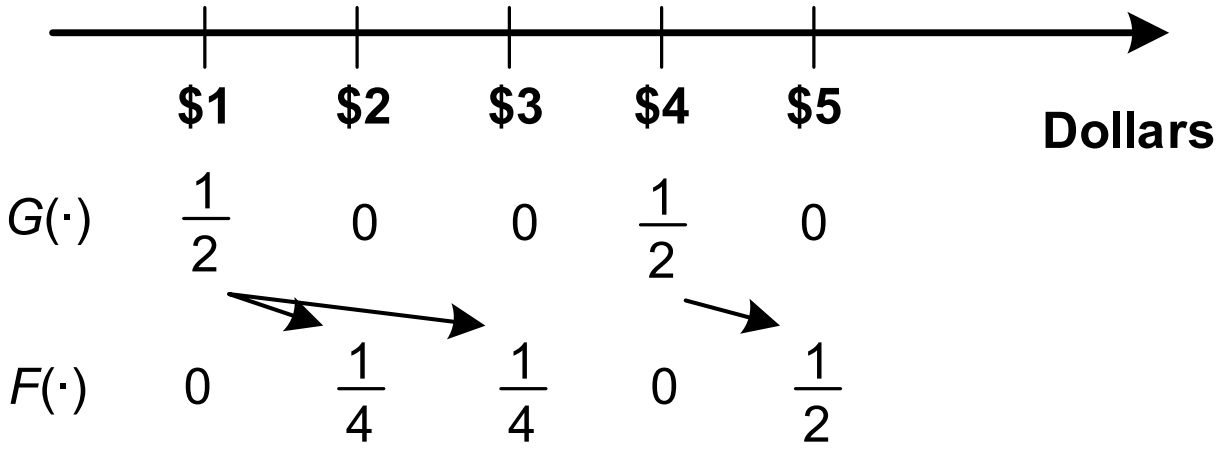
Comparison of Payoff Distributions

- At any given outcome x, the probability of obtaining prizes above x is higher with lottery $F(\cdot)$ than with lottery $G(\cdot)$, i.e., $1 - F(x) \geq 1 - G(x)$.



Comparison of Payoff Distributions

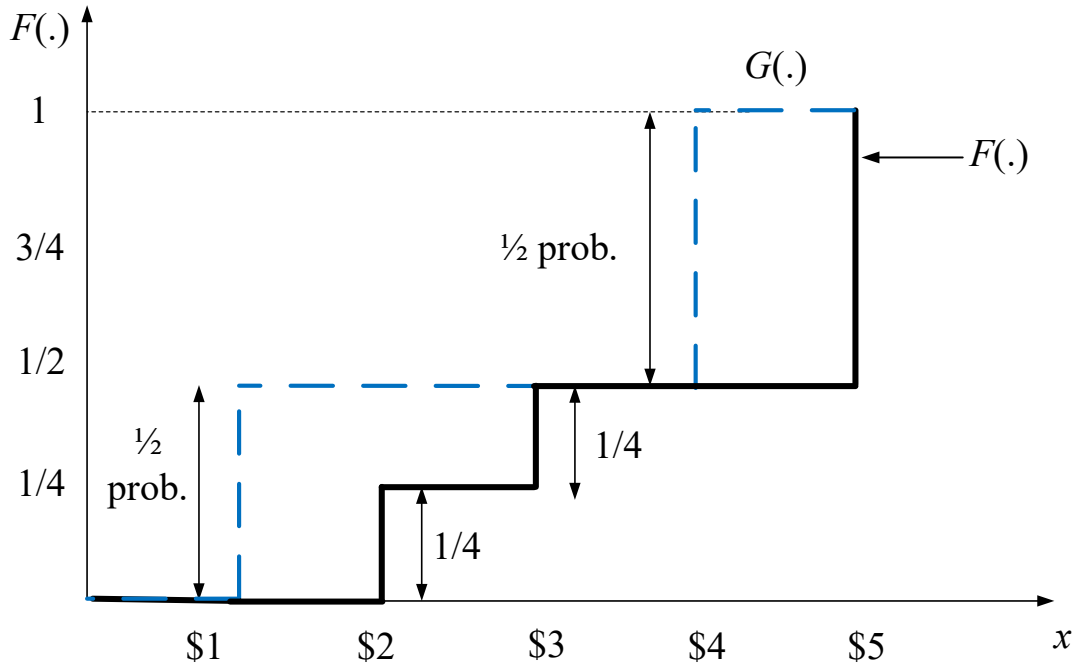
- ***Example***:
  - Let us take lotteries $F(\cdot)$ and $G(\cdot)$ over discrete outcomes.



How can we know if $F(\cdot)$ FOSD $G(\cdot)$?

Comparison of Payoff Distributions

- ***Example*** (continued):
  - $F(\cdot)$ lies below lottery $G(\cdot)$. Hence, $F(\cdot)$ concentrates more probability weight on higher monetary outcomes.
  - Thus, $F(\cdot)$ FOSD $G(\cdot)$.

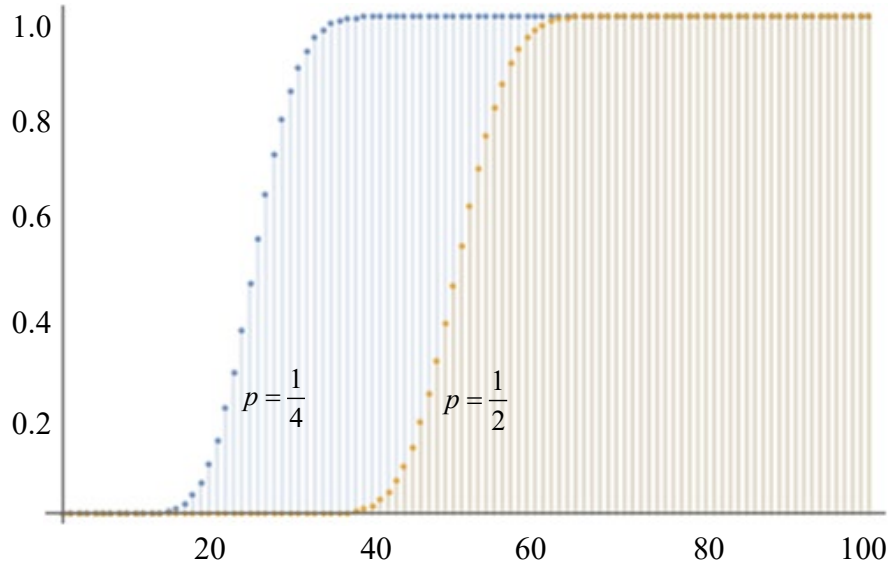


Comparison of Payoff Distributions

- ***Example*** (Binomial distribution):
  - Consider the binomial distribution

$$F(x;N,p)=\binom{N}{p}p^{x}(1-p)^{N-x}$$

  - where $x \in [0,N]$. Assuming $N=100$ and parameter p increasing from $p=\frac{1}{4}$ to $p=\frac{1}{2}$. Then, $F(x;100,1/2)$ FOSD $F(x;100,1/4)$.



Comparison of Payoff Distributions

- ***Example*** (advertising): consider the CDF
  $$F(x; a_1, a_2) = (1 + a_1 - a_2)x - (a_1 - a_2)x^2$$
  - a_1 and a_2: advertising by firms 1 and 2.
  - $x \in [0,1]$: the consumer's preference. Firm 1's product is at $x = 0$ and firm 2's at $x = 1$, so a low x means the consumer prefers firm 1's product.
- It is a CDF on [0, 1]:
  $$F(0) = 0,$$
  $$F(1) = (1 + a_1 - a_2) - (a_1 - a_2) = 1$$

Comparison of Payoff Distributions

- ***Density***: differentiating with respect to x,

$$f(x; a_1, a_2) = \frac{\partial F}{\partial x} = 1 + (a_1 - a_2) - 2(a_1 - a_2)x$$

- The density is linear in x, with endpoints

$$f(0) = 1 + (a_1 - a_2), \qquad f(1) = 1 - (a_1 - a_2)$$

- Hence it is a valid density, f ≥ 0 on [0, 1], if and only if $|a_1 - a_2| \leq 1$. It integrates to one:

$$\int_0^1 f(x)dx = F(1) - F(0) = 1$$

Comparison of Payoff Distributions

- ***Equal advertising***, $a_1 = a_2$:

$$F(x) = x, \qquad f(x) = 1$$

  - The uniform distribution: the two campaigns cancel out.
- Otherwise, the curvature of the CDF is

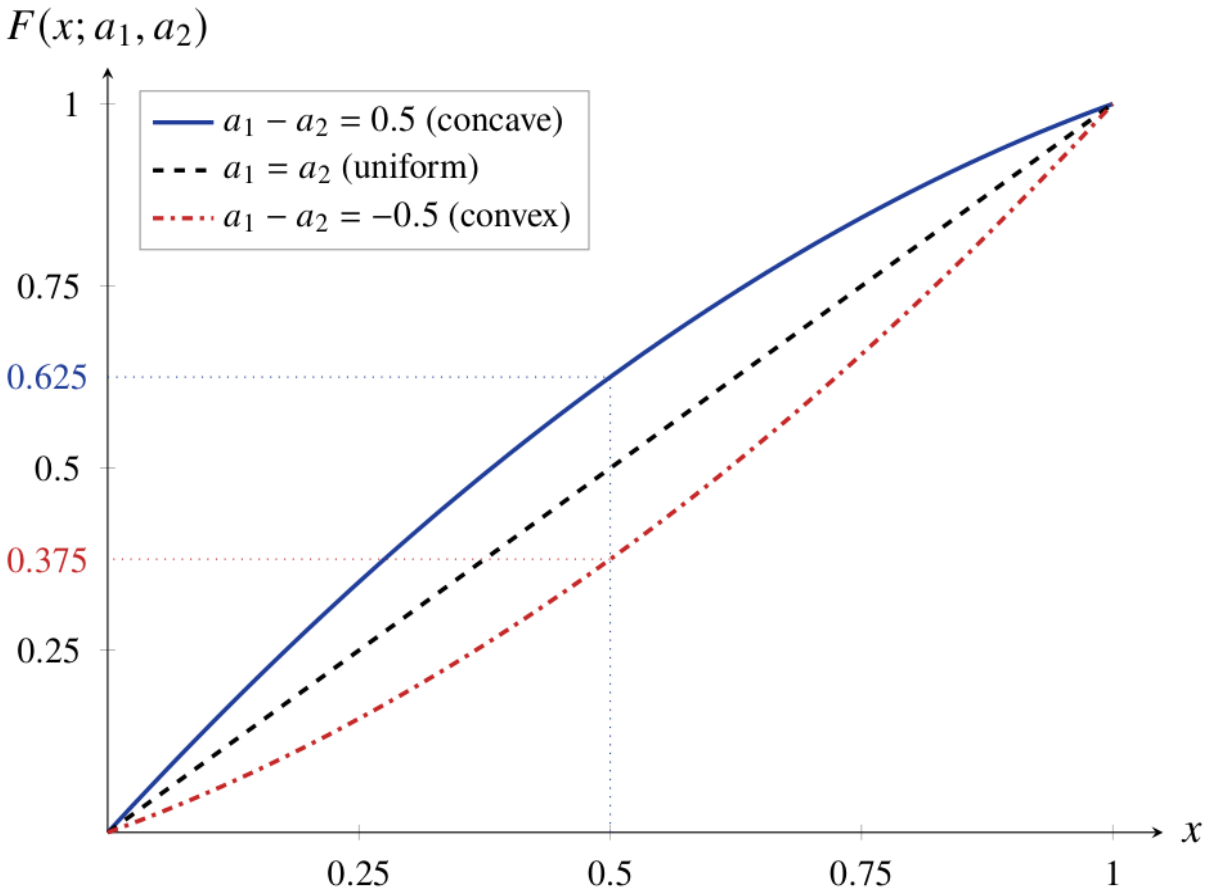
$$F''(x) = -2(a_1 - a_2)$$

Comparison of Payoff Distributions

- ***Firm 1 advertises more***, $a_1 > a_2$: F″ < 0, so the CDF is ***concave***.
  - The density decreases in x: more consumers near firm 1's product.
- ***Firm 2 advertises more***, $a_1 < a_2$: F″ > 0, so the CDF is ***convex***.
  - The density increases in x: more consumers near firm 2's product.

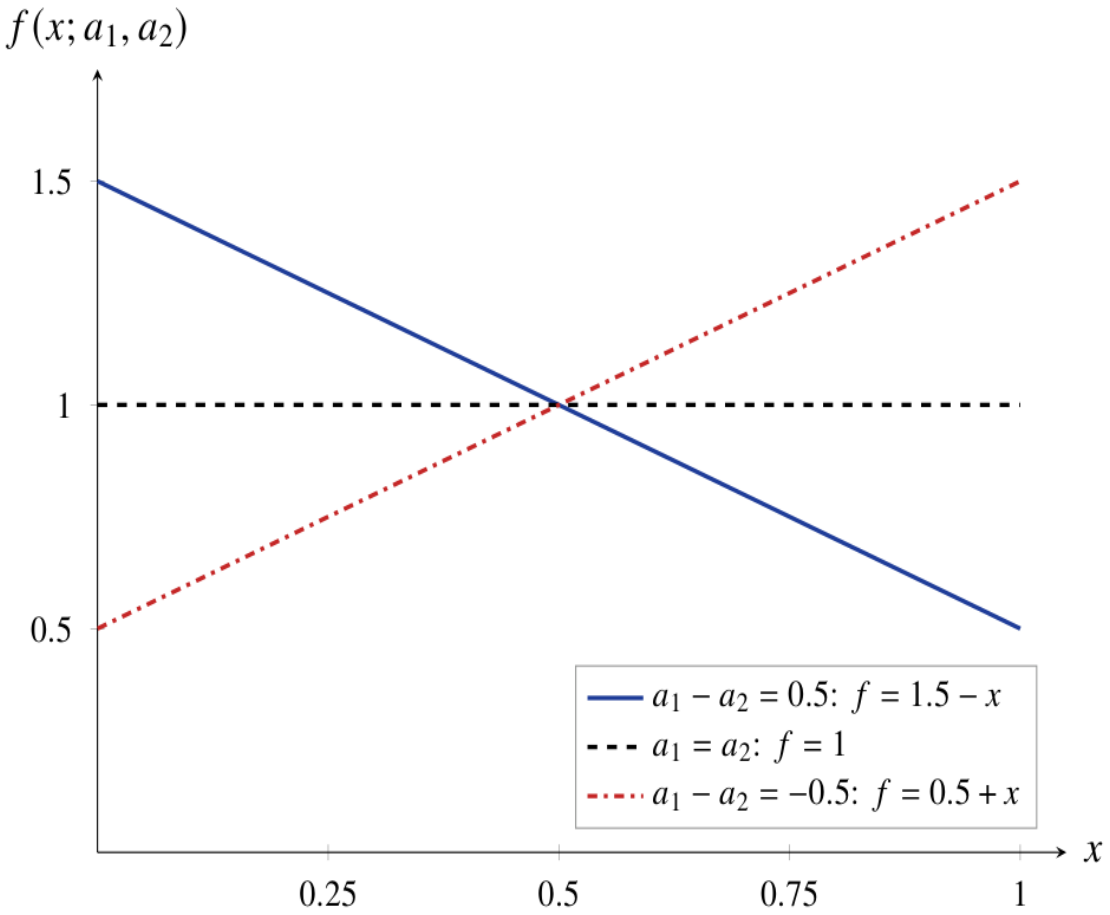
Comparison of Payoff Distributions

- CDFs with $a_1 - a_2$ equal to 0.5, 0, and –0.5:



Comparison of Payoff Distributions

- The corresponding densities:



Comparison of Payoff Distributions

- ***FOSD***: comparing with the uniform CDF,

$$F(x; a_1, a_2) - x = (a_1 - a_2)x(1 - x)$$

  - If $a_1 > a_2$, then $F(x) \geq x$ for every x: the uniform distribution FOSD F.
    - Preferences shift toward firm 1 (lower x).
  - If $a_1 < a_2$, then $F(x) \leq x$ for every x: F FOSD the uniform distribution.
    - Preferences shift toward firm 2.

Comparison of Payoff Distributions

- More generally, more advertising by firm 1 raises the CDF at every x:

$$\frac{\partial F}{\partial a_1} = x(1-x) \geq 0$$

- Average preference:

$$E[x] = \int_0^1 x\, f(x)dx = \frac{1 + a_1 - a_2}{2} - \frac{2(a_1 - a_2)}{3}$$

$$= \frac{1}{2} - \frac{a_1 - a_2}{6}$$

Comparison of Payoff Distributions

- Share of consumers closer to firm 1 than to firm 2 (those with x < 1/2):

$$F(\frac{1}{2}; a_1, a_2) = \frac{1}{2} + \frac{a_1 - a_2}{4}$$

- ***Example***: with $a_1 - a_2$ = 0.5,

$$F(\frac{1}{2}) = \frac{1}{2} + \frac{0.5}{4} = 0.625$$

  - Firm 1's extra advertising moves the share of consumers on its side from 50 to 62.5 percent.

Comparison of Payoff Distributions

- Numerical values:

$a_1 - a_2$	$F(0.25)$	$F(0.5)$	$F(0.75)$	$E[x]$
0.5 (concave)	0.344	0.625	0.844	0.417
0 (uniform)	0.250	0.500	0.750	0.500
−0.5 (convex)	0.156	0.375	0.656	0.583

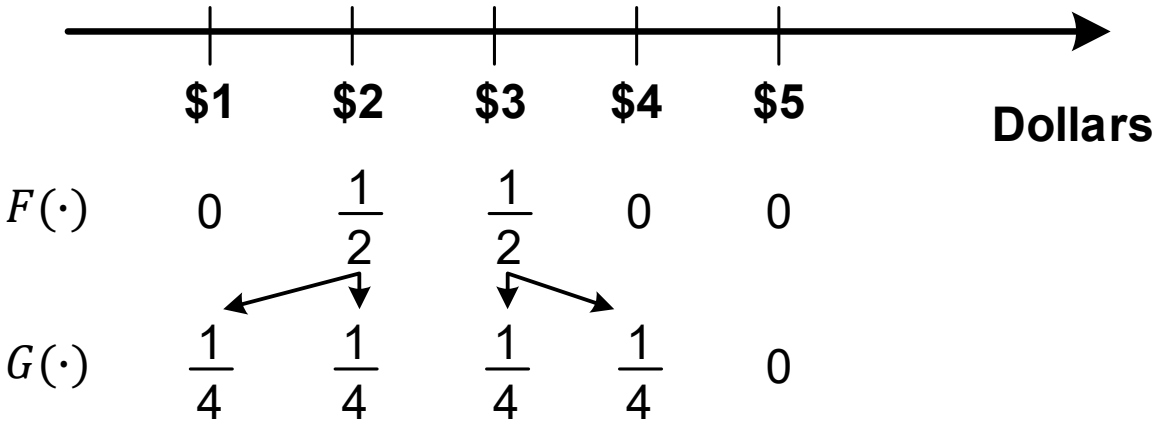
Comparison of Payoff Distributions

- We now focus on the *riskiness* or *dispersion* of a lottery, as opposed to higher/lower returns of lottery (FOSD).
- To focus on riskiness, we assume that the CDFs we compare have the *same mean* (i.e., same expected return).
- ***SOSD***: $F(\cdot)$ SOSD $G(\cdot)$ if, for every non-decreasing and concave function u: $\mathbb{R} \to \mathbb{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

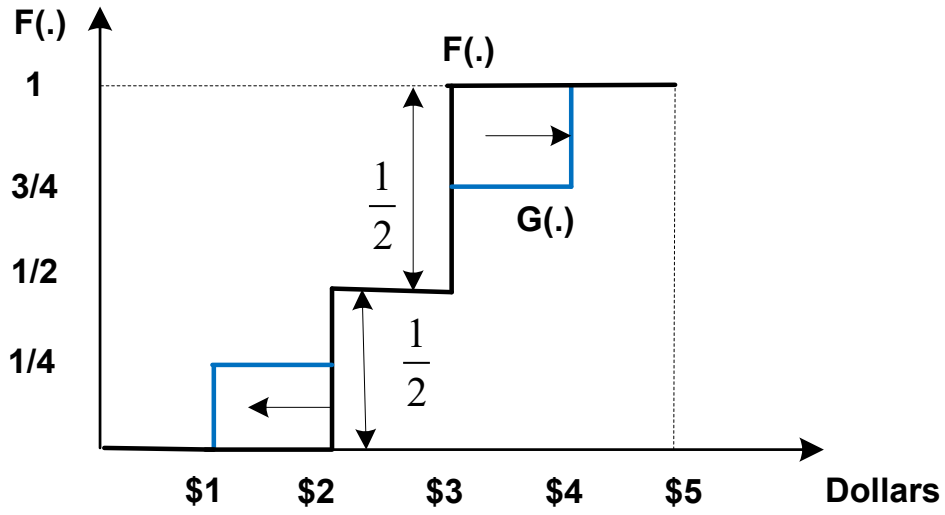
Comparison of Payoff Distributions

- ***Example*** (Mean-Preserving Spread):
  - Let us take lotteries $F(\cdot)$ and $G(\cdot)$ over discrete outcomes.
  - Lottery $G(\cdot)$ spreads the probability weight of lottery $F(\cdot)$ over a larger set of monetary outcomes.
  - The mean is nonetheless unaltered (2.5).
  - For these two reasons, we say that a CDF is a mean-preserving spread of the other.



Comparison of Payoff Distributions

- $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$, but it is riskier than $F(\cdot)$ in the SOSD sense.
- Note that neither FOSD the other
  - $F(\cdot)$ is not above/below $G(\cdot)$ for all x



Comparison of Payoff Distributions

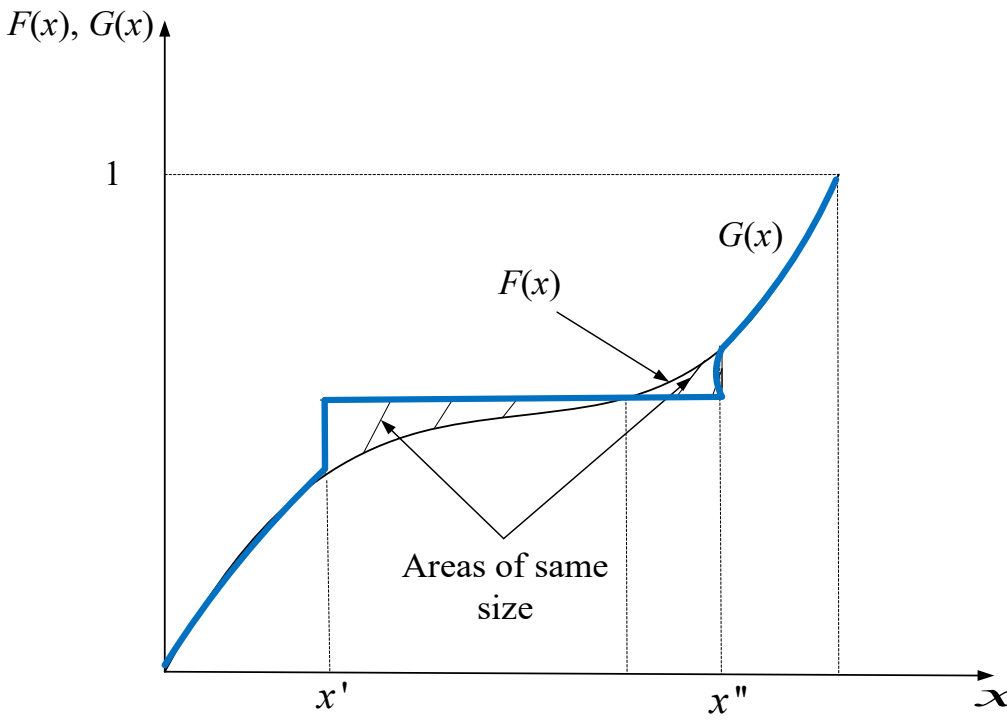
- ***Example*** (Elementary increase in risk):
  - $G(\cdot)$ is an *Elementary Increase in Risk* (EIR) of another CDF $F(\cdot)$ if $G(\cdot)$ takes all the probability weight of an interval $[x', x'']$ and transfers it to the *end points* of this interval, x' and x'', such that the mean of the original lottery is preserved.
  - EIR is a mean-preserving spread (MPS), but the converse is not necessarily true:

$$EIR \underset{\nLeftarrow}{\Rightarrow} MPS$$

  - Hence, if $G(\cdot)$ is an EIR of $F(\cdot)$, then $F(\cdot)$ SOSD $G(\cdot)$.

Comparison of Payoff Distributions

- ***Example*** (continued):
  - both CDFs $F(\cdot)$ and $G(\cdot)$ maintain the same mean.
  - $G(\cdot)$ concentrates more probability at the end points of the interval $[x', x'']$ than $F(\cdot)$.



Comparison of Payoff Distributions

- ***Hazard rate dominance***: The hazard rate of lottery $F(x)$ is

$$HR_F(x) = \frac{f(x)}{1 - F(x)}$$

  - *Intuition*: It measures the instantaneous probability of an event happening at time x given that it did not happen before x.
  - *Example*: a computer stops working at exactly x
  - If $HR_F(x) \leq HR_G(x)$, lottery $F(x)$ dominates $G(x)$ in terms of the hazard rate.

Comparison of Payoff Distributions

- Since $-HR_F(x)$ can be expressed as

$$-HR_F(x) = \frac{d}{dx}\ln\big(1 - F(x)\big)$$

- Solving for $F(x)$,

$$F(x) = 1 - \exp\left(-\int_0^x HR_F(t)dt\right)$$

- Then,

$$F(x) = 1 - \exp\left(-\int_0^x HR_F(t)dt\right)$$

$$\leq 1 - \exp\left(-\int_0^x HR_G(t)dt\right) = G(x)$$

- Thus, $HR_F(x) \leq HR_G(x)$ implies that $F(x)$ FOSD $G(x)$.

Comparison of Payoff Distributions

- ***Reverse hazard rate***: The reverse hazard rate of lottery $F(x)$ is

$$RHR_F(x) = \frac{f(x)}{F(x)}$$

  - *Intuition*: It measures the probability that, conditional on the realized payoff in the lottery being equal or lower than x, the payoff you receive is exactly x.
  - If $RHR_F(x) \geq RHR_G(x)$, lottery $F(x)$ dominates $G(x)$ in terms of the reverse hazard rate.

Comparison of Payoff Distributions

- Integrating both sides, we obtain

$$\int_x^\infty RHR(t)dt = \int_x^\infty \frac{d}{dt}\ln(F(t))\,dt$$
$$= \ln F(\infty) - \ln F(x) = -\ln F(x).$$

- where the last steps use $F(\infty) = 1$ and $\ln(1) = 0$. Solving for $F(x)$,we have

$$F(x) = \exp\left(-\int_x^\infty RHR(t)dt\right)$$

- Therefore, if $RHR_F(x) \geq RHR_G(x)$, then

$$F(x) = \exp\left(-\int_x^\infty RHR_F(t)dt\right) \leq \exp\left(-\int_x^\infty RHR_G(t)dt\right) = G(x)$$

- which simplifies to $F(x) \leq G(x)$.
- That is, RHR dominance implies FOSD dominance; but the converse is not necessarily true.

Comparison of Payoff Distributions

- ***Likelihood ratio***: The likelihood ratio of a lottery $F(x)$ is

$$LR_F = \frac{f(y)}{f(x)}$$

for any two payoffs x and y, where $y > x$.

– $F(x)$ dominates $G(x)$ in terms of likelihood ratio if

$$\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)}$$

Comparison of Payoff Distributions

- LR dominance implies HR dominance:
  - Let us rewrite LR dominance as

$$\frac{g(y)}{g(x)} \leq \frac{f(y)}{f(x)}$$

  - Then, for all x,

$$\int_x^\infty \frac{g(y)}{g(x)}\,dy \leq \int_x^\infty \frac{f(y)}{f(x)}\,dy$$

  - Simplifying

$$\frac{1-G(x)}{g(x)} \leq \frac{1-F(x)}{f(x)} \text{ or } \frac{f(x)}{1-F(x)} \leq \frac{g(x)}{1-G(x)}$$

    which implies $HR_F(x) \leq HR_G(x)$.

Comparison of Payoff Distributions

- Summary:
  - LR dominance implies HR dominance
  - HR and RHR dominance imply FOSD.

Appendix 5.1:
State-Dependent Utility

State-Dependent Utility

- So far the decision maker only cared about the payoff arising from every outcome of the lottery.
- Now we assume that the decision maker cares not only about his monetary outcomes, but also about the ***state of nature*** that causes every outcome.
  - That is, $u_{\text{state } 1}(x) \neq u_{\text{state } 2}(x)$ for given x.

State-Dependent Utility

- Let us assume that each of the possible monetary payoffs in a lottery is generated by an underlying cause (i.e., an underlying state of nature).
- ***Examples***:
  - The monetary payoff of an insurance policy is generated by a car accident
    - State of nature = {car accident, no car accident}
  - The monetary payoff of a corporate stock is generated by the state of the economy
    - State of nature = {economic growth, economic depression}

State-Dependent Utility

- Generally, let $s \in S$ denote a state of nature, where S is a finite set.
- Every state s has a well-defined, objective probability $\pi_s \geq 0$.
- A random variable is function $g: S \to \mathbb{R}$, that maps states into monetary payoffs.

State-Dependent Utility

- ***Examples*** (revisited):
  - *Car accident*: the random variable assigns a monetary value to the state of nature car accident, and to the state of nature no accident.

State of nature	Probability	Monetary payoff
Car accident	π_{accident}	Damage + Deductible – Premium = $1,000
No car accident	$\pi_{\text{no accident}}$	Premium = -$50

State-Dependent Utility

- ***Examples*** (revisited):
  - *Corporate stock*: the random variable assigns a monetary value to the state of nature economic growth, and to the state of nature economic depression.

State of nature	Probability	Monetary payoff
Economic growth	π_{growth}	Dividends, higher price of shares = $250
Economic depression	$\pi_{\text{depression}}$	No dividends, loss if we sell shares = -$125

State-Dependent Utility

- Every random variable $g(\cdot)$ can be used to represent lottery $F(\cdot)$ over monetary payoffs as

$$F(x) = \sum_{\{s:\, g(s) \leq x\}} \pi_s$$

where $\{s:\ g(s) \leq x\}$ represents all those states of nature s that generate a monetary payoff $g(s) \in \mathbb{R}$ below a cutoff payoff x.

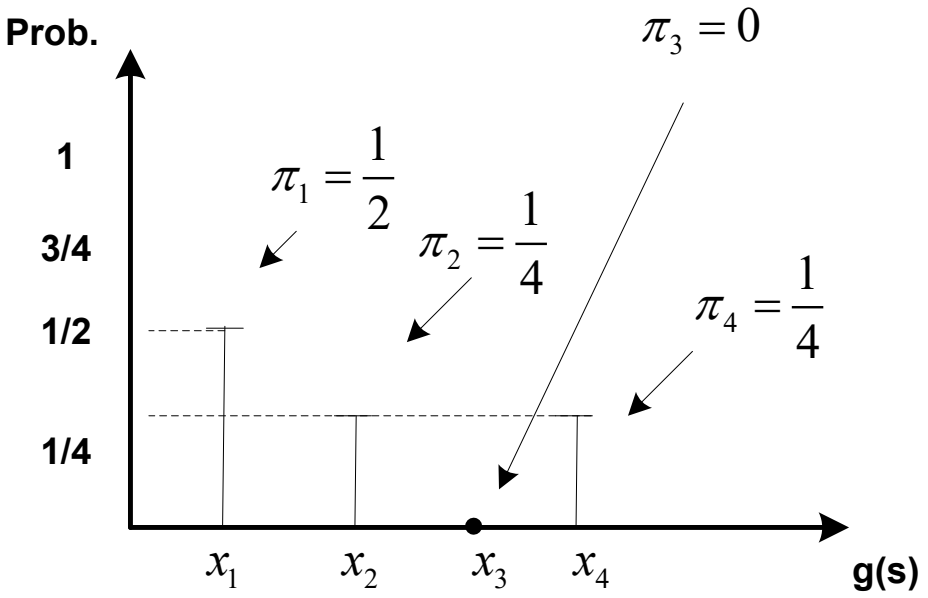
- The random variable $g(\cdot)$ generates a monetary payoff for every state of nature $s \in S$, and since S is finite, we can represent this list of monetary payoffs as

$$(x_1, x_2, \ldots, x_S) \in \mathbb{R}_+^S$$

where x_s is the monetary payoff corresponding to state of nature s.

State-Dependent Utility

- ***Example***:
  - A random variable $g(\cdot)$ describes the monetary outcome associated to the four states of nature $S = \{1,2,3,4\}$.
  - Outcomes are ordered from low to high, i.e., $x_1 \leq x_2 \leq x_3 \leq x_4$.



State-Dependent Utility

- ***Example*** (continued):
  - Hence,

$$F(x_1) = \pi_1 = \frac{1}{2}$$
$$F(x_2) = \pi_1 + \pi_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$F(x_3) = \pi_1 + \pi_2 + \pi_3 = \frac{1}{2} + \frac{1}{4} + 0 = \frac{3}{4}$$
$$F(x_4) = \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

- Disadvantage of $F(x)$:
  - For a given x, we cannot keep track of which state(s) of nature that generated x.

State-Dependent Utility: Extended EU representation

- We now have a preference relation $\succsim$ ranks lists of monetary payoffs $(x_1, x_2, \ldots, x_S) \in \mathbb{R}_+^S$.
- Note the similarity of this setting with that in consumer theory:
  - Preferences over bundles then, preferences over lists of monetary payoffs here.
  - Since $(x_1, x_2, \ldots, x_S) \in \mathbb{R}_+^S$ specifies one payoff for each state of nature, this list is also referred to as *contingent commodities*.

State-Dependent Utility: Extended EU representation

- Preference relation $\gtrsim$ has an ***Extended EU representation*** if for every $s \in S$, there is a function $u_S: \mathbb{R}_+ \to \mathbb{R}$ (mapping the monetary outcome of state s, x_s, into a utility value in $\mathbb{R}$), such that for any two lists of monetary outcomes $(x_1, x_2, \ldots, x_S) \in \mathbb{R}_+^S$ and $(x_1', x_2', \ldots, x_S') \in \mathbb{R}_+^S$,

$$(x_1, x_2, \ldots, x_S) \gtrsim (x_1', x_2', \ldots, x_S') \text{ iff}$$

$$\sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(x_s')$$

- The main difference with the previous sections is that now the Bernoulli utility function is *state-dependent*, $u_s(\cdot)$, whereas in the previous sections it was *state-independent*, $u(\cdot)$.

State-Dependent Utility: Extended EU representation

- Graphical representation:
  - First, at the "certainty line" the decision maker receives the same monetary amount, regardless the state of nature, $x_1 = x_2$.
  - Second, all the (x_1, x_2) pairs on a given ind. curve satisfy $\pi_1 \cdot u_1(x_1) + \pi_2 \cdot u_2(x_2) = \bar{U}$
  - Third, the upper contour set of an ind. curve that passes through point $(\bar{x}_1, \bar{x}_2)$ satisfy

$$\pi_1 \cdot u_1(x_1) + \pi_2 \cdot u_2(x_2) \geq \pi_1 \cdot u_1(\bar{x}_1) + \pi_2 \cdot u_2(\bar{x}_2)$$

    or, more generally, $\sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(\bar{x}_s)$.

State-Dependent Utility: Extended EU representation

- Graphical representation:
  - Fourth, movement along a given ind. curve does not change the decision maker's utility level. Hence, totally differentiating

$$\pi_1 \cdot \frac{\partial u_1(\bar{x}_1)}{\partial x_1} dx_1 + \pi_2 \cdot \frac{\partial u_2(\bar{x}_2)}{\partial x_2} dx_2 = 0$$

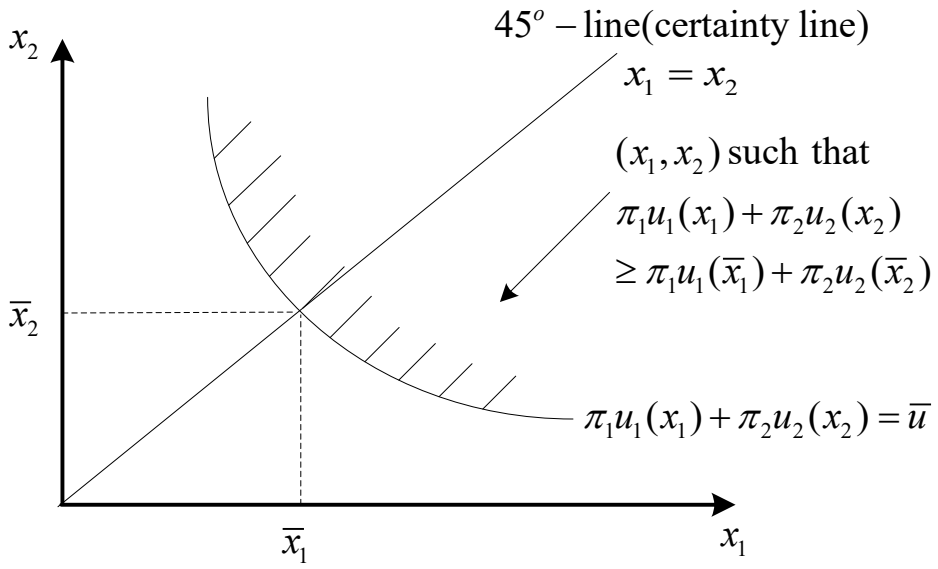
and re-arranging,

$$\frac{dx_2}{dx_1} = -\frac{\pi_1 \cdot \frac{\partial u_1(\bar{x}_1)}{\partial x_1}}{\pi_2 \cdot \frac{\partial u_2(\bar{x}_2)}{\partial x_2}} = -\frac{\pi_1 \cdot u_1'(\bar{x}_1)}{\pi_2 \cdot u_2'(\bar{x}_2)}$$

which represents the slope of the ind. curve, evaluated at point $(\bar{x}_1, \bar{x}_2)$. This is really similar to MRS.

State-Dependent Utility: Extended EU representation

- Graphical representation:
  - The slope of the ind. curve at $(\bar{x}_1, \bar{x}_2)$ is
    $$\frac{dx_2}{dx_1} = -\frac{\pi_1 \cdot u_1'(\bar{x}_1)}{\pi_2 \cdot u_2'(\bar{x}_2)}$$
  - If the Bernoulli utility is state-independent, i.e., $u_1(\cdot) = u_2(\cdot) = \cdots = u_S(\cdot)$, then the slope is
    $$\frac{dx_2}{dx_1} = -\frac{\pi_1}{\pi_2}$$



State-Dependent Utility: Extended EU representation

- ***Example*** (Insurance with state-dependent utility):
  - Start from an initial situation of $(w, w - D)$ without insurance, where D is loss from accident.
  - After insurance is purchased, the decision maker gets a payment of z_1 in state 1, and z_2 in state 2, where $z_1 \leqslant 0$ and $z_2 \leqslant 0$,

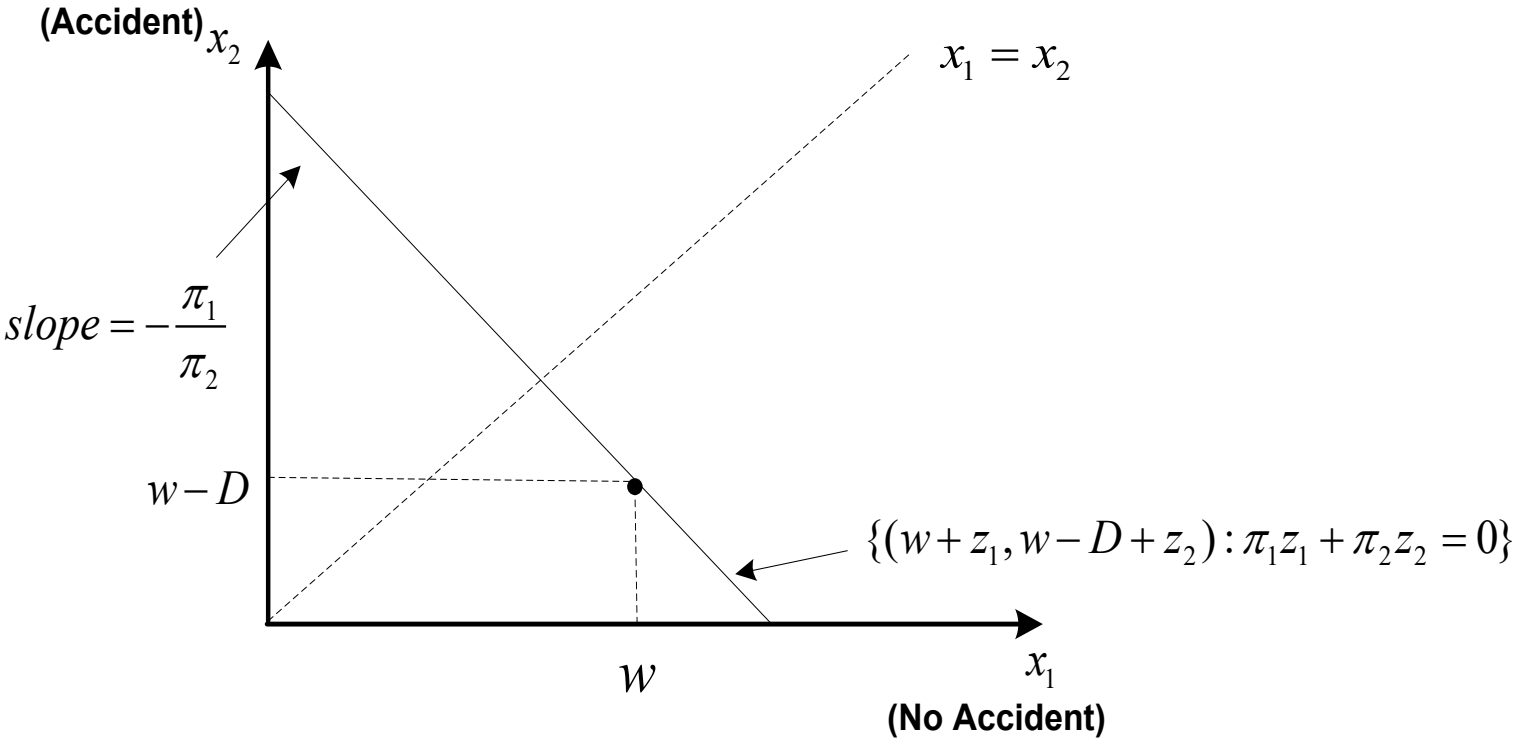
$$(w + z_1, w - D + z_2)$$

  - Moreover, if the policy is actuarially fair, then its expected payoff is zero,

$$\pi_1 z_1 + \pi_2 z_2 = 0$$

State-Dependent Utility: Extended EU representation

- ***Example*** (continued):
  - The budget line is $z_2 = -\frac{\pi_1}{\pi_2} z_1$



State-Dependent Utility: Extended EU representation

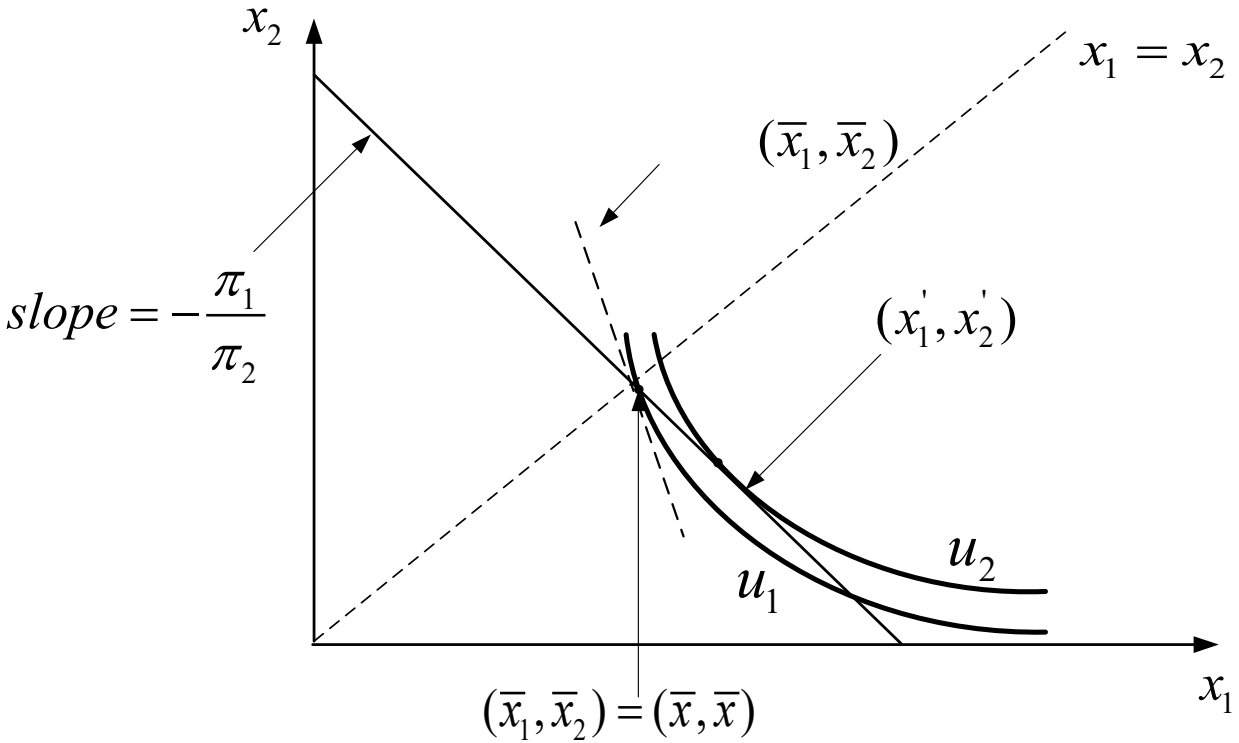
- Without state dependency:
  - Indifference curves are tangent to the budget line at the certainty line, since the slope of the indifference curve is $-\frac{\pi_1}{\pi_2}$.
  - Hence, the decision maker would insure completely since his consumption level is unaffected by the possibility of suffering an accident.

State-Dependent Utility: Extended EU representation

- With state dependency:
  - Indifference curves are NOT tangent to the budget line at the certainty line.
- ***Example*** (continued):
  - The decision-maker prefers a point such as (x_1', x_2') to the certain outcome $(\bar{x}, \bar{x})$.
  - That is, at $(\bar{x}, \bar{x})$ he prefers higher payoffs in state 1 than in state 2 if $u_1'(\bar{x}) > u_2'(\bar{x})$. Otherwise, he would prefer higher payoffs in state 2 than in state 1.

State-Dependent Utility: Extended EU representation

– Note that $u_1'(\bar{x}) > u_2'(\bar{x})$ implies that $\frac{u_1'(\bar{x})}{u_2'(\bar{x})} > 1$ and $-\frac{\pi_1 \cdot u_1'(\bar{x})}{\pi_2 \cdot u_2'(\bar{x})} < -\frac{\pi_1}{\pi_2}$.



State-Dependent Utility: Extended EU representation

- Let us now allow for the possibility that the monetary payoff under state s, x_s, is not a certain amount of money, but a random amount with distribution function $F_s(\cdot)$.
- Hence, all monetary outcomes arising from the S states of world can be described as a lottery $L = (F_1, F_2, \ldots, F_S)$.
- Given this "extended" definition of lotteries, we can then re-write the IA, as the "extended" IA.

State-Dependent Utility: Extended EU representation

- ***Extended IA***: The preference relation satisfies the extended IA if, for any three lotteries L, L', and L'' and $\alpha \in (0,1)$, we have that

$$L \gtrsim L' \text{ iff}$$
$$\alpha L + (1-\alpha)L'' \gtrsim \alpha L' + (1-\alpha)L''$$

- Hence, the "extended" IA is a mere extension of the standard IA to the case of "extended" lotteries $L = (F_1, F_2, \ldots, F_S)$.

State-Dependent Utility: Extended EU representation

- ***Extended EU theorem***: Suppose preferences relation satisfies continuity and the extended IA. Then we can assign a utility function $u_s(\cdot)$ for money in every state s such that for any two lotteries $L = (F_1, F_2, \ldots, F_S)$ and $L' = (F_1', F_2', \ldots, F_S')$ we have

$$L \gtrsim L' \text{ iff}$$

$$\sum_s \left(\int u_s(x_s) dF_s(x_s) \right) \geq \sum_s \left(\int u_s(x_s) dF_s'(x_s) \right)$$

Appendix 5.2:
Subjective Probability Theory

Subjective Probability Theory

- So far we were assuming that probabilities were objective and observable.
- This is not the case in certain cases. People might instead hold probabilistic *beliefs* about the likelihood of a certain event: ***subjective probability***.

Subjective Probability Theory

- Can we deduce subjective probability from actual behavior? Yes!
- Imagine a decision maker who prefers a gamble

($1 in state 1, $0 in state 2) $\succsim$

($0 in state 1, $1 in state 2)

- If the value of money is the same across states, then he must be assigning a higher subjective probability to state 1 than to state 2.

Subjective Probability Theory

- Let us start with some definitions.
- First, we define state s preferences, $\succsim_s$, on state s lotteries $F_s(\cdot)$ by $F_s(\cdot) \succsim F_s'(\cdot)$ if

$$\int u_s(x_s)dF_s(x_s) \geq \int u_s(x_s)dF_s'(x_s)$$

- Hence, the state preferences $(\succsim_1, \succsim_2, \ldots, \succsim_S)$ on state lotteries $(F_1, F_2, \ldots, F_S)$ are ***state uniform*** if

$\succsim_s = \succsim_{s'}$ for any two states s and s'

Subjective Probability Theory

- That is, preferences over lotteries are state uniform if for any two states s and s', the ranking of any two lotteries $F_s(\cdot)$ and $F_s'(\cdot)$ coincides in both states, i.e.,

$$F_s(\cdot) \gtrsim F_s'(\cdot) \text{ or}$$

$$F_s'(\cdot) \gtrsim F_s(\cdot) \text{ or}$$

$$F_s(\cdot) \sim F_s'(\cdot)$$

Subjective Probability Theory

- With state uniformity, $u_s(\cdot)$ and $u_{s'}(\cdot)$ can differ only up to an increasing linear transformation.
- That is, there is a utility function $u(\cdot)$ such that

$$u_s(\cdot) = \pi_s u(\cdot) + \beta_s$$
$$u_{s'}(\cdot) = \pi_{s'} u(\cdot) + \beta_{s'}$$

  for every state s and s', and for every $\pi_s, \pi_{s'} > 0$ and $\beta_s, \beta_{s'} > 0$.
- In words, the ranking between the expected utility of state s and s' remains unaffected.

Subjective Probability Theory

- ***Subjective probabilities EU theorem***:
  - Suppose that a preference relation satisfies continuity and the extended IA, and that preferences over lotteries are state uniform.
  - Then, there are subjective probabilities $(\pi_1, \pi_2, \ldots, \pi_S) \gg 0$ and a utility function $u(\cdot)$ on certain amounts of money, such that for any two lists of monetary amounts $(x_1, x_2, \ldots, x_S)$ and $(x_1', x_2', \ldots, x_S')$,

$$(x_1, x_2, \ldots, x_S) \gtrsim (x_1', x_2', \ldots, x_S') \text{ iff}$$

$$\sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(x_s')$$

Subjective Probability Theory

- *Intuition*: a decision maker prefers the first list of monetary outcomes to the second if the "subjective" expected utility from the first list is larger than or equal to that from the second.
- The predictions of the subjective EU theorem are not necessarily satisfied in all experimental settings.
  - Example: *Ellsberg paradox*

Subjective Probability Theory

- ***Ellsberg paradox***:
  - An urn contains 300 balls: 100 are red and the remaining 200 are either blue or green.
  - We first present the following two gambles to a group of students, asking each of them to choose either gamble A or B.
    - *Gamble A*: $1000 if the ball is red
    - *Gamble B*: $1000 if the ball is blue
  - We next present the following two gambles to the same group of students, asking each of them to choose either gamble C or D.
    - Gamble C: $1000 if the ball is not red
    - Gamble D: $1000 if the ball is not blue

Subjective Probability Theory

- ***Ellsberg paradox*** (continued):
  - Common choices: people choose A to B, and C to D.
  - But these choices violate subjective EU theory!
  - We know that

$$p(\text{Red}) = 1 - p(\text{not Red})$$
$$p(\text{Blue}) = 1 - p(\text{not Blue})$$

  - If gamble A is preferred to B, then we must have

$$p(\text{Red})u(\$1000) > p(\text{Blue})u(\$1000) \Longrightarrow$$
$$p(\text{Red}) > p(\text{Blue})$$

  - And if gamble C is preferred to D, then we must have

$$p(\text{not Red})u(\$1000) > p(\text{not Blue})u(\$1000) \Longrightarrow$$
$$p(\text{not Red}) > p(\text{not Blue})$$

  - But the above two expressions are incompatible.

Appendix 5.3: Ambiguity and Ambiguity Aversion

Ambiguity and Ambiguity Aversion

- Alternative theories that account for the anomaly in the Ellsberg paradox:
  1) expected utility theory with multiple priors (also referred to as *maxmin* expected utility)
  2) rank-dependent expected utility (or *Choquet* expected utility)
- Individuals have ambiguous (unclear) beliefs, rather than objective or subjective beliefs.
- Let f denote an act $f: s \rightarrow x$ from the set of states to the set of outcomes.

Ambiguity and Ambiguity Aversion

- ***Maxmin expected utility*** (MEU):
  - If subjects have too little information to form their priors, one could alternatively allow them to consider a set of priors.
  - If an individual is uncertainty averse, he will choose lottery f over another lottery g if the former provides a higher expected utility than the latter according to his worst possible prior.

Ambiguity and Ambiguity Aversion

- ***Uncertainty aversion***: Consider an individual who is indifferent between two lotteries f and g. Then, he is *uncertainty averse* if he weakly prefers the compound lottery $\alpha f + (1-\alpha)g$ to lottery f, where $\alpha \in (0,1)$.
  - *Intuition*: a decision maker who is uncertainty averse has a preference for mixing (or hedging), since the compound lottery becomes at least as valuable as either of the two lotteries alone.

Ambiguity and Ambiguity Aversion

- ***Certainty-independence***: For any two lotteries f and g and a constant act k (i.e., a certain outcome or a lottery that remains constant across all states), the decision maker weakly prefers lottery f to g if and only if he prefers $\alpha f + (1-\alpha)k$ to $\alpha g + (1-\alpha)k$, where $\alpha \in (0,1)$.
  - Certainty-independence axiom relaxes the IA as it only requires that preferences over two lotteries to be unaffected when each lottery is mixed with a certain outcome k.

Ambiguity and Ambiguity Aversion

- A decision maker weakly prefers lottery f to g if and only if

$$\min_{p \in C} \int_S u\big(f(s)\big) dp(s) \geq \min_{p \in C} \int_S u\big(g(s)\big) dp(s)$$

- That is, the individual evaluates the expected utility of lotteries f and g according to each of his multiple priors $p \in C$, and then selects the lottery that yields the highest of the worst possible expected utilities.

Ambiguity and Ambiguity Aversion

- ***Example***:
  - Consider a decision maker with Bernoulli utility function $u(x) = \sqrt{x}$, where $x \geq 0$ denotes monetary amounts.
  - Assume that the decision maker faces two lotteries

$$L_A = (\$1, \$100)$$
$$L_B = (\$3, \$5)$$

  - Also, assume that the decision maker's priors are

$$(p_A, 1 - p_A) \text{ for } L_A$$
$$(p_B, 1 - p_B) \text{ for } L_B$$

Ambiguity and Ambiguity Aversion

- ***Example*** (continued):
  - According to MEU, the decision maker chooses lottery L_B if

$$\min_{p_B}[p_B\sqrt{3} + (1 - p_B)\sqrt{5}]$$
$$\geq \min_{p_A}[p_A\sqrt{1} + (1 - p_A)\sqrt{100}]$$

  - If the decision maker does not have any available information with which to update his priors, priors can take values $(p_A, p_B) \in [0,1]$.
  - It is possible that in his most pessimistic belief, he receives the lowest monetary amount with probability one.

Ambiguity and Ambiguity Aversion

- ***Example*** (continued):
  - Then, with argmin $p_B = 1$,

$$\min_{p_B}[p_B\sqrt{3} + (1 - p_B)\sqrt{5}] = \sqrt{3}$$

  - Similarly, with argmin $p_A = 1$,

$$\min_{p_A}[p_A\sqrt{1} + (1 - p_A)\sqrt{100}] = 1$$

  - Hence a decision maker with MEU preferences selects lottery L_B because $\sqrt{3} \geq 1$.

Ambiguity and Ambiguity Aversion

- ***Choquet expected utility*** (CEU):
  - Define beliefs with the use of capacities.
  - A capacity is defined as a real-valued function $v(\cdot)$ from a subset of the state space S to $[0,1]$, with the normalization $v(\emptyset) = 0$ and $v(S) = 1$.
  - If the capacity $v(\cdot)$ satisfies monotonicity, $v(A) \geq v(B)$, where A is a superset of B.
  - We cannot use a standard integral over states since the capacity $v(\cdot)$ does not correspond to our notion of beliefs.

Ambiguity and Ambiguity Aversion

– A decision maker weakly prefers f to g if the *Choquet integrals* satisfy

$$\int_S u\big(f(S)\big)dv(S) \geq \int_S u\big(g(S)\big)dv(S)$$

– The CEU and MEU models are connected if we impose the uncertainty aversion axiom in CEU context. For that we need that capacity $v(\cdot)$ satisfies *supermodularity*, i.e.,

$$v(A \cup B) - v(B) \geq v(A \cup C) - v(C)$$

where C is a subset of B, i.e., $C \subset B$.

Ambiguity and Ambiguity Aversion

- ***Example***:
  - While the use of Choquet integrals is involved, the literature often uses "simple" capacities.
  - A simple capacity on state space S can be understood as a convex combination between two extreme capacities:
    1. a standard probability weight on A, $p(A) \in [0,1]$.
    2. the "complete ignorance" capacity w, where $w(S) = 1$ and $w(A) = 0$ for every $A \subseteq S$.

Ambiguity and Ambiguity Aversion

- ***Example*** (continued):
  - Formally, simple capacities are defined as
    $$v(A) = \lambda p(A) + (1 - \lambda)w(A)$$
    for every $A \subseteq S$ and where $\lambda \in [0,1]$.
  - Parameter λ denotes the individual's degree of confidence on $p(A)$, while $(1 - \lambda)$ captures his degree of ambiguity about $p(A)$.
  - For further reading, see Haller (2000) and Aflaki (2013).

Ambiguity and Ambiguity Aversion

- Further reading:
  - Choquet, G. (1953). Theory of capacities. *Ann. Inst. Fourier* (*Grenoble*) 5 131-295.
  - Dow, J. and S. Werlang. (1992). Uncertainty aversion, risk aversion, and the optimal choice of portfolio. *Econometrica,* (1), 197.
  - Epstein, L. and T. Wang. (1994). Intertemporal Asset Pricing under Knightian Uncertainty. *Econometrica,* (2), 283-322.
  - Hansen, L., Sargent, T. (2001). “Robust Control and Model Uncertainty”. American Economic Review 91, 60-66.
  - Machina, M. (2014). *Handbook of the economics of risk and uncertainty* (First edition). Elsevier.
  - Mukerji, S. and J. Tallon (2004). Ambiguity aversion and the absence of wage indexation. *Journal of Monetary Economics,* (3), 653-670.
  - Nishimura, K. and H. Ozaki. (2004). Search and knightian uncertainty. *Journal of Economic Theory,* (2), 299-333.
  - Schmeidler, D. (1989). Subjective Probability and Expected Utility Without Additivity. *Econometrica*, 57, 571-587.
  - Uppal, R. and T. Wang. (2003). Model Misspecification and Underdiversification. *Journal of Finance,* (6), 2465-2486.

Advanced Microeconomic Theory

Subjective Probability Theory

Outline

- Objective and subjective probabilities
- States, acts, and bets
- Eliciting beliefs from choices
- Subjective expected utility
- Applications: insurance and betting
- The Ellsberg paradox

Subjective Probability Theory

- So far, probabilities were ***objective***: known to the individual and to the analyst.
  - A fair coin, a roulette wheel, a lottery ticket.
- Most uncertainty in real life comes without known probabilities.
  - Will it rain tomorrow?
  - Will a new drug pass its clinical trials?
  - Will a start-up survive its first year?

Subjective Probability Theory

- Can we still use expected utility when nobody hands the individual a probability?
- ***Subjective probability theory*** (Savage, 1954; Anscombe and Aumann, 1963): yes, if preferences satisfy a few axioms.
- Probabilities are then ***derived from choices***, not assumed.

States, Acts, and Bets

Subjective Probability Theory

- ***States of the world***: $s = 1, 2, \ldots, S$.
  - Exactly one state occurs, and the individual does not control which one.
- ***Act***: a list of outcomes, one for each state,

$$f = (x_1, x_2, \ldots, x_S)$$

  - Choosing an act is choosing a random variable whose probabilities are not given.

Subjective Probability Theory

- ***Example***: two states, rain and sun. Should you carry an umbrella? (Payoffs in utility units.)

Act	**Rain**	**Sun**
Carry umbrella	8	8
Leave it home	0	10

Subjective Probability Theory

- If π is the probability of rain the individual has in mind, carrying the umbrella is better when
  $$8 > \pi \cdot 0 + (1 - \pi) \cdot 10 \quad \text{if and only if} \quad \pi > 0.2$$
- The choice reveals whether the individual believes rain is more or less likely than 0.2.
  - Choices carry information about beliefs.

Subjective Probability Theory

- A ***bet*** on an event pays a prize if the event occurs, and zero otherwise.
- If the individual prefers a bet on rain to a bet on sun with the same prize, the individual acts as if rain is more likely.
- Comparing bets therefore ranks events by their likelihood.

Eliciting Beliefs from Choices

Subjective Probability Theory

- ***Example***: a bet pays \$100 if it rains, \$0 otherwise.
- The individual is indifferent between the bet and \$36 for sure.
- If $u(x) = \sqrt{x}$, then

$$\pi\sqrt{100} + (1-\pi)\sqrt{0} = \sqrt{36}$$

$$10\pi = 6, \qquad \text{so} \quad \pi = 0.6$$

Subjective Probability Theory

- A risk-neutral individual who is indifferent between the same bet and \$40 holds

$$100\pi = 40, \qquad \text{so} \quad \pi = 0.4$$

- The certainty equivalent alone does not reveal π.
  - We need the individual's utility function too: beliefs and risk attitudes are intertwined.

Subjective Probability Theory

- Once π is known, every act can be ranked.
- ***Example***: with π = 0.6 and $u(x) = \sqrt{x}$, the act paying \$64 if rain and \$16 if sun has certainty equivalent

$$\text{CE} = [0.6\sqrt{64} + 0.4\sqrt{16}]^2 = 6.4^2 = 40.96$$

  - Below its expected value, 0.6(64) + 0.4(16) = \$44.80, because of risk aversion.

Subjective Expected Utility

Subjective Probability Theory

- Anscombe and Aumann (1963) allow the outcome in each state to be an objective lottery L_s.
  - Example: in state s, a ticket in a raffle with known odds.
- An act is then a list of lotteries,

$$(L_1, \ldots, L_S)$$

Subjective Probability Theory

- If preferences over acts satisfy continuity and the independence axiom, there are functions u_s such that

$$U(L_1, \ldots, L_S) = \sum_{s=1}^{S} \sum_{x} L_s(x) u_s(x)$$

- This is ***extended expected utility*** (MWG, Section 6.F).
  - The utility function u_s can depend on the state.

Subjective Probability Theory

- Problem: state-dependent utility cannot be separated from probabilities.
- ***Example***: these two specifications rank every act in the same way:

$$\pi = (\frac{1}{2}, \frac{1}{2}), \qquad u_1(x) = 2\sqrt{x}, \qquad u_2(x) = \sqrt{x}$$

$$\pi = (\frac{2}{3}, \frac{1}{3}), \qquad u(x) = \sqrt{x}$$

Subjective Probability Theory

- Indeed, for every act,

$$\frac{1}{2}2\sqrt{x_1}+\frac{1}{2}\sqrt{x_2}=\frac{3}{2}\left[\frac{2}{3}\sqrt{x_1}+\frac{1}{3}\sqrt{x_2}\right]$$

- The right side is a positive multiple of the second specification, so both rank acts identically.
  - Choices cannot tell us whether the individual believes rain is more likely or enjoys money more when it rains.

Subjective Probability Theory

- ***State uniformity***: the ranking of lotteries is the same in every state,

  $L \gtrsim_s L'$ for some s implies $L \gtrsim_{s'} L'$ for every s'

- Under state uniformity, every u_s is a positive affine transformation of a single u,

$$u_s(x) = \pi_s u(x) + \beta_s$$

  – where $\pi_s > 0$ and β_s is any real number.

Subjective Probability Theory

- Substituting into extended expected utility,

$$\sum_s \sum_x L_s(x) u_s(x) = \sum_s \pi_s \sum_x L_s(x) u(x) + \sum_s \beta_s$$

- The last sum is a constant, so it does not affect the ranking of acts.
- Dividing every π_s by their sum does not affect it either:

$$\pi_s \text{ becomes } \frac{\pi_s}{\sum_k \pi_k}$$

Subjective Probability Theory

- ***Subjective expected utility theorem***: under the axioms and state uniformity,

$$U(L_1, \ldots, L_S) = \sum_{s=1}^{S} \pi_s \sum_x L_s(x) u(x)$$

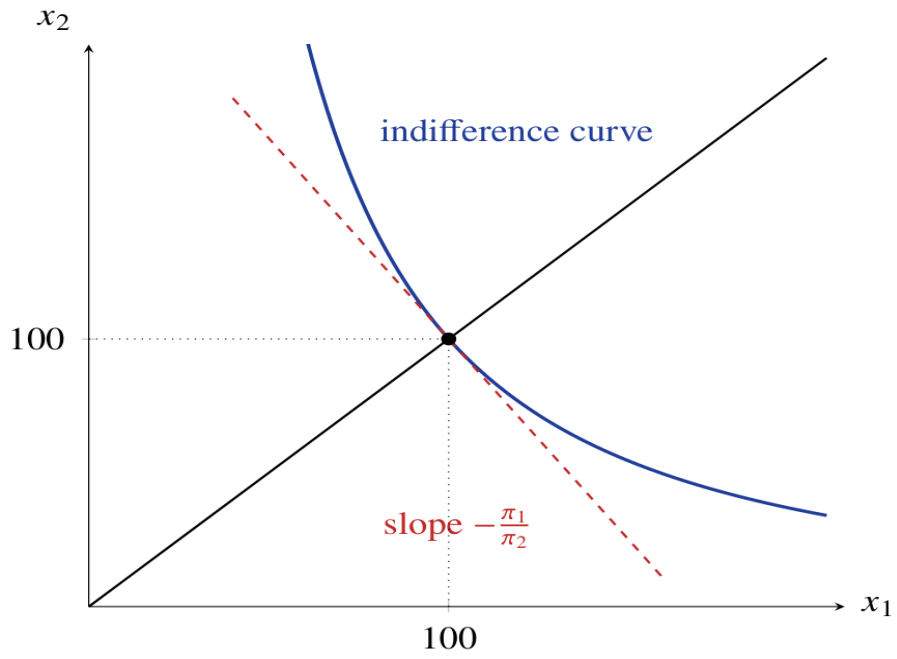
  - with $\pi_s > 0$ and $\pi_1 + \cdots + \pi_S = 1$.
- π_s is the individual's ***subjective probability*** of state s.
  - It is unique, and u is unique up to a positive affine transformation.

Subjective Probability Theory

- With two states, an indifference curve is
$$\pi_1 u(x_1) + \pi_2 u(x_2) = \bar{u}$$
- Its slope is
$$\text{MRS} = \frac{\pi_1 u'(x_1)}{\pi_2 u'(x_2)}$$
- Along the certainty line, $x_1 = x_2$, the slope is $\text{MRS} = \frac{\pi_1}{\pi_2}$: beliefs alone.

Subjective Probability Theory

- A more optimistic belief about state 1 makes indifference curves steeper at the certainty line.



Applications: Insurance and Betting

Insurance with Subjective Beliefs

- Wealth W; a loss L occurs with subjective probability π.
- Coverage c costs qc, where q is the price per dollar of coverage.
- The individual solves

$$\max_{c \geq 0} \pi\, u(W - L + (1-q)c) + (1-\pi)u(W - qc)$$

Insurance with Subjective Beliefs

- The first-order condition is

$$\frac{\pi(1-q)}{W-L+(1-q)c} = \frac{(1-\pi)q}{W-qc}$$

- Solving for c,

$$c^* = \frac{\pi(1-q)W-(1-\pi)q(W-L)}{q(1-q)}$$

  – when the right side is positive; otherwise, $c^* = 0$.

Insurance with Subjective Beliefs

- ***Example***:

$W = 100, \qquad L = 50, \qquad q = 0.2, \qquad u(x) = \ln x$

- Then

$$c^* = \frac{90\pi - 10}{0.16}$$

- Full insurance, $c^* = L$, arises only when $\pi = q$: the belief equals the price.

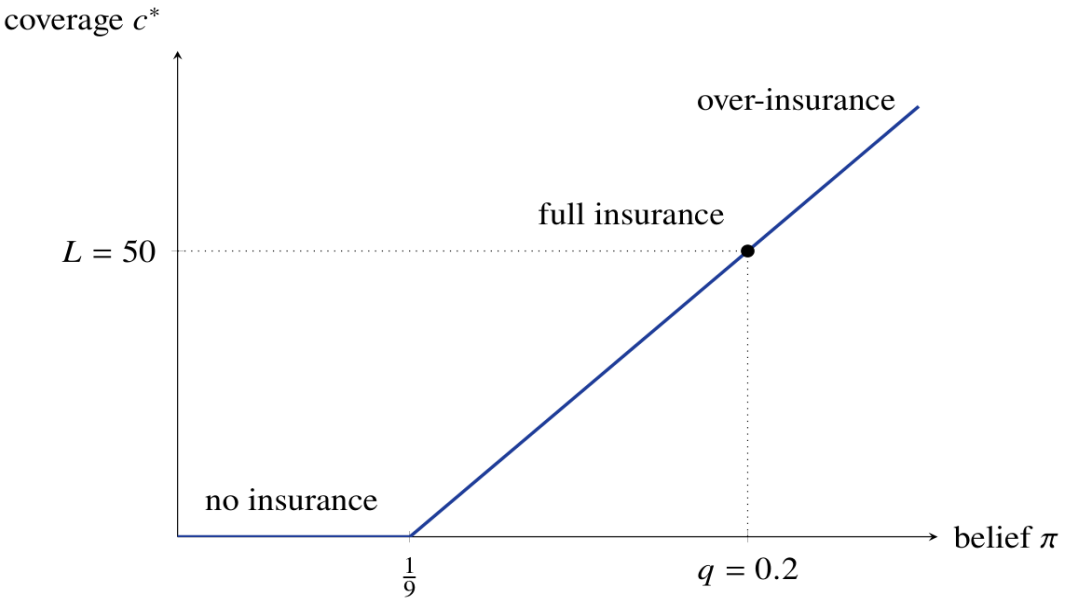
Insurance with Subjective Beliefs

- Same price, different beliefs, different coverage:

Belief π	**0.10**	**0.12**	**0.15**	**0.18**	**0.20**	**0.22**
Coverage c^*	0	5.00	21.88	38.75	50.00	61.25

Insurance with Subjective Beliefs

- No coverage if $\pi \leq \frac{1}{9}$; partial coverage between 1/9 and q; over-insurance if $\pi > q$.



Betting with Different Beliefs

- Two investors, each with $100 in state 1 and $100 in state 2, and $u(x) = \ln x$.
- A bet of size b gives investor 1 an extra b in state 1 and takes b in state 2.
- Investor 1 believes state 1 has probability 0.6; investor 2 believes it has probability 0.3.

Betting with Different Beliefs

- An investor with belief π on state 1 solves

$$\max_b \pi \ln(100 + b) + (1 - \pi)\ln(100 - b)$$

$$b^* = 100(2\pi - 1)$$

- Investor 1 wants $b^* = 20$.
- Investor 2 wants $b^* = -40$: a bet of 40 on state 2.

Betting with Different Beliefs

- At $b = 20$, both investors gain relative to not betting:

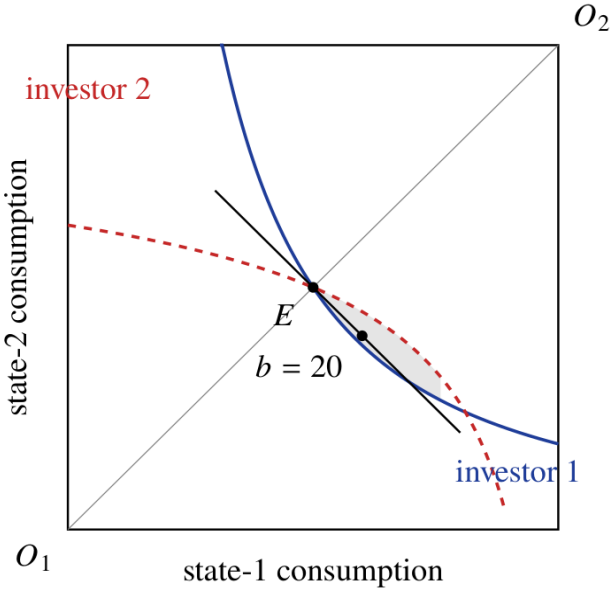
  $0.6\ln 120 + 0.4\ln 80 \approx 4.625 > \ln 100 \approx 4.605$

  $0.3\ln 80 + 0.7\ln 120 \approx 4.666 > \ln 100 \approx 4.605$

- Trade arises from different beliefs alone: same wealth, same utility function, no risk to share.

Betting with Different Beliefs

- Slopes of the indifference curves through E: $-\frac{0.6}{0.4} = -1.5$ and $-\frac{0.3}{0.7} \approx -0.43$.
  - The shaded lens contains the bets both investors prefer.



The Ellsberg Paradox

The Ellsberg Paradox

- An urn has 300 balls: 100 red, and 200 blue or green in an unknown proportion.
- Bets pay $1,000 if:
  - A: red. B: blue.
  - C: blue or green. D: red or green.
- Most people choose A over B and C over D (Ellsberg, 1961).

The Ellsberg Paradox

- Payoffs, in dollars:

Bet	Red (100)	Blue	Green
A	1,000	0	0
B	0	1,000	0
C	0	1,000	1,000
D	1,000	0	1,000

The Ellsberg Paradox

- Can a single prior explain these choices? With $p_R = \frac{1}{3}$, SEU requires
  - $A \succ B$:

$$p_R > p_B$$

  - $C \succ D$:

$$p_B + p_G > p_R + p_G, \qquad \text{so } p_B > p_R$$

- A contradiction: no subjective probabilities rationalize both choices.

The Ellsberg Paradox

- Bets A and B differ from C and D only in the green column, where each pair pays the same.
  - SEU (the sure-thing principle) says that common column should not matter.
- Yet A and C have known probabilities, 1/3 and 2/3, while B and D do not.
- People dislike unknown probabilities: ***ambiguity aversion***.

Summary

- Subjective probabilities are read from choices over bets and acts.
- State uniformity separates beliefs from tastes, giving subjective expected utility.
- Different beliefs alone change insurance demand and create trade.
- The Ellsberg paradox shows choices that no single prior can explain.

References

- Anscombe, F. J., and R. J. Aumann (1963) "A Definition of Subjective Probability." Annals of Mathematical Statistics 34(1): 199–205.
- Ellsberg, D. (1961) "Risk, Ambiguity, and the Savage Axioms." Quarterly Journal of Economics 75(4): 643–669.
- Mas-Colell, A., M. D. Whinston, and J. R. Green (1995) Microeconomic Theory. Oxford University Press, Section 6.F.
- Savage, L. J. (1954) The Foundations of Statistics. Wiley.

Advanced Microeconomic Theory

Alternatives to Subjective Expected Utility

Outline

- Risk and ambiguity
- Maxmin expected utility
- Choquet expected utility
- The smooth ambiguity model
- Comparing the models

Risk and Ambiguity

- ***Risk***: the probabilities of outcomes are known (a fair coin).
- ***Ambiguity***: they are not (an urn with an unknown mix of balls).
  - Knight (1921) calls this uncertainty.
- Subjective expected utility (SEU) treats both alike: every event gets one probability.

Risk and Ambiguity

- In the Ellsberg urn, most people choose A over B and C over D.
  - No single prior explains both choices.
- Three ways to accommodate these choices:
  - Many priors, evaluated at the worst one (maxmin).
  - Non-additive probabilities (Choquet).
  - A prior over priors (smooth ambiguity).

Maxmin Expected Utility

Maxmin Expected Utility

- Gilboa and Schmeidler (1989): the individual holds a closed, convex set of priors C and evaluates each act f by

$$U(f) = \min_{p \in C} \sum_{s} p_s u(f(s))$$

- Each act is judged by its worst expected utility in C.
  - If C has a single prior, MEU coincides with SEU.

Maxmin Expected Utility

- MEU keeps weak order, continuity, and monotonicity, and weakens independence to:
- ***Certainty independence***: for every constant act h and $\alpha \in [0,1]$, $f \gtrsim g$ if and only if

$$\alpha f + (1-\alpha)h \gtrsim \alpha g + (1-\alpha)h$$

  - Independence is required only when mixing with an act whose payoff does not depend on the state.

Maxmin Expected Utility

- ***Uncertainty aversion***: for every $\alpha \in [0,1]$,
$$f \sim g \quad \text{implies} \quad \alpha f + (1-\alpha) g \gtrsim f$$
- Mixing two equally good acts can only help: the mixture ***hedges*** ambiguity.
  - An SEU individual is indifferent to such mixtures.

Hedging in the Ellsberg Urn

- Set of priors:

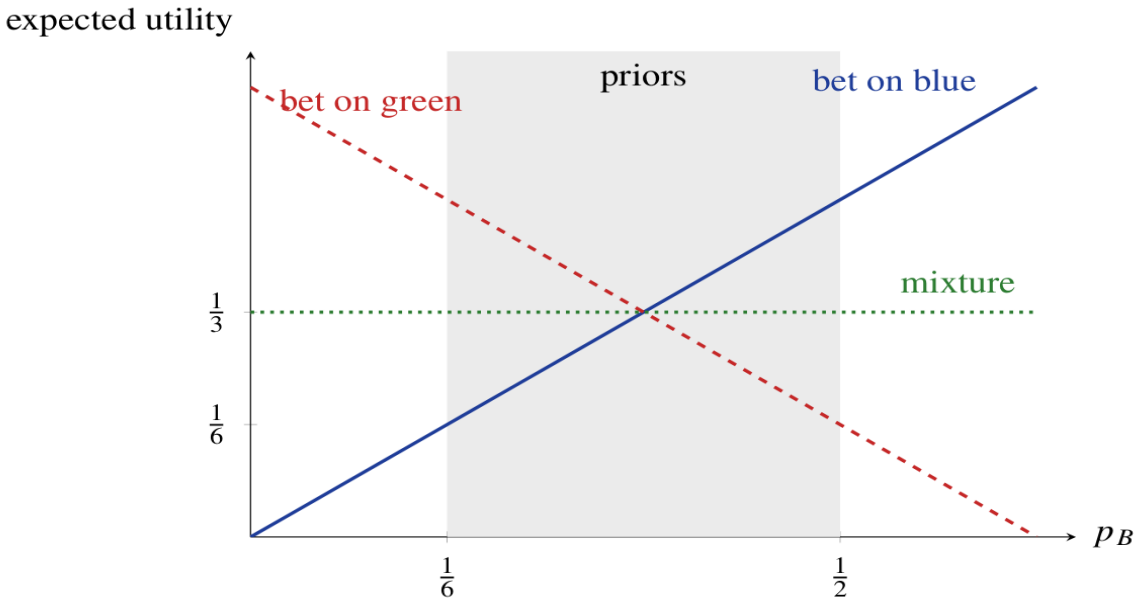
$$p_R = \frac{1}{3}, \qquad p_B \in [\frac{1}{6}, \frac{1}{2}], \qquad p_G = \frac{2}{3} - p_B$$

- Bet on blue: worst case $\min_{p_B} p_B = \frac{1}{6}$.
- Bet on green: worst case $\min_{p_B} p_G = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.
- Flip a fair coin to decide which bet to hold:

$$\frac{1}{2} p_B + \frac{1}{2} p_G = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

Hedging in the Ellsberg Urn

- The mixture yields 1/3 under every prior, more than either bet's worst case, 1/6.



MEU and the Ellsberg Paradox

- With $u(\$1{,}000) = 1, \;\; u(\$0) = 0$ and the same set of priors:

$$U(A) = \frac{1}{3}$$

$$U(B) = \min_{p \in C} p_B = \frac{1}{6}$$

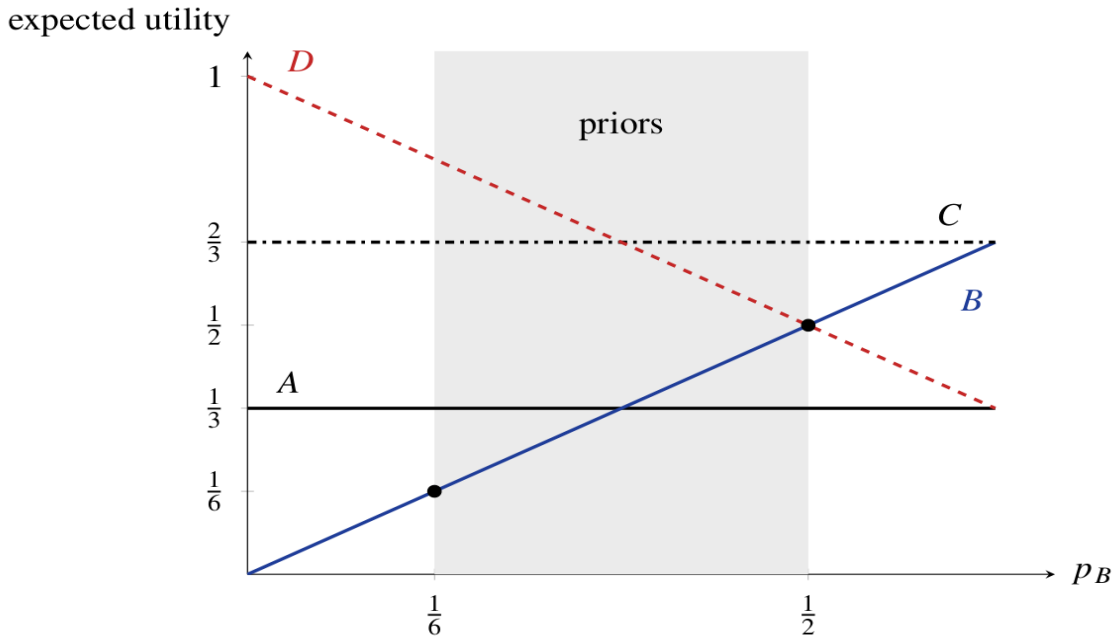
$$U(C) = \frac{2}{3}$$

$$U(D) = \frac{1}{3} + \min_{p \in C} p_G = \frac{1}{2}$$

- So A $\succ$ B and C $\succ$ D, as most people choose.

MEU and the Ellsberg Paradox

- Bets B and D are evaluated at opposite ends of the set of priors.



MEU: The Size of the Set of Priors

- Two states, low and high; p is the probability of low; $u(x) = x$.
  - L_A pays \$1 if low and \$100 if high.
  - L_B pays \$3 if low and \$5 if high.
- If $C = [0.3, 0.7]$, the worst prior for both is $p = 0.7$:

$$U(L_A) = 0.7(1) + 0.3(100) = 30.7$$
$$U(L_B) = 0.7(3) + 0.3(5) = 3.6$$

MEU: The Size of the Set of Priors

- A larger set of priors makes the individual more cautious:

Set of priors	$U(L_A)$	$U(L_B)$	**Choice**
$C = \{0.5\}$ (SEU)	50.5	4.0	L_A
$C = [0.3, 0.7]$	30.7	3.6	L_A
$C = [0, 1]$	1	3	L_B

MEU: The Size of the Set of Priors

- With complete ignorance, $C = [0, 1]$, MEU ranks acts by their worst outcome (maximin).
- The set of priors measures ambiguity, and the worst-case rule is the attitude toward it.
  - MEU combines both into a single object, C.

MEU and No Trade

- Dow and Werlang (1992): an asset pays \$1 in state H and \$0 otherwise, at price q.
- A risk-neutral individual with priors $p \in [0.4, 0.6]$ on state H.
  - Buy one unit, worst case:

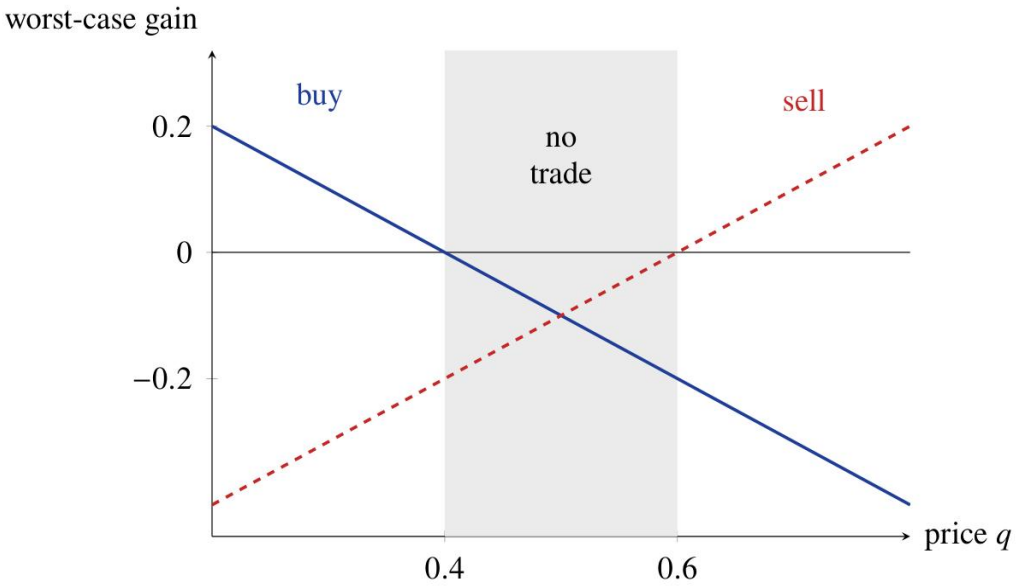
$$\min_p (p - q) = 0.4 - q$$

  - Sell one unit, worst case:

$$\min_p (q - p) = q - 0.6$$

MEU and No Trade

- For $0.4 < q < 0.6$, both positions lose in the worst case: the individual neither buys nor sells.
  - An SEU individual trades at every $q \neq p$.



Choquet Expected Utility

Choquet Expected Utility

- Schmeidler (1989) replaces probabilities with a ***capacity*** v, which assigns a number to each event:

$$v(\emptyset) = 0, \qquad v(S) = 1$$
$$A \subseteq B \quad \text{implies} \quad v(A) \leq v(B)$$

- v need not be additive: for disjoint events, $v(A \cup B)$ can differ from $v(A) + v(B)$.

Choquet Expected Utility

- With two states and $x_a \geq x_b$, the Choquet integral is

$$CEU(f) = v(a)u(x_a) + [1 - v(a)]u(x_b)$$

- The better outcome gets weight $v(a)$; the worse outcome gets the rest.
  - If v is additive, this is SEU.

Choquet Expected Utility

- In general, order outcomes from best to worst,

$$x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(S)}$$

- let E_k be the states with the k best outcomes, and $E_0 = \emptyset$:

$$CEU(f) = \sum_{k=1}^{S} [v(E_k) - v(E_{k-1})]u(x_{(k)})$$

Choquet Expected Utility

- ***Example***: $v(a) = 0.4, \quad v(b) = 0.5$, with $u(x) = x$.
  - \$100 in state a, \$0 in b: $CEU = 0.4(100) = 40$.
  - \$100 in state b, \$0 in a: $CEU = 0.5(100) = 50$.
  - \$100 in a, \$60 in b: $CEU = 0.4(100) + 0.6(60) = 76$.
  - \$60 in a, \$100 in b: $CEU = 0.5(100) + 0.5(60) = 80$.
- Since $v(a) + v(b) = 0.9 < 1$, the missing 0.1 always goes to the worse outcome: pessimism.

Choquet Expected Utility

- A capacity is ***convex*** (supermodular) if, for all events A and B,

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$$

- Schmeidler (1989): with CEU, uncertainty aversion holds if and only if v is convex.
  - In the example,

$$v(a) + v(b) = 0.9 \leq v(S) + v(\emptyset) = 1$$

CEU and the Ellsberg Paradox

- Take the capacity

$$v(R) = \frac{1}{3}, \qquad v(B) = v(G) = \frac{1}{6}$$

$$v(B \cup G) = \frac{2}{3}, \qquad v(R \cup B) = v(R \cup G) = \frac{1}{2}$$

- It is monotone and convex.

CEU and the Ellsberg Paradox

- Each bet pays $1,000 on one event, so its CEU is the capacity of that event:

Bet	Pays on	CEU
A	Red	1/3
B	Blue	1/6
C	Blue or green	2/3
D	Red or green	1/2

CEU and MEU

- The ***core*** of a capacity is the set of priors that give every event at least its capacity,
  $\text{core}(v) = \{p : p(E) \geq v(E) \text{ for every event } E\}$
- If v is convex, CEU equals MEU with $C = \text{core}(v)$.
  - In the Ellsberg example, the core is

$$\{(\frac{1}{3}, p_B, \frac{2}{3} - p_B) : p_B \in [\frac{1}{6}, \frac{1}{2}]\}$$

CEU and MEU

- This is the set of priors used in the MEU slides, which is why CEU and MEU give the same values for A, B, C, and D.
- Not every MEU set of priors is the core of a capacity, so MEU is more general than convex CEU.

A Simple Capacity

- Start from a prior p and a confidence level $\lambda \in [0, 1]$:

$$v(A) = \lambda p(A) \text{ for } A \neq S, \qquad v(S) = 1$$

- Then

$$CEU(f) = \lambda E_p[u(f)] + (1 - \lambda) \min_s u(f(s))$$

  - $\lambda = 1$: SEU. $\lambda = 0$: worst outcome only.

A Simple Capacity

- ***Example***: $\lambda = 0.8, \quad p = 0.5$, and an act paying \$100 if rain and \$60 if sun:

$$0.8(0.5 \cdot 100 + 0.5 \cdot 60) + 0.2 \cdot 60 = 64 + 12 = 76$$

- Below its SEU of 80: the individual shifts weight 0.2 to the worst outcome.

The Smooth Ambiguity Model

Smooth Ambiguity

- Klibanoff, Marinacci, and Mukerji (2005):

$$U(f) = \sum_{\theta} \mu(\theta) \varphi\left(\sum_{s} p_{\theta}(s) u(f(s))\right)$$

  - p_{θ}: possible priors; μ: the individual's belief over them.
  - φ: attitude toward ambiguity.

Smooth Ambiguity

- $\varphi(t) = t$: SEU with the average prior.
- φ concave: ambiguity aversion, since spreads in expected utility across priors are penalized.
- As φ becomes infinitely concave, the model approaches MEU over the priors in the support of μ.
- Ambiguity (μ) and the attitude toward it (φ) are separate objects.

Smooth Ambiguity and Ellsberg

- Let p_B be 1/6 or 1/2 with equal probability, and
$$\varphi(t) = -e^{-\alpha t}, \qquad \alpha > 0$$
- The certainty equivalent of bet B, in units of u, solves
$$\varphi(t_B) = \frac{1}{2}\varphi(\frac{1}{6}) + \frac{1}{2}\varphi(\frac{1}{2})$$

Smooth Ambiguity and Ellsberg

- Solving,

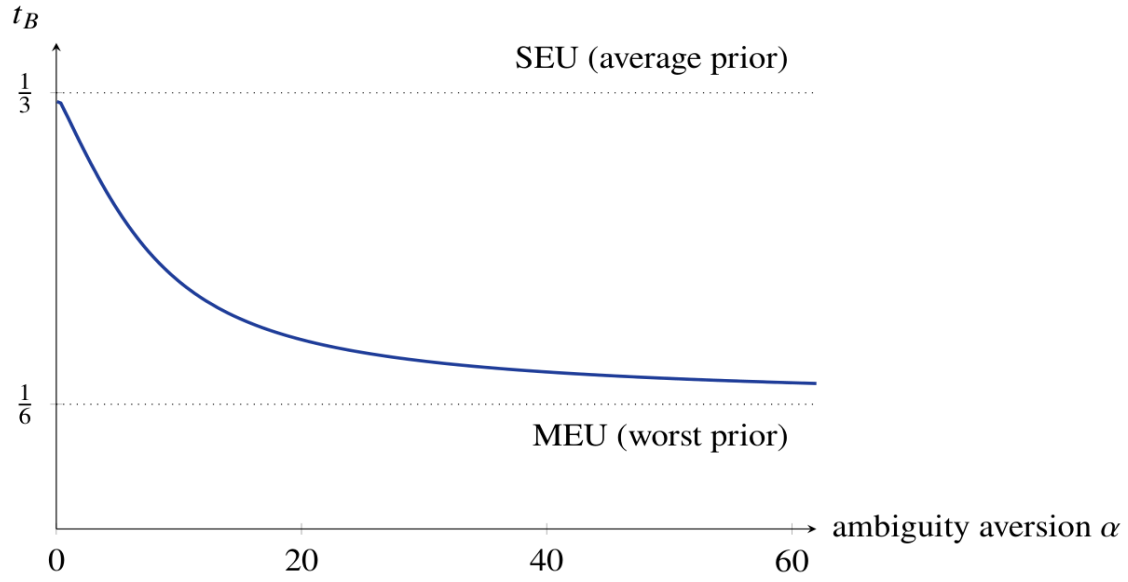
$$t_B = -\frac{1}{\alpha}\ln[\frac{1}{2}e^{-\frac{\alpha}{6}} + \frac{1}{2}e^{-\frac{\alpha}{2}}]$$

- Bet A has a known value of 1/3; bet C, of 2/3.

α	**0.01**	**1**	**5**	**20**	**100**
t_B	0.333	0.320	0.271	0.201	0.174
t_D	0.667	0.653	0.604	0.535	0.507

Smooth Ambiguity and Ellsberg

- More ambiguity aversion moves the value of bet B from the SEU value, 1/3, to the MEU value, 1/6.



Smooth Ambiguity: Investment

- Wealth 1; invest x in an asset with gross return R:

$$w = 1 + (R - 1)x, \qquad R \in \{0.75, 2\}$$

$$u(w) = 10w - w^2$$

- Two priors on the low return, equally likely:

$$\Pr(R = 0.75) \in \{0.25, 0.9\}$$

Smooth Ambiguity: Investment

- Expected net return per dollar:
  - Good prior:
$$0.25(-0.25) + 0.75(1) = 0.6875$$
  - Bad prior:
$$0.9(-0.25) + 0.1(1) = -0.125$$
- With $\varphi(t) = t^{\rho}, \quad \rho \in (0, 1]$, the individual maximizes
$$V(x) = \frac{1}{2}[E_{0.25}u(w)]^{\rho} + \frac{1}{2}[E_{0.9}u(w)]^{\rho}$$

Smooth Ambiguity: Investment

- Relative ambiguity aversion:

$$-\frac{t\varphi''(t)}{\varphi'(t)} = 1 - \rho$$

  - Values of x^* above 1 mean the individual borrows to invest.

ρ	**1 (SEU)**	**0.5**	**0.25**	**0.1**
x^*	2.44	1.92	1.69	1.58

Smooth Ambiguity: Investment

- More ambiguity aversion (smaller ρ) lowers the investment.
- Under MEU with the same two priors, the bad prior has a negative expected return, so $x^* = 0$.
  - The smooth model gives an interior answer between these two extremes.

Comparing the Models

Comparing the Models

- Four ways to evaluate an act:

Model	Beliefs	Evaluation
SEU	One prior	Expected utility
MEU	Set of priors	Worst prior
CEU	Capacity	Choquet integral
Smooth	Prior over priors	Concave ϕ of expected utility

Summary

- Ellsberg choices reveal aversion to unknown probabilities.
- MEU: many priors, worst case.
- CEU: non-additive probabilities; convex capacities give MEU.
- Smooth ambiguity separates the amount of ambiguity from the attitude toward it.

References

- Dow, J., and S. R. da Costa Werlang (1992) "Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio." Econometrica 60(1): 197–204.
- Gilboa, I., and D. Schmeidler (1989) "Maxmin Expected Utility with Non-Unique Prior." Journal of Mathematical Economics 18(2): 141–153.
- Klibanoff, P., M. Marinacci, and S. Mukerji (2005) "A Smooth Model of Decision Making under Ambiguity." Econometrica 73(6): 1849–1892.
- Knight, F. H. (1921) Risk, Uncertainty and Profit. Houghton Mifflin.
- Schmeidler, D. (1989) "Subjective Probability and Expected Utility without Additivity." Econometrica 57(3): 571–587.

Advanced Microeconomic Theory

Chapter 6: General Equilibrium

Outline

- General equilibrium without production
- Competitive markets and the welfare theorems
- General equilibrium with production
- Comparative statics
- Taxes in general equilibrium
- Appendix: Large economies and the core

General Equilibrium

General Equilibrium

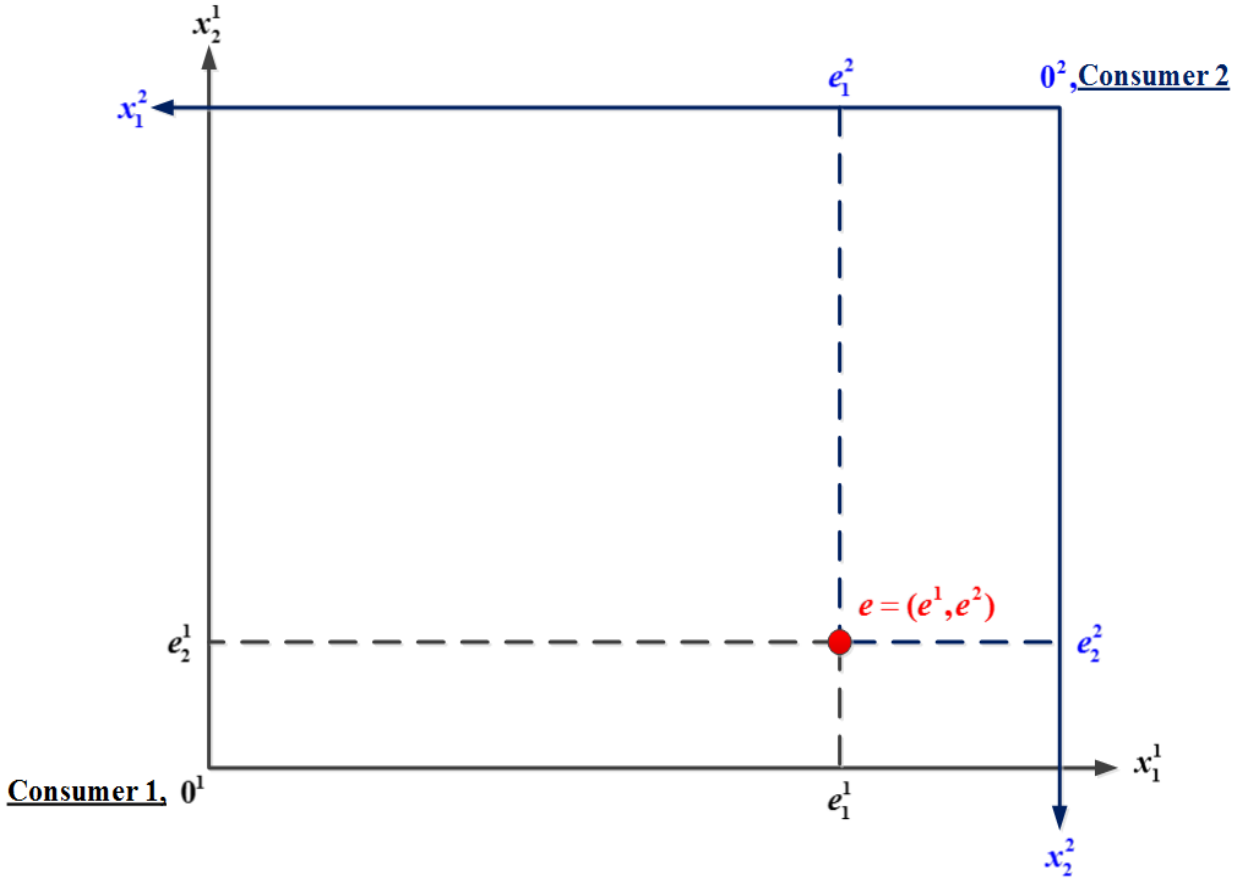
- So far, we explored equilibrium conditions in:
  - a single market with
  - a single type of consumer.
- Now we examine settings with markets for:
  - different goods and
  - multiple consumers.

General Equilibrium: No Production

- Consider an economy with two goods and two consumers, $i \in \{1,2\}$.
- Each consumer i is initially endowed with $\mathbf{e}^i \equiv (e_1^i, e_2^i)$ units of goods 1 and 2.
- Any other bundle for consumer i is denoted by $\mathbf{x}^i \equiv (x_1^i, x_2^i)$.

General Equilibrium: No Production

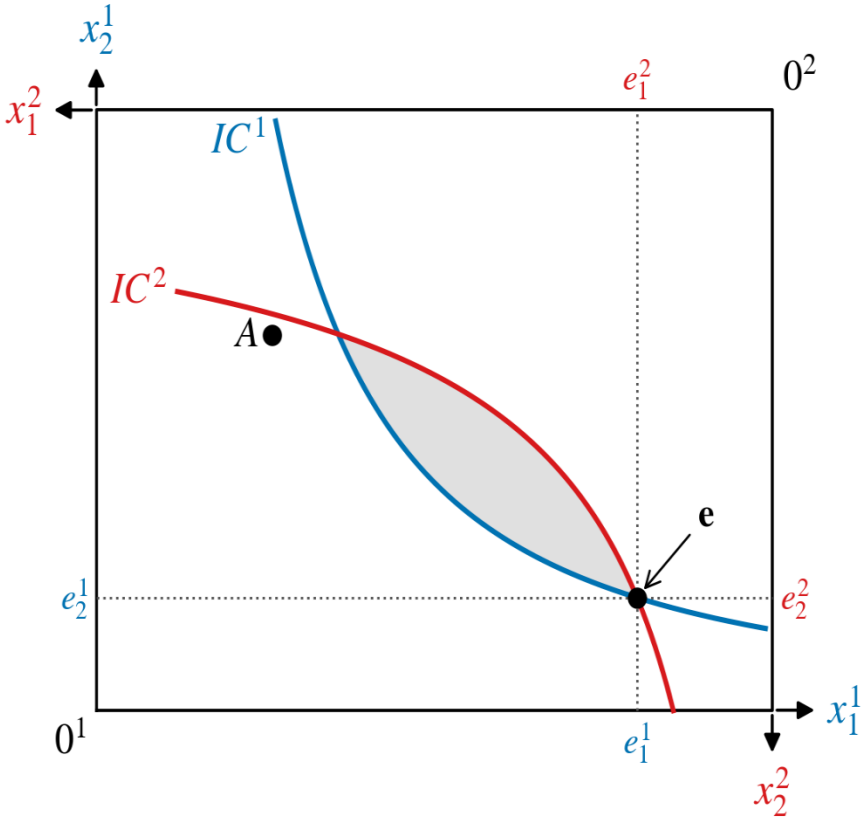
- ***Edgeworth box***:



General Equilibrium: No Production

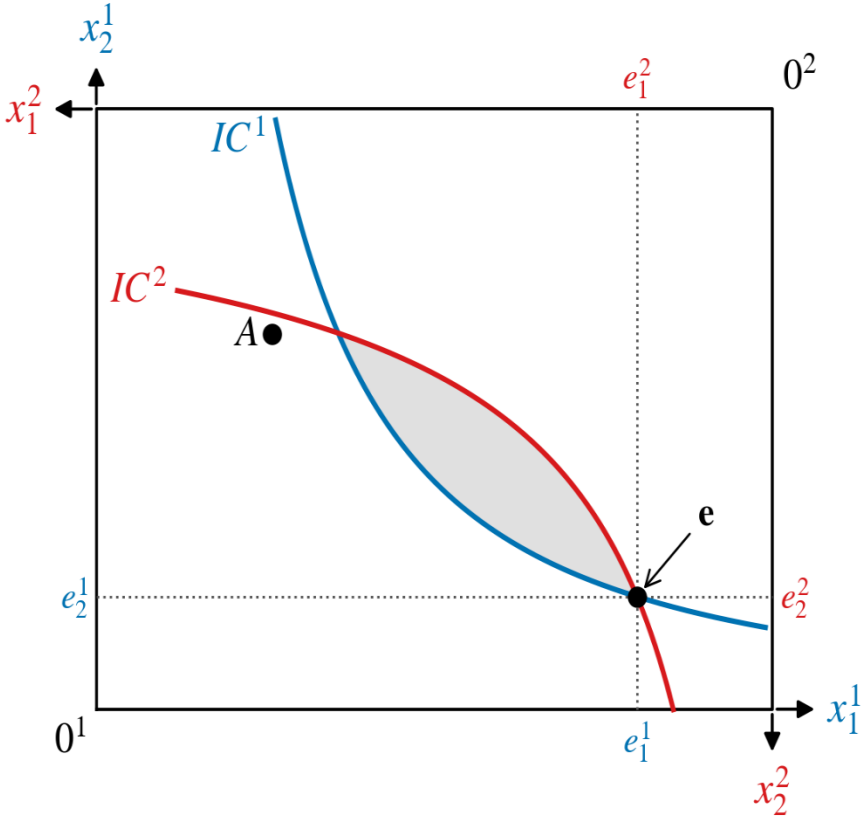
- IC^i is the indifference curve of consumer i, which passes through his endowment point $\mathbf{e}^i$.
- The shaded area represents the set of allocations $(\mathbf{x}^1, \mathbf{x}^2)$ satisfying

$$u^1(x_1^1, x_2^1) \geq u^1(e_1^1, e_2^1)$$
$$u^2(x_1^2, x_2^2) \geq u^2(e_1^2, e_2^2)$$



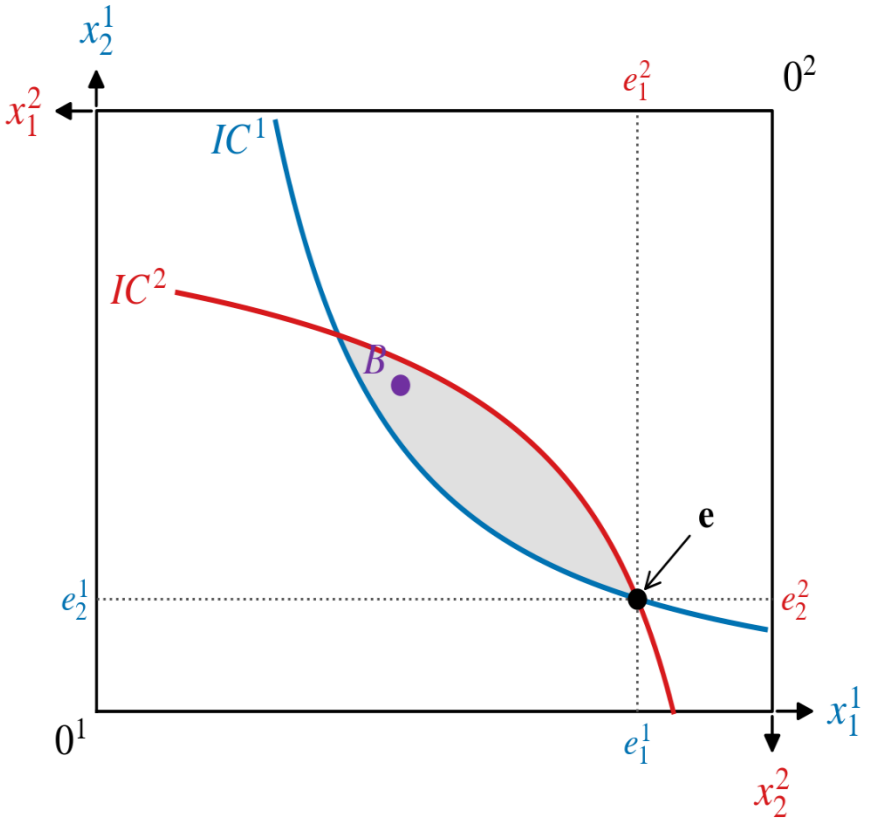
General Equilibrium: No Production

- Allocation A cannot be a barter equilibrium (defined formally below):
  - Consumer 1 prefers his endowment $\mathbf{e}^1$ to his bundle at A (A lies below IC^1), so he does not trade to A.



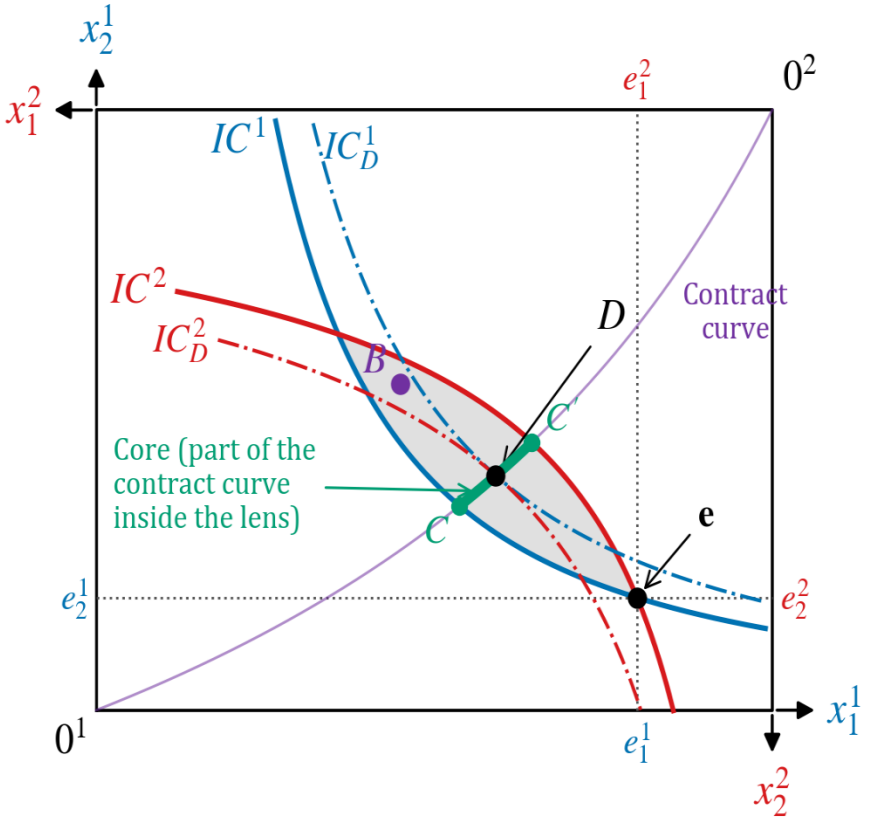
General Equilibrium: No Production

- Not all points in the lens-shaped area are barter equilibria!
- Allocation B lies inside the lens-shaped area
  - Thus, it yields a higher utility level than the initial endowment **e** for both consumers.



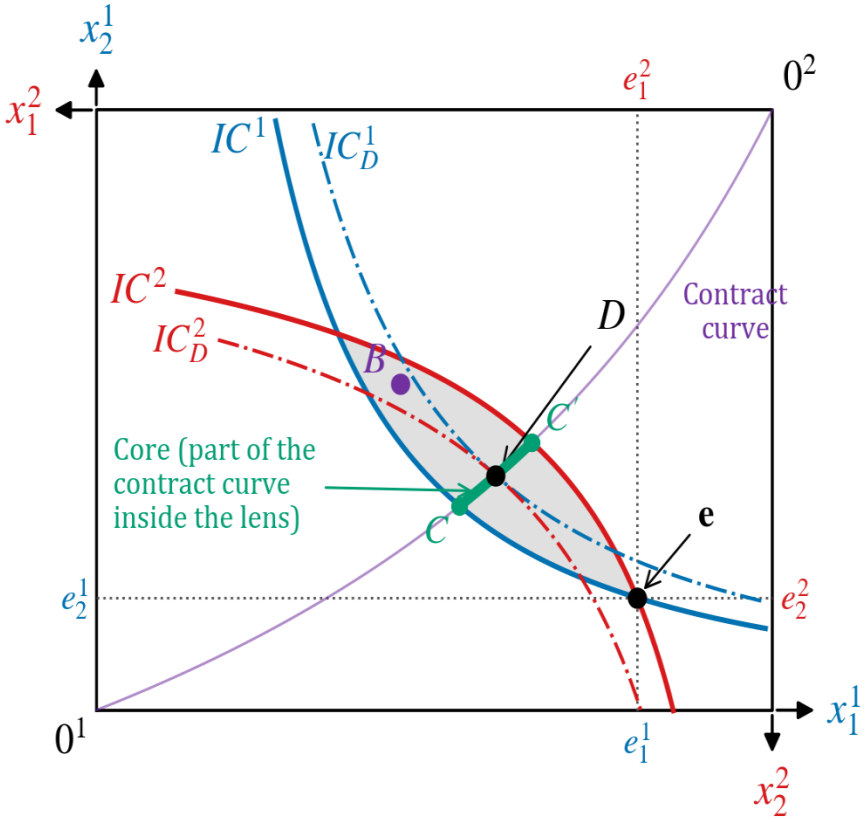
General Equilibrium: No Production

- Allocation D, however, makes both consumers better off than allocation B.
  - It lies on the ***contract curve***, in which the indifference curves are tangent to one another.



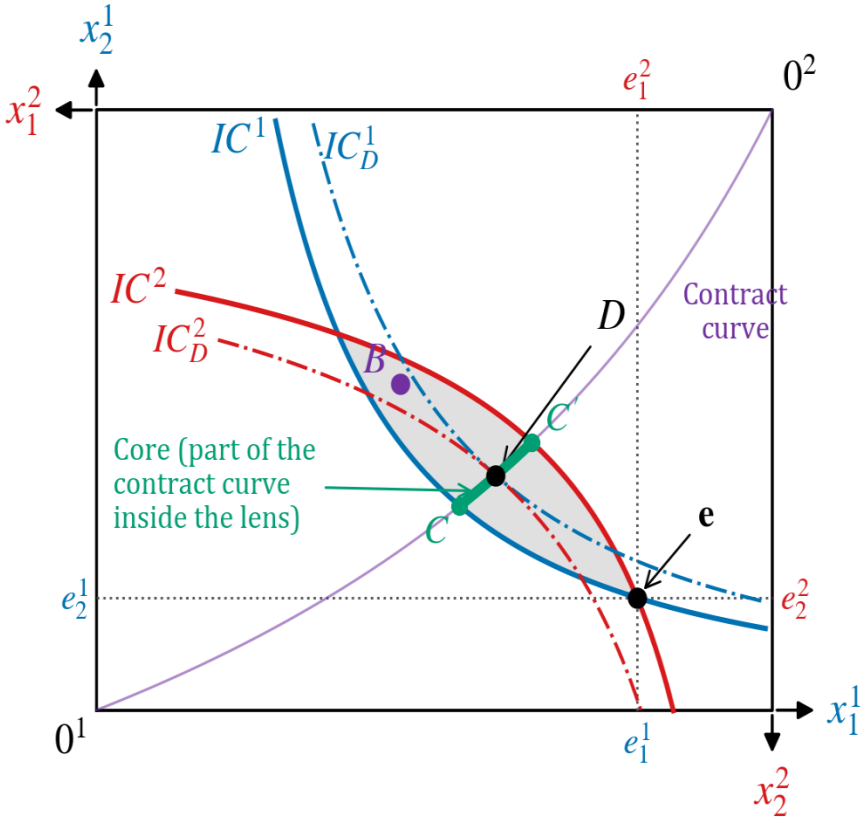
General Equilibrium: No Production

- Allocation D is an equilibrium: it lies in the lens (no consumer prefers his endowment) and no Pareto improvement is possible.



General Equilibrium: No Production

- Being in the lens-shaped area is then a necessary, but not sufficient, condition for an allocation to be a barter equilibrium.



General Equilibrium: No Production

- ***Feasible allocation***:
  - An allocation $\mathbf{x} \equiv (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$ is *feasible* if it satisfies

$$\sum_{i=1}^{I} \mathbf{x}^i \leq \sum_{i=1}^{I} \mathbf{e}^i$$

  - In words, the aggregate amount of goods in allocation $\mathbf{x}$ does not exceed the aggregate initial endowment $\mathbf{e} \equiv \sum_{i=1}^{I} \mathbf{e}^i$.

General Equilibrium: No Production

- ***Pareto-efficient allocations***: A feasible allocation $\mathbf{x}$ is Pareto efficient if there is no other feasible allocation $\mathbf{y}$ which is:
  - weakly preferred by all consumers, $\mathbf{y}^i \succsim^i \mathbf{x}^i$ for all $i \in \{1, \ldots, I\}$, and
  - strictly preferred by at least one consumer, $\mathbf{y}^i \succ^i \mathbf{x}^i$ for some i.

General Equilibrium: No Production

- That is, allocation $\mathbf{x}$ is Pareto efficient if there is no other feasible allocation $\mathbf{y}$ making:
  - all individuals at least as well off as under $\mathbf{x}$, and
  - at least one individual strictly better off.

General Equilibrium: No Production

- ***Pareto-efficient allocations***:
  - Consider the program

$$\max_{\mathbf{x}^1,\ldots,\mathbf{x}^I \geq \mathbf{0}} u^i(\mathbf{x}^i)$$

$$\text{s.t. } u^j(\mathbf{x}^j) \geq \overline{u}^j \text{ for every } j \neq i, \quad \text{and}$$

$$\sum_{h=1}^{I} \mathbf{x}^h \leq \sum_{h=1}^{I} \mathbf{e}^h \quad \text{(feasibility)}$$

  - where $\mathbf{x}^i \equiv (x_1^i, x_2^i)$.

General Equilibrium: No Production

- With strictly increasing utilities, an allocation $(\mathbf{x}^1, \ldots, \mathbf{x}^I)$ is Pareto efficient if and only if it solves this program for some levels $\overline{u}^j$, $j \neq i$; varying $\overline{u}^j$ traces out the set.
- That is, a Pareto-efficient allocation maximizes individual i's utility without reducing the utility of any other individual j below a given level $\overline{u}^j$, and satisfies feasibility.

General Equilibrium: No Production

- The Lagrangian is

$$\mathcal{L} = u^i(\mathbf{x}^i) + \sum_{j \neq i} \lambda^j [u^j(\mathbf{x}^j) - \overline{u}^j]$$
$$+ \sum_{k=1}^{2} \mu_k [\sum_{h=1}^{I} e_k^h - \sum_{h=1}^{I} x_k^h]$$

  - where μ_k is the multiplier on the feasibility constraint of good k, so we have one multiplier for each good.

General Equilibrium: No Production

- FOC with respect to $\mathbf{x}^i = (x_1^i, x_2^i)$:

$$\frac{\partial \mathcal{L}}{\partial x_k^i} = \frac{\partial u^i(\mathbf{x}^i)}{\partial x_k^i} - \mu_k \leq 0$$

  - for every good k of consumer i.

- For any individual $j \neq i$, the FOCs become

$$\frac{\partial \mathcal{L}}{\partial x_k^j} = \frac{\lambda^j \partial u^j(\mathbf{x}^j)}{\partial x_k^j} - \mu_k \leq 0$$

General Equilibrium: No Production

- FOCs with respect to λ^j and μ_k yield $u^j(\mathbf{x}^j) \geq \overline{u}^j$ and $\sum_{h=1}^{I} x_k^h \leq \sum_{h=1}^{I} e_k^h$, respectively.
- Interior solutions: the FOCs of consumer i hold with equality,

$$\frac{\partial u^i(\mathbf{x}^i)}{\partial x_1^i} = \mu_1 \quad \text{and} \quad \frac{\partial u^i(\mathbf{x}^i)}{\partial x_2^i} = \mu_2$$

General Equilibrium: No Production

- Dividing the first condition by the second,

$$\frac{\frac{\partial u^i(\mathbf{x}^i)}{\partial x_1^i}}{\frac{\partial u^i(\mathbf{x}^i)}{\partial x_2^i}} = \frac{\mu_1}{\mu_2}$$

- Similarly, for every consumer $j \neq i$,

$$\frac{\lambda^j \partial u^j(\mathbf{x}^j)}{\partial x_1^j} = \mu_1 \quad \text{and} \quad \frac{\lambda^j \partial u^j(\mathbf{x}^j)}{\partial x_2^j} = \mu_2$$

General Equilibrium: No Production

- Dividing the first condition by the second, λ^j cancels out,

$$\frac{\frac{\partial u^j(\mathbf{x}^j)}{\partial x_1^j}}{\frac{\partial u^j(\mathbf{x}^j)}{\partial x_2^j}} = \frac{\mu_1}{\mu_2}$$

- Both ratios are equal to μ_1/μ_2.

General Equilibrium: No Production

- Hence, a compact condition for Pareto efficiency is

$$MRS_{1,2}^{i} = \frac{\mu_1}{\mu_2} = MRS_{1,2}^{j}$$

  - for every consumer $j \neq i$.
- Graphically, consumers' indifference curves are tangent to one another at the Pareto-efficient allocations.

General Equilibrium: No Production

- ***Example 6.5*** (Pareto efficiency):
  - We now call the two consumers A and B.
  - Consider a barter economy with two goods, 1 and 2, and two consumers, A and B, with initial endowments
    $$\mathbf{e}^A = (100,350) \quad \text{and} \quad \mathbf{e}^B = (100,50)$$
  - Each consumer's utility function is Cobb-Douglas,
    $$u^i(x_1^i, x_2^i) = x_1^i x_2^i \quad \text{for every} \quad i \in \{A, B\}$$
  - Let us find the set of Pareto efficient allocations.

General Equilibrium: No Production

- ***Example 6.5*** (continued):
  - Pareto efficient allocations are reached at points where $MRS_{1,2}^{A} = MRS_{1,2}^{B}$. Hence,

$$MRS_{1,2}^{A} = MRS_{1,2}^{B}, \qquad \text{that is,} \qquad \frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B}$$

  - or, cross-multiplying,

$$x_2^A x_1^B = x_2^B x_1^A$$

General Equilibrium: No Production

- ***Example 6.5*** (continued):
  - Using the feasibility constraints for goods 1 and 2 (feasibility binds because utility is strictly increasing),

$$e_1^A + e_1^B = x_1^A + x_1^B$$
$$e_2^A + e_2^B = x_2^A + x_2^B$$

  - we obtain

$$x_1^B = e_1^A + e_1^B - x_1^A$$
$$x_2^B = e_2^A + e_2^B - x_2^A$$

General Equilibrium: No Production

- ***Example 6.5*** (continued):
  - Combining the tangency condition and the feasibility constraints yields

$$x_2^A(e_1^A + e_1^B - x_1^A) = (e_2^A + e_2^B - x_2^A)x_1^A$$

  - Expanding both sides,

$$x_2^A(e_1^A + e_1^B) - x_1^A x_2^A = x_1^A(e_2^A + e_2^B) - x_1^A x_2^A$$

General Equilibrium: No Production

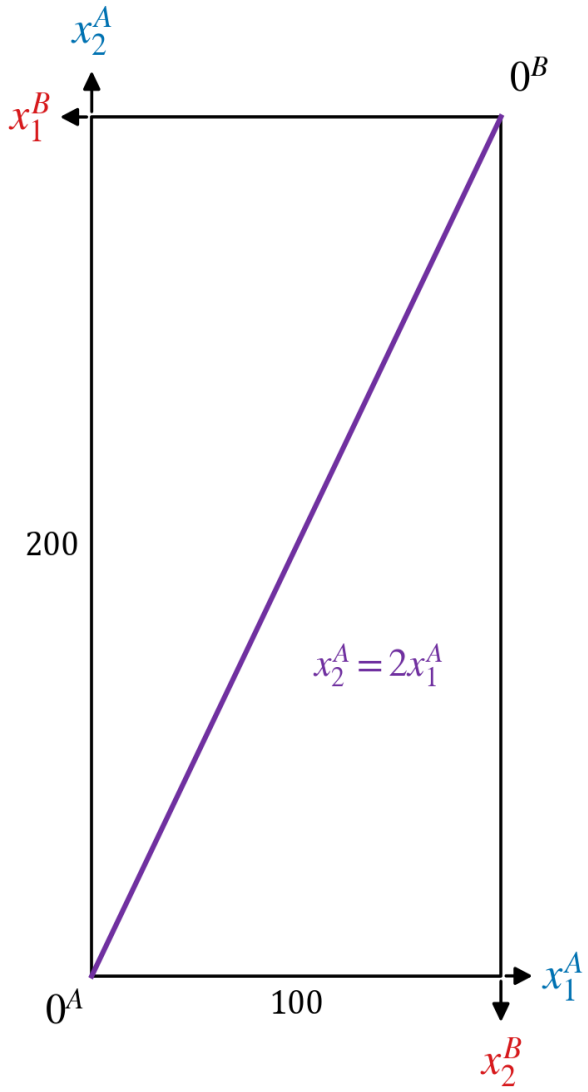
- ***Example 6.5*** (continued):
  - The $x_1^A x_2^A$ terms cancel, so

$$x_2^A = \frac{e_2^A + e_2^B}{e_1^A + e_1^B} x_1^A = \frac{350 + 50}{100 + 100} x_1^A = 2x_1^A$$

  - for all $x_1^A \in [0,200]$.

General Equilibrium: No Production

- ***Example 6.5*** (continued):
  - The set of Pareto-efficient allocations is the diagonal of the Edgeworth box, $x_2^A = 2x_1^A$.
  - It connects the origins of consumers A and B, since the aggregate endowment is (200, 400).



General Equilibrium: No Production

- ***Blocking coalitions***: Consider a nonempty coalition $S \subseteq \{1, \ldots, I\}$. We say that S blocks the feasible allocation $\mathbf{x}$ if there are bundles $(\mathbf{y}^i)_{i \in S}$ such that:
  - (1) they are feasible for S:

$$\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$$

  - (2) $\mathbf{y}^i \gtrsim^i \mathbf{x}^i$ for all $i \in S$, and $\mathbf{y}^i \succ^i \mathbf{x}^i$ for some $i \in S$.

General Equilibrium: No Production

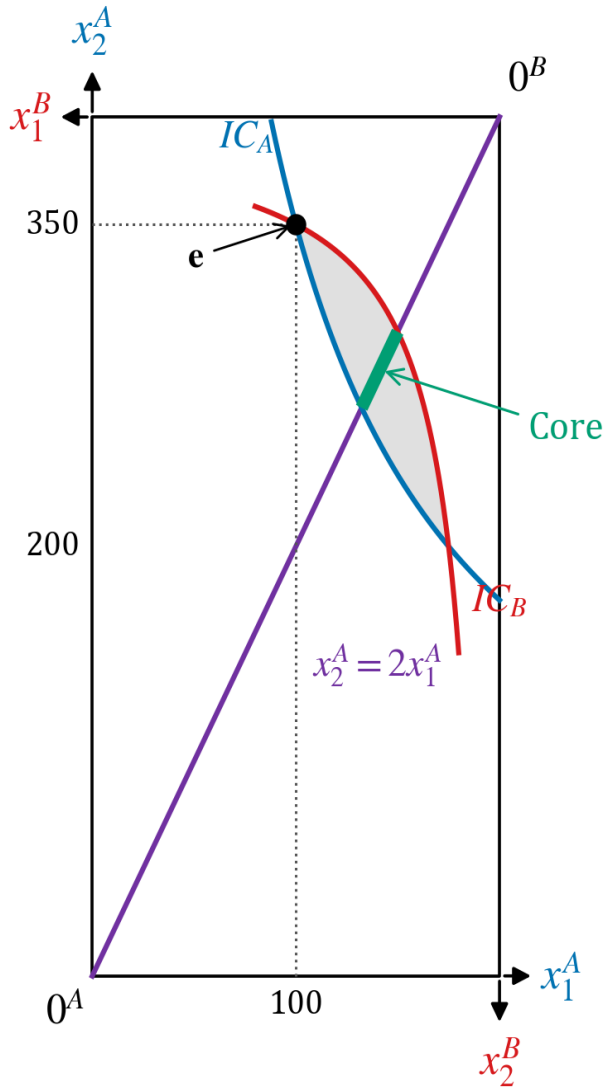
- ***Core***: The core of an exchange economy with endowment $\mathbf{e}$, denoted $C(\mathbf{e})$, is the set of all unblocked feasible allocations.
- ***Equilibrium in a barter economy***: An allocation is an equilibrium of the barter economy if it is in the core $C(\mathbf{e})$, the set of feasible allocations that no coalition blocks.

General Equilibrium: No Production

- Core allocations:
  - a) leave no consumer worse off than at his endowment, since a single consumer would otherwise block (in the box: the lens, including its boundary);
  - b) do not allow for further Pareto improvements (the grand coalition cannot block), so they lie on the contract curve.

General Equilibrium: No Production

- ***Example 6.5*** (continued):
  - The indifference curves through the endowment point form the lens-shaped (shaded) area.
  - The core is the segment of the contract curve inside this area.



General Equilibrium: Competitive Markets

- The barter economy did not require prices for an equilibrium to arise.
- Now we explore the equilibrium in economies where we allow prices to emerge.
- Order of analysis:
  - consumers' preferences
  - the excess demand function
  - the equilibrium allocations in competitive markets (i.e., *Walrasian equilibrium allocations*)

General Equilibrium: Competitive Markets

- ***Consumers***:
  - Consider each consumer's utility function to be continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}_+^L$.
  - Hence, for every price vector $\mathbf{p} \gg \mathbf{0}$, the UMP of every consumer i, with budget constraint
$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i$$
  - yields a unique solution, the Walrasian demand $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$.

General Equilibrium: Competitive Markets

- A solution exists since the budget set is compact for $\mathbf{p} \gg \mathbf{0}$, and it is unique by strict quasiconcavity.
- $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ is continuous in $\mathbf{p}$ (theorem of the maximum).

General Equilibrium: Competitive Markets

– Intuitively, individual i's income comes from selling his endowment $\mathbf{e}^i$ at market prices $\mathbf{p}$, producing $\mathbf{p} \cdot \mathbf{e}^i = p_1 e_1^i + \cdots + p_L e_L^i$ dollars to be used in the purchase of bundle $\mathbf{x}^i$.

General Equilibrium: Competitive Markets

- ***Excess demand***:
  - Summing the Walrasian demands $x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ for good k of every individual in the economy, we obtain the aggregate demand for good k.
  - The difference between the aggregate demand and the aggregate endowment of good k yields the excess demand of good k:

$$z_k(\mathbf{p}) = \sum_{i=1}^{I} x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - \sum_{i=1}^{I} e_k^i$$

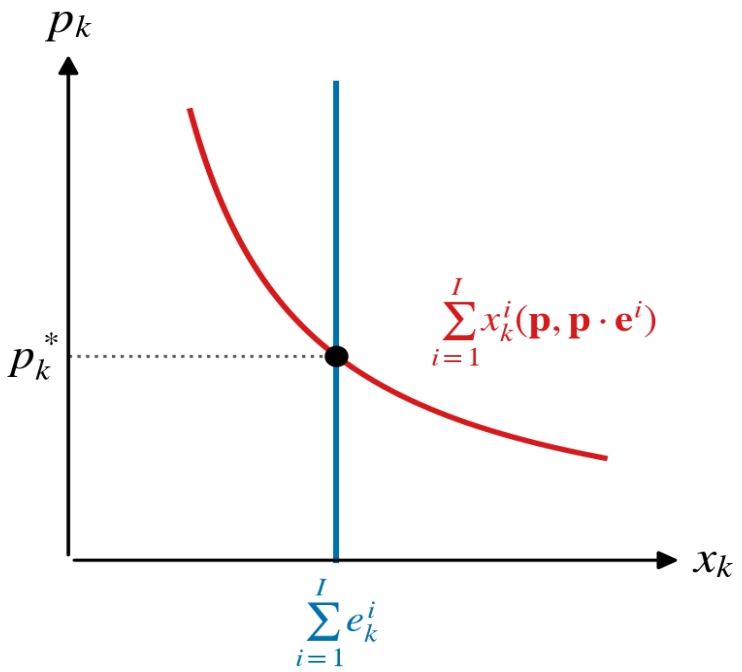
  - where $z_k(\mathbf{p}) \in \mathbb{R}$.

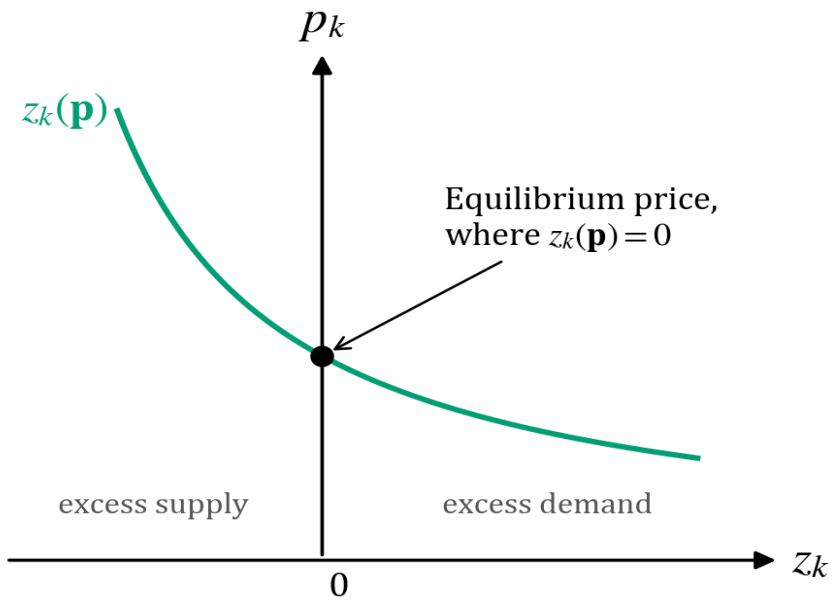
General Equilibrium: Competitive Markets

- When $z_k(\mathbf{p}) > 0$, the aggregate demand for good k exceeds its aggregate endowment.
  - Excess demand of good k
- When $z_k(\mathbf{p}) < 0$, the aggregate demand for good k falls short of its aggregate endowment.
  - Excess supply of good k

General Equilibrium: Competitive Markets

- Aggregate demand, aggregate endowment, and excess demand of good k (other prices held fixed)





General Equilibrium: Competitive Markets

- The excess demand function $\mathbf{z}(\mathbf{p}) \equiv (z_1(\mathbf{p}), \ldots, z_L(\mathbf{p}))$ satisfies the following properties:
- ***Walras' law***: $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$.
  - Since utility is strictly increasing, every consumer exhausts his income at every $\mathbf{p} \gg \mathbf{0}$,

$$\sum_{k=1}^{L} p_k x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) = \sum_{k=1}^{L} p_k e_k^i$$

General Equilibrium: Competitive Markets

- ***Walras' law*** (continued):
  - Equivalently, for every consumer i,

$$\sum_{k=1}^{L} p_k [x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - e_k^i] = 0$$

General Equilibrium: Competitive Markets

– Summing over all individuals,

$$\sum_{i=1}^{I}\sum_{k=1}^{L} p_k\left[x_k^i\left(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i\right) - e_k^i\right] = 0$$

– We can re-write the above expression as

$$\sum_{k=1}^{L}\sum_{i=1}^{I} p_k\left[x_k^i\left(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i\right) - e_k^i\right] = 0$$

which is equivalent to

$$\sum_{k=1}^{L} p_k \underbrace{\left(\sum_{i=1}^{I}\left[x_k^i\left(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i\right)\right] - \sum_{i=1}^{I} e_k^i\right)}_{z_k(\mathbf{p})} = 0$$

– Hence,

$$\sum_{k=1}^{L} p_k \cdot z_k(\mathbf{p}) = \mathbf{p}\cdot\mathbf{z}(\mathbf{p}) = 0$$

General Equilibrium: Competitive Markets

- In a two-good economy, Walras' law implies
$$p_1 z_1(\mathbf{p}) = -p_2 z_2(\mathbf{p})$$
- Intuition: if there is excess demand in market 1, $z_1(\mathbf{p}) > 0$, there must be excess supply in market 2, $z_2(\mathbf{p}) < 0$.

General Equilibrium: Competitive Markets

- Hence, if market 1 is in equilibrium, $z_1(\mathbf{p}) = 0$, then so is market 2, $z_2(\mathbf{p}) = 0$.
- More generally, if the markets of $L - 1$ goods are in equilibrium, then so is the Lth market (since $p_L > 0$, Walras' law gives $p_L z_L(\mathbf{p}) = 0$, so $z_L(\mathbf{p}) = 0$).

General Equilibrium: Competitive Markets

- ***Continuity***: $\mathbf{z}(\mathbf{p})$ is continuous at every $\mathbf{p} \gg \mathbf{0}$.
  - This follows from individual Walrasian demands being continuous in prices.
- ***Homogeneity***: $\mathbf{z}(\lambda\mathbf{p}) = \mathbf{z}(\mathbf{p})$ for all $\lambda > 0$.
  - This follows from Walrasian demands being homogeneous of degree zero in prices.
  - In other words, Walrasian demand is unaffected by a common change in prices, λ, so the excess demand function is not affected either.

General Equilibrium: Competitive Markets

- In summary, the excess demand function satisfies three properties: Walras' law, continuity, and homogeneity of degree zero.
- We now use excess demand to define a Walrasian equilibrium allocation.

General Equilibrium: Competitive Markets

- ***Walrasian equilibrium***:
  - A price vector $\mathbf{p}^* \gg \mathbf{0}$ is a Walrasian equilibrium if aggregate excess demand is zero at that price vector, $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.
  - In words, price vector $\mathbf{p}^*$ clears all markets.
  - Alternatively, $\mathbf{p}^* \gg \mathbf{0}$ is a Walrasian equilibrium if each consumer solves his UMP and aggregate demand equals aggregate supply,

$$\sum_{i=1}^{I} \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i) = \sum_{i=1}^{I} \mathbf{e}^i$$

General Equilibrium: Competitive Markets

- The allocation

$$\mathbf{x}(\mathbf{p}^*) = (\mathbf{x}^1(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^1), \ldots, \mathbf{x}^I(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^I))$$

- is a ***Walrasian equilibrium allocation*** (WEA).
- $W(\mathbf{e})$ denotes the set of WEAs.

General Equilibrium: Competitive Markets

- ***Existence of a Walrasian equilibrium***:
  - Under the assumptions on preferences above, and if the aggregate endowment is strictly positive, $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, the excess demand function $\mathbf{z}(\mathbf{p})$:
    - is continuous and homogeneous of degree zero, and satisfies Walras' law;
    - is bounded below; and
    - satisfies a boundary condition: if some price approaches zero, excess demand for some good grows without bound.

General Equilibrium: Competitive Markets

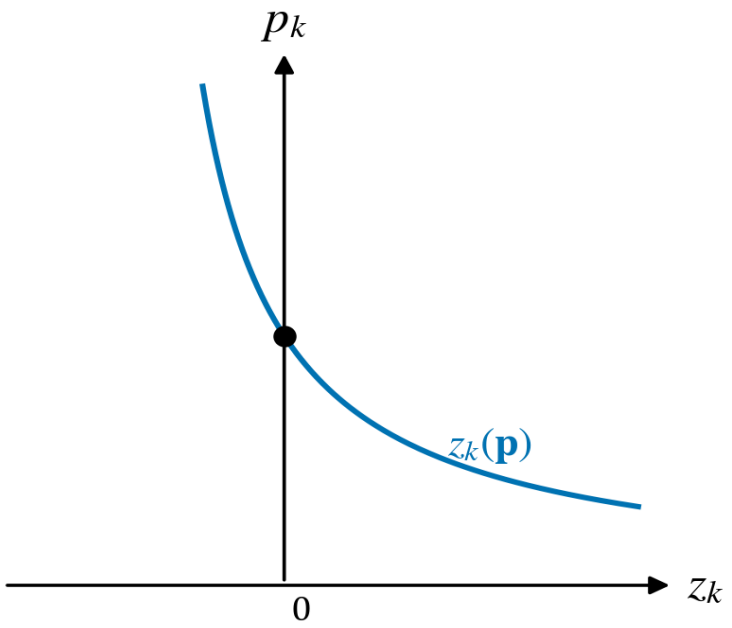
- ***Existence of a Walrasian equilibrium*** (continued):
  - Then, there is a price vector $\mathbf{p}^* \gg \mathbf{0}$ with $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.
  - See Mas-Colell, Whinston and Green (1995, Propositions 17.B.2 and 17.C.1); see also Varian (1992).

General Equilibrium: Competitive Markets

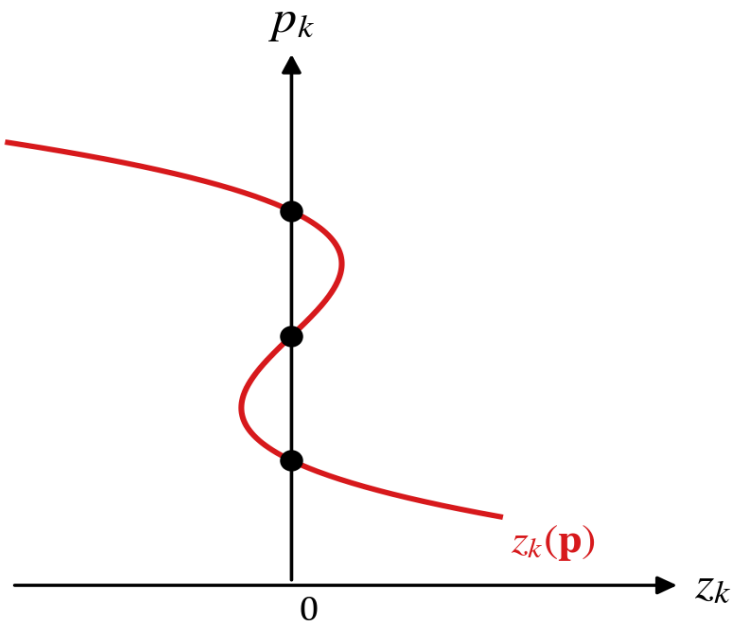
- ***Uniqueness of equilibrium prices***:
  - The equilibrium price is unique when excess demand crosses zero only once.
  - For instance, when all goods are gross substitutes: $z_k(\mathbf{p})$ is increasing in p_j for every $j \neq k$.

General Equilibrium: Competitive Markets

- ***Uniqueness of equilibrium prices***:



Crosses zero once:
unique equilibrium price



Crosses zero three times:
three equilibrium prices

General Equilibrium: Competitive Markets

- ***Example 6.6*** (Walrasian equilibrium allocation):
  - Consider the economy in Example 6.5. Consumer A solves

$$\max_{x_1^A, x_2^A} x_1^A x_2^A$$

  - subject to

$$p_1 x_1^A + p_2 x_2^A = 100p_1 + 350p_2$$

  - where the right side is the market value of his endowment, $\mathbf{e}^A$ = (100, 350).

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - The Lagrangian is
    $$\mathcal{L} = x_1^A x_2^A + \lambda_A[100p_1 + 350p_2 - p_1 x_1^A - p_2 x_2^A]$$
  - FOCs:
    $$x_2^A = \lambda_A p_1 \quad \text{and} \quad x_1^A = \lambda_A p_2$$
  - Dividing the first FOC by the second,
    $$MRS_{1,2}^A = \frac{x_2^A}{x_1^A} = \frac{p_1}{p_2}$$
  - or, rearranging,
    $$p_1 x_1^A = p_2 x_2^A$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Substituting this into consumer A's budget constraint yields

$$p_1 x_1^A + p_1 x_1^A = p_1 \cdot 100 + p_2 \cdot 350$$

$$\text{or } x_1^A = 50 + 175\frac{p_2}{p_1}$$

    which is consumer A's Walrasian demand for good 1.
  - Plugging this value back into $p_1 x_1^A = p_2 x_2^A$ yields

$$p_1\left(50 + 175\frac{p_2}{p_1}\right) = p_2 x_2^A$$

$$\text{or } x_2^A = 175 + 50\frac{p_1}{p_2}$$

    which is consumer A's Walrasian demand for good 2.

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Consumer B solves the same problem, but with endowment $\mathbf{e}^B$ = (100, 50):

$$\max_{x_1^B, x_2^B} x_1^B x_2^B$$

  - subject to

$$p_1 x_1^B + p_2 x_2^B = 100p_1 + 50p_2$$

  - The Lagrangian is

$$\mathcal{L} = x_1^B x_2^B + \lambda_B [100p_1 + 50p_2 - p_1 x_1^B - p_2 x_2^B]$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - FOCs:

$$x_2^B = \lambda_B p_1 \quad \text{and} \quad x_1^B = \lambda_B p_2$$

  - Dividing the first FOC by the second,

$$MRS_{1,2}^B = \frac{x_2^B}{x_1^B} = \frac{p_1}{p_2}$$

  - or, rearranging,

$$p_1 x_1^B = p_2 x_2^B$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Substituting $p_1 x_1^B = p_2 x_2^B$ into consumer B's budget constraint yields

$$p_1 x_1^B + p_1 x_1^B = 100p_1 + 50p_2$$

  - or

$$x_1^B = 50 + 25\frac{p_2}{p_1}$$

  - which is consumer B's Walrasian demand for good 1.

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Plugging this value back into $p_1 x_1^B = p_2 x_2^B$ yields

$$p_1(50 + 25\frac{p_2}{p_1}) = p_2 x_2^B$$

  - or

$$x_2^B = 25 + 50\frac{p_1}{p_2}$$

  - which is consumer B's Walrasian demand for good 2.

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - For good 1, the feasibility constraint is
$$x_1^A + x_1^B = 100 + 100$$
  - Substituting the Walrasian demands,

$$(50 + 175\frac{p_2}{p_1}) + (50 + 25\frac{p_2}{p_1}) = 200, \qquad \text{so} \quad \frac{p_2}{p_1} = \frac{1}{2}$$

  - Plugging the relative prices into the Walrasian demands yields the WEA:

$$(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2}) = (137.5{,}275; 62.5{,}125; 2)$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Where do these numbers come from? The feasibility constraint for good 1 simplifies to

$$100 + 200\frac{p_2}{p_1} = 200, \qquad \text{so} \quad \frac{p_2}{p_1} = \frac{1}{2} \quad \text{and} \quad \frac{p_1}{p_2} = 2$$

  - Plugging the price ratio into consumer A's Walrasian demands:

$$x_1^A = 50 + 175(\frac{1}{2}) = 137.5$$

$$x_2^A = 175 + 50(2) = 275$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Similarly, for consumer B:

$$x_1^B = 50 + 25(\frac{1}{2}) = 62.5$$

$$x_2^B = 25 + 50(2) = 125$$

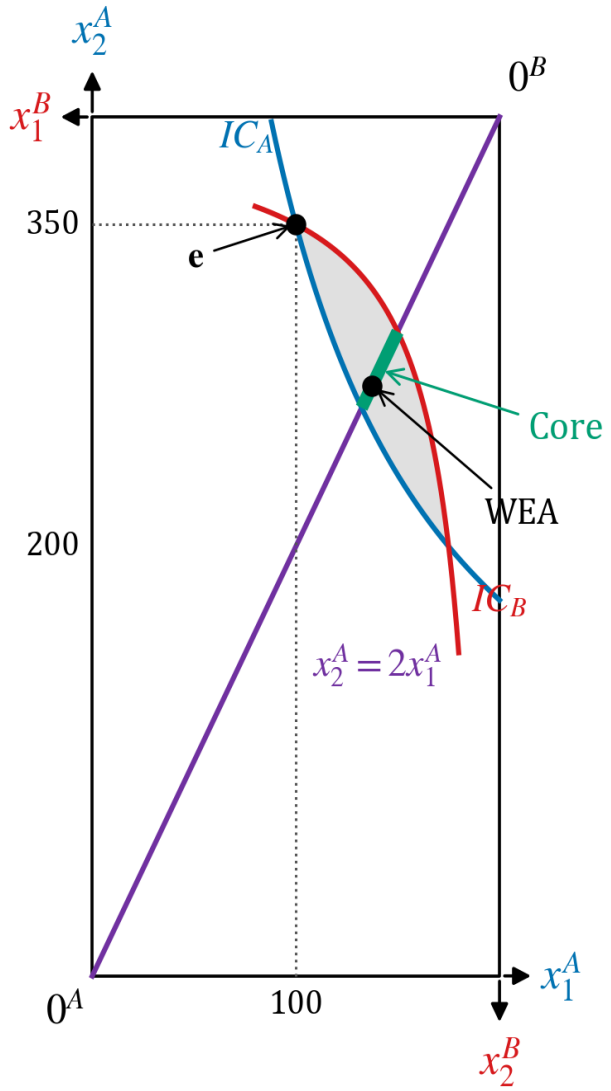
  - The market for good 2 clears as well,

$$x_2^A + x_2^B = 275 + 125 = 400 = 350 + 50$$

  - as Walras' law predicts.

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Initial allocation,
  - core allocation, and
  - Walrasian equilibrium allocations (WEA).



General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Core allocations lie on the contract curve, $x_2^A = 2x_1^A$, where both consumers are at least as well off as at their endowments.
  - ***Lower bound***: consumer A must reach his utility at $\mathbf{e}^A$, $u^A = x_1^A(2x_1^A) = 2(x_1^A)^2 \geq 100 \cdot 350 = 35{,}000$
  - which yields

$$x_1^A \geq \sqrt{17{,}500} \approx 132.3, \qquad \text{so} \quad x_2^A \geq 264.6$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - On the contract curve, consumer B's bundle also lies on the ray from his origin,
    $$x_2^B = 400 - 2x_1^A = 2(200 - x_1^A) = 2x_1^B$$
  - ***Upper bound***: B must reach his utility at $\mathbf{e}^B$,
    $$u^B = x_1^B(2x_1^B) = 2(x_1^B)^2 \geq 100 \cdot 50 = 5{,}000$$
  - which yields
    $$x_1^B \geq \sqrt{2{,}500} = 50, \qquad \text{so} \quad x_2^B \geq 100$$

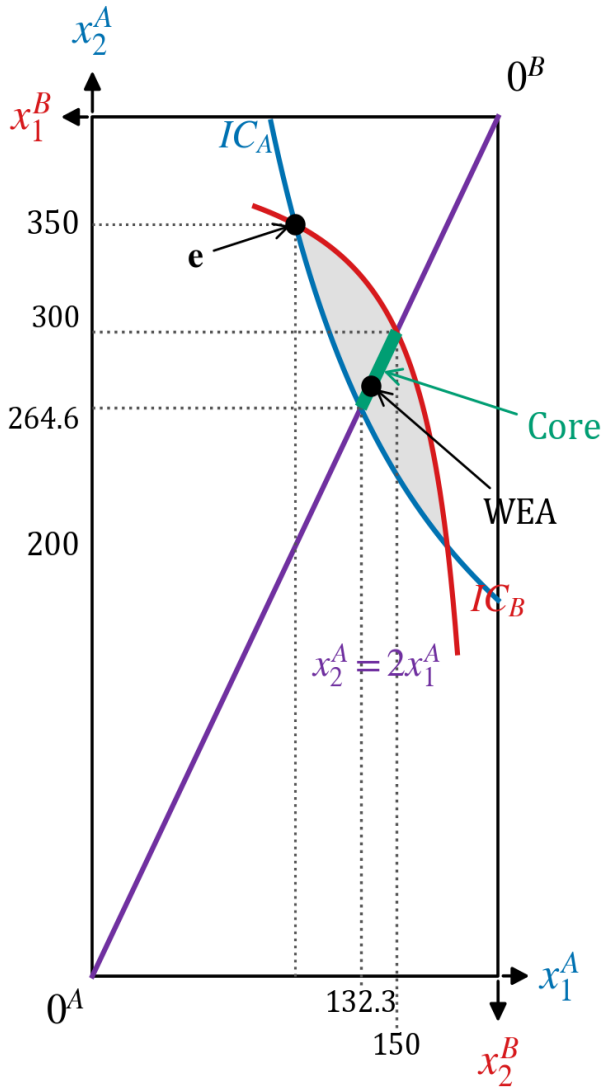
General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Since consumer A receives the rest of the aggregate endowment, the upper bound for consumer A is

$$x_1^A = 200 - x_1^B \leq 200 - 50 = 150$$
$$x_2^A = 400 - x_2^B \leq 400 - 100 = 300$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Hence, the core is
    $$132.3 \leq x_1^A \leq 150,$$
    $$\text{with} \quad x_2^A = 2x_1^A$$
  - where consumer B receives the rest of the endowment.
  - The WEA, $x_1^A = 137.5$, lies in the core.



General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Consumer A: utility $137.5 \times 275 = 37{,}812.5$ at the WEA, and $100 \times 350 = 35{,}000$ at $\mathbf{e}^A$,
    $$u^A(WEA) = 37{,}812.5 > 35{,}000 = u^A(\mathbf{e}^A)$$
  - Consumer B: utility $62.5 \times 125 = 7{,}812.5$ at the WEA, and $100 \times 50 = 5{,}000$ at $\mathbf{e}^B$,
    $$u^B(WEA) = 7{,}812.5 > 5{,}000 = u^B(\mathbf{e}^B)$$
  - Both consumers gain from trade, as every core allocation requires.

General Equilibrium: Competitive Markets

- ***Equilibrium allocations must be in the core***:
  - Under the assumptions on preferences above, every WEA is in the core, $W(\mathbf{e}) \subseteq C(\mathbf{e})$.
- *Proof* (by contradiction):
  - Take a WEA $\mathbf{x}(\mathbf{p}^*)$ with equilibrium price $\mathbf{p}^*$, but $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$.
  - Since $\mathbf{x}(\mathbf{p}^*)$ is a WEA, it must be feasible.

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - If $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$, we can find a coalition S and bundles $(\mathbf{y}^i)_{i \in S}$ such that
    $$u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)) \quad \text{for all} \quad i \in S$$
  - with strict inequality for at least one $i \in S$,
    $$u^i(\mathbf{y}^i) > u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i))$$

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - Allocation $\mathbf{y}$ is feasible for the coalition,

$$\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$$

  - Multiplying both sides of the feasibility condition by $\mathbf{p}^*$ yields

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$$

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - Since $\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)$ maximizes u^i on consumer i's budget set, any bundle he strictly prefers costs more: $\mathbf{p}^* \cdot \mathbf{y}^i > \mathbf{p}^* \cdot \mathbf{e}^i$ if $u^i(\mathbf{y}^i) > u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i))$

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - If $u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i))$ but $\mathbf{p}^* \cdot \mathbf{y}^i < \mathbf{p}^* \cdot \mathbf{e}^i$, strict monotonicity gives an affordable bundle strictly better than $\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)$, a contradiction.
  - Hence,

$$\mathbf{p}^* \cdot \mathbf{y}^i \geq \mathbf{p}^* \cdot \mathbf{e}^i \quad \text{for all} \quad i \in S$$

  - with strict inequality for at least one $i \in S$.

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - Hence, summing over all consumers in the coalition S, we obtain

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i > \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{x}^i\left(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i\right) = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$$

    which contradicts $\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$.

  - Therefore, all WEAs must be part of the core, i.e.,

$$\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$$

General Equilibrium: Competitive Markets

- ***Remarks***:
  - Since a WEA exists (existence result above) and every WEA is in the core, the core is nonempty.
  - All core allocations are Pareto efficient (the grand coalition cannot block a core allocation): we cannot increase the welfare of one consumer without decreasing that of at least one other consumer.
  - Hence, all WEAs (which are part of the core) are also Pareto efficient.

General Equilibrium: Competitive Markets

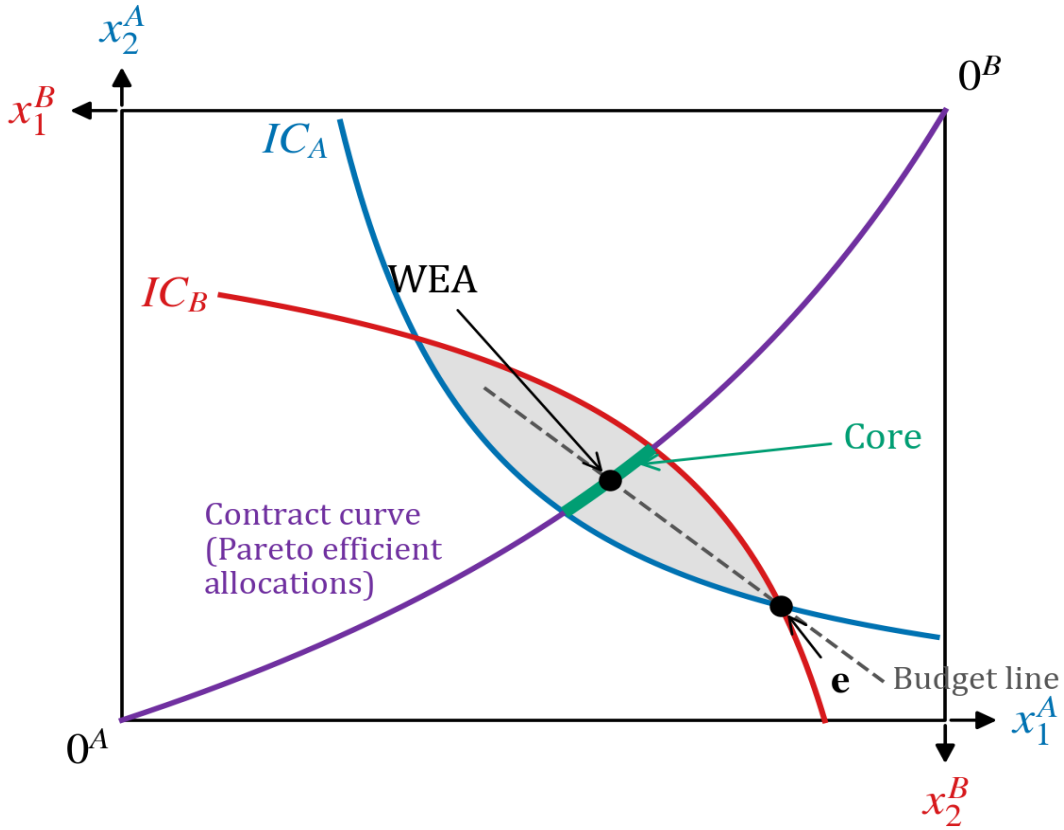
- ***First welfare theorem***: Under the assumptions on preferences above (strictly increasing utility is the key one; local nonsatiation is enough), every WEA is Pareto efficient.
  - Every WEA is in the core (previous result), and no core allocation can be Pareto dominated, since the grand coalition would block it.

General Equilibrium: Competitive Markets

- ***First welfare theorem*** (continued):
  - The WEA lies in the core (the segment of the contract curve within the lens-shaped area).
  - The core is a subset of the set of Pareto efficient allocations.

General Equilibrium: Competitive Markets

- First welfare theorem



General Equilibrium: Competitive Markets

- ***Second welfare theorem***:
  - Under the assumptions on preferences above, let $\overline{\mathbf{x}} \gg \mathbf{0}$ be a Pareto-efficient allocation.
  - Let $\mathbf{p}^* \gg \mathbf{0}$ be the price vector with

$$\frac{{p_1}^*}{{p_2}^*} = MRS^i_{1,2}(\overline{\mathbf{x}}^i)$$

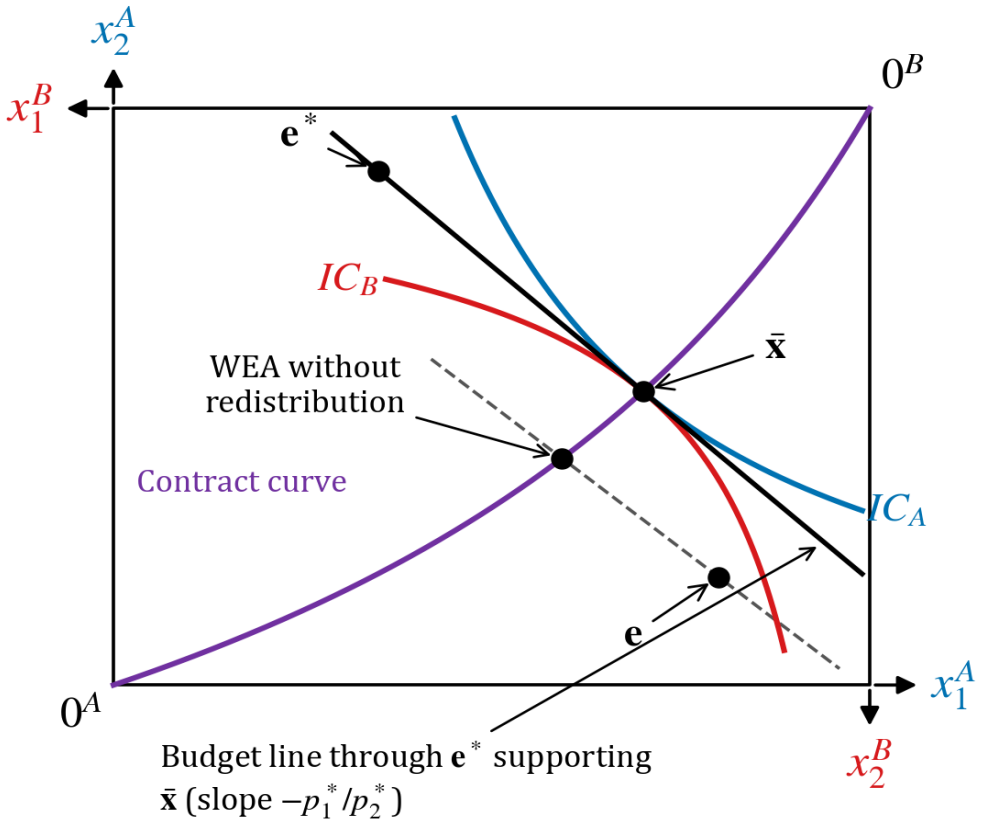
  - for every consumer i.

General Equilibrium: Competitive Markets

- ***Second welfare theorem*** (continued):
  - If endowments are redistributed to any $\mathbf{e}^{*i}$ with $\sum_{i=1}^{I} \mathbf{e}^{*i} = \sum_{i=1}^{I} \mathbf{e}^{i}$ and $\mathbf{p}^{*} \cdot \mathbf{e}^{*i} = \mathbf{p}^{*} \cdot \bar{\mathbf{x}}^{i}$ for every i (for example, $\mathbf{e}^{*i} = \bar{\mathbf{x}}^{i}$),
  - then $(\bar{\mathbf{x}}, \mathbf{p}^{*})$ is a Walrasian equilibrium of the economy with endowments $\mathbf{e}^{*}$.

General Equilibrium: Competitive Markets

- Second welfare theorem



General Equilibrium: Competitive Markets

- ***Example 6.7*** (WEA and second welfare theorem):
  - Consider an economy with utility functions
    $$u^A = x_1^A x_2^A \quad \text{and} \quad u^B = \min\{x_1^B, x_2^B\}$$
  - and initial endowments
    $$\mathbf{e}^A = (3,1) \quad \text{and} \quad \mathbf{e}^B = (1,3)$$
  - Good 2 is the numeraire, $p_2 = 1$.
  - Consumer B's utility is not strictly increasing, but it is locally nonsatiated, which is all the first welfare theorem needs.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  1) Pareto efficient allocations (PEAs):

  – At a Pareto efficient allocation, B cannot hold more of one good than the other: the excess could go to A, raising u^A without lowering u^B. Hence,

$$x_1^B = x_2^B$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Given the feasibility constraints

$$x_1^A + x_1^B = 4$$
$$x_2^A + x_2^B = 4$$

  - substitute x_2^B for x_1^B in the first constraint to get

$$x_2^B = 4 - x_1^A$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  1) Pareto efficient allocations:

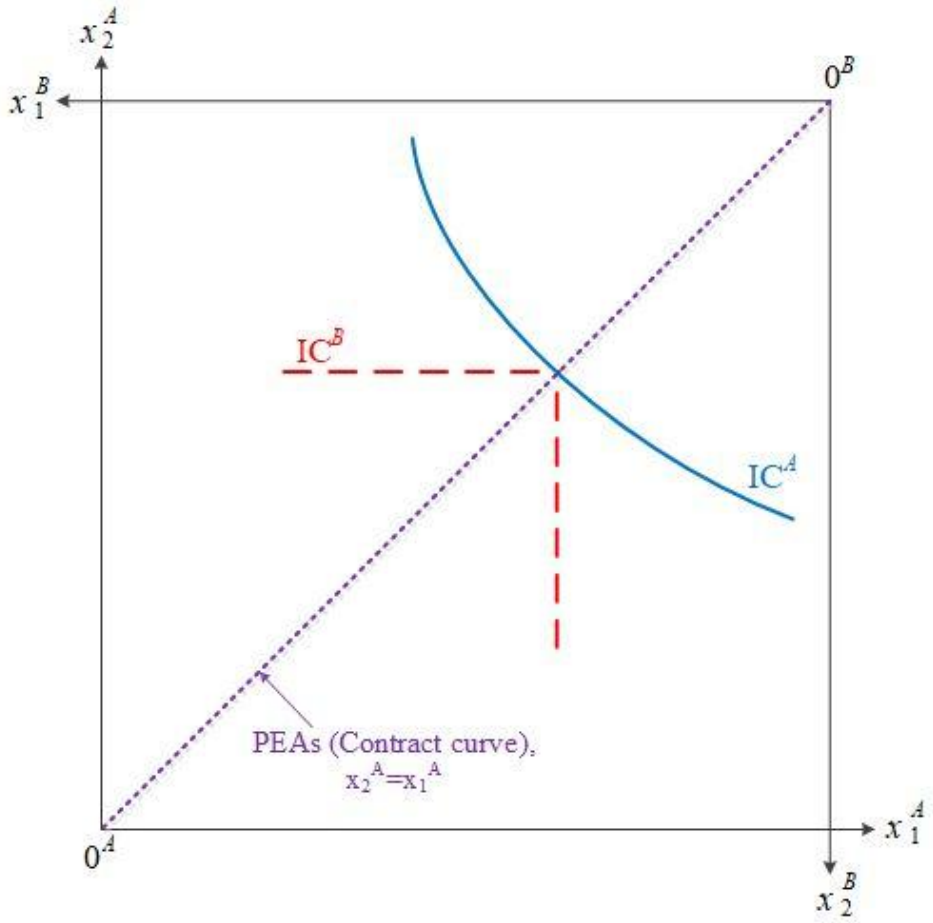
  – Substituting the above expression in the second constraint yields

$$x_2^A + \underbrace{(4 - x_1^A)}_{x_2^B} = 4 \iff x_2^A = x_1^A$$

  – This defines the contract curve, i.e., the set of Pareto efficient allocations.

General Equilibrium: Competitive Markets

- Contract curve (PEAs)



General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  2) WEA:

  – Consumer A's maximization problem is

$$\max_{x_1^A,\, x_2^A} \; x_1^A x_2^A$$

$$\text{s.t. } p_1 x_1^A + x_2^A \leq p_1 \cdot 3 + 1$$

  – FOCs:

$$x_2^A - \lambda p_1 = 0$$

$$x_1^A - \lambda = 0$$

$$p_1 x_1^A + x_2^A = 3p_1 + 1$$

  where λ is the Lagrange multiplier.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Combining the first two FOCs,

$$\lambda = \frac{x_2^A}{p_1} = x_1^A, \qquad \text{so} \quad x_2^A = p_1 x_1^A$$

  - Hence, consumer A spends the same amount on each good. Substituting into the budget constraint,

$$p_1 x_1^A + p_1 x_1^A = 3p_1 + 1$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Hence, with income $m^A = 3p_1 + 1$ (the value of his endowment), consumer A's Walrasian demands are

$$x_1^A(p_1) = \frac{m^A}{2p_1} = \frac{3p_1 + 1}{2p_1}$$

$$x_2^A(p_1) = \frac{m^A}{2} = \frac{3p_1 + 1}{2}$$

  - Next, we find consumer B's Walrasian demands.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Consumer B solves

$$\max_{x_1^B, x_2^B} \min\{x_1^B, x_2^B\} \quad \text{s.t.} \quad p_1 x_1^B + x_2^B \leq p_1 \cdot 1 + 3$$

  - He consumes at the kink, $x_1^B = x_2^B$, so

$$x_1^B = x_2^B = \frac{p_1 + 3}{p_1 + 1}$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Market clearing for good 1 requires
$$\frac{3p_1 + 1}{2p_1} + \frac{p_1 + 3}{p_1 + 1} = 4$$
  - which simplifies to
$$3{p_1}^2 - 2p_1 - 1 = (3p_1 + 1)(p_1 - 1) = 0$$
  - whose unique positive root is $p_1 = 1$.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Substituting $p_1 = 1$ into the Walrasian demands,

$$x_1^A = \frac{3+1}{2} = 2 \quad \text{and} \quad x_2^A = \frac{3+1}{2} = 2$$

$$x_1^B = x_2^B = \frac{1+3}{1+1} = 2$$

  - The market for good 2 clears as well, as Walras' law predicts,

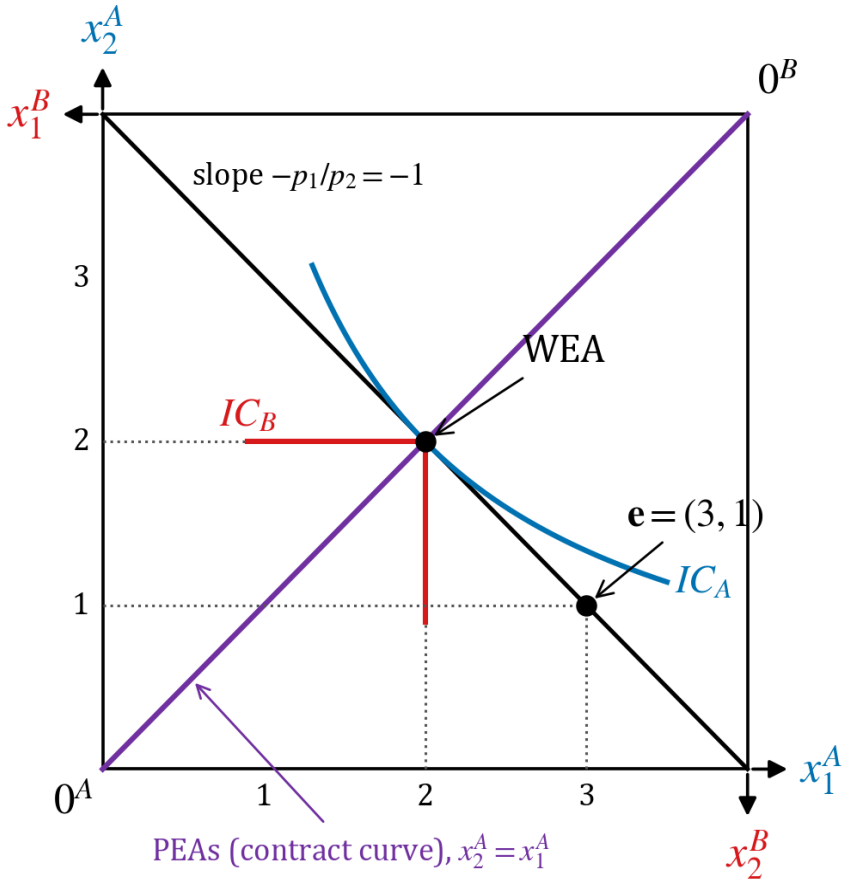
$$x_2^A + x_2^B = 2 + 2 = 4 = 1 + 3$$

  - Thus, the WEA is

$$(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2}) = (2,2; 2,2; 1)$$

General Equilibrium: Competitive Markets

- Second welfare theorem: WEA and PEAs



General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued): Second welfare theorem.
  - Suppose that we want to reach the Pareto-efficient allocation

$$\hat{\mathbf{x}}^A = (1,1) \quad \text{and} \quad \hat{\mathbf{x}}^B = (3,3)$$

  - which lies on the contract curve and favors consumer B.
  - Without transfers, the market yields the WEA

$$(x_1^A, x_2^A; x_1^B, x_2^B) = (2,2; 2,2)$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - ***Step 1***: find the price ratio that supports $\hat{\mathbf{x}}^A$. Consumer A's MRS at (1,1) is

$$MRS_{1,2}^A = \frac{x_2^A}{x_1^A} = \frac{1}{1} = 1, \qquad \text{so} \quad \frac{p_1}{p_2} = 1$$

  - Consumer B's indifference curve has a kink at $\hat{\mathbf{x}}^B$, so he demands $x_1^B = x_2^B$ at any prices, including $p_1 = p_2 = 1$.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - ***Step 2***: find the lump-sum transfers T_A and T_B, with $T_A = -T_B$, so that each consumer can afford his bundle in $\hat{\mathbf{x}}^A$ and $\hat{\mathbf{x}}^B$.
  - At $p_1 = p_2 = 1$, the endowments are worth

$$\mathbf{p} \cdot \mathbf{e}^A = 1(3) + 1(1) = 4 \quad \text{and} \quad \mathbf{p} \cdot \mathbf{e}^B = 1(1) + 1(3) = 4$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Consumer A's income after the transfer must cover the cost of $\hat{\mathbf{x}}^A$:

$$4 + T_A = \mathbf{p} \cdot \hat{\mathbf{x}}^A = 1(1) + 1(1) = 2, \qquad \text{so} \quad T_A = -2$$

  - Similarly, for consumer B:

$$4 + T_B = \mathbf{p} \cdot \hat{\mathbf{x}}^B = 1(3) + 1(3) = 6, \qquad \text{so} \quad T_B = 2$$

  - Hence, $T_A = -T_B$: consumer A transfers 2 to consumer B, so the transfers add up to zero.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - ***Step 3***: check that $\hat{\mathbf{x}}^A$ and $\hat{\mathbf{x}}^B$ form a WEA given the transfers. Using consumer A's Walrasian demands with income $m^A = 4 + T_A = 2$,

$$x_k^A = \frac{m^A}{2p_k} = \frac{2}{2(1)} = 1 \quad \text{for} \quad k = 1,2$$

  - Using consumer B's Walrasian demand with income $m^B = 4 + T_B = 6$,

$$x_1^B = x_2^B = \frac{m^B}{p_1 + p_2} = \frac{6}{1 + 1} = 3$$

General Equilibrium: Competitive Markets

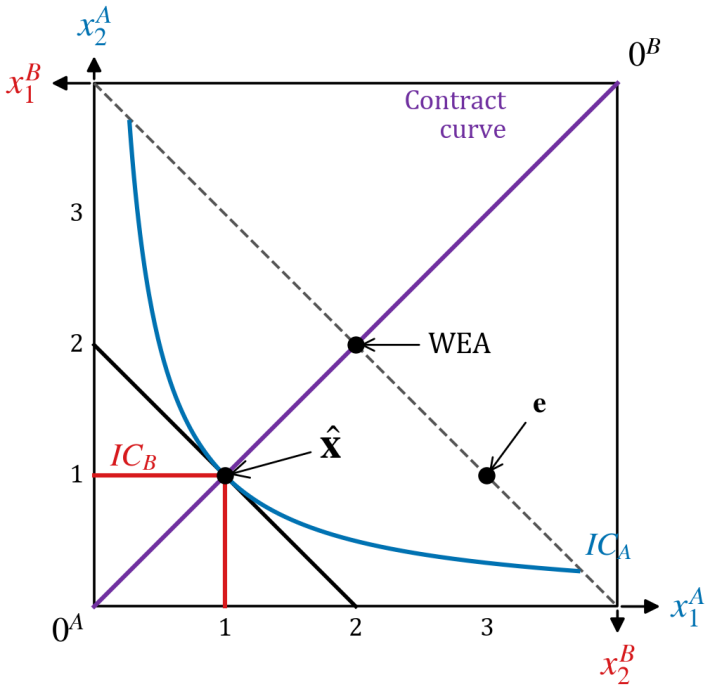
- ***Example 6.7*** (continued):
  - Both markets clear,

$$x_1^A + x_1^B = 1 + 3 = 4 = 3 + 1$$
$$x_2^A + x_2^B = 1 + 3 = 4 = 1 + 3$$

  - Hence, the Pareto-efficient allocation is a WEA with transfers $T_A = -2$ and $T_B = 2$: consumer A sells two units of good 1 to finance his transfer.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Dashed line: no transfers, WEA at (2,2).
  - Solid line: budget line after the transfer.



General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - ***Redistributing goods***: suppose instead that the planner reaches the same allocation by moving units of the goods, not money.
  - Consumer A gives t_1 units of good 1 and t_2 units of good 2 to consumer B (a negative t_k means that B gives to A), so the new endowments are

$$\hat{\mathbf{e}}^A = (3 - t_1, 1 - t_2)$$
$$\hat{\mathbf{e}}^B = (1 + t_1, 3 + t_2)$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - We look for t_1 and t_2 such that the WEA from these new endowments is

$$\hat{\mathbf{x}}^A = (1,1) \quad \text{and} \quad \hat{\mathbf{x}}^B = (3,3)$$

  - As in Step 1, consumer A's MRS at (1,1) equals 1, so the supporting prices are $p_1 = p_2 = 1$.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Consumer A's income is now the value of his new endowment,
    $$m^A = p_1(3 - t_1) + p_2(1 - t_2) = 4 - t_1 - t_2$$
  - His Walrasian demands give $\hat{\mathbf{x}}^A = (1,1)$ only if
    $$x_k^A = \frac{m^A}{2p_k} = \frac{4 - t_1 - t_2}{2} = 1$$
  - that is,
    $$t_1 + t_2 = 2$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Consumer B's income is then
    $$m^B = (1 + t_1) + (3 + t_2) = 4 + t_1 + t_2 = 6$$
  - so his Walrasian demand is
    $$x_1^B = x_2^B = \frac{m^B}{p_1 + p_2} = \frac{6}{2} = 3$$
  - as desired, and both markets clear.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Hence, any transfer with $t_1 + t_2 = 2$ works, as long as consumer A owns enough of each good,

$$t_1 \leq 3 \quad \text{and} \quad t_2 \leq 1$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - For instance,

$$(t_1, t_2) = (2,0): \quad \hat{\mathbf{e}}^A = (1,1)$$
$$(t_1, t_2) = (1,1): \quad \hat{\mathbf{e}}^A = (2,0)$$
$$(t_1, t_2) = (3,-1): \quad \hat{\mathbf{e}}^A = (0,2)$$

  - In the first case no trade is needed; in the other two, A trades one unit of a good for one unit of the other.

General Equilibrium: Production

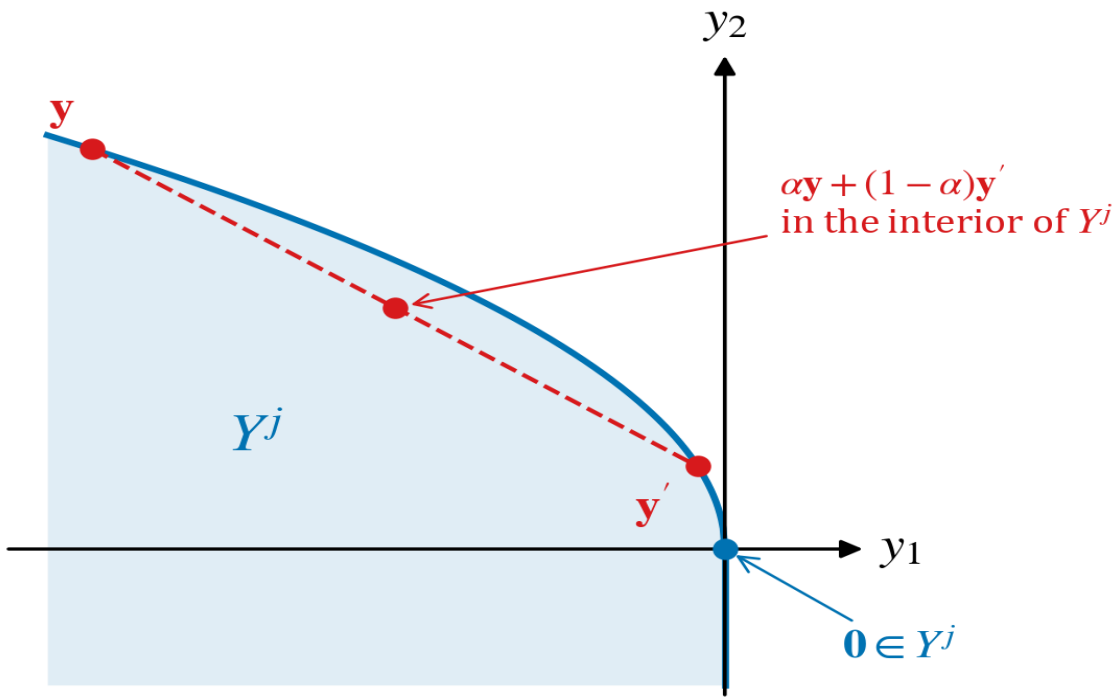
- Let us now extend our previous results to a setting where firms are also active.
- Assume J firms in the economy, each with production set Y^j, which satisfies:
  - Inaction is possible, $\mathbf{0} \in Y^j$.

General Equilibrium: Production

- Production set Y^j (continued):
  - Y^j is closed and bounded (compact), so a profit-maximizing plan exists at every $\mathbf{p}$; closedness also puts the frontier in Y^j.
  - Y^j is strictly convex, so convex combinations of two production plans lie in the interior of the production set.

General Equilibrium: Production

- Production set Y^j for firm j



General Equilibrium: Production

- Every firm j facing a fixed price vector $\mathbf{p} \gg \mathbf{0}$ independently and simultaneously solves

$$\max_{\mathbf{y}^j \in Y^j} \mathbf{p} \cdot \mathbf{y}^j$$

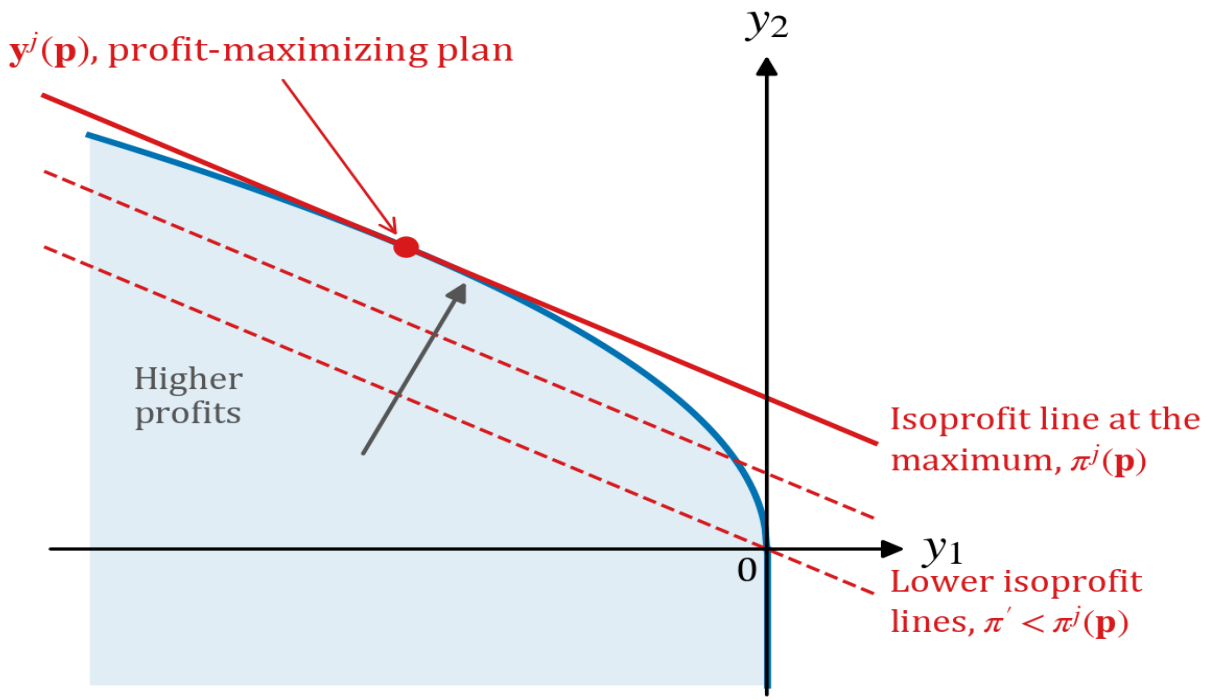
- A profit-maximizing production plan $\mathbf{y}^j(\mathbf{p})$ exists (Y^j is compact) and is unique (Y^j is strictly convex).

General Equilibrium: Production

- By the theorem of the maximum, both the argmax, $\mathbf{y}^j(\mathbf{p})$, and the value function, $\pi^j(\mathbf{p}) \equiv \mathbf{p} \cdot \mathbf{y}^j(\mathbf{p})$, are continuous in $\mathbf{p}$.

General Equilibrium: Production

- $\mathbf{y}^j(\mathbf{p})$ exists and is unique



General Equilibrium: Production

- ***Aggregate production set***:
  - The aggregate production set is the set of all sums of feasible plans,

$$Y = \{\mathbf{y}: \mathbf{y} = \sum_{j=1}^{J} \mathbf{y}^j, \qquad \text{where } \mathbf{y}^j \in Y^j \text{ for every } j\}$$

General Equilibrium: Production

- Define

$$\mathbf{y}(\mathbf{p}) = \sum_{j=1}^{J} \mathbf{y}^j(\mathbf{p})$$

- Then $\mathbf{y}(\mathbf{p})$ maximizes $\mathbf{p} \cdot \mathbf{y}$ over Y: aggregate profit maximization equals firm-by-firm maximization.

General Equilibrium: Production

- In an economy with J firms, each of them earning $\pi^j(\mathbf{p})$ profits in equilibrium, how are profits distributed?
- Assume that each individual i owns a share θ^{ij} of firm j's profits, where

$$0 \leq \theta^{ij} \leq 1 \text{ and } \sum_{i=1}^{I} \theta^{ij} = 1$$

General Equilibrium: Production

- This allows for multiple sharing profiles:
  - $\theta^{ij} = 1$: individual i owns all shares of firm j;
  - $\theta^{ij} = \frac{1}{I}$: all individuals hold equal shares of firm j.

General Equilibrium: Production

- Consumer i's budget constraint becomes

$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + \sum_{j=1}^{J} \theta^{ij} \pi^j(\mathbf{p})$$

- where $\sum_{j=1}^{J} \theta^{ij} \pi^j(\mathbf{p})$ is new relative to the standard budget constraint.

General Equilibrium: Production

- Let us express the budget constraint using

$$m^i(\mathbf{p}) \equiv \mathbf{p} \cdot \mathbf{e}^i + \sum_{j=1}^{J} \theta^{ij} \pi^j(\mathbf{p})$$

- that is, $\mathbf{p} \cdot \mathbf{x}^i \leq m^i(\mathbf{p})$, where $m^i(\mathbf{p}) > 0$, since $\pi^j(\mathbf{p}) \geq 0$ (inaction is possible) and $\mathbf{p} \cdot \mathbf{e}^i > 0$.

General Equilibrium: Production

- ***Equilibrium with production***:
  - We start by defining excess demand functions and use such a definition to identify the set of equilibrium allocations.
  - *Excess demand*: The excess demand function for good k is

$$z_k(\mathbf{p}) \equiv \sum_{i=1}^{I} x_k^i(\mathbf{p}, m^i(\mathbf{p})) - \sum_{i=1}^{I} e_k^i - \sum_{j=1}^{J} y_k^j(\mathbf{p})$$

    where $\sum_{j=1}^{J} y_k^j(\mathbf{p})$ is a new term relative to the analysis of general equilibrium without production.

General Equilibrium: Production

- Hence, the aggregate excess demand vector is

$$\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), z_2(\mathbf{p}), \dots, z_L(\mathbf{p}))$$

General Equilibrium: Production

- ***WEA with production***: $\mathbf{p}^* \gg \mathbf{0}$ is a Walrasian equilibrium price vector, and $(\mathbf{x}(\mathbf{p}^*), \mathbf{y}(\mathbf{p}^*))$ the WEA, if:
  - 1) $\mathbf{x}^i(\mathbf{p}^*, m^i(\mathbf{p}^*))$ solves consumer i's UMP;
  - 2) $\mathbf{y}^j(\mathbf{p}^*)$ solves firm j's PMP;
  - 3) markets clear,

$$\sum_{i=1}^{I} \mathbf{x}^i(\mathbf{p}^*, m^i(\mathbf{p}^*)) = \sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j(\mathbf{p}^*)$$

  - that is, $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

General Equilibrium: Production

- Setup: two firms; firm k produces good k with $F_k(K_k, L_k)$; input prices r and w; $F_{kK} = \frac{\partial F_k}{\partial K_k}$, and similarly for labor (here L denotes labor).
- From every consumer solving his UMP, the marginal rate of substitution between goods 1 and 2 satisfies, for any consumers i and h,

$$MRS_{1,2}^{i} = MRS_{1,2}^{h} = \frac{p_1}{p_2}$$

General Equilibrium: Production

- Similarly, from every firm solving its PMP, we obtain

$$p_1 F_{1K} = r \quad \text{and} \quad p_1 F_{1L} = w$$

  - for firm 1, and

$$p_2 F_{2K} = r \quad \text{and} \quad p_2 F_{2L} = w$$

  - for firm 2.

General Equilibrium: Production

- Dividing these expressions yields

$$MRTS^1_{L,K} \equiv \frac{F_{1L}}{F_{1K}} = \frac{w}{r}$$

  - for firm 1, and

$$MRTS^2_{L,K} \equiv \frac{F_{2L}}{F_{2K}} = \frac{w}{r}$$

  - for firm 2.

General Equilibrium: Production

- Therefore, $MRTS^1_{L,K} = MRTS^2_{L,K}$.
- Moving dK units of capital from firm 2 to firm 1 changes outputs by

$$dy_1 = F_{1K}dK \quad \text{and} \quad dy_2 = -F_{2K}dK$$

  - so the marginal rate of transformation is

$$MRT^K_{1,2} = -\frac{dy_2}{dy_1} = \frac{F_{2K}}{F_{1K}}$$

General Equilibrium: Production

- Dividing $p_1 F_{1K} = r$ by $p_2 F_{2K} = r$ yields

$$\frac{p_1 F_{1K}}{p_2 F_{2K}} = \frac{r}{r} = 1, \qquad \text{so} \quad \frac{p_1}{p_2} = \frac{F_{2K}}{F_{1K}} = MRT_{1,2}^{K}$$

- Similarly, dividing $p_1 F_{1L} = w$ by $p_2 F_{2L} = w$ yields

$$\frac{p_1}{p_2} = \frac{F_{2L}}{F_{1L}} = MRT_{1,2}^{L}$$

General Equilibrium: Production

- Combining $MRS_{1,2}^{i} = \frac{p_1}{p_2}$ for every consumer i with $\frac{p_1}{p_2} = MRT_{1,2}^{m}$ for every input $m \in \{K, L\}$ yields

$$MRS_{1,2}^{i} = MRT_{1,2}^{m} = \frac{p_1}{p_2}$$

General Equilibrium: Production

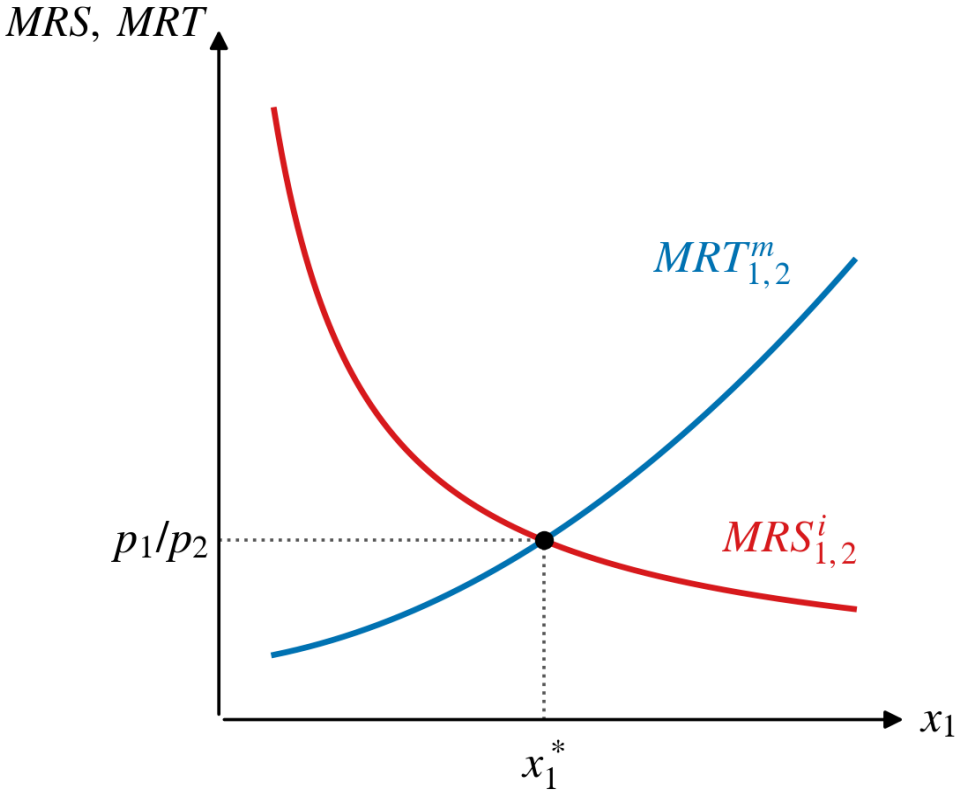
- Inspecting $MRS^i_{1,2} = MRT^m_{1,2} = \frac{p_1}{p_2}$, we see that
  - $MRS^i_{1,2} = \frac{MU^i_1}{MU^i_2}$ decreases in x_1 along an indifference curve, by strict quasiconcavity of u^i.
  - In contrast, $MRT^m_{1,2} = \frac{F_{2m}}{F_{1m}}$ increases in x_1 (by diminishing marginal products).

General Equilibrium: Production

- Intuitively, to increase x_1, we move units of input m from firm 2 to firm 1, which raises the marginal product of this input in firm 2 and lowers it in firm 1.
- Plotting $MRS^i_{1,2}$ and $MRT^m_{1,2}$ as functions of x_1, they cross at the equilibrium level of good 1, x_1^*, at a height of $\frac{p_1}{p_2}$.

General Equilibrium: Production

- Equilibrium with production



General Equilibrium: Production

- ***Existence***: Under the assumptions on preferences and production sets above, assume that the aggregate endowment is strictly positive in every good,

$$\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$$

- Then, there is a price vector $\mathbf{p}^* \gg \mathbf{0}$ such that a WEA exists, that is, $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

General Equilibrium: Production

- ***Example 6.8*** (Equilibrium with production):
  - Consider a two-consumer, two-good economy where consumer $i \in \{A, B\}$ has utility function $u^i = x_1^i x_2^i$.
  - There are two firms in this economy; each uses capital (K) and labor (L) to produce one of the two goods.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Firm 1 produces good 1 according to $y_1 = K_1^{0.75} L_1^{0.25}$.
  - Firm 2 produces good 2 according to $y_2 = K_2^{0.25} L_2^{0.75}$.
  - Consumer A is endowed with $(K^A, L^A) = (1,1)$, while consumer B is endowed with $(K^B, L^B) = (2,1)$.
  - Let us find a WEA in this economy with production.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - UMPs: Consumer i's maximization problem is

$$\max_{x_1^i, x_2^i} x_1^i x_2^i$$

  - subject to

$$p_1 x_1^i + p_2 x_2^i = rK^i + wL^i$$

  - where r and w are the prices of capital and labor, respectively.
  - Consumers are not endowed with goods 1 and 2. Each consumer i is only endowed with capital and labor, K^i and L^i, which he sells to the firms, earning rK^i from capital and wL^i from labor.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Both technologies exhibit constant returns to scale, so firms earn zero profits and income is $rK^i + wL^i$.
  - FOC:

$$\frac{p_1}{p_2} = MRS^i_{1,2} = \frac{x^i_2}{x^i_1}, \qquad \text{so} \quad p_1 x^i_1 = p_2 x^i_2$$

  - for $i \in \{A, B\}$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Taking the above equation for consumers A and B, and adding them together yields

$$p_1(x_1^A + x_1^B) = p_2(x_2^A + x_2^B)$$

    where $x_1^A + x_1^B$ is the left side of the feasibility condition $x_1^A + x_1^B = y_1 = K_1^{0.75} L_1^{0.25}$.
  - Substituting both feasibility conditions into the above expression, and re-arranging, yields

$$\frac{p_1}{p_2} = \frac{K_2^{0.25} L_2^{0.75}}{K_1^{0.75} L_1^{0.25}}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - PMPs: Firm 1's maximization problem is

$$\max_{K_1, L_1} p_1 K_1^{0.75} L_1^{0.25} - rK_1 - wL_1$$

  - FOCs:

$$r = 0.75 p_1 K_1^{-0.25} L_1^{0.25}$$
$$w = 0.25 p_1 K_1^{0.75} L_1^{-0.75}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Dividing the second FOC by the first gives the tangency condition for profit maximization

$$MRTS_{L,K}^{1} = \frac{w}{r}, \qquad \text{so} \quad \frac{w}{r} = \frac{1}{3}\frac{K_1}{L_1}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Likewise, firm 2's PMP gives the following FOCs:

$$r = 0.25 p_2 K_2^{-0.75} L_2^{0.75}$$
$$w = 0.75 p_2 K_2^{0.25} L_2^{-0.25}$$

  - Dividing the second FOC by the first gives the tangency condition for profit maximization

$$MRTS_{L,K}^2 = \frac{w}{r}, \qquad \text{so} \quad \frac{w}{r} = 3\frac{K_2}{L_2}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Combining both $MRTS_{L,K}$ yields

$$\frac{1}{3}\frac{K_1}{L_1} = 3\frac{K_2}{L_2}, \qquad \text{so} \quad \frac{K_1}{L_1} = 9\frac{K_2}{L_2}$$

  - Hence, firm 1 is more capital intensive than firm 2, that is, its capital-to-labor ratio is higher.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Setting both firms' price of capital, r, equal to each other yields

$$0.75p_1K_1^{-0.25}L_1^{0.25} = 0.25p_2K_2^{-0.75}L_2^{0.75}$$

  - so

$$\frac{p_1}{p_2} = \frac{1}{3}\left(\frac{K_1}{L_1}\right)^{0.25}\left(\frac{K_2}{L_2}\right)^{-0.75}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Setting both firms' price of labor, w, equal to each other yields

$$0.25p_1K_1^{0.75}L_1^{-0.75} = 0.75p_2K_2^{0.25}L_2^{-0.25}$$

  - so

$$\frac{p_1}{p_2} = 3\left(\frac{K_1}{L_1}\right)^{-0.75}\left(\frac{K_2}{L_2}\right)^{0.25}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Setting the price ratio from consumers' UMP equal to the first price ratio from firms' PMP yields

$$\frac{K_2^{0.25} L_2^{0.75}}{K_1^{0.75} L_1^{0.25}} = \frac{1}{3}\left(\frac{K_1}{L_1}\right)^{0.25}\left(\frac{K_2}{L_2}\right)^{-0.75}$$

  - Since $L_1^{-0.25}$ and $L_2^{0.75}$ appear on both sides, they cancel:

$$\frac{1}{3} K_1^{0.25} K_2^{-0.75} = K_2^{0.25} K_1^{-0.75}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Multiplying both sides by $3K_1^{0.75}K_2^{0.75}$ yields

$$K_1 = 3K_2, \qquad \text{that is,} \qquad K_2 = \frac{K_1}{3}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - By the feasibility conditions, we know that $K_1 + K_2 = K^A + K^B = 3$ or $K_2 = 3 - K_1$.
  - Substituting the above expression into $K_1 = 3K_2$, we find the equilibrium capital used by firms 1 and 2:

$$K_1 = 3(3 - K_1) \implies K_1^* = \frac{9}{4}$$

$$K_2^* = \frac{1}{3}K_1^* = \frac{3}{4}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Setting price ratio from consumers' UMP equal to the second price ratio from firms' PMP yields

$$\frac{K_2^{0.25}L_2^{0.75}}{K_1^{0.75}L_1^{0.25}} = 3\left(\frac{K_1}{L_1}\right)^{-0.75}\left(\frac{K_2}{L_2}\right)^{0.25} \Longrightarrow L_1 = \frac{1}{3}L_2$$

  - By the feasibility conditions, we know that $L_1 + L_2 = L^A + L^B = 2$ or $L_2 = 2 - L_1$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Substituting the above expression into $L_1 = \frac{1}{3}L_2$, we find the equilibrium labor used by firms 1 and 2:

$$L_1 = \frac{1}{3}(2 - L_1) \implies L_1^* = \frac{1}{2}$$

$$L_2^* = 3L_1^* = \frac{3}{2}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Substituting the capital and labor used by firms 1 and 2 into the price ratio from consumers' UMP yields

$$\frac{p_1}{p_2} = \frac{\left(\frac{3}{4}\right)^{0.25}\left(\frac{3}{2}\right)^{0.75}}{\left(\frac{9}{4}\right)^{0.75}\left(\frac{1}{2}\right)^{0.25}} = \sqrt{\frac{2}{3}} \approx 0.82$$

  - Normalizing $p_2 = 1$ gives $p_1 = 0.82$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Furthermore, substituting our calculated values into firm 1's FOCs for r and w yields

$$r^* = 0.75\sqrt{\frac{2}{3}}\left(\frac{9}{4}\right)^{-0.25}\left(\frac{1}{2}\right)^{0.25} = \frac{2^{3/4}}{4} \approx 0.42$$

$$w^* = 0.25\sqrt{\frac{2}{3}}\left(\frac{9}{4}\right)^{0.75}\left(\frac{1}{2}\right)^{-0.75} = \frac{3 \cdot 2^{3/4}}{8} \approx 0.63$$

  - so that $\frac{w^*}{r^*} = \frac{3}{2}$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Using consumer A's tangency condition, $p_1 x_1^A = p_2 x_2^A$.
  - Substituting this into consumer A's budget constraint yields

$$2p_1 x_1^A = rK^A + wL^A$$

  - so consumer A spends half of his income on each good.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Plugging in our calculated values,

$$x_1^{A*} = \frac{rK^A + wL^A}{2p_1} = \frac{0.42 + 0.63}{2(0.82)} = 0.64$$

$$x_2^{A*} = \frac{rK^A + wL^A}{2p_2} = \frac{0.42 + 0.63}{2} = 0.53$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Performing the same steps for consumer B yields

$$x_1^{B*} = \frac{rK^B + wL^B}{2p_1} = \frac{2(0.42) + 0.63}{2(0.82)} = 0.90$$

$$x_2^{B*} = \frac{rK^B + wL^B}{2p_2} = \frac{2(0.42) + 0.63}{2} = 0.74$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Thus, our WEA is
    $$(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}) = (0.64,0.53; 0.90,0.74)$$
  - with firms using
    $$(K_1^*, L_1^*; K_2^*, L_2^*) = \left(\frac{9}{4}, \frac{1}{2}; \frac{3}{4}, \frac{3}{2}\right)$$
  - and prices
    $$\frac{p_1}{p_2} = 0.82 \quad \text{and} \quad (r^*, w^*) = (0.42,0.63)$$

General Equilibrium: Production

- ***Equilibrium with production: welfare***:
  - We extend the First and Second Welfare Theorems to economies with production, connecting WEA and Pareto efficient allocations.
  - ***Pareto efficiency***: A feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto efficient if there is no other feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that

$$u^i(\hat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i) \quad \text{for every} \quad i = 1, \ldots, I$$

  - with $u^i(\hat{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$ for at least one consumer.

General Equilibrium: Production

- In an economy with two goods, two consumers (i and h), two firms and two inputs (labor and capital), every Pareto efficient allocation solves, for some $\overline{u}^h$,

$$\max_{x_1^i, x_2^i, x_1^h, x_2^h, L_1, K_1, L_2, K_2 \geq 0} u^i(x_1^i, x_2^i)$$
$$\text{s.t. } u^h(x_1^h, x_2^h) \geq \overline{u}^h$$

General Equilibrium: Production

- and the feasibility constraints:
  - technological feasibility,

$$x_1^i + x_1^h \leq F_1(L_1, K_1), \qquad x_2^i + x_2^h \leq F_2(L_2, K_2)$$

  - input feasibility,

$$L_1 + L_2 \leq \overline{L}, \qquad K_1 + K_2 \leq \overline{K}$$

General Equilibrium: Production

- The Lagrangian is

$$\mathcal{L} = u^i(x_1^i, x_2^i) + \lambda[u^h(x_1^h, x_2^h) - \overline{u}^h]$$
$$+\mu_1[F_1(L_1, K_1) - x_1^i - x_1^h] + \mu_2[F_2(L_2, K_2) - x_2^i - x_2^h]$$
$$+\delta_L[\overline{L} - L_1 - L_2] + \delta_K[\overline{K} - K_1 - K_2]$$

General Equilibrium: Production

- In the case of interior solutions, the FOCs with respect to x_k^i and x_k^h yield

$$\frac{\partial u^i}{\partial x_k^i} = \mu_k \quad \text{and} \quad \lambda \frac{\partial u^h}{\partial x_k^h} = \mu_k$$

- Hence, as in barter economies, efficiency in consumption requires

$$MRS_{1,2}^i = MRS_{1,2}^h = \frac{\mu_1}{\mu_2}$$

General Equilibrium: Production

- FOCs with respect to L_j and K_j, where $j \in \{1,2\}$, yield

$$\mu_1 \frac{\partial F_1}{\partial L} = \delta_L = \mu_2 \frac{\partial F_2}{\partial L}$$
$$\mu_1 \frac{\partial F_1}{\partial K} = \delta_K = \mu_2 \frac{\partial F_2}{\partial K}$$

- Dividing the first condition by the second,

$$\frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}} = \frac{\delta_L}{\delta_K} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

General Equilibrium: Production

- This is a condition for efficiency that we encountered in production theory: the $MRTS_{L,K}$ between labor and capital must coincide across firms.
- Otherwise, welfare could be increased by assigning more labor to the firm with the higher $MRTS_{L,K}$ (and more capital to the other firm), which raises the output of both goods.

General Equilibrium: Production

- Combining the above two conditions for efficiency in consumption and production, we obtain

$$\frac{\frac{\partial u^i}{\partial x_1^i}}{\frac{\partial u^i}{\partial x_2^i}} = \frac{\mu_1}{\mu_2} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_1}{\partial L}}$$

General Equilibrium: Production

- That is, $MRS^{i}_{1,2}$ must coincide with the rate at which units of good 2 can be transformed into units of good 1, that is, the marginal rate of transformation $MRT_{1,2}$.
- Moving one unit of labor from firm 2 to firm 1 raises the output of good 1 by $\frac{\partial F_1}{\partial L}$ and lowers the output of good 2 by $\frac{\partial F_2}{\partial L}$.

General Equilibrium: Production

- Hence, one more unit of good 1 costs $MRT_{1,2}$ units of good 2, where

$$MRT_{1,2} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_1}{\partial L}}$$

- *Intuition*: for an allocation to be efficient, the rate at which consumers are willing to substitute goods 1 and 2 must coincide with the rate at which good 2 can be transformed into good 1.

General Equilibrium: Production

- ***First welfare theorem with production***: Under the assumptions on preferences above (strictly increasing utility is the key one), every WEA is Pareto efficient.
- *Proof* (by contradiction):
  - Suppose that $(\mathbf{x}, \mathbf{y})$ is a WEA at prices $\mathbf{p}^*$, but is not Pareto efficient.

General Equilibrium: Production

- Since $(\mathbf{x}, \mathbf{y})$ is a WEA, it is feasible:

$$\sum_{i=1}^{I} \mathbf{x}^i = \sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j$$

General Equilibrium: Production

- Because $(\mathbf{x}, \mathbf{y})$ is not Pareto efficient, there exists some other feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that

$$u^i(\hat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i) \quad \text{for every} \quad i = 1, \ldots, I$$

- with $u^i(\hat{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$ for at least one consumer.
- That is, $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ makes at least one consumer strictly better off than the WEA, without making anyone worse off.

General Equilibrium: Production

- But this implies that bundle $\hat{\mathbf{x}}^i$ is at least as costly as $\mathbf{x}^i$,

$$\mathbf{p}^* \cdot \hat{\mathbf{x}}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i$$

- for every consumer i (with at least one strict inequality):
  - if $u^i(\hat{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$, then $\mathbf{p}^* \cdot \hat{\mathbf{x}}^i > \mathbf{p}^* \cdot \mathbf{x}^i$; otherwise, i would have chosen $\hat{\mathbf{x}}^i$;
  - if $u^i(\hat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$ and $\mathbf{p}^* \cdot \hat{\mathbf{x}}^i < \mathbf{p}^* \cdot \mathbf{x}^i$, strict monotonicity gives an affordable bundle strictly better than $\mathbf{x}^i$.

General Equilibrium: Production

- Summing over all consumers yields

$$\mathbf{p}^* \cdot \sum_{i=1}^{I} \hat{\mathbf{x}}^i > \mathbf{p}^* \cdot \sum_{i=1}^{I} \mathbf{x}^i$$

- Since $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is feasible,

$$\sum_{i=1}^{I} \hat{\mathbf{x}}^i \leq \sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \hat{\mathbf{y}}^j$$

General Equilibrium: Production

- Because $\mathbf{p}^* \gg \mathbf{0}$,

$$\mathbf{p}^* \cdot \left(\sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \hat{\mathbf{y}}^j \right) \geq \mathbf{p}^* \cdot \sum_{i=1}^{I} \hat{\mathbf{x}}^i$$

$$> \mathbf{p}^* \cdot \sum_{i=1}^{I} \mathbf{x}^i = \mathbf{p}^* \cdot \left(\sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j \right)$$

General Equilibrium: Production

- or, rearranging,

$$\mathbf{p}^* \cdot \sum_{j=1}^{J} \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \sum_{j=1}^{J} \mathbf{y}^j$$

- However, this result implies that $\mathbf{p}^* \cdot \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \mathbf{y}^j$ for some firm j.

General Equilibrium: Production

- This indicates that production plan $\mathbf{y}^j$ was not profit-maximizing and, as a consequence, it cannot be part of a WEA.
- We therefore reached a contradiction.
- This implies that the original statement was true: if an allocation $(\mathbf{x}, \mathbf{y})$ is a WEA, it must also be Pareto efficient.

General Equilibrium: Production

- ***Example 6.9*** (WEA and PEA with production):
  - Consider the setting described in Example 6.8.
  - The set of Pareto efficient allocations must satisfy

$$MRS_{1,2}^{A} = MRS_{1,2}^{B}$$
$$MRTS_{L,K}^{1} = MRTS_{L,K}^{2}$$
$$MRS_{1,2}^{i} = MRT_{1,2}$$

General Equilibrium: Production

- ***Example 6.9*** (continued):
  - By consumer A's tangency condition,

$$MRS_{1,2}^{A} = \frac{x_2^A}{x_1^A} = \frac{p_1}{p_2} = 0.82$$

  - and, by consumer B's tangency condition,

$$MRS_{1,2}^{B} = \frac{x_2^B}{x_1^B} = \frac{p_1}{p_2} = 0.82$$

  - which implies that $MRS_{1,2}^{A} = MRS_{1,2}^{B}$.

General Equilibrium: Production

- ***Example 6.9*** (continued):
  - From Example 6.8, $\frac{K_1}{L_1} = \frac{9}{2}$ and $\frac{K_2}{L_2} = \frac{1}{2}$, so

$$MRTS_{L,K}^{1} = \frac{1}{3}\frac{K_1}{L_1} = \frac{1}{3} \cdot \frac{9}{2} = \frac{3}{2}$$

$$MRTS_{L,K}^{2} = 3\frac{K_2}{L_2} = 3 \cdot \frac{1}{2} = \frac{3}{2}$$

  - which implies that $MRTS_{L,K}^{1} = MRTS_{L,K}^{2}$.

General Equilibrium: Production

- ***Example 6.9*** (continued):
  - Finally, the marginal rate of transformation is

$$MRT_{1,2} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_1}{\partial L}} = \frac{0.75\left(\frac{K_2}{L_2}\right)^{0.25}}{0.25\left(\frac{K_1}{L_1}\right)^{0.75}}$$

$$= \frac{0.75\left(\frac{1}{2}\right)^{0.25}}{0.25\left(\frac{9}{2}\right)^{0.75}} \approx 0.82$$

General Equilibrium: Production

- ***Example 6.9*** (continued):
  - Summarizing,

$$MRS_{1,2}^{A} = MRS_{1,2}^{B} = 0.82$$

$$MRTS_{L,K}^{1} = MRTS_{L,K}^{2} = \frac{3}{2}$$

$$MRS_{1,2}^{i} = MRT_{1,2} = 0.82$$

  - Since all three conditions hold, our WEA from Example 6.8 is Pareto efficient.

General Equilibrium: Production

- ***Second welfare theorem with production***:
  - Consider the assumptions on consumers and producers described above.
  - Then, for every Pareto efficient allocation $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ we can find:
  - (a) a profile of income transfers $(T_1, T_2, \ldots, T_I)$ redistributing income among consumers, that is, satisfying $\sum_{i=1}^{I} T_i = 0$;
  - (b) a price vector $\overline{\mathbf{p}}$, such that:

General Equilibrium: Production

- (1) Bundle $\overline{\mathbf{x}}^i$ solves the UMP

$$\max_{\mathbf{x}^i} u^i(\mathbf{x}^i)$$

$$\text{s.t. } \overline{\mathbf{p}} \cdot \mathbf{x}^i \leq m^i(\overline{\mathbf{p}}) + T_i \quad \text{for every} \quad i = 1, \ldots, I$$

  - where individual i's original income $m^i(\overline{\mathbf{p}})$ is increased (decreased) if the transfer T_i is positive (negative).

General Equilibrium: Production

- (2) Production plan $\overline{\mathbf{y}}^j$ solves the PMP

$$\max_{\mathbf{y}^j} \overline{\mathbf{p}} \cdot \mathbf{y}^j$$

$$\text{s.t.} \quad \mathbf{y}^j \in Y^j \quad \text{for every} \quad j = 1, \ldots, J$$

General Equilibrium: Production

- ***Example 6.10*** (Second welfare theorem with production):
  - Consider Example 6.8. Both consumers have identical Cobb-Douglas preferences, so every Pareto efficient allocation uses the production plan of Example 6.8 and gives each consumer the same share of y_1 and y_2.

General Equilibrium: Production

- ***Example 6.10*** (continued):
  - Consider, for instance, the Pareto efficient allocation

$$(\overline{x}_1^A, \overline{x}_2^A; \overline{x}_1^B, \overline{x}_2^B) = (0.75, 0.61; 0.79, 0.65)$$

  - which is feasible since

$$0.75 + 0.79 = 1.54 = y_1$$
$$0.61 + 0.65 = 1.26 = y_2$$

General Equilibrium: Production

- ***Example 6.10*** (continued):
  - Prices remain unchanged: firms' input choices do not change, and aggregate demand depends only on aggregate income, since each consumer spends half of his income on each good.
  - Hence,

$$(p_1, p_2; r, w) = (0.82,1; 0.42,0.63)$$

General Equilibrium: Production

- ***Example 6.10*** (continued):
  - ***Transfers***: with a lump-sum transfer, consumer A's budget constraint becomes

$$p_1\overline{x}_1^A + p_2\overline{x}_2^A = rK^A + wL^A + T_A$$

  - Since $(K^A, L^A) = (1,1)$, at the prices above his income is

$$0.42(1) + 0.63(1) + T_A = 1.05 + T_A$$

General Equilibrium: Production

- ***Example 6.10*** (continued):
  - Consumer A can afford $(\overline{x}_1^A, \overline{x}_2^A) = (0.75, 0.61)$ only if his income equals its cost,
    $$0.82(0.75) + 1(0.61) \approx 1.22 = 1.05 + T_A$$
  - so $T_A \approx 0.17$.
  - This bundle is also his best choice: with Cobb-Douglas utility, he spends half of his income on each good, and indeed
    $$p_1\overline{x}_1^A = 0.82(0.75) \approx 0.61 = p_2\overline{x}_2^A$$

General Equilibrium: Production

- ***Example 6.10*** (continued):
  - Similarly, consumer B's income is
    $$0.42(2) + 0.63(1) + T_B = 1.47 + T_B$$
  - so that he can afford $(\overline{x}_1^B, \overline{x}_2^B) = (0.79, 0.65)$ only if
    $$0.82(0.79) + 1(0.65) \approx 1.30 = 1.47 + T_B$$
  - so $T_B \approx -0.17$.

General Equilibrium: Production

- ***Example 6.10*** (continued):
  - Consumer B also spends half of his income on each good,

$$p_1\overline{x}_1^B = 0.82(0.79) \approx 0.65 = p_2\overline{x}_2^B$$

  - Hence, $T_A + T_B = 0$ (up to rounding): consumer B transfers 0.17 to consumer A.
  - With these transfers, the Pareto efficient allocation is a WEA with transfers at the same prices.

Comparative Statics

Comparative Statics

- We analyze how equilibrium outcomes are affected by an increase in:
  - the price of one good;
  - the endowment of one input.
- Consider a setting with two goods, each produced with two factors under constant returns to scale (CRS): factor 1 is capital ($w_1 = r$) and factor 2 is labor ($w_2 = w$).

Comparative Statics

- If both goods are produced, a necessary condition for input prices (w_1^*, w_2^*) to be in equilibrium is

$$c_1(w_1, w_2) = p_1 \quad \text{and} \quad c_2(w_1, w_2) = p_2$$

  - where $c_j(w_1, w_2)$ is firm j's unit cost function.
- Under constant returns, marginal cost equals unit cost, so price equals unit cost and profits are zero.

Comparative Statics

- Let $z_{1j}(w)$ denote firm j's demand for factor 1, and $z_{2j}(w)$ its demand for factor 2, when producing one unit of output.
- These are the conditional factor demands evaluated at $q = 1$ (unit factor demands); under constant returns, $z(w, q) = q \cdot z(w, 1)$.

Comparative Statics

- The production of good 1 is more intensive in factor 1 than is the production of good 2 if

$$\frac{z_{11}(w)}{z_{21}(w)} > \frac{z_{12}(w)}{z_{22}(w)}$$

  - where $\frac{z_{1j}(w)}{z_{2j}(w)}$ represents firm j's demand for input 1 relative to that of input 2.

Comparative Statics: Price Change

1) ***Changes in the price of one good,*** p_j **(Stolper-Samuelson theorem)**:
   - Consider an economy with two consumers and two firms satisfying the above factor intensity assumption.
   - If the price of good j, p_j, increases, then:
     a) the equilibrium price of the factor more intensively used in the production of good j increases; while
     b) the equilibrium price of the other factor decreases.

Comparative Statics: Price Change

- *Proof*:
  - Let us first take the equilibrium conditions

$$c_1(w_1, w_2) = p_1 \text{ and } c_2(w_1, w_2) = p_2$$

  - Differentiating them yields

$$\frac{\partial c_1(w_1,w_2)}{\partial w_1} dw_1 + \frac{\partial c_1(w_1,w_2)}{\partial w_2} dw_2 = dp_1$$

$$\frac{\partial c_2(w_1,w_2)}{\partial w_1} dw_1 + \frac{\partial c_2(w_1,w_2)}{\partial w_2} dw_2 = dp_2$$

  - Applying Shephard's lemma, we obtain

$$z_{11}(w)dw_1 + z_{21}(w)dw_2 = dp_1$$

$$z_{12}(w)dw_1 + z_{22}(w)dw_2 = dp_2$$

Comparative Statics: Price Change

- If only price p_1 varies, then $dp_2 = 0$. Hence, we can rewrite the second expression as

$$z_{12}(w)dw_1 + z_{22}(w)dw_2 = 0, \qquad \text{so} \quad dw_2 = -\frac{z_{12}}{z_{22}}dw_1$$

- Substituting $dw_2 = -\frac{z_{12}}{z_{22}}dw_1$ into the first expression yields

$$\left[z_{11} - \frac{z_{21}z_{12}}{z_{22}}\right]dw_1 = dp_1$$

Comparative Statics: Price Change

- Solving for the ratio and multiplying its numerator and denominator by z_{22},

$$\frac{dw_1}{dp_1} = \frac{z_{22}}{z_{11}z_{22} - z_{12}z_{21}}$$

- Substituting this into $dw_2 = -\frac{z_{12}}{z_{22}} dw_1$ yields

$$\frac{dw_2}{dp_1} = -\frac{z_{12}}{z_{11}z_{22} - z_{12}z_{21}}$$

Comparative Statics: Price Change

- From the factor intensity condition,

$$\frac{z_{11}(w)}{z_{21}(w)} > \frac{z_{12}(w)}{z_{22}(w)}$$

- cross-multiplying (all factor demands are positive),

$$z_{11}z_{22} - z_{12}z_{21} > 0$$

Comparative Statics: Price Change

- Hence, the denominator in both $\frac{dw_1}{dp_1}$ and $\frac{dw_2}{dp_1}$ is positive.
- Since $z_{22} > 0$ and $z_{12} > 0$ (factor demands), $\frac{dw_1}{dp_1} > 0$ and $\frac{dw_2}{dp_1} < 0$.

Comparative Statics: Price Change

- ***Example 6.11***:
  - Let us find the unit factor demands in Example 6.8, where factor 1 is capital ($w_1 = r$) and factor 2 is labor ($w_2 = w$).
  - From firm 1's tangency condition,

$$MRTS_{L,K}^{1} = \frac{1}{3}\frac{K_1}{L_1} = \frac{w}{r}, \qquad \text{so} \quad L_1 = \frac{rK_1}{3w}$$

  - Substituting into $y_1 = K_1^{0.75} L_1^{0.25}$,

$$y_1 = K_1^{0.75}\left(\frac{rK_1}{3w}\right)^{0.25} = K_1\left(\frac{r}{3w}\right)^{0.25}$$

Comparative Statics: Price Change

- ***Example 6.11*** (continued):
  - Setting $y_1 = 1$ and using $L_1 = \frac{rK_1}{3w}$, firm 1's unit factor demands are

$$z_{11} = z_{K1} = \left(\frac{r}{3w}\right)^{-1/4} \quad \text{and} \quad z_{21} = z_{L1} = \left(\frac{r}{3w}\right)^{3/4}$$

  - From firm 2's tangency condition,

$$MRTS^2_{L,K} = 3\frac{K_2}{L_2} = \frac{w}{r}, \qquad \text{so} \quad K_2 = \frac{wL_2}{3r}$$

Comparative Statics: Price Change

- ***Example 6.11*** (continued):
  - Substituting into $y_2 = K_2^{0.25} L_2^{0.75}$,

$$y_2 = \left(\frac{wL_2}{3r}\right)^{0.25} L_2^{0.75} = L_2 \left(\frac{w}{3r}\right)^{0.25}$$

  - Setting $y_2 = 1$ and using $K_2 = \frac{wL_2}{3r}$, firm 2's unit factor demands are

$$z_{12} = z_{K2} = \left(\frac{w}{3r}\right)^{3/4} \quad \text{and} \quad z_{22} = z_{L2} = \left(\frac{w}{3r}\right)^{-1/4}$$

Comparative Statics: Price Change

- ***Example 6.11*** (continued):
  - Hence,

$$z_{11}z_{22} - z_{12}z_{21} = \left(\frac{r}{3w}\right)^{-1/4}\left(\frac{w}{3r}\right)^{-1/4} - \left(\frac{w}{3r}\right)^{3/4}\left(\frac{r}{3w}\right)^{3/4}$$

$$= \left(\frac{1}{9}\right)^{-1/4} - \left(\frac{1}{9}\right)^{3/4} = \sqrt{3} - \frac{1}{3\sqrt{3}} = \frac{8}{3\sqrt{3}} \approx 1.54 > 0$$

  - for every (r, w): firm 1 is more capital intensive than firm 2 at all factor prices.

Comparative Statics: Price Change

- ***Example 6.11*** (continued):
  - Using the expressions from the proof, with $w_1 = r$ and $w_2 = w$: at $(r, w) = (0.42, 0.63)$, $\frac{w}{3r} = \frac{1}{2}$, so

$$z_{22} = 2^{1/4} \approx 1.19 \quad \text{and} \quad z_{12} = 2^{-3/4} \approx 0.59$$

$$\frac{dr}{dp_1} = \frac{z_{22}}{z_{11}z_{22} - z_{12}z_{21}} = \frac{2^{1/4} \cdot 3\sqrt{3}}{8} \approx 0.77 > 0$$

$$\frac{dw}{dp_1} = -\frac{z_{12}}{z_{11}z_{22} - z_{12}z_{21}} = -\frac{2^{-3/4} \cdot 3\sqrt{3}}{8} \approx -0.39 < 0$$

Comparative Statics: Endowment Change

2) ***Changes in endowments*** **(Rybczynski theorem)**:
   - Consider an economy with two consumers and two firms satisfying the above factor intensity assumption.
   - Holding goods prices (p_1, p_2), and hence factor prices, constant, if the endowment of one factor increases while the other stays fixed, then
     a) production of the good that uses this factor more intensively increases; whereas
     b) the production of the other good decreases.

Comparative Statics: Endowment Change

- *Proof*:
  - Consider an economy with two factors, labor and capital, and two goods, 1 and 2.
  - Let $z_{Lj}(w)$ denote firm j's factor demand for labor (when producing one unit of output).
  - Similarly, let $z_{Kj}(w)$ denote firm j's factor demand for capital per unit of output.

Comparative Statics: Endowment Change

- Since goods prices, and hence factor prices, are fixed, the z's are constant.
- Then, factor feasibility requires

$$L = z_{L1}(w) \cdot y_1 + z_{L2}(w) \cdot y_2$$
$$K = z_{K1}(w) \cdot y_1 + z_{K2}(w) \cdot y_2$$

Comparative Statics: Endowment Change

- Differentiating the first condition (the z's are constant),

$$dL = z_{L1}dy_1 + z_{L2}dy_2$$

- Dividing both sides by L, and multiplying and dividing the term of each good j by y_j, we obtain

$$\frac{dL}{L} = \left(\frac{z_{L1}y_1}{L}\right)\left(\frac{dy_1}{y_1}\right) + \left(\frac{z_{L2}y_2}{L}\right)\left(\frac{dy_2}{y_2}\right)$$

Comparative Statics: Endowment Change

- We can express:
  - $\frac{z_{Lj}(w)y_j}{L} \equiv \gamma_{Lj}$, the share of labor used by firm j;
  - $\frac{dy_j}{y_j} \equiv \%\Delta y_j$, the percentage change in the output of good j;
  - $\frac{dL}{L} \equiv \%\Delta L$, the percentage increase in the endowment of labor in the economy.

Comparative Statics: Endowment Change

- Hence, the above expression becomes

$$\%\Delta L = \gamma_{L1} \cdot \%\Delta y_1 + \gamma_{L2} \cdot \%\Delta y_2$$

- A similar expression can be obtained for the endowment of capital:

$$\%\Delta K = \gamma_{K1} \cdot \%\Delta y_1 + \gamma_{K2} \cdot \%\Delta y_2$$

Comparative Statics: Endowment Change

- Note that $\gamma_{L1}, \gamma_{L2} \in (0,1)$ and $\gamma_{L1} + \gamma_{L2} = 1$ (full employment).
- Hence, $\%\Delta L$ is a weighted average of $\%\Delta y_1$ and $\%\Delta y_2$, so it lies between them.
- The same argument applies to $\%\Delta K$, where $\gamma_{K1}, \gamma_{K2} \in (0,1)$ and $\gamma_{K1} + \gamma_{K2} = 1$.

Comparative Statics: Endowment Change

- Capital is assumed to be more intensively used in firm 1,

$$\frac{K_1}{L_1} > \frac{K_2}{L_2}$$

- Since $K_2 = K - K_1$ and $L_2 = L - L_1$, this holds if and only if $K_1 L > L_1 K$, that is,

$$\gamma_{K1} = \frac{K_1}{K} > \frac{L_1}{L} = \gamma_{L1}$$

  - where $\gamma_{Kj} = \frac{z_{Kj}(w) y_j}{K} = \frac{K_j}{K}$; hence, $\gamma_{K2} < \gamma_{L2}$ for firm 2.

Comparative Statics: Endowment Change

- Subtracting the expression for $\%\Delta L$ from that for $\%\Delta K$, and using $\gamma_{K2} - \gamma_{L2} = -(\gamma_{K1} - \gamma_{L1})$,

$$\%\Delta K - \%\Delta L = (\gamma_{K1} - \gamma_{L1})(\%\Delta y_1 - \%\Delta y_2)$$

- Hence, if capital becomes relatively more abundant than labor, $\%\Delta K > \%\Delta L$, then $\%\Delta y_1 > \%\Delta y_2$.

Comparative Statics: Endowment Change

- If, in addition, the endowment of labor stays fixed, $\%\Delta L = 0$, then

$$0 = \gamma_{L1} \cdot \%\Delta y_1 + \gamma_{L2} \cdot \%\Delta y_2$$

  - so $\%\Delta y_1$ and $\%\Delta y_2$ have opposite signs, that is, $\%\Delta y_1 > 0 > \%\Delta y_2$.
- Since $\%\Delta K$ is a weighted average of $\%\Delta y_1$ and $\%\Delta y_2$,

$$\%\Delta y_1 > \%\Delta K > 0 > \%\Delta y_2$$

Comparative Statics: Endowment Change

- That is,

$$\%\Delta L = \gamma_{L1} \cdot \%\Delta y_1 + \gamma_{L2} \cdot \%\Delta y_2$$
$$\%\Delta K = \gamma_{K1} \cdot \%\Delta y_1 + \gamma_{K2} \cdot \%\Delta y_2$$

  - where $\gamma_{K1} > \gamma_{L1}$ and $\gamma_{K2} < \gamma_{L2}$: relative to labor, capital puts more weight on $\%\Delta y_1$.
- *Intuition*: the change in the input endowment produces a more-than-proportional increase in the output of the good whose production is intensive in that input, and the output of the other good falls.

Comparative Statics: Endowment Change

- ***Example 6.12*** (Rybczynski theorem):
  - Consider Example 6.8, where the aggregate endowments are $\overline{K} = 3$ and $\overline{L} = 2$.
  - The endowment of capital increases to $\overline{K} = 5$, while $\overline{L} = 2$ stays fixed.
  - Goods prices stay at $p_1 = 0.82$ and $p_2 = 1$, so factor prices stay at $(r, w) = (0.42, 0.63)$, with $\frac{r}{w} = \frac{2}{3}$.

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - From Example 6.11, with $\frac{r}{3w} = \frac{2}{9}$ and $\frac{w}{3r} = \frac{1}{2}$, the unit factor demands are

$$z_{K1} = \left(\frac{2}{9}\right)^{-1/4} \approx 1.46, \qquad z_{L1} = \left(\frac{2}{9}\right)^{3/4} \approx 0.32$$

$$z_{K2} = \left(\frac{1}{2}\right)^{3/4} \approx 0.59, \qquad z_{L2} = \left(\frac{1}{2}\right)^{-1/4} \approx 1.19$$

  - Since prices are fixed, these do not change.

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Factor feasibility with $\overline{K} = 5$ and $\overline{L} = 2$ requires

$$5 = 1.46y_1 + 0.59y_2$$
$$2 = 0.32y_1 + 1.19y_2$$

  - Each firm keeps its capital-labor ratio,

$$\frac{K_1}{L_1} = \frac{z_{K1}}{z_{L1}} = \frac{9}{2} \quad \text{and} \quad \frac{K_2}{L_2} = \frac{z_{K2}}{z_{L2}} = \frac{1}{2}$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Solving, the inputs used by each firm are

$$(K_1, L_1; K_2, L_2) = \left(\frac{9}{2}, 1; \frac{1}{2}, 1\right)$$

  - so outputs become

$$y_1 = \left(\frac{9}{2}\right)^{3/4} \approx 3.09 \quad \text{and} \quad y_2 = \left(\frac{1}{2}\right)^{1/4} \approx 0.84$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Relative to Example 6.8 ($y_1 \approx 1.54$ and $y_2 \approx 1.26$), the output of good 1 doubles (+100%), while that of good 2 falls by one third (−33.3%).
  - Firm 1 uses capital more intensively than firm 2: $\frac{K_1}{L_1} = \frac{9}{2} > \frac{1}{2} = \frac{K_2}{L_2}$.
  - If, instead, goods prices adjusted to a new WEA (Example 6.8 with $\overline{K} = 5$), then $\frac{p_1}{p_2} \approx 0.63$ and both outputs would rise, to $y_1 \approx 2.27$ and $y_2 \approx 1.43$.

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Let us now apply the proof. Since goods prices are fixed, the z's are those above.
  - Using $\gamma_{Kj} = \frac{K_j}{\overline{K}}$ and $\gamma_{Lj} = \frac{L_j}{\overline{L}}$ and the values from Example 6.8, we can assign the following values:

$$\gamma_{K1} = \frac{2.25}{3} = 0.75 \quad \text{and} \quad \gamma_{K2} = \frac{0.75}{3} = 0.25$$

$$\gamma_{L1} = \frac{0.5}{2} = 0.25 \quad \text{and} \quad \gamma_{L2} = \frac{1.5}{2} = 0.75$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Our two equations then become

$$0 = 0.25 \cdot \%\Delta y_1 + 0.75 \cdot \%\Delta y_2$$

$$\frac{2}{3} = 0.75 \cdot \%\Delta y_1 + 0.25 \cdot \%\Delta y_2$$

  - where $\%\Delta K = \frac{5-3}{3} = \frac{2}{3}$. Solving them simultaneously yields

$$\%\Delta y_1 = 1 = 100\% \quad \text{and} \quad \%\Delta y_2 = -\frac{1}{3} \approx -33.3\%$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Hence, an increase in the endowment of capital by 67% raises good 1's output by 100%, while that of good 2 decreases by 33.3%:

$$\%\Delta y_1 = 100\% > \%\Delta K \approx 67\% > 0 > \%\Delta y_2 \approx -33.3\%$$

  - These are the output changes found above (magnification effect).

Introducing Taxes

Introducing Taxes: Tax on Goods

- Assume that a sales tax t_k is imposed on good k. Then the price paid by consumers rises to $p_k^C = (1 + t_k)p_k^P$, where p_k^P is the price received by producers.
- If goods 1 and 2 face the same tax, $t_1 = t_2$, consumers and producers face the same price ratio:

$$\frac{p_1^C}{p_2^C} = \frac{(1 + t_1)p_1^P}{(1 + t_2)p_2^P} = \frac{p_1^P}{p_2^P}$$

Introducing Taxes: Tax on Goods

- Hence, with inputs in fixed supply and tax revenue returned lump-sum, the after-tax allocation is still Pareto efficient.
- However, if only good 1 is affected by the tax, i.e., $t_1 > 0$ while $t_2 = 0$, then the allocation will not be Pareto efficient.

Introducing Taxes: Tax on Goods

- In this setting, each firm's $MRTS_{L,K}$ still equals the common input price ratio $\frac{w_L}{w_K}$:

$$\frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}} = \frac{w_L}{w_K} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

- Therefore, the allocation of inputs still achieves productive efficiency.

Introducing Taxes: Tax on Goods

- Similarly, the $MRT_{1,2}$ still coincides with the price ratio of goods 1 and 2:

$$MRT_{1,2} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_1}{\partial L}} = \frac{p_1^P}{p_2} = \frac{\frac{\partial F_2}{\partial K}}{\frac{\partial F_1}{\partial K}}$$

  - where p_1^P is the price that producers of good 1 receive, net of the tax (since $t_2 = 0$, $p_2 = p_2^C = p_2^P$).

Introducing Taxes: Tax on Goods

- However, while $MRS_{1,2}$ equals the price ratio that consumers face, $\frac{p_1^C}{p_2} = \frac{(1+t_1)p_1^P}{p_2}$, it now becomes larger than the price ratio that producers face, $\frac{p_1^P}{p_2}$:

$$MRS_{1,2} = \frac{p_1^C}{p_2} = \frac{(1+t_1)p_1^P}{p_2} > \frac{p_1^P}{p_2} = MRT_{1,2}$$

Introducing Taxes: Tax on Goods

- *Intuition*: The rate at which consumers are willing to give up good 2 for an additional unit of good 1 is larger than the rate at which firms must give up good 2 to produce an additional unit of good 1.
- Thus, efficiency calls for producing more of good 1 and less of good 2: the tax leads to too little production of good 1.

Introducing Taxes: Tax on Inputs

- Similar arguments extend to the introduction of taxes on inputs.
- Assume that only inputs used by firm 2 are taxed: firm 2 pays $w_m^P = (1 + t_m)w_m^C$ for input $m \in \{L, K\}$, while firm 1 pays w_m^C.

Introducing Taxes: Tax on Inputs

- If both inputs are subject to the same tax, $t_L = t_K = t$, the input price ratios that firms 1 and 2 face coincide:

$$\frac{w_L^P}{w_K^P} = \frac{(1+t)w_L^C}{(1+t)w_K^C} = \frac{w_L^C}{w_K^C}$$

- Hence, the efficiency condition is unaffected.

Introducing Taxes: Tax on Inputs

- However, when taxes differ, $t_L \neq t_K$, productive efficiency no longer holds:
  - While firm 1 (untaxed) satisfies

$$\frac{w_L^C}{w_K^C} = \frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}}$$

  - and firm 2 (taxed) satisfies

$$\frac{w_L^P}{w_K^P} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

Introducing Taxes: Tax on Inputs

- Hence, the input price ratios that firms 1 and 2 face do not coincide:

$$MRTS_{L,K}^{1} = \frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}} = \frac{w_L^C}{w_K^C}$$

$$MRTS_{L,K}^{2} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}} = \frac{w_L^P}{w_K^P} = \frac{(1+t_L)w_L^C}{(1+t_K)w_K^C}$$

- That is, the two firms' $MRTS_{L,K}$ differ whenever $t_L \neq t_K$.

Introducing Taxes: Tax on Inputs

- For instance, if $t_L > t_K$, the $MRTS_{L,K}$ is larger for firm 2 than for firm 1:

$$MRTS^2_{L,K} - MRTS^1_{L,K} = \frac{w_L^C}{w_K^C}\frac{t_L - t_K}{1 + t_K} > 0$$

- Thus, the allocation of inputs is inefficient: labor is relatively more productive in firm 2 than in firm 1.
  - Moving labor from firm 1 to firm 2, and capital from firm 2 to firm 1, would raise the output of both goods.

Appendix:
Large Economies and the Core

Large Economies and the Core

- We know that Walrasian equilibrium allocations (WEAs) are part of the core.
- We now show that, as the economy becomes larger, the core shrinks and, in the limit, only WEAs remain in the core of every replica economy.

Large Economies and the Core

- Let us first consider an economy with I consumers, each with utility function u^i and endowment vector $\mathbf{e}^i$.
- Consider this economy's replica by doubling the number of consumers to $2I$, each of them still with utility function u^i and endowment vector $\mathbf{e}^i$.
  - There are now two consumers of each type, i.e., "twins," having identical preferences and endowments.

Large Economies and the Core

- Define an r-fold replica economy $\mathcal{E}_r$, having r consumers of each type, for a total of rI consumers.
- For any consumer type $i \in \{1, \ldots, I\}$, all r consumers of that type share the common utility function u^i and have identical endowments $\mathbf{e}^i \gg \mathbf{0}$.
- Preferences satisfy the assumptions above. We also assume that indifference curves are differentiable.

Large Economies and the Core

- When comparing two replica economies, $\mathcal{E}_r$ is larger than $\mathcal{E}_{r'}$ when $r > r'$, since it has more consumers of every type.
- Bundle $\mathbf{x}^{iq}$ indicates the vector of goods for the qth consumer of type i.
- The feasibility condition is

$$\sum_{i=1}^{I} \sum_{q=1}^{r} \mathbf{x}^{iq} = r \sum_{i=1}^{I} \mathbf{e}^{i}$$

Large Economies and the Core

- ***Equal treatment in the core***: Under the assumptions on preferences above (strict quasiconcavity is the key one), if $\mathbf{x}$ is in the core of $\mathcal{E}_r$, then every consumer of type i receives the same bundle,

$$\mathbf{x}^{iq} = \mathbf{x}^{iq'}$$

  - for every two consumers $q \neq q' \in \{1,2,\ldots,r\}$ of type i, and for every type $i \in \{1,\ldots,I\}$.

Large Economies and the Core

- That is, in the r-fold replica economy, consumers of the same type not only start with the same endowment vector $\mathbf{e}^i$, but they also end up with the same bundle in every core allocation.

Large Economies and the Core

- *Proof* (by contradiction):
  - Consider a two-fold replica economy $\mathcal{E}_2$. The argument extends to r-fold replicas (see below).
  - Suppose that allocation $\mathbf{x} \equiv (\mathbf{x}^{11}, \mathbf{x}^{12}, \mathbf{x}^{21}, \mathbf{x}^{22})$ is in the core of $\mathcal{E}_2$, $\mathbf{x} \in C_2$.
  - Since $\mathbf{x}$ is in the core, it must be feasible:

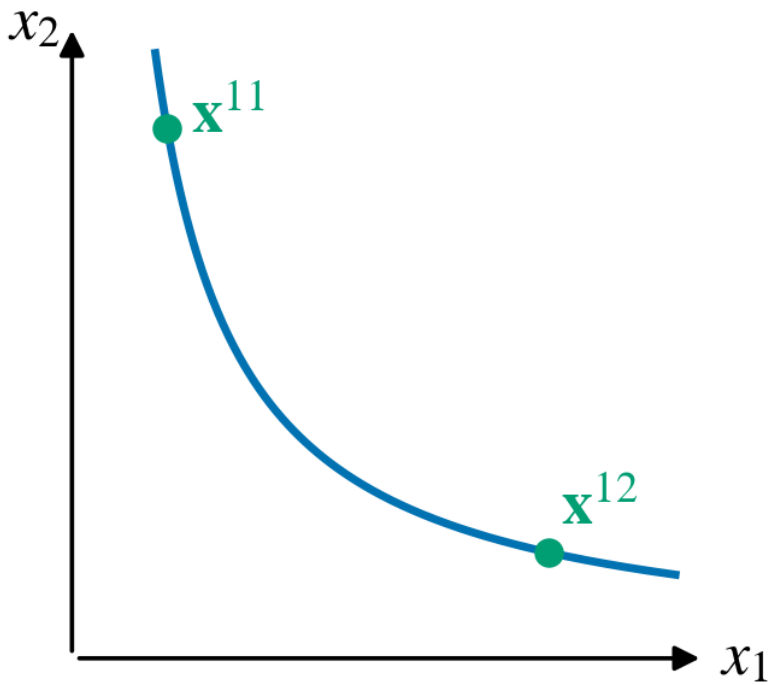
$$\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22} = 2\mathbf{e}^1 + 2\mathbf{e}^2$$

Large Economies and the Core

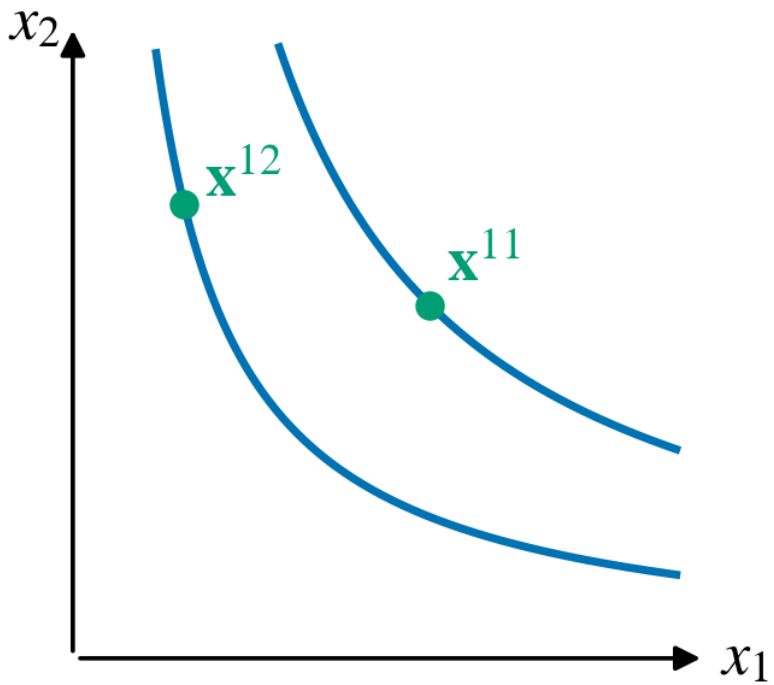
- Assume that allocation $\mathbf{x}$ does not assign the same consumption vectors to the two type-1 twins, $\mathbf{x}^{11} \neq \mathbf{x}^{12}$.
- Without loss of generality, label the type-1 twins so that $\mathbf{x}^{11} \gtrsim^{1} \mathbf{x}^{12}$.
  - This is true for both type-1 consumers, since they have the same preferences.
  - The other case follows by relabeling the twins.

Large Economies and the Core

- Type-1 twins treated unequally in the core



$\mathbf{x}^{11} \sim {}^1\mathbf{x}^{12}$
(same indifference curve)



$\mathbf{x}^{11} \succ {}^1\mathbf{x}^{12}$

Large Economies and the Core

- Similarly, label the type-2 twins so that $\mathbf{x}^{21} \succsim^{2} \mathbf{x}^{22}$.
- Hence, consumer 12 is the worst-off type-1 consumer and consumer 22 is the worst-off type-2 consumer.
- Let us take these two worst-off consumers, one of each type, and check if they can form a blocking coalition to oppose allocation $\mathbf{x}$.

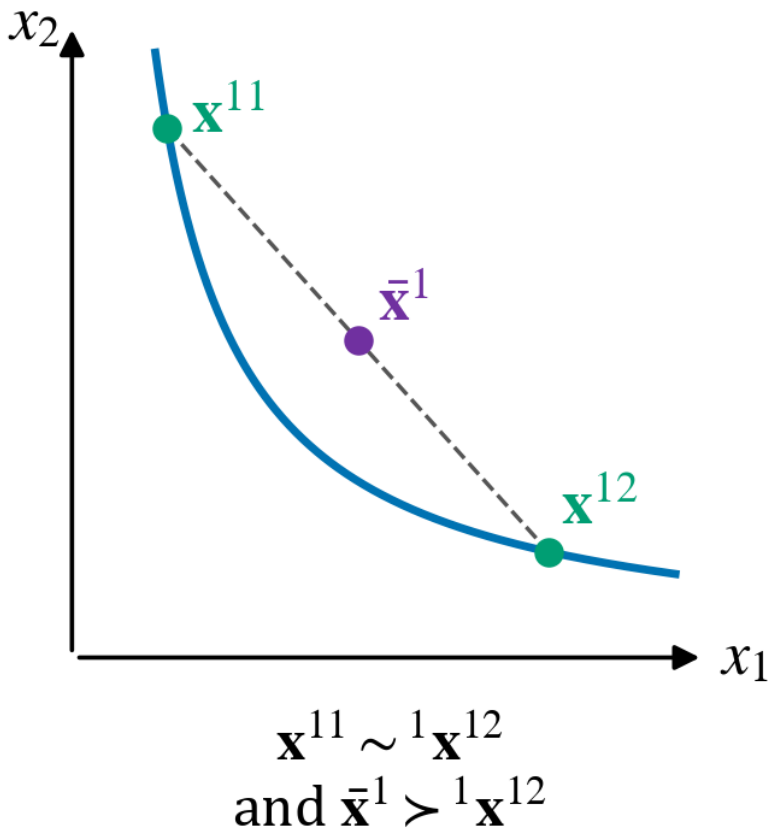
Large Economies and the Core

- The average bundles for type-1 and type-2 consumers are

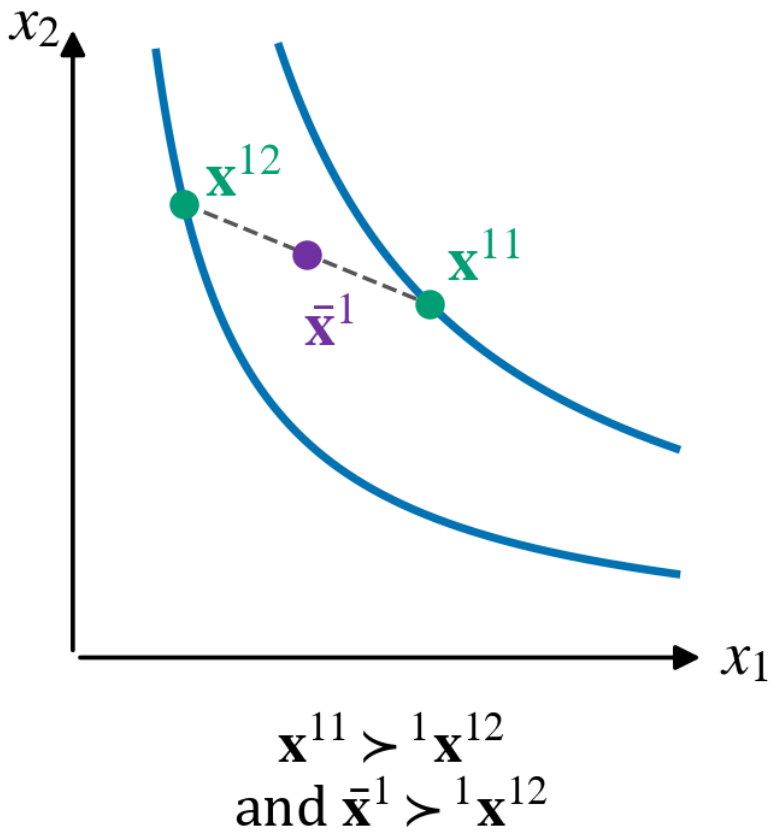
$$\bar{\mathbf{x}}^1 = \frac{\mathbf{x}^{11} + \mathbf{x}^{12}}{2} \quad \text{and} \quad \bar{\mathbf{x}}^2 = \frac{\mathbf{x}^{21} + \mathbf{x}^{22}}{2}$$

Large Economies and the Core

- Average bundles leading to a blocking coalition



$\mathbf{x}^{11} \sim {}^1\mathbf{x}^{12}$
and $\bar{\mathbf{x}}^1 \succ {}^1\mathbf{x}^{12}$



$\mathbf{x}^{11} \succ {}^1\mathbf{x}^{12}$
and $\bar{\mathbf{x}}^1 \succ {}^1\mathbf{x}^{12}$

Large Economies and the Core

- *Desirability*. By the strict quasiconcavity assumed above, and since $\mathbf{x}^{11} \neq \mathbf{x}^{12}$, the worst-off type-1 consumer strictly prefers $\overline{\mathbf{x}}^{1}$ to $\mathbf{x}^{12}$:

$$\overline{\mathbf{x}}^{1} \succ^{1} \mathbf{x}^{12}$$

  - That is, a convex combination (the midpoint) of $\mathbf{x}^{11}$ and $\mathbf{x}^{12}$ is strictly preferred to the original bundle $\mathbf{x}^{12}$.

Large Economies and the Core

- A similar argument applies to the worst-off type-2 consumer, $\bar{\mathbf{x}}^2 \gtrsim^2 \mathbf{x}^{22}$ (weakly, since $\mathbf{x}^{21} = \mathbf{x}^{22}$ is possible).
- Hence, $(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2)$ makes consumer 12 strictly better off and leaves consumer 22 at least as well off as at the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$.

Large Economies and the Core

- *Feasibility*. Can consumers 12 and 22 achieve $\left(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2\right)$?
- Summing the bundles that consumers 12 and 22 need, we obtain

$$\bar{\mathbf{x}}^1 + \bar{\mathbf{x}}^2 = \frac{\mathbf{x}^{11} + \mathbf{x}^{12}}{2} + \frac{\mathbf{x}^{21} + \mathbf{x}^{22}}{2}$$

$$= \frac{1}{2}(\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22}) = \frac{1}{2}(2\mathbf{e}^1 + 2\mathbf{e}^2) = \mathbf{e}^1 + \mathbf{e}^2$$

- Hence, the pair of bundles $\left(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2\right)$ is feasible.

Large Economies and the Core

- In summary, the pair of bundles $\left(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2\right)$:
  - makes consumer 12 strictly better off and consumer 22 no worse off than the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$;
  - is feasible for consumers 12 and 22.
- Thus, coalition {12, 22} blocks $\mathbf{x}$, contradicting $\mathbf{x} \in C_2$.
- Therefore, if an allocation is in the core of the replica economy, it must give consumers of the same type the same bundle.

Large Economies and the Core

- The argument extends to $\mathcal{E}_r$: give the worst-off consumer of each type i the average bundle

$$\bar{\mathbf{x}}^i = \frac{1}{r}\sum_{q=1}^{r} \mathbf{x}^{iq}$$

- These bundles are feasible for the coalition of worst-off consumers:

$$\sum_{i=1}^{I} \bar{\mathbf{x}}^i = \frac{1}{r}\sum_{i=1}^{I}\sum_{q=1}^{r} \mathbf{x}^{iq} = \sum_{i=1}^{I} \mathbf{e}^i$$

  - Each of them weakly prefers $\bar{\mathbf{x}}^i$, and strictly so for a type with unequal bundles.

Large Economies and the Core

- If $\mathbf{x}$ is in the core of an r-fold replica economy $\mathcal{E}_r$, $\mathbf{x} \in C_r$, then (by the equal treatment property) allocation $\mathbf{x}$ must be of the form

$$\mathbf{x} = \left(\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r \text{ times}}, \underbrace{\mathbf{x}^2, \ldots, \mathbf{x}^2}_{r \text{ times}}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r \text{ times}} \right)$$

  - All consumers of the same type must receive the same bundle.

Large Economies and the Core

- Core allocations in $\mathcal{E}_r$ are r-fold copies of feasible allocations in $\mathcal{E}_1$, $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$.
- Dividing the feasibility condition in $\mathcal{E}_r$ by r,

$$r\sum_{i=1}^{I}\mathbf{x}^i = r\sum_{i=1}^{I}\mathbf{e}^i, \qquad \text{so} \quad \sum_{i=1}^{I}\mathbf{x}^i = \sum_{i=1}^{I}\mathbf{e}^i$$

Large Economies and the Core

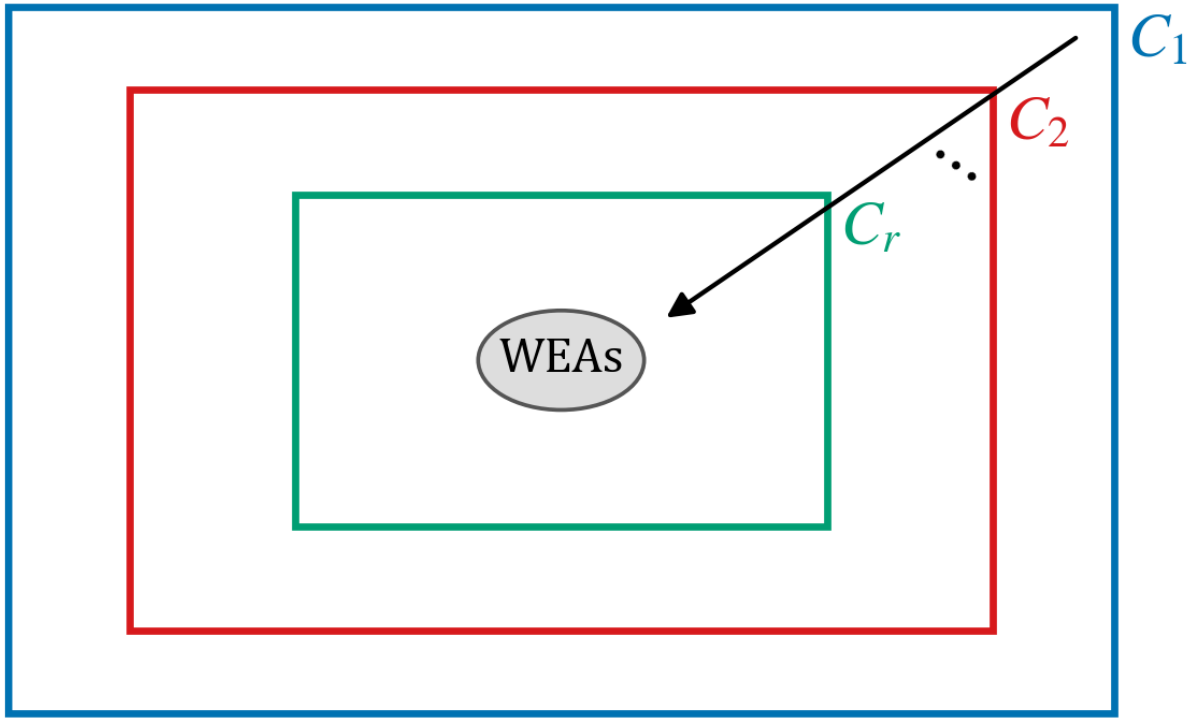
- The core shrinks as the economy enlarges: the sequence of core sets is nested,

$$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_r$$

  - where each core allocation is viewed as its type profile $(\mathbf{x}^1, \ldots, \mathbf{x}^I)$.
- That is, the core of the original (un-replicated) economy, C_1, is a superset of the core of the 2-fold replica economy, C_2; C_2 is a superset of the core of the 3-fold replica economy, C_3; etc.

Large Economies and the Core

- The core shrinks as r increases



$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_r$: as $r \to \infty$, C_r shrinks toward the WEAs

Large Economies and the Core

- *Proof*:
  - Since we seek to show that $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_{r-1} \supseteq C_r$, it suffices to show that, for every $r \geq 2$, $C_{r-1} \supseteq C_r$.
  - Suppose that allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I) \in C_r$.
  - There is no blocking coalition to $\mathbf{x}$ in the r-fold replica economy $\mathcal{E}_r$.
  - The $(r-1)$-fold copy of $\mathbf{x}$ is feasible in $\mathcal{E}_{r-1}$. We then need to show that $\mathbf{x}$ cannot be blocked by any coalition in $\mathcal{E}_{r-1}$ either.

Large Economies and the Core

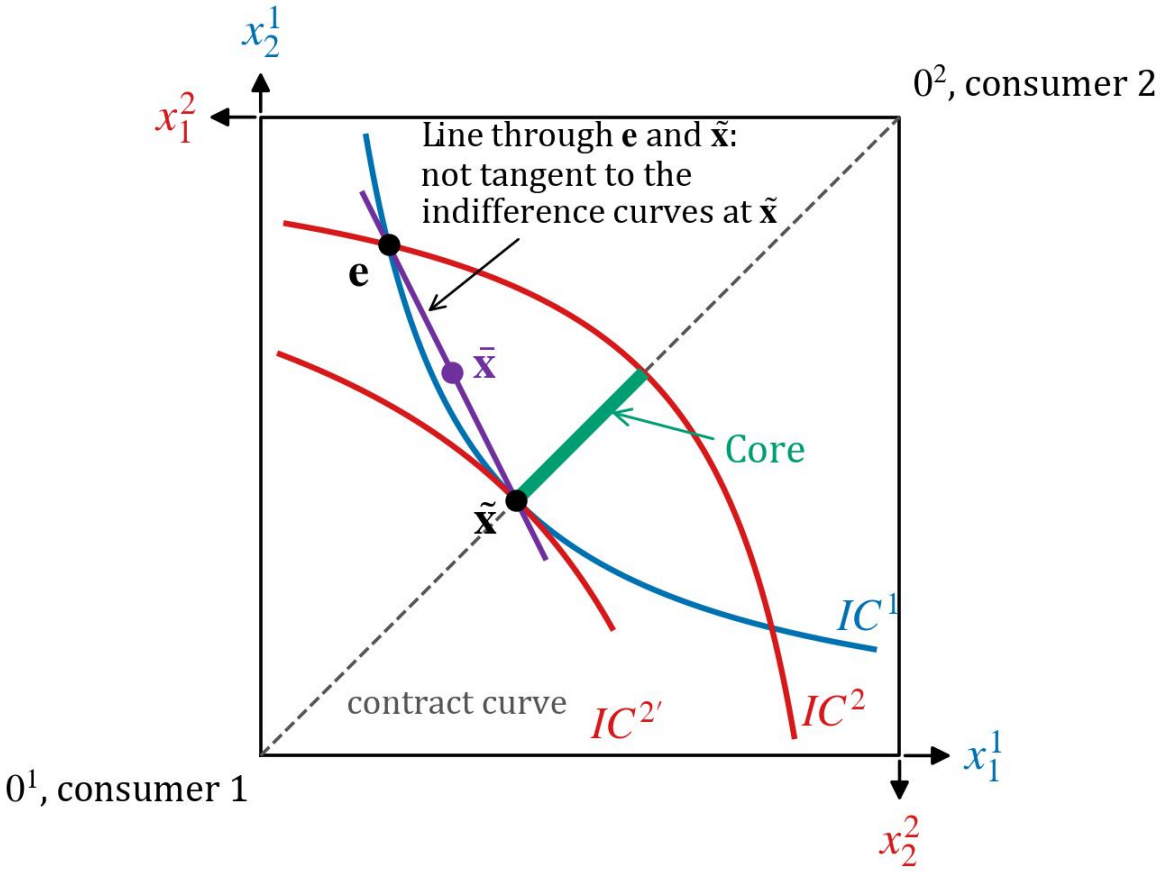
- If we could find a coalition S that blocks $\mathbf{x}$ in $\mathcal{E}_{r-1}$ with bundles $\mathbf{y}$, then we could also find a blocking coalition in $\mathcal{E}_r$:
  - all members of $\mathcal{E}_{r-1}$ are also present in $\mathcal{E}_r$, and their endowments have not changed.
- The same coalition S with the same bundles $\mathbf{y}$ is feasible in $\mathcal{E}_r$, which contradicts $\mathbf{x} \in C_r$. Hence, $\mathbf{x} \in C_{r-1}$.

Large Economies and the Core

- We now show that the core shrinks strictly: some allocations in C_1 are not in C_2.
- Consider an economy with two consumers and two goods. We show that the endpoint $\tilde{\mathbf{x}}$ of C_1, where consumer 1 is indifferent between $\tilde{\mathbf{x}}^1$ and $\mathbf{e}^1$, does not belong to the core of the 2-fold replica economy, C_2.

Large Economies and the Core

- Un-replicated economy $\mathcal{E}_1$



$\tilde{\mathbf{x}}$ gives consumer 1 the lowest utility among core allocations

Large Economies and the Core

- $\tilde{\mathbf{x}}$ is the endpoint of the core at which consumer 1 is indifferent to $\mathbf{e}$.
- $\tilde{\mathbf{x}}$ is not a WEA, since the line through $\tilde{\mathbf{x}}$ and $\mathbf{e}$ is not tangent to the consumers' indifference curves at $\tilde{\mathbf{x}}$.
- If the core shrinks as the economy enlarges, we should be able to show that $\tilde{\mathbf{x}} \notin C_2$.
- Let us build a blocking coalition against $\tilde{\mathbf{x}}$.

Large Economies and the Core

- *Desirability*. In $\mathcal{E}_2$, $\tilde{\mathbf{x}}$ gives $\tilde{\mathbf{x}}^{11} = \tilde{\mathbf{x}}^{12} = \tilde{\mathbf{x}}^{1}$ to the type-1 consumers and $\tilde{\mathbf{x}}^{21} = \tilde{\mathbf{x}}^{22} = \tilde{\mathbf{x}}^{2}$ to the type-2 consumers.
- Consider the coalition $S = \{11,12,21\}$ and the midpoint $\frac{1}{2}\mathbf{e}^1 + \frac{1}{2}\tilde{\mathbf{x}}^1$.
- Since $u^1(\mathbf{e}^1) = u^1(\tilde{\mathbf{x}}^1)$ and $\mathbf{e}^1 \neq \tilde{\mathbf{x}}^1$, strict quasiconcavity gives

$$u^1\left(\frac{1}{2}\mathbf{e}^1 + \frac{1}{2}\tilde{\mathbf{x}}^1\right) > u^1(\tilde{\mathbf{x}}^1)$$

Large Economies and the Core

- Hence, the midpoint is strictly preferred by both type-1 consumers.
- If the midpoint is offered to consumers 11 and 12, while consumer 21 keeps $\tilde{\mathbf{x}}^2$, they will all accept it:

$$\bar{\mathbf{x}}^{11} \equiv \frac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{11}) \succ^1 \tilde{\mathbf{x}}^{11}$$

$$\bar{\mathbf{x}}^{12} \equiv \frac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{12}) \succ^1 \tilde{\mathbf{x}}^{12}$$

- Consumer 21 receives $\bar{\mathbf{x}}^{21} = \tilde{\mathbf{x}}^2$, so he is indifferent.

Large Economies and the Core

- *Feasibility*. Since $\bar{\mathbf{x}}^{11} = \bar{\mathbf{x}}^{12}$, the sum of the proposed bundles is

$$\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \bar{\mathbf{x}}^{21} = 2 \times \frac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{11}) + \tilde{\mathbf{x}}^{21}$$
$$= \mathbf{e}^1 + \tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21}$$

- Recall that $\tilde{\mathbf{x}} \in C_1$, so it is feasible in $\mathcal{E}_1$:

$$\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2 = \mathbf{e}^1 + \mathbf{e}^2$$

- Therefore, since $\tilde{\mathbf{x}}^{11} = \tilde{\mathbf{x}}^1$ and $\tilde{\mathbf{x}}^{21} = \tilde{\mathbf{x}}^2$,

$$\tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21} = \mathbf{e}^1 + \mathbf{e}^2$$

Large Economies and the Core

- We can thus rewrite the above equality as

$$\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \bar{\mathbf{x}}^{21} = \mathbf{e}^1 + \underbrace{(\tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21})}_{=\mathbf{e}^1+\mathbf{e}^2} = 2\mathbf{e}^1 + \mathbf{e}^2$$

- This confirms feasibility.

Large Economies and the Core

- Hence, the endpoint $\tilde{\mathbf{x}}$ in the core of the un-replicated economy does not belong to the core of the two-fold replica economy, $\tilde{\mathbf{x}} \notin C_2$.
- There is a blocking coalition $S = \{11,12,21\}$ and an alternative allocation $\overline{\mathbf{x}} = \left(\overline{\mathbf{x}}^{11}, \overline{\mathbf{x}}^{12}, \overline{\mathbf{x}}^{21}\right)$ that makes consumers 11 and 12 strictly better off and leaves consumer 21 indifferent, and that is feasible for the coalition members.

Large Economies and the Core

- WEA in replicated economies:
  - Consider a WEA in the un-replicated economy $\mathcal{E}_1$, $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$, at prices $\mathbf{p}^*$.
  - $\mathbf{x}$ is a WEA of $\mathcal{E}_1$ at prices $\mathbf{p}^*$ if and only if its r-fold copy is a WEA of $\mathcal{E}_r$ at $\mathbf{p}^*$:

$$\mathbf{x} = \left(\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r \text{ times}}, \underbrace{\mathbf{x}^2, \ldots, \mathbf{x}^2}_{r \text{ times}}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r \text{ times}} \right)$$

Large Economies and the Core

- Each type-i consumer faces the same budget and has the same demand, so excess demand in $\mathcal{E}_r$ is r times that in $\mathcal{E}_1$.
- Hence, the r-fold copy of a WEA also belongs to the core of $\mathcal{E}_r$ (because every WEA is in the core, $W(\mathbf{e}) \subseteq C(\mathbf{e})$, shown earlier).

Large Economies and the Core

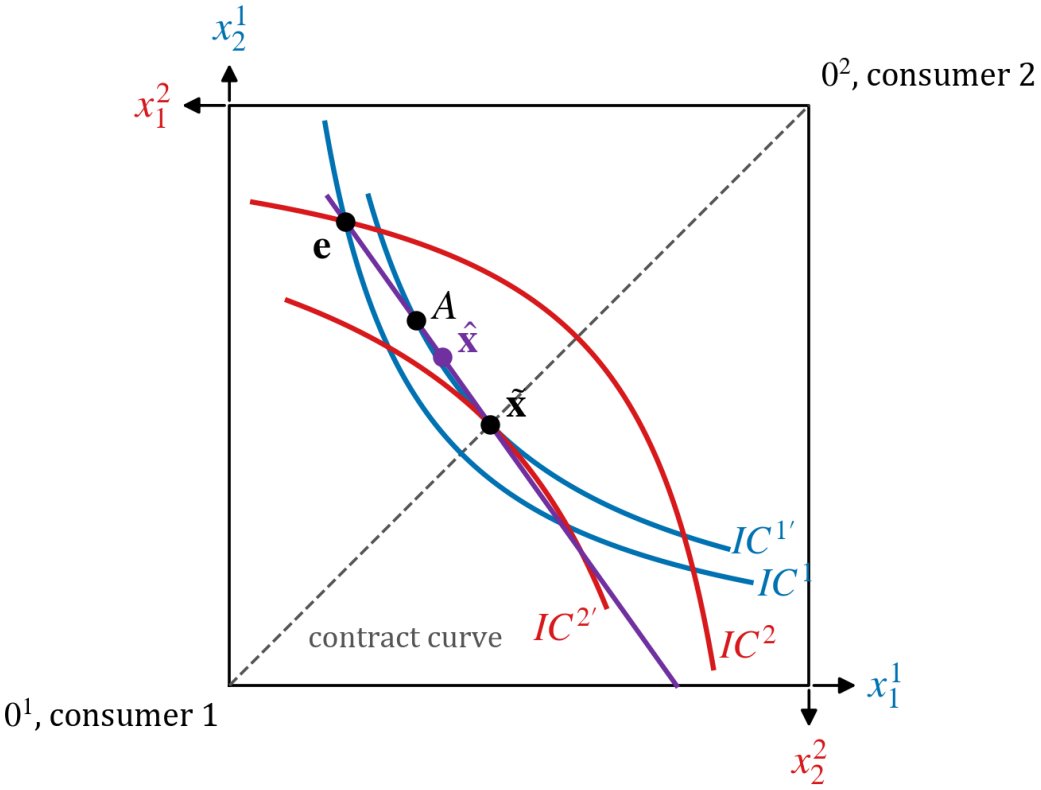
- ***A limit theorem on the core***: If $\mathbf{x}$ is an allocation of $\mathcal{E}_1$ whose r-fold copy belongs to C_r for every $r \geq 1$, then $\mathbf{x}$ must be a WEA of the un-replicated economy $\mathcal{E}_1$.
- We prove it for two types and two goods, with differentiable indifference curves (the general result, Debreu and Scarf 1963, uses a separating-hyperplane argument).

Large Economies and the Core

- *Proof* (by contradiction):
  - Consider that $\tilde{\mathbf{x}}$ belongs to C_r for every $r \geq 1$, but is not a WEA.
  - Since $\tilde{\mathbf{x}} \in C_r$ and $C_r \subseteq C_1$, $\tilde{\mathbf{x}} \in C_1$.
  - Allocation $\tilde{\mathbf{x}}$ must then be within the lens-shaped area and on the contract curve.

Large Economies and the Core

- A core allocation $\tilde{\mathbf{x}}$ that is not a WEA
  - A: the second intersection of the line through $\mathbf{e}$ and $\tilde{\mathbf{x}}$ with consumer 1's indifference curve through $\tilde{\mathbf{x}}$.



Large Economies and the Core

- Consider now the line connecting $\tilde{\mathbf{x}}$ and $\mathbf{e}$, and let $\frac{p_1}{p_2}$ be its slope (in absolute value).
- Since $\tilde{\mathbf{x}}$ is on the contract curve, $MRS^1_{1,2}(\tilde{\mathbf{x}}^1) = MRS^2_{1,2}(\tilde{\mathbf{x}}^2)$.
- Since $\tilde{\mathbf{x}}$ is not a WEA, $\frac{p_1}{p_2} \neq MRS^1_{1,2}(\tilde{\mathbf{x}}^1)$, so the line is tangent to neither indifference curve.
- Relabel types if needed so that the line cuts consumer 1's indifference curve at $\tilde{\mathbf{x}}$, as in the figure.

Large Economies and the Core

- Can allocation $\tilde{\mathbf{x}}$ be in C_r for every r and yet not be a WEA?
- Let us show that if $\tilde{\mathbf{x}}$ is not a WEA, it cannot be in C_r for every r.
- To demonstrate that $\tilde{\mathbf{x}} \notin C_r$ for r large enough, let us find a blocking coalition.

Large Economies and the Core

- By strict convexity of preferences, we can find a set of bundles (such as those between A and $\tilde{\mathbf{x}}^1$) that type-1 consumers strictly prefer to $\tilde{\mathbf{x}}^1$:

$$\hat{\mathbf{x}}^1 \equiv \frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\tilde{\mathbf{x}}^1$$

  - where r is large enough that $\hat{\mathbf{x}}^1$ lies between A and $\tilde{\mathbf{x}}^1$.

Large Economies and the Core

- Consider a coalition S with all r type-1 consumers and $r - 1$ type-2 consumers.
- Let us now show that allocation $\hat{\mathbf{x}}$ satisfies the properties of desirability and feasibility for the blocking coalition S.

Large Economies and the Core

- *Desirability*. If we give every type-1 consumer the bundle $\hat{\mathbf{x}}^1$, then $\hat{\mathbf{x}}^1 \succ^1 \tilde{\mathbf{x}}^1$.
- Each type-2 consumer in S keeps $\hat{\mathbf{x}}^2 = \tilde{\mathbf{x}}^2$ and is indifferent.

Large Economies and the Core

- *Feasibility*. The coalition's endowment is
$$r\mathbf{e}^1 + (r-1)\mathbf{e}^2$$
- Summing over the consumers in coalition S, their aggregate allocation is
$$r\hat{\mathbf{x}}^1 + (r-1)\hat{\mathbf{x}}^2 = r\left(\frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\tilde{\mathbf{x}}^1\right) + (r-1)\tilde{\mathbf{x}}^2$$
$$= \mathbf{e}^1 + (r-1)(\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2)$$

Large Economies and the Core

- Since $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2)$ is in the core of $\mathcal{E}_1$, it must be feasible:

$$\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2 = \mathbf{e}^1 + \mathbf{e}^2$$

- Combining the above two results, we find that

$$r\hat{\mathbf{x}}^1 + (r-1)\hat{\mathbf{x}}^2 = \mathbf{e}^1 + (r-1)\underbrace{(\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2)}_{=\mathbf{e}^1+\mathbf{e}^2}$$

$$= \mathbf{e}^1 + (r-1)(\mathbf{e}^1 + \mathbf{e}^2) = r\mathbf{e}^1 + (r-1)\mathbf{e}^2$$

Large Economies and the Core

- This confirms feasibility.
- Hence, r type-1 consumers and $r - 1$ type-2 consumers can form a coalition that blocks $\tilde{\mathbf{x}}$ in $\mathcal{E}_r$, so $\tilde{\mathbf{x}} \notin C_r$.

Large Economies and the Core

- Thus, if $\tilde{\mathbf{x}}$ is not a WEA, then $\tilde{\mathbf{x}} \notin C_r$ for every r large enough.
- As a consequence, if $\tilde{\mathbf{x}} \in C_r$ for all $r \geq 1$, then $\tilde{\mathbf{x}}$ must be a WEA.
- Together with the result on WEAs of replica economies, the allocations that remain in C_r for every r are the WEAs.

Advanced Microeconomic Theory

Chapter 6: Partial and General Equilibrium

Outline

- Partial Equilibrium Analysis
- General Equilibrium Analysis
- Comparative Statics
- Welfare Analysis

Partial Equilibrium Analysis

- In a competitive equilibrium (CE), all agents must select an optimal allocation given their resources:
  - Firms choose profit-maximizing production plans given their technology;
  - Consumers choose utility-maximizing bundles given their budget constraint.
- A competitive equilibrium allocation will emerge at a price that makes consumers' purchasing plans coincide with the firms' production decisions.

Partial Equilibrium Analysis

- ***Firm***:
  - Given the price vector p^*, firm j's equilibrium output level q_j^* must solve

$$\max_{q_j \geq 0} \; p^* q_j - c_j(q_j)$$

    which yields the necessary and sufficient condition

$$p^* \leq c_j'(q_j^*), \text{ with equality if } q_j^* > 0$$

  - That is, every firm j produces until the point in which its marginal cost, $c_j'(q_j^*)$, coincides with the current market price.

Partial Equilibrium Analysis

- ***Consumers***:
  - Consider a quasilinear utility function

$$u_i(m_i, x_i) = m_i + \phi_i(x_i)$$

where m_i denotes the numeraire, and $\phi_i'(x_i) > 0$, $\phi_i''(x_i) < 0$ for all $x_i > 0$.

  - Normalizing, $\phi_i(0) = 0$. Recall that with quasilinear utility functions, the wealth effects for all non-numeraire commodities are zero.

Partial Equilibrium Analysis

– Consumer i's UMP is

$$\max_{m_i \in \mathbb{R}_+,\, x_i \in \mathbb{R}_+} \; m_i + \phi_i(x_i)$$

$$\text{s.t.} \quad \underbrace{m_i + p^* x_i}_{\text{Total expend.}} \leq \underbrace{w_{m_i} + \sum_{j=1}^{J} \theta_{ij} (\underbrace{p^* q_j^* - c_j(q_j^*)}_{\text{Profits}})}_{\text{Total resources (endowment+profits)}}$$

– The budget constraint must hold with equality (by Walras' law). Hence,

$$m_i = -p^* x_i + \left[w_{m_i} + \sum_{j=1}^{J} \theta_{ij} \left(p^* q_j^* - c_j\left(q_j^*\right) \right) \right]$$

Partial Equilibrium Analysis

– Substituting the budget constraint into the objective function,

$$\max_{x_i \in \mathbb{R}_+} \quad \phi_i(x_i) - p^* x_i + \left[w_{m_i} + \sum_{j=1}^{J} \theta_{ij} \left(p^* q_j^* - c_j(q_j^*) \right) \right]$$

– FOCs wrt x_i yields

$$\phi_i'(x_i^*) \leq p^*, \text{ with equality if } x_i^* > 0$$

– That is, consumer increases the amount he buys of good x until the point in which the marginal utility he obtains exactly coincides with the market price he has to pay for it.

Partial Equilibrium Analysis

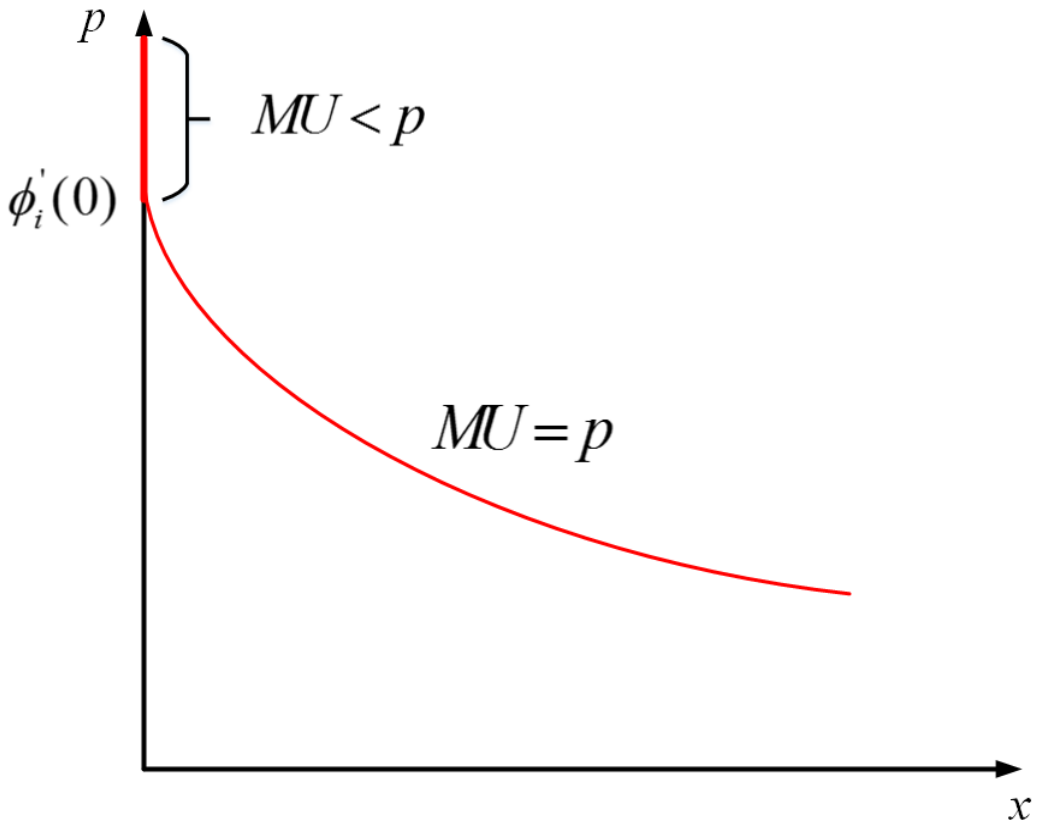
– Hence, an allocation $(x_1^*, x_2^*, \ldots, x_I^*, q_1^*, q_2^*, \ldots, q_J^*)$ and a price vector $p^* \in \mathbb{R}^L$ constitute a CE if:

$$p^* \leq c_j'(q_j^*), \text{ with equality if } q_j^* > 0$$
$$\phi_i'(x_i^*) \leq p^*, \text{ with equality if } x_i^* > 0$$
$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$$

– Note that these conditions do not depend upon the consumer's initial endowment.

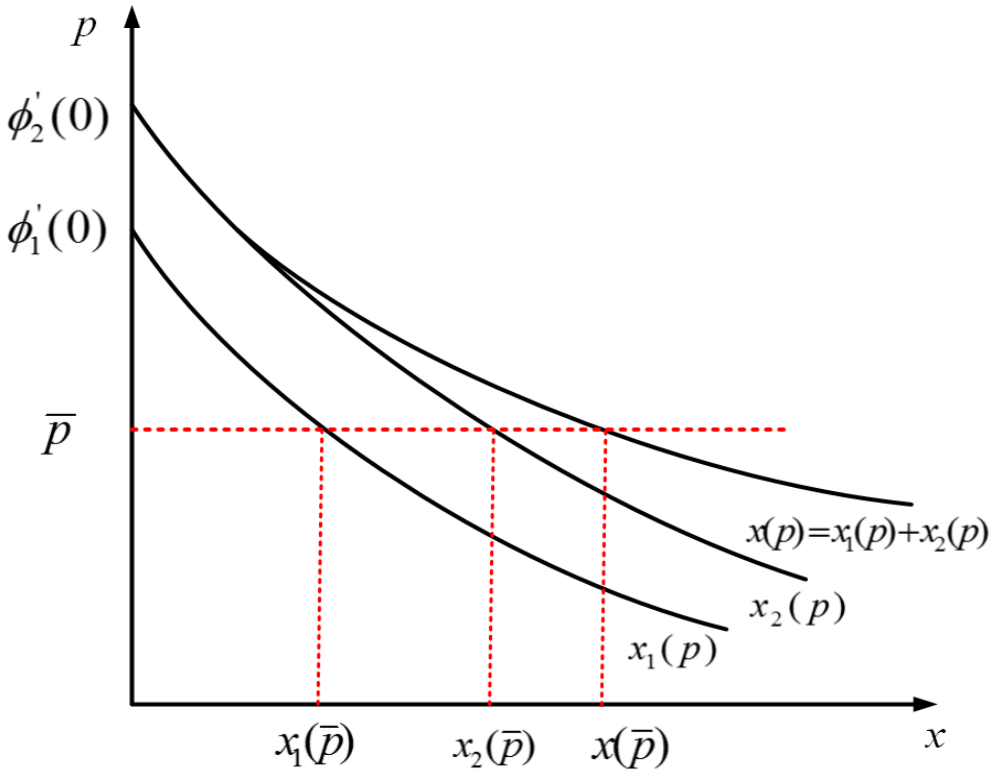
Partial Equilibrium Analysis

- The individual demand curve, where $\phi_i'(x_i^*) \leq p^*$



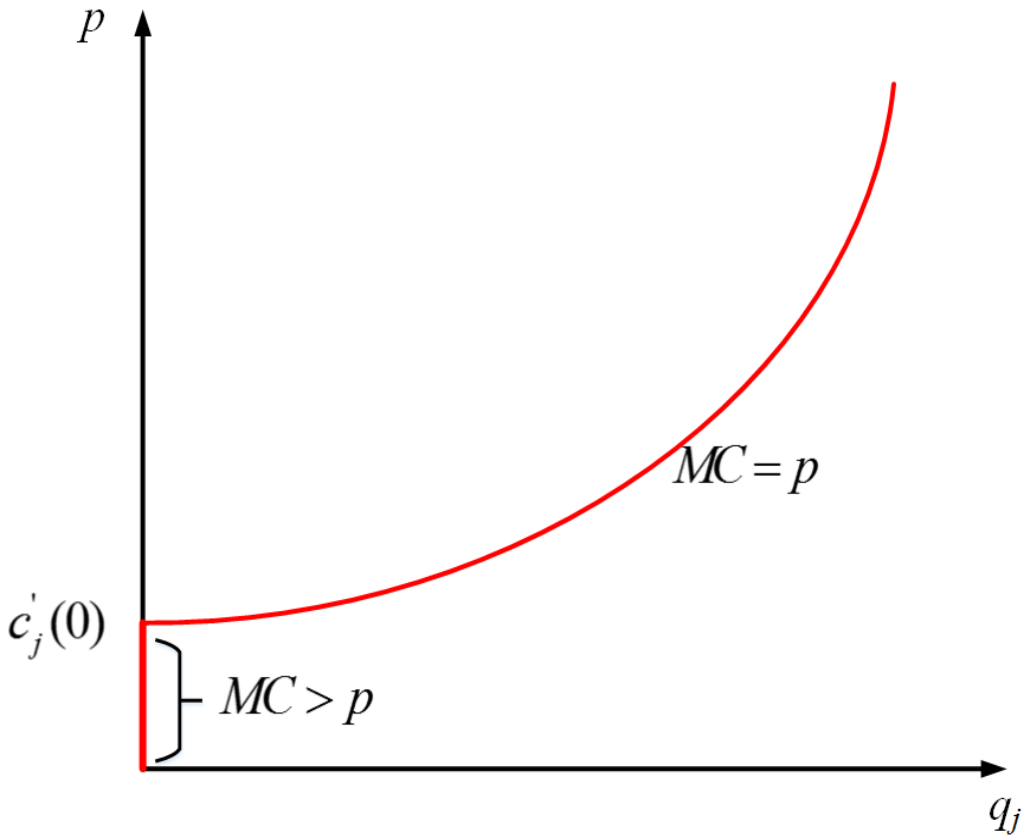
Partial Equilibrium Analysis

- Horizontally summing individual demand curves yields the aggregate demand curve.



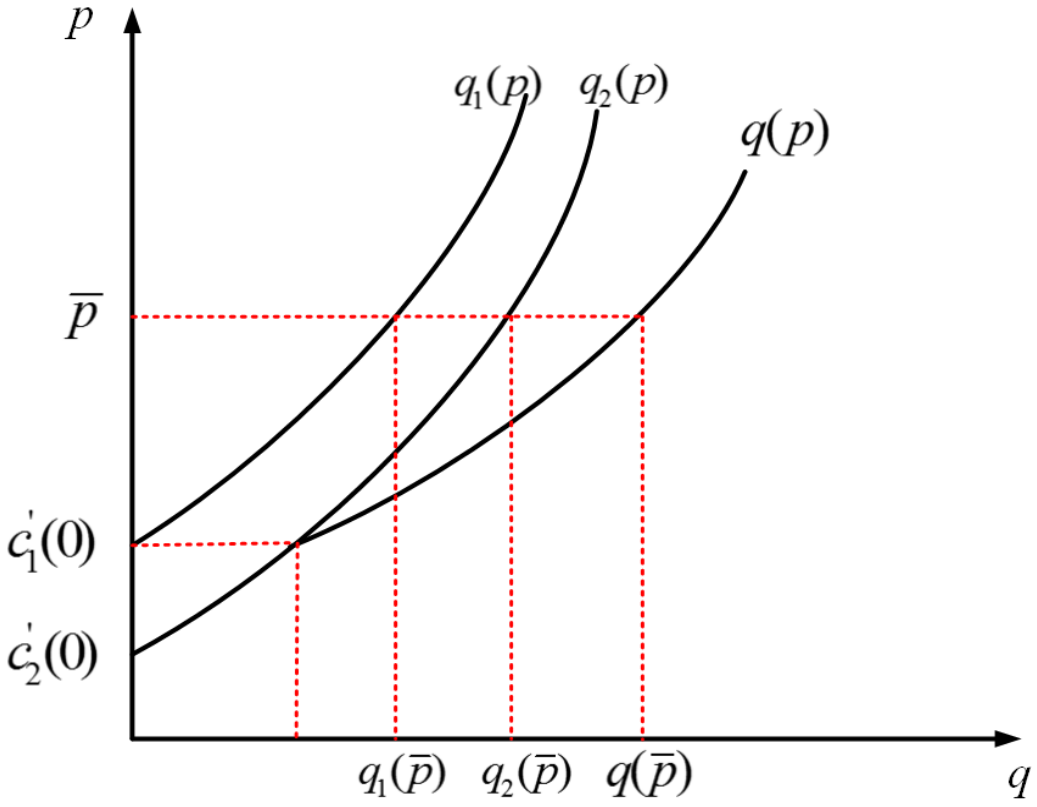
Partial Equilibrium Analysis

- The individual supply curve, where $p^* \leq c_j'(q_j^*)$



Partial Equilibrium Analysis

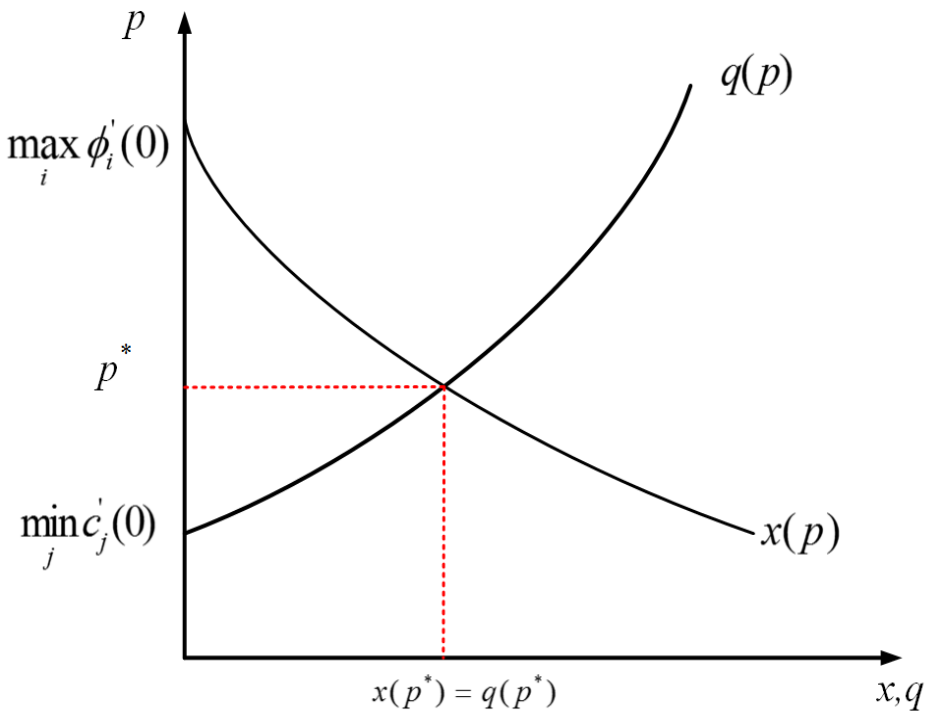
- Horizontally summing individual supply curves yields the aggregate supply curve.



Partial Equilibrium Analysis

- Superimposing aggregate demand and aggregate supply curves, we obtain the CE allocation of good x.
- To guarantee that a CE exists, the equilibrium price p^* must satisfy

$$\max_i \phi_i'(0) \geq p^* \geq \min_j c_j'(0)$$

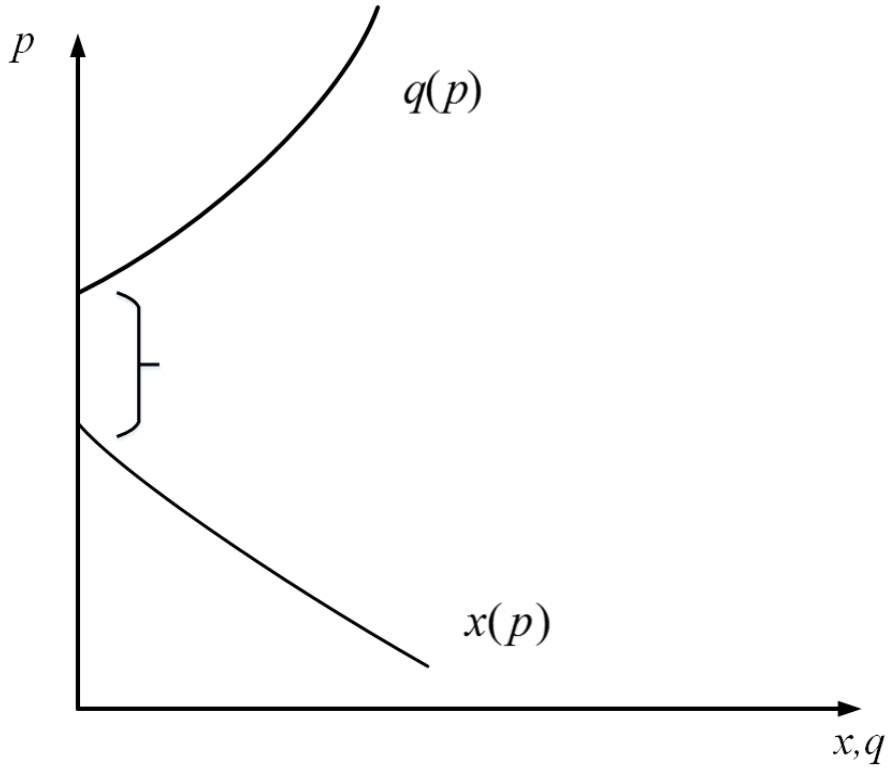


Partial Equilibrium Analysis

- If we have

$$\max_i \ \phi_i'(0) < \min_j \ c_j'(0),$$

then there is *no* positive production or consumption of good x.



Partial Equilibrium Analysis

- Also, since $\phi_i'(x_i)$ is downward sloping in x_i, and $c_j'(q_j)$ is upward sloping in q_j, then aggregate demand and supply cross at a unique point.
  - Hence, the CE allocation is unique.

Partial Equilibrium Analysis

- ***Example 6.1***:
  - Assume a perfectly competitive industry consisting of two types of firms: 100 firms of type *A* and 30 firms of type *B.*
  - Short-run supply curve of type *A* firm is
$$s_A(p) = 2p$$
  - Short-run supply curve of type *B* firm is
$$s_B(p) = 10p$$
  - The Walrasian market demand curve is
$$x(p) = 5000 - 500p$$

Partial Equilibrium Analysis

- ***Example 6.1*** (continued):
  - Summing the individual supply curves of the 100 type-*A* firms and the 30 type-*B* firms,
    $$S(p) = 100 \cdot 2p + 30 \cdot 10p = 500p$$
  - The short-run equilibrium occurs at the price at which quantity demanded equals quantity supplied,
    $$5000 - 500p = 500p, \text{ or } p = 5$$
  - Each type-*A* firm supplies: $s_A(p) = 2 \cdot 5 = 10$
  - Each type-*B* firm supplies: $s_B(p) = 10 \cdot 5 = 50$

Comparative Statics

Comparative Statics

- Let us assume that the consumer's preferences are affected by a vector of parameters $\alpha \in \mathbb{R}^M$, where $M \leq L$.
  - Then, consumer i's utility from good x is $\phi_i(x_i, \alpha)$.
- Similarly, firms' technology is affected by a vector of parameters $\beta \in \mathbb{R}^S$, where $S \leq L$.
  - Then, firm j's cost function is $c_j(q_j, \beta)$.
- Notation:
  - $\hat{p}_i(p,t)$ is the effective price paid by the consumer
  - $\hat{p}_j(p,t)$ is the effective price received by the firm
  - Per unit tax: $\hat{p}_i(p,t) = p + t$.
    - Example: $t = \$2$, regardless of the price p
  - Ad valorem tax (sales tax): $\hat{p}_i(p,t) = p + pt = p(1+t)$
    - Example: $t = 0.1$ (10%).

Comparative Statics

- If consumption and production are strictly positive in the CE, then

$$\phi_i'(x_i^*, \alpha) = \hat{p}_i(p^*, t) \text{ for every consumer } i$$

$$c_j'(q_j^*, \beta) = \hat{p}_j(p^*, t) \text{ for every firm } j$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$$

- Then we have $I + J + 1$ equations, which depend on parameter values α, β and t.
- In order to understand how x_i^* or q_j^* depends on parameters α and β, we can use the ***Implicit Function Theorem***.
  - The above functions have to be differentiable.

Comparative Statics

- ***Implicit Function Theorem***:
  - Let $u(x, y)$ be a utility function, where x and y are amounts of two goods.
  - If $\frac{\partial u(\bar{x},\bar{y})}{\partial y} \neq 0$ when evaluated at $(\bar{x}, \bar{y})$, then

$$\frac{\partial u(\bar{x},\bar{y})}{\partial x} dx + \frac{\partial u(\bar{x},\bar{y})}{\partial y} dy = 0$$

    which yields

$$\frac{dy(\bar{x})}{dx} = -\frac{\frac{\partial u(\bar{x}, \bar{y})}{\partial x}}{\frac{\partial u(\bar{x}, \bar{y})}{\partial y}}$$

Comparative Statics

– Similarly, if $\frac{\partial u(\bar{x},\bar{y})}{\partial x} \neq 0$ when evaluated at $(\bar{x}, \bar{y})$, then

$$\frac{dx(\bar{y})}{dy} = -\frac{\frac{\partial u(\bar{x},\bar{y})}{\partial y}}{\frac{\partial u(\bar{x},\bar{y})}{\partial x}}$$

for all $(\bar{x}, \bar{y})$.

Comparative Statics

– Similarly, if $u(x, \alpha)$ describes the consumption of a single good x,where α determines the consumer's preference for x, and $\frac{\partial u(x,\alpha)}{\partial x} \neq 0$, then

$$\frac{dx(\alpha)}{d\alpha} = -\frac{\frac{\partial u(x,\alpha)}{\partial \alpha}}{\frac{\partial u(x,\alpha)}{\partial x}}$$

– The left-hand side is unknown.

– The right-hand side is, however, easier to find.

Comparative Statics

- ***Sales tax*** (Example 6.2):
  - The expression of the aggregate demand is now $x(p+t)$, because the effective price that the consumer pays is actually $p+t$.
  - In equilibrium, the market price after imposing the tax, $p^*(t)$, must hence satisfy
$$x(p^*(t)+t) = q\big(p^*(t)\big)$$
  - If the sales tax is marginally increased, and functions are differentiable at $p = p^*(t)$,
$$x'(p^*(t)+t)\cdot(p^{*\prime}(t)+1) = q'\big(p^*(t)\big)\cdot p^{*\prime}(t)$$

Comparative Statics

– Rearranging, we obtain

$$p^{*\prime}(t) \cdot \left[x'(p^*(t)+t) - q'\left(p^*(t)\right)\right] = -x'(p^*(t)+t)$$

– Hence,

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t) - q'\left(p^*(t)\right)}$$

– Since $x(p)$ is decreasing in prices, $x'(p^*(t)+t) < 0$, and $q(p)$ is increasing in prices, $q'\left(p^*(t)\right) > 0$,

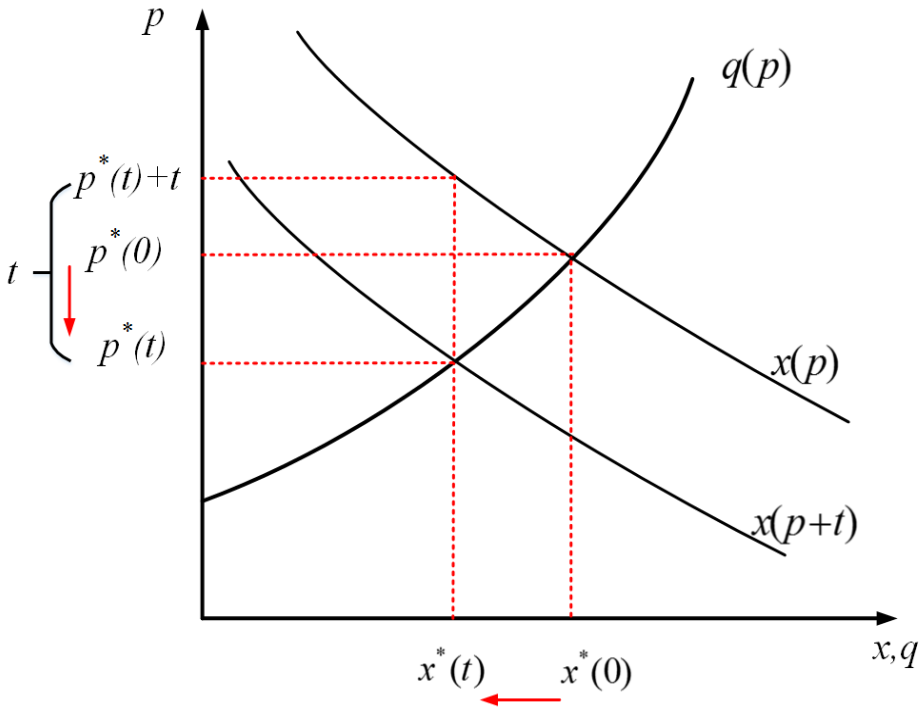
$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{\underbrace{x'(p^*(t)+t)}_{-} - \underbrace{q'\left(p^*(t)\right)}_{+}} = -\frac{-}{-} = -$$

Comparative Statics

- Hence, $p^{*\prime}(t) \leq 0$.
- Moreover, $p^{*\prime}(t) \in (-1,0]$.
- Therefore, $p^*(t)$ decreases in t.
  - That is, the price received by producers falls in the tax, but less than proportionally.
- Additionally, since $p^*(t) + t$ is the price paid by consumers, then $p^{*\prime}(t) + 1$ is the marginal increase in the price paid by consumers when the tax marginally increases.
  - Since $p^{*\prime}(t) > -1$, then $p^{*\prime}(t) + 1 < 1$, and consumers' cost of the product also raises less than proportionally.

Comparative Statics

- *No tax:*
  - CE occurs at $p^*(0)$ and $x^*(0)$
- *Tax:*
  - x^* decreases from $x^*(0)$ to $x^*(t)$
  - Consumers now pay $p^*(t) + t$
  - Producers now receive $p^*(t)$ for the $x^*(t)$ units they sell.



Comparative Statics

- ***Sales Tax*** (Extreme Cases):
  - *a) The supply is very responsive to price changes, i.e., $q'\left(p^*(t)\right)$ is large.*

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t)-q'\left(p^*(t)\right)} \to 0$$

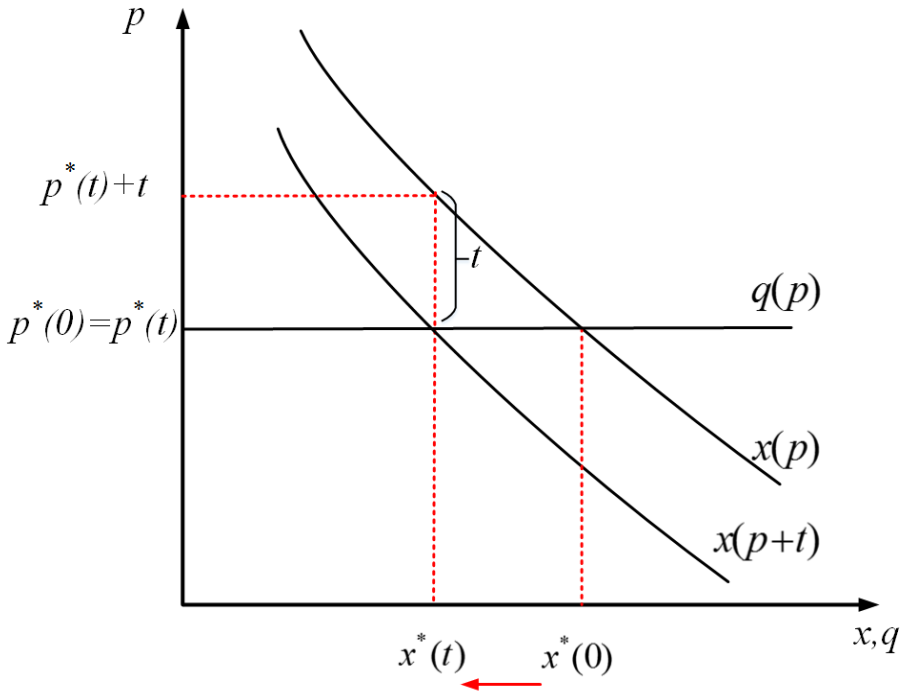
  - Therefore, $p^{*\prime}(t) \to 0$, and the price received by producers does not fall.
  - However, consumers still have to pay $p^*(t) + t$.
    - A marginal increase in taxes therefore provides an increase in the consumer's price of

$$p^{*\prime}(t) + 1 = 0 + 1 = 1$$

    - The tax is solely borne by consumers.

Comparative Statics

- A very elastic supply curve
  - The price received by producers almost does not fall.
  - But, the price paid by consumers increases by exactly the amount of the tax.



Comparative Statics

b) *The supply is not responsive to price changes, i.e., $q'\left(p^*(t)\right)$ is close to zero.*

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t)-q'\left(p^*(t)\right)} = -1$$

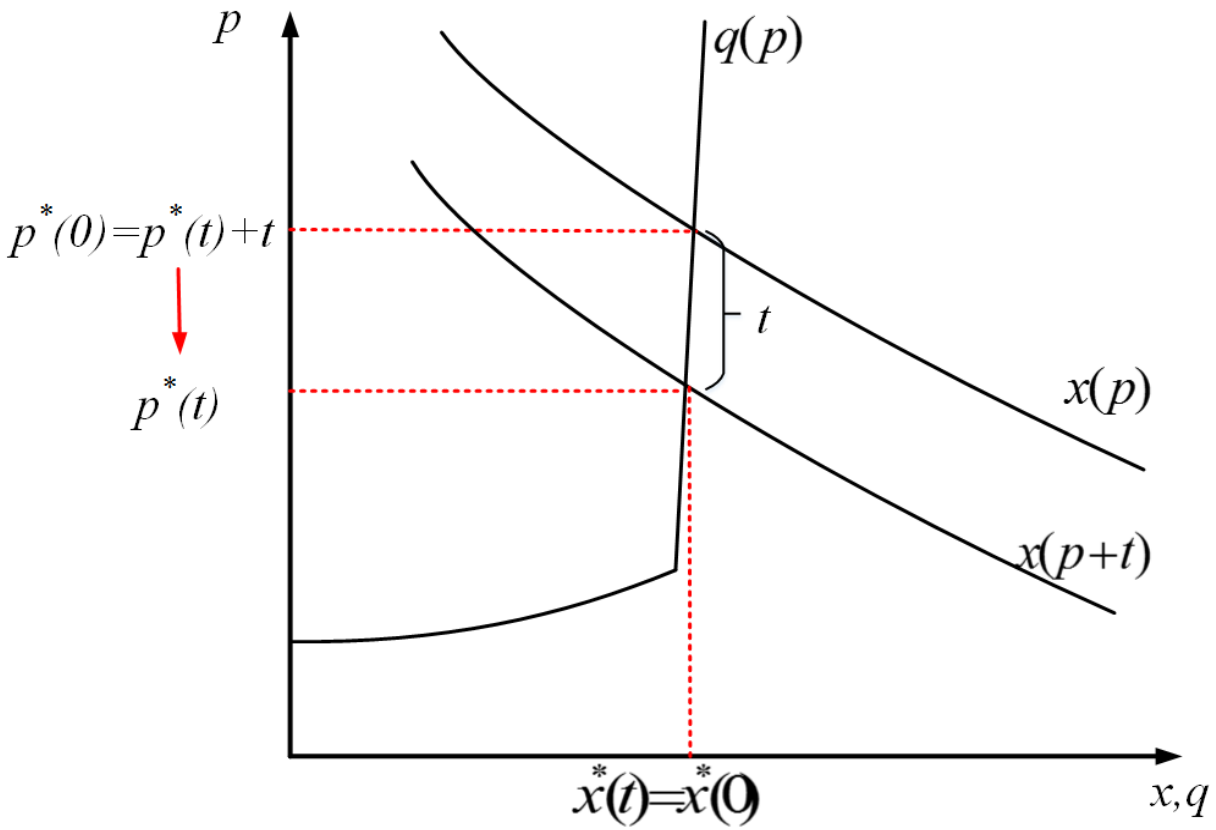
- Therefore, $p^{*\prime}(t) = -1$, and the price received by producers falls by \$1 for every extra dollar in taxes.
  - Producers bear most of the tax burden
- In contrast, consumers pay $p^*(t) + t$
  - A marginal increase in taxes produces an increase in the consumer's price of

$$p^{*\prime}(t) + 1 = -1 + 1 = 0$$

  - Consumers do not bear the tax burden at all.

Comparative Statics

- Inelastic supply curve



Comparative Statics

- ***Example 6.3***:
  - Consider a competitive market in which the government will be imposing a sales tax t per unit.
  - Aggregate demand curve is $x(p) = Ap^{\varepsilon}$, where $A > 0$ and $\varepsilon < 0$, and aggregate supply curve is $q(p) = ap^{\gamma}$, where $a > 0$ and $\gamma > 0$.
  - Let us evaluate how the equilibrium price is affected by a marginal increase in the tax.

Comparative Statics

- ***Example 6.3*** (continued):
  - The change in the price received by producers at $t = 0$ is

$$p^{*\prime}(0) = -\frac{x'(p^*)}{x'(p^*) - q'(p^*)}$$

$$= -\frac{A\varepsilon p^{*\varepsilon-1}}{A\varepsilon p^{*\varepsilon-1} - a\gamma p^{*\gamma-1}} = -\frac{A\varepsilon p^{*\varepsilon}}{A\varepsilon p^{*\varepsilon} - a\gamma p^{*\gamma}}$$

$$= -\frac{\varepsilon x(p^*)}{\varepsilon x(p^*) - \gamma q(p^*)} = -\frac{\varepsilon}{\varepsilon - \gamma}$$

  - The change in the price paid by consumers at $t = 0$ is

$$p^{*\prime}(0) + 1 = -\frac{\varepsilon}{\varepsilon - \gamma} + 1 = -\frac{\gamma}{\varepsilon - \gamma}$$

Comparative Statics

- ***Example 6.3*** (continued):
  - When $\gamma = 0$ (i.e., supply is perfectly inelastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax since $p^{*\prime}(0) = -\frac{\varepsilon}{\varepsilon - 0} = -1$.
    - That is, producers bear the full effect of the tax.
  - When $\varepsilon = 0$ (i.e., demand is perfectly inelastic), the price received by producers is unchanged, and the price paid by consumers increases by the amount of the tax because $p^{*\prime}(0) + 1 = -\frac{\gamma}{0 - \gamma} = 1$.
    - That is, consumers bear the full burden of the tax.

Comparative Statics

- ***Example 6.3*** (continued):
  - When $\varepsilon \to -\infty$ (i.e., demand is perfectly elastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax since

$$\lim_{\varepsilon \to -\infty} p^{*\prime}(0) = -\frac{1}{1 - \lim_{\varepsilon \to -\infty} \frac{\gamma}{\varepsilon}} = -1$$

  - When $\gamma \to +\infty$ (i.e., supply is perfectly elastic), the price received by producers is unchanged, and the price paid by consumers increases by the amount of the tax because

$$\lim_{\gamma \to +\infty} p^{*\prime}(0) + 1 = -\frac{1}{\lim_{\gamma \to +\infty} \frac{\varepsilon}{\gamma} - 1} = 1$$

Welfare Analysis

Welfare Analysis

- Let us now measure the changes in the aggregate social welfare due to a change in the competitive equilibrium allocation.
- Consider the aggregate surplus

$$S = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$$

- Take a differential change in the quantity of good k that individuals consume and that firms produce such that $\sum_{i=1}^{I} dx_i = \sum_{j=1}^{J} dq_j$.
- The change in the aggregate surplus is

$$dS = \sum_{i=1}^{I} \phi_i'(x_i)dx_i - \sum_{j=1}^{J} c_j'(q_j)dq_j$$

Welfare Analysis

- Since
  - $\phi_i'(x_i) = P(x)$ for all consumers; and
    - That is, every individual consumes until MB=p.
  - $c_j'(q_j) = C'(q)$ for all firms
    - That is, every firm's MC coincides with aggregate MC

  then the change in surplus can be rewritten as

$$dS = \sum_{i=1}^{I} P(x)dx_i - \sum_{j=1}^{J} C'(q)dq_j$$

$$= P(x)\sum_{i=1}^{I} dx_i - C'(q)\sum_{j=1}^{J} dq_j$$

Welfare Analysis

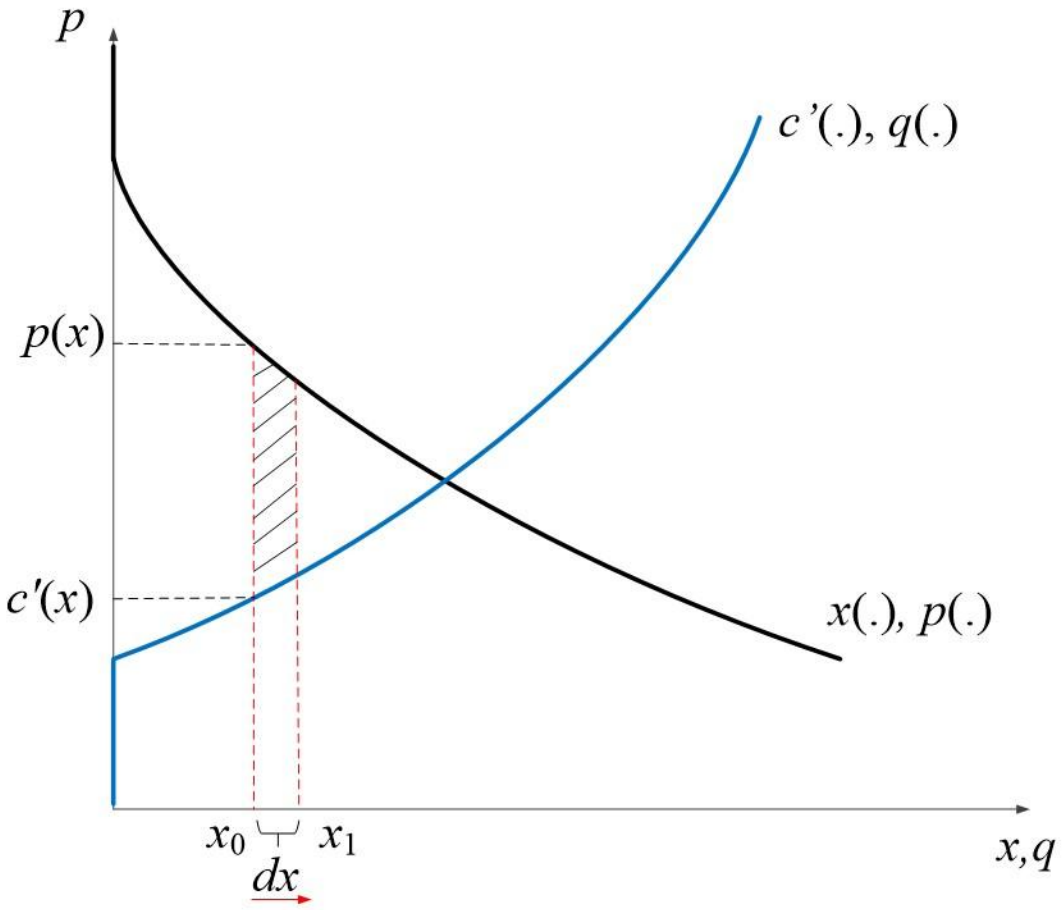
- But since $\sum_{i=1}^{I} dx_i = \sum_{j=1}^{J} dq_j = dx$, and $x = q$ by market feasibility, then

$$dS = [P(x) - C'(q)]dx$$

- *Intuition*:
  - The change in surplus of a marginal increase in consumption (and production) reflects the difference between the consumers' additional utility and the firms' additional cost of production.

Welfare Analysis

- Differential change in surplus



Welfare Analysis

- We can also integrate the above expression, eliminating the differentials, in order to obtain the total surplus for an aggregate consumption level of x:

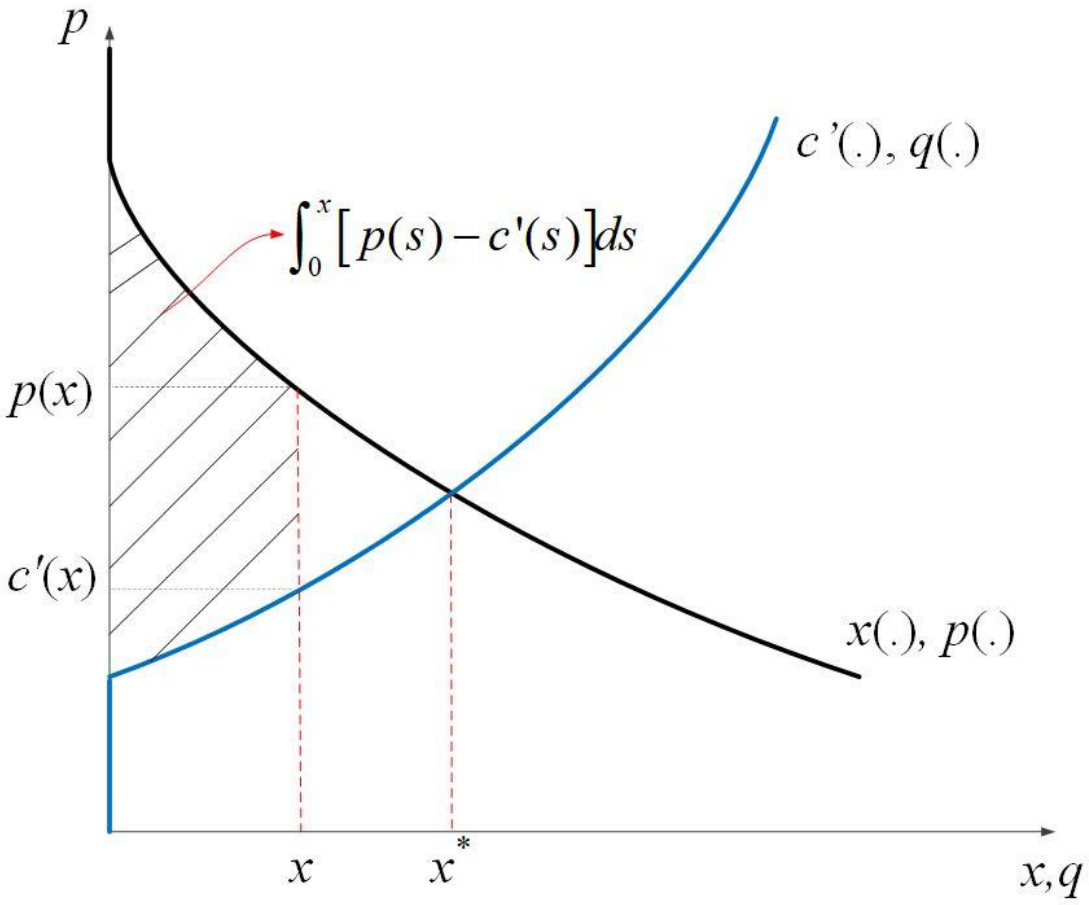
$$S(x) = S_0 + \int_0^x [P(s) - C'(s)]ds$$

where $S_0 = S(0)$ is the constant of integration, and represents the aggregate surplus when aggregate consumption is zero, $x = 0$.

  - $S_0 = 0$ if the intercept of the marginal cost function satisfies $c_j'(0) = 0$ for all J firms.

Welfare Analysis

- Surplus at aggregate consumption x



Welfare Analysis

- For which consumption level is aggregate surplus $S(x)$ maximized?
  - Differentiating $S(x)$ with respect to x,

$$S'(x) = P(x^*) - C'(x^*) \leq 0$$

$$\text{or,} \quad P(x^*) \leq C'(x^*)$$

  - The second order (sufficient) condition is

$$S''(x) = \underbrace{P'(x^*)}_{-} - \underbrace{C''(x^*)}_{+} < 0$$

    - Hence, $S(x^*)$ is concave.
    - Then, when $x^* > 0$, aggregate surplus $S(x)$ is maximized at $P(x^*) = C'(x^*)$.

Welfare Analysis

- Therefore, the CE allocation maximizes aggregate surplus.
- This is the ***First Welfare Theorem***:
  - Every CE is Pareto optimal (PO).

Welfare Analysis

- ***Example 6.4***:
  - Consider an aggregate demand $x(p) = a - bp$ and aggregate supply $y(p) = J \cdot \frac{p}{2}$, where J is the number of firms in the industry.
  - The CE price solves

$$a - bp = J \cdot \frac{p}{2} \quad \text{or} \quad p = \frac{2a}{2b+J}$$

  - Intuitively, as demand increases (the number of firms) increases, the equilibrium price increases (decreases, respectively).

Welfare Analysis

- ***Example 6.4*** (continued):
  - Therefore, equilibrium output is

$$x^* = a - b\frac{2a}{2b+J} = \frac{aJ}{2b+J}$$

  - Surplus is

$$S(x^*) = \int_0^{x^*} \big(p(x) - C'(x)\big)dx$$

where $p(x) = \frac{a-x}{b}$ and since $y = \frac{pJ}{2}$, solving for p and rearranging yields the marginal cost function $p = C'(x) = \frac{2x}{J}$, so that

$$S(x^*) = \int_0^{x^*} \left(\frac{a-x}{b} - \frac{2x}{J}\right)dx = \frac{a^2J}{4b^2+2bJ} = \frac{a^2}{2b\left(1+\frac{2b}{J}\right)}$$

which is increasing in the number of firms J.

General Equilibrium

General Equilibrium

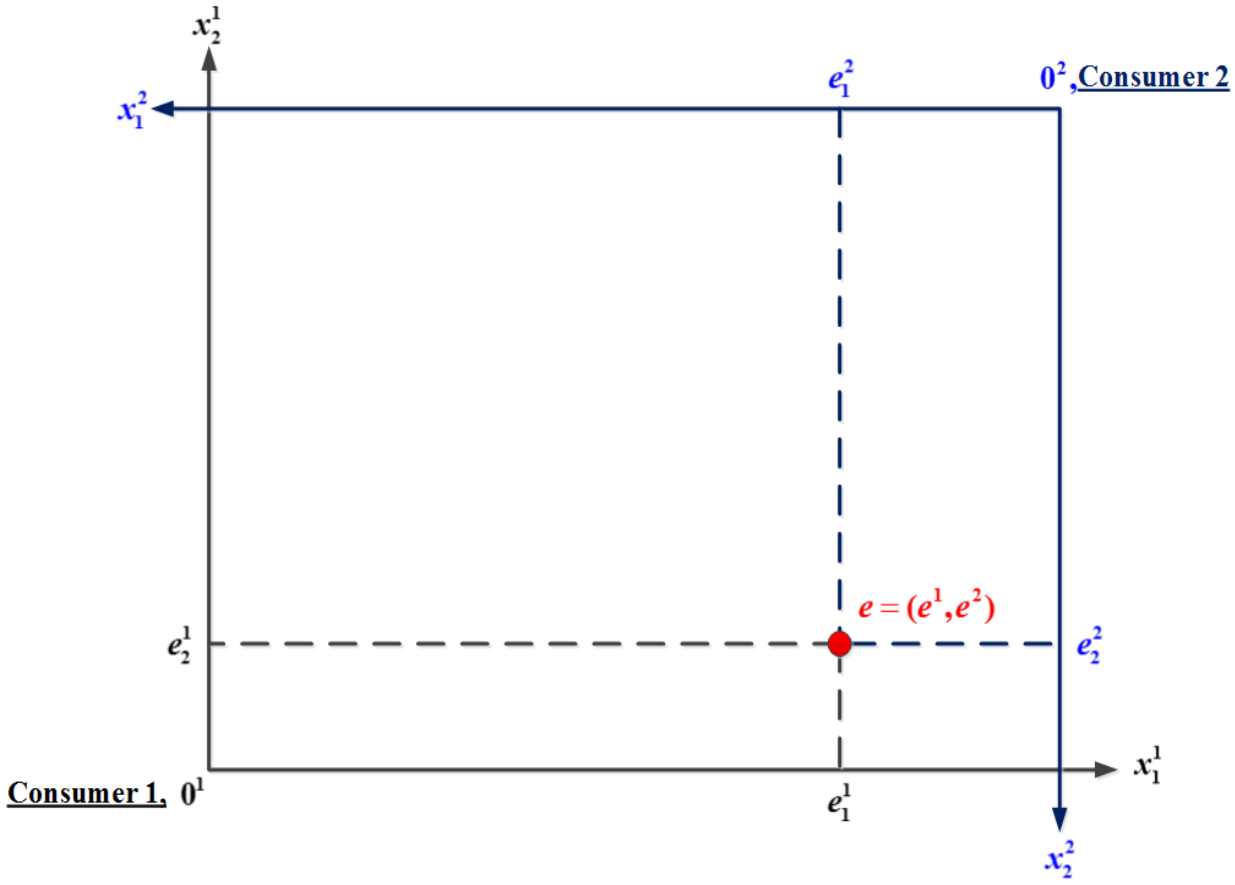
- So far, we explored equilibrium conditions in:
  - a single market with
  - a single type of consumer.
- Now we examine settings with markets for:
  - different goods and
  - multiple consumers.

General Equilibrium: No Production

- Consider an economy with two goods and two consumers, $i = \{1,2\}$.
- Each consumer i is initially endowed with $\mathbf{e}^i \equiv (e_1^i, e_2^i)$ units of good 1 and 2.
- Any other allocation for individual i is denoted by $\mathbf{x}^i \equiv (x_1^i, x_2^i)$.

General Equilibrium: No Production

- ***Edgeworth box***:

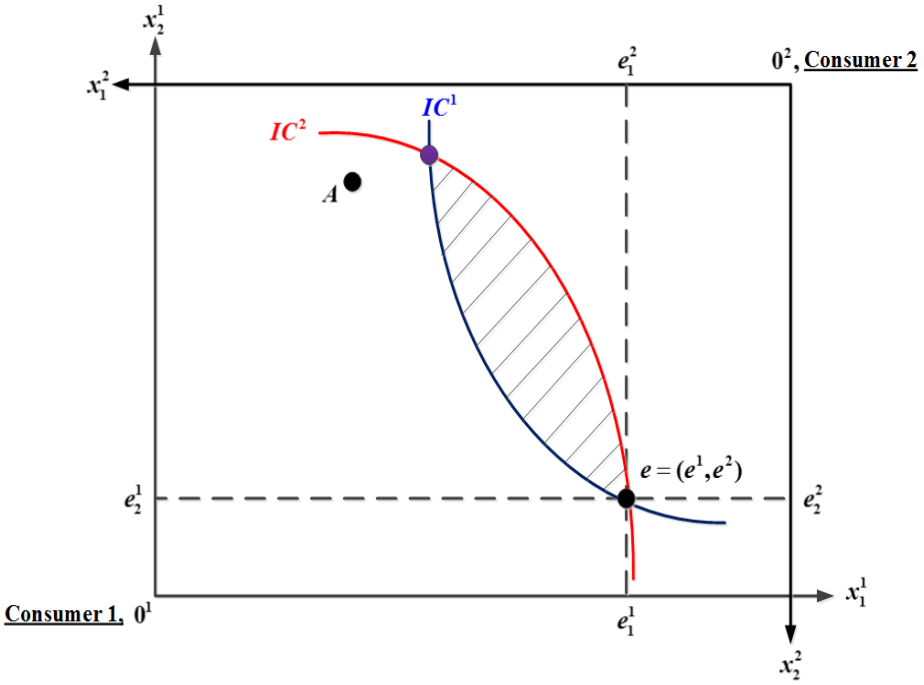


General Equilibrium: No Production

- IC^i is the indifference curve of consumer i, which passes through his endowment point $\mathbf{e}^i$.
- The shaded area represents the set of bundles (x_1^i, x_2^i) for consumer i satisfying

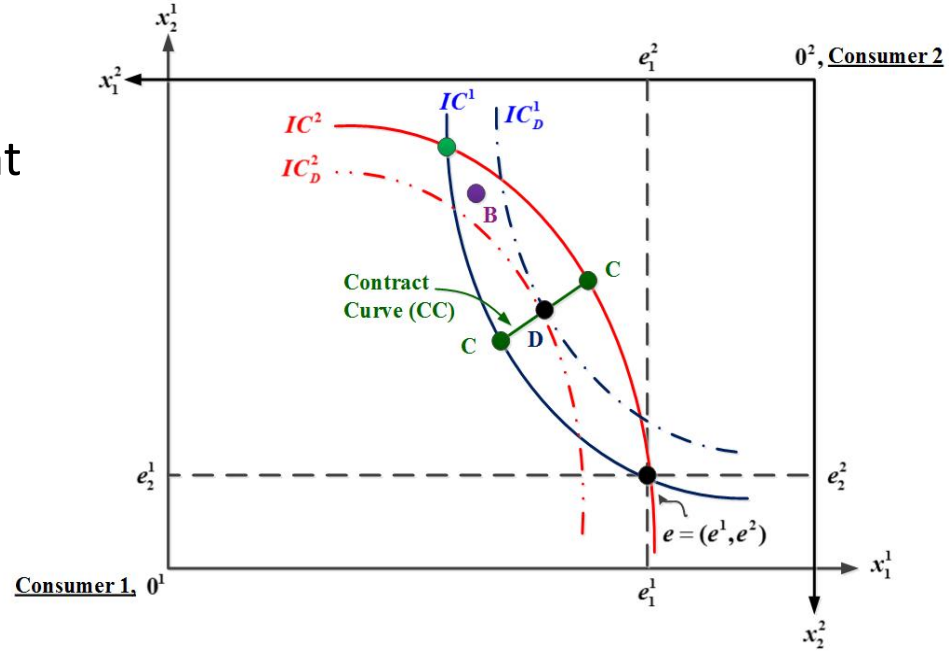
$$u^1(x_1^1, x_2^1) \geq u^1(e_1^1, e_2^1)$$
$$u^2(x_1^2, x_2^2) \geq u^2(e_1^2, e_2^2)$$

- Bundle A cannot be a barter equilibrium:
  - Consumer 1 does not exchange $\mathbf{e}^i$ for A.



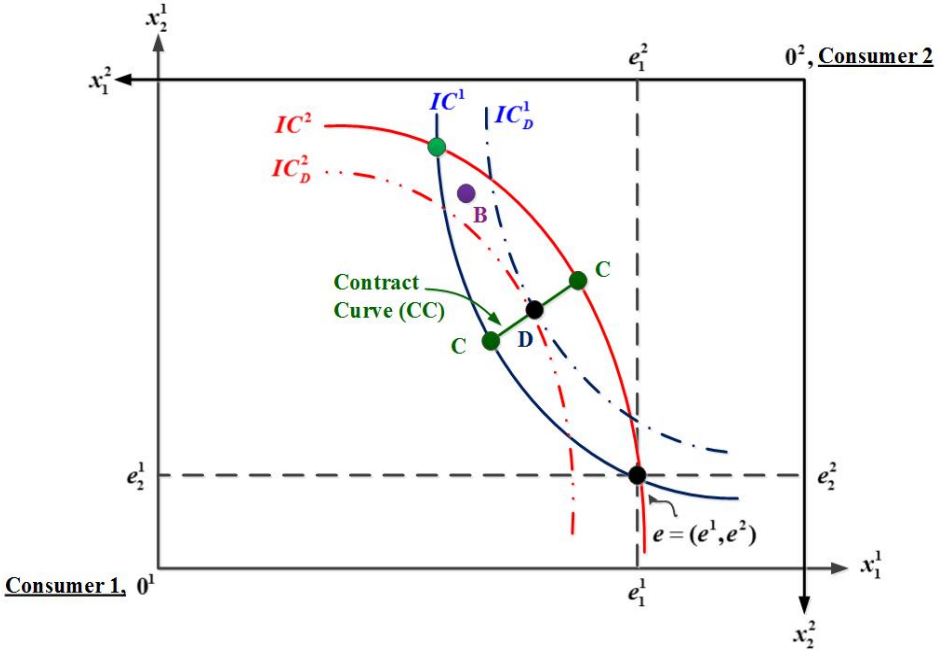
General Equilibrium: No Production

- Not all points in the lens-shaped area are barter equilibria!
- Bundle B lies inside the lens-shaped area
  - Thus, it yields a higher utility level than the initial endowment **e** for both consumers.



General Equilibrium: No Production

- Not all points in the lens-shaped area are barter equilibria!
- Bundle D, however, makes both consumers better off than bundle B.
  - It lies on "***contract curve***," in which the indifference curves are tangent to one another.
  - It is an equilibrium, since Pareto improvements are no longer possible
- Being in the lens-shaped is then a necessary, but not sufficient, condition for an allocation to be a barter equilibrium.



General Equilibrium: No Production

- ***Feasible allocation***:
  - An allocation $\mathbf{x} \equiv (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$ is *feasible* if it satisfies

$$\sum_{i=1}^{I} \mathbf{x}^i \leq \sum_{i=1}^{I} \mathbf{e}^i$$

  - In words, the aggregate amount of goods in allocation $\mathbf{x}$ does not exceed the aggregate initial endowment $\mathbf{e} \equiv \sum_{i=1}^{I} \mathbf{e}^i$.

General Equilibrium: No Production

- ***Pareto-efficient allocations***:
  - A feasible allocation $\mathbf{x}$ is *Pareto efficient* if there is no other feasible allocation $\mathbf{y}$ which is:
    - weakly preferred by all consumers, i.e.,$\mathbf{y}^i \succsim \mathbf{x}^i$ for all $i \in I$, and
    - strictly preferred by at least one consumer, $\mathbf{y}^i \succ \mathbf{x}^i$.
  - That is, allocation $\mathbf{x}$ is Pareto efficient if there is no other feasible allocation $\mathbf{y}$ making:
    - all individuals at least as well off as under $\mathbf{x}$, and
    - making at least one individual strictly better off.

General Equilibrium: No Production

- ***Pareto-efficient allocations***:
  - The set of Pareto efficient allocations $(\mathbf{x}^1, \ldots, \mathbf{x}^I)$ solves

$$\max_{\mathbf{x}^1,\ldots,\mathbf{x}^I \geq 0} u^i(\mathbf{x}^i)$$
$$\text{s.t.} \quad u^j(\mathbf{x}^j) \geq \bar{u}^j \text{ for } j \neq i, \text{ and}$$
$$\textstyle\sum_{i=1}^{I} \mathbf{x}^i \leq \sum_{i=1}^{I} \mathbf{e}^i \quad \text{(feasibility)}$$

    where $\mathbf{x}^i = (x_1^i, x_2^i)$.

  - That is, allocations $(\mathbf{x}^1, \ldots, \mathbf{x}^I)$ are Pareto efficient if they maximize individual i's utility without reducing the utility of all other individuals below a given level $\bar{u}^j$, and satisfying feasibility.

General Equilibrium: No Production

- The Lagrangian is

$$L\left(\mathbf{x}^1, \ldots, \mathbf{x}^I; \lambda^1, \ldots, \lambda^{i-1}, \lambda^{i+1}, \ldots, \lambda^I, \mu\right) =$$

$$u^i\left(\mathbf{x}^i\right) + \sum_{j \neq i}^{I} \lambda^j \left[u^j\left(\mathbf{x}^j\right) - \bar{u}^j\right] + \cdots$$

$$+\mu\left[\sum_{i=1}^{I} \mathbf{e}^i - \sum_{i=1}^{I} \mathbf{x}^i\right]$$

- FOC wrt $\mathbf{x}^i = (x_1^i, x_2^i)$ yields

$$\frac{\partial L}{\partial x_k^i} = \frac{\partial u^i(\mathbf{x}^i)}{\partial x_k^i} - \mu \leq 0$$

  for every good k of consumer i.
- For any individual $j \neq i$, the FOCs become

$$\frac{\partial L}{\partial x_k^j} = \lambda^j \frac{\partial u^j(\mathbf{x}^j)}{\partial x_k^j} - \mu \leq 0$$

General Equilibrium: No Production

- FOCs wrt λ^j and μ yield $u^j(\mathbf{x}^j) \geq \bar{u}^j$ and $\sum_{i=1}^{I} \mathbf{x}^i \leq \sum_{i=1}^{I} \mathbf{e}^i$, respectively.
- In the case of interior solutions, a compact condition for Pareto efficiency is

$$\frac{\frac{\partial u^i(\mathbf{x}^i)}{\partial x_1^i}}{\frac{\partial u^i(\mathbf{x}^i)}{\partial x_2^i}} = \frac{\frac{\partial u^j(\mathbf{x}^j)}{\partial x_1^j}}{\frac{\partial u^j(\mathbf{x}^j)}{\partial x_2^j}} \text{ or } MRS_{1,2}^i = MRS_{1,2}^j$$

  for every consumer $j \neq i$.
- Graphically, consumers' indifference curves become tangent to one another at the Pareto efficient allocations.

General Equilibrium: No Production

- ***Example 6.5*** (Pareto efficiency):
  - Consider a barter economy with two goods, 1 and 2, and two consumers, A and B, each with the initial endowments of
    - $\mathbf{e}^A = (100, 350)$ and
    - $\mathbf{e}^B = (100,50)$, respectively.
  - Each consumer's utility function is a Cobb-Douglas type given by
    - $u^i\left(x_1^i, x_2^i\right) = x_1^i x_2^i$ for every individual $i = \{A, B\}$.
  - Let us find the set of Pareto efficient allocations.

General Equilibrium: No Production

- ***Example*** (continued):
  - Pareto efficient allocations are reached at points where the $MRS^A_{1,2} = MRS^B_{1,2}$. Hence,

$$MRS^A_{1,2} = MRS^B_{1,2} \Longrightarrow \frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} \quad \text{or} \quad x_2^A x_1^B = x_2^B x_1^A$$

  - Using the feasibility constraints for good 1 and 2, i.e.,

$$e_1^A + e_1^B = x_1^A + x_1^B$$
$$e_2^A + e_2^B = x_2^A + x_2^B$$

    we obtain

$$x_1^B = e_1^A + e_1^B - x_1^A$$
$$x_2^B = e_2^A + e_2^B - x_2^A$$

General Equilibrium: No Production

- ***Example*** (continued):
  - Combining the tangency condition and feasibility constraints yields

$$x_2^A \underbrace{(e_1^A + e_1^B - x_1^A)}_{x_1^B} = \underbrace{(e_2^A + e_2^B - x_2^A)}_{x_2^B} x_1^A$$

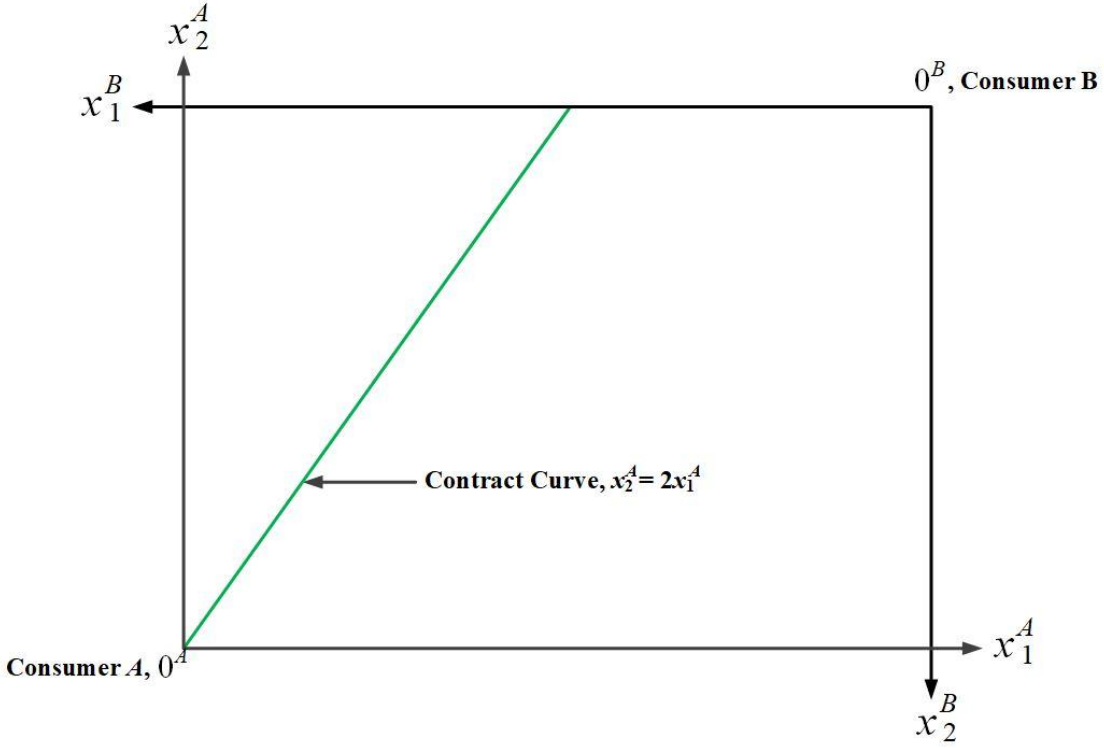
which can be re-written as

$$x_2^A = \frac{e_2^A + e_2^B}{e_1^A + e_1^B} x_1^A = \frac{350+50}{100+100} x_1^A = 2x_1^A$$

for all $x_1^A \in [0,200]$.

General Equilibrium: No Production

- ***Example*** (continued):
  - The line representing the set of Pareto efficient allocations



General Equilibrium: No Production

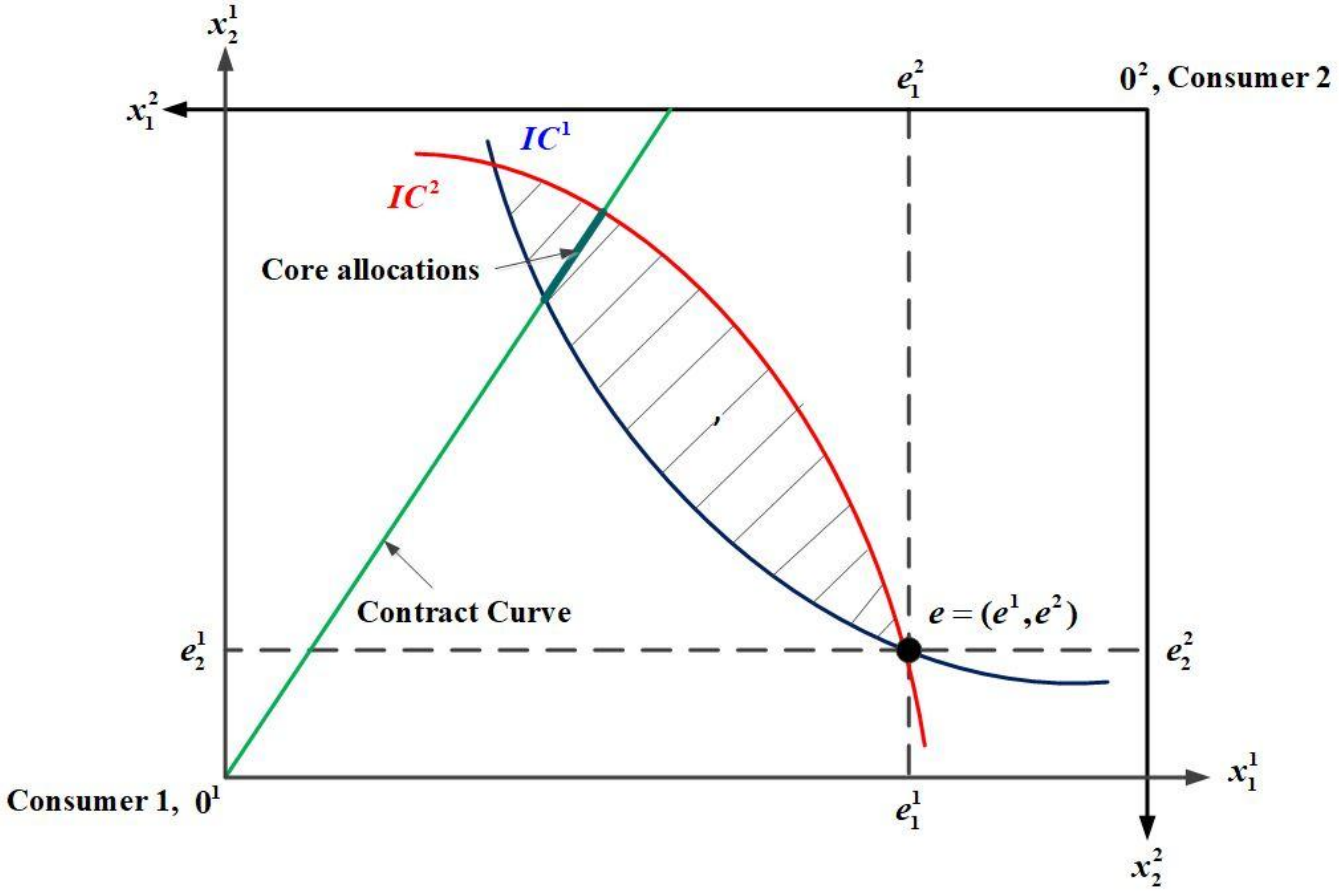
- ***Blocking coalitions***: Let $S \subset I$ denote a coalition of consumers. We say that S *blocks* the feasible allocation $\mathbf{x}$ if there is an allocation $\mathbf{y}$ such that:

  1) *Allocation is feasible for S*. The aggregate amount of goods that individuals in S enjoy in allocation $\mathbf{y}$ coincides with their aggregate initial endowment, i.e., $\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$; and
  2) *Preferable*. Allocation $\mathbf{y}$ makes all individuals in the coalition weakly better off than under $\mathbf{x}$, i.e., $\mathbf{y}^i \succsim \mathbf{x}^i$ where $i \in S$, but makes at least one individual strictly better off, i.e., $\mathbf{y}^i \succ \mathbf{x}^i$.

General Equilibrium: No Production

- ***Equilibrium in a barter economy***: A feasible allocation $\mathbf{x}$ is an equilibrium in the exchange economy with initial endowment $\mathbf{e}$ if $\mathbf{x}$ is *not blocked* by any coalition of consumers.
- ***Core***: The core of an exchange economy with endowment $\mathbf{e}$, denoted $C(\mathbf{e})$, is the set of all *unblocked* feasible allocations $\mathbf{x}$.
  - Such allocations:
    a) are mutually beneficial for all individuals (i.e., they lie in the lens-shaped area)
    b) do not allow for further Pareto improvements (i.e., they lie on the contract curve)

General Equilibrium: No Production



General Equilibrium: Competitive Markets

- Barter economy did not require prices for an equilibrium to arise.
- Now we explore the equilibrium in economies where we allow prices to emerge.
- Order of analysis:
  - consumers' preferences
  - the excess demand function
  - the equilibrium allocations in competitive markets (i.e., *Walrasian equilibrium allocations*)

General Equilibrium: Competitive Markets

- ***Consumers***:
  - Consider consumers' utility function to be continuous, strictly increasing, and strictly quasiconcave in $\mathbb{R}_+^L$.
  - Hence the UMP of every consumer i, when facing a budget constraint

    $$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i \text{ for all price vector } \mathbf{p} \gg \mathbf{0}$$

    yields a unique solution, which is the Walrasian demand $\mathbf{x}(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$.
  - $\mathbf{x}(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ is continuous in the price vector $\mathbf{p}$.

General Equilibrium: Competitive Markets

– Intuitively, individual i's income comes from selling his endowment $\mathbf{e}^i$ at market prices $\mathbf{p}$, producing $\mathbf{p} \cdot \mathbf{e}^i = p_1 e_1^i + \cdots + p_L e_L^i$ dollars to be used in the purchase of allocation $\mathbf{x}^i$.

General Equilibrium: Competitive Markets

- ***Excess demand***:
  - Summing the Walrasian demand $\mathbf{x}(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ for good k of every individual in the economy, we obtain the *aggregate demand* for good k.
  - The difference between the aggregate demand and the aggregate endowment of good k yields the *excess demand of good* k**:**

$$z_k(\mathbf{p}) = \sum_{i=1}^{I} x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - \sum_{i=1}^{I} e_k^i$$

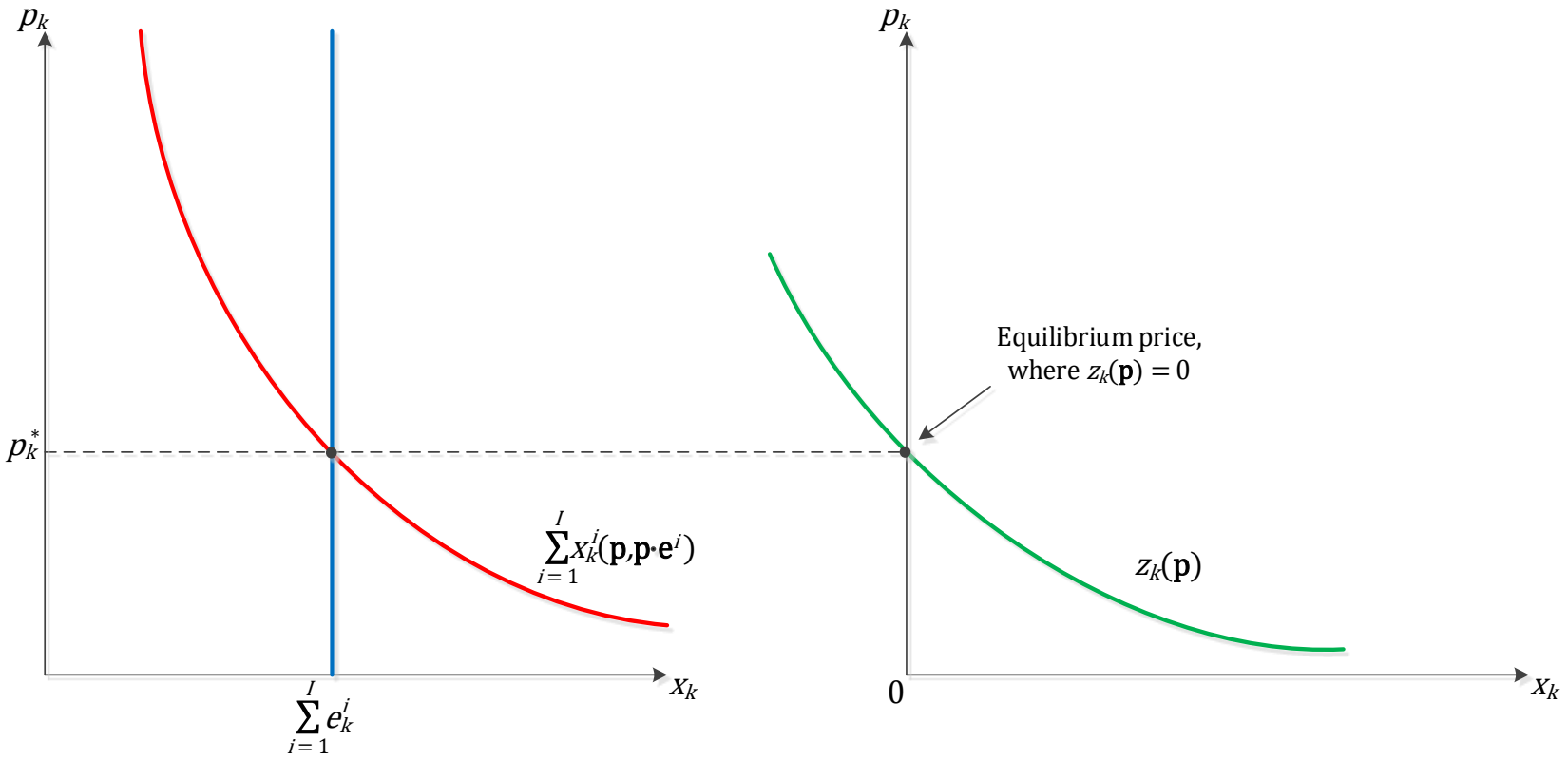
where $z_k(\mathbf{p}) \in \mathbb{R}$.

General Equilibrium: Competitive Markets

- When $z_k(\mathbf{p}) > 0$, the aggregate demand for good k exceeds its aggregate endowment.
  - Excess demand of good k
- When $z_k(\mathbf{p}) < 0$, the aggregate demand for good k falls short of its aggregate endowment.
  - Excess supply of good k

General Equilibrium: Competitive Markets

- Difference in demand and supply, and excess demand



General Equilibrium: Competitive Markets

- The excess demand function $\mathbf{z}(\mathbf{p}) \equiv \left(z_1(\mathbf{p}), \dots, z_k(\mathbf{p}), \dots, z_L(\mathbf{p})\right)$ satisfies following properties:

  1) ***Walras' law***: $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0.$

     – Since every consumer $i \in I$ exhausts all his income,

$$\sum_{k=1}^{L} p_k \cdot x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) = \sum_{k=1}^{L} p_k e_k^i \;\Leftrightarrow$$

$$\sum_{k=1}^{L} p_k \left[x_k^i\left(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i\right) - e_k^i\right] = 0$$

General Equilibrium: Competitive Markets

– Summing over all individuals,

$$\sum_{i=1}^{I}\sum_{k=1}^{L} p_k\left[x_k^i\left(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i\right) - e_k^i\right] = 0$$

– We can re-write the above expression as

$$\sum_{k=1}^{L}\sum_{i=1}^{I} p_k\left[x_k^i\left(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i\right) - e_k^i\right] = 0$$

which is equivalent to

$$\sum_{k=1}^{L} p_k \underbrace{\left(\sum_{i=1}^{I}\left[x_k^i\left(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i\right)\right] - \sum_{i=1}^{I} e_k^i\right)}_{z_k(\mathbf{p})} = 0$$

– Hence,

$$\sum_{k=1}^{L} p_k \cdot z_k(\mathbf{p}) = \mathbf{p}\cdot\mathbf{z}(\mathbf{p}) = 0$$

General Equilibrium: Competitive Markets

- In a two-good economy, Walras' law implies

$$p_1 \cdot z_1(\mathbf{p}) = -p_2 \cdot z_2(\mathbf{p})$$

- *Intuition*: if there is excess demand in market 1, $z_1(\mathbf{p}) > 0$, there must be excess supply in market 2, $z_2(\mathbf{p}) < 0$.

- Hence, if market 1 is in equilibrium, $z_1(\mathbf{p}) = 0$, then so is market 2, $z_2(\mathbf{p}) = 0$.

- More generally, if the markets of $L - 1$ goods are in equilibrium, then so is the L^{th} market.

General Equilibrium: Competitive Markets

2) ***Continuity***: $\mathbf{z}(\mathbf{p})$ is continuous at $\mathbf{p}$.
   - This follows from individual Walrasian demands being continuous in prices.

3) ***Homogeneity***: $\mathbf{z}(\lambda\mathbf{p}) = \mathbf{z}(\mathbf{p})$ for all $\lambda > 0$.
   - This follows from Walrasian demands being homogeneous of degree zero in prices.

- We now use excess demand to define a Walrasian equilibrium allocation.

General Equilibrium: Competitive Markets

- ***Walrasian equilibrium***:
  - A price vector $\mathbf{p}^* \gg 0$ is a Walrasian equilibrium if aggregate excess demand is zero at that price vector, $\mathbf{z}(\mathbf{p}^*) = 0$.
    - In words, price vector $\mathbf{p}^*$ clears all markets.
  - Alternatively, $\mathbf{p}^* \gg 0$ is a Walrasian equilibrium if:
    1) Each consumer solves his UMP, and
    2) Aggregate demand equals aggregate supply

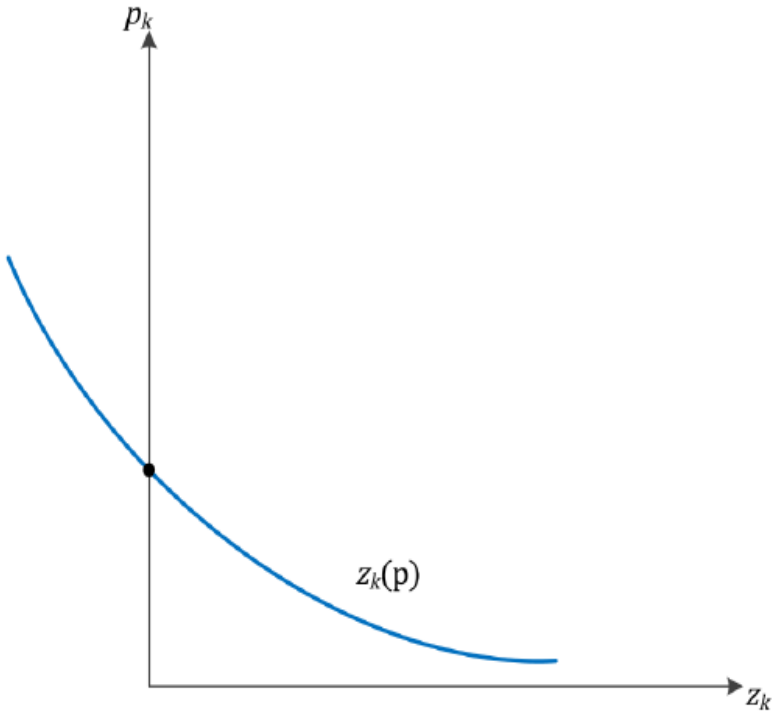
$$\sum_{i=1}^{I} x^i\left(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i\right) = \sum_{i=1}^{I} \mathbf{e}^i$$

General Equilibrium: Competitive Markets

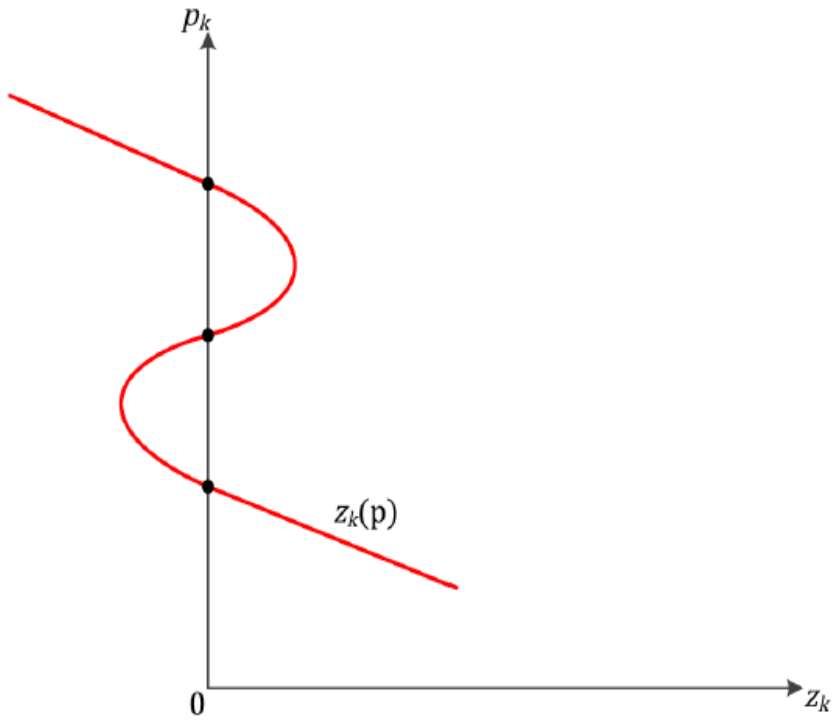
- ***Existence of a Walrasian equilibrium***:
  - A Walrasian equilibrium price vector $\mathbf{p}^* \in \mathbb{R}^L_{++}$, i.e., $\mathbf{z}(\mathbf{p}^*) = 0$, exists if the excess demand function $\mathbf{z}(\mathbf{p})$ satisfies *continuity* and *Walras' law* (Varian, 1992).

General Equilibrium: Competitive Markets

- ***Uniqueness of equilibrium prices***:



satisfied



violated

General Equilibrium: Competitive Markets

- ***Example 6.6*** (Walrasian equilibrium allocation):
  - In example 6.5, we determined that

$$MRS_{1,2}^{A} = MRS_{1,2}^{B} = \frac{p_1}{p_2}$$

$$\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} = \frac{p_1}{p_2}$$

  - Let us determine the Walrasian demands of each good for each consumer.
  - Rearranging the second equation above, we get

$$p_1 x_1^A = p_2 x_2^A$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Substituting this into consumer A's budget constraint yields

$$p_1 x_1^A + p_1 x_1^A = p_1 \cdot 100 + p_2 \cdot 350$$

$$\text{or } x_1^A = 50 + 175\frac{p_2}{p_1}$$

    which is consumer A's Walrasian demand for good 1.
  - Plugging this value back into $p_1 x_1^A = p_2 x_2^A$ yields

$$p_1\left(50 + 175\frac{p_2}{p_1}\right) = p_2 x_2^A$$

$$\text{or } x_2^A = 175 + 50\frac{p_1}{p_2}$$

    which is consumer A's Walrasian demand for good 2.

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - We can obtain consumer B's demand in an analogous way. In particular, substituting $p_1 x_1^B = p_2 x_2^B$ into consumer B's budget constraint yields

$$p_1 x_1^B + p_1 x_1^B = p_1 \cdot 100 + p_2 \cdot 50$$

$$\text{or } x_1^B = 50 + 25\frac{p_2}{p_1}$$

    which is consumer B's Walrasian demand for good 1.
  - Plugging this value back into $p_1 x_1^B = p_2 x_2^B$ yields

$$p_1\left(50 + 25\frac{p_2}{p_1}\right) = p_2 x_2^B$$

$$\text{or } x_2^B = 25 + 50\frac{p_1}{p_2}$$

    which is consumer B's Walrasian demand for good 2.

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - For good 1, the feasibility constraint is

$$x_1^A + x_1^B = 100 + 100$$

$$\left(50 + 175\frac{p_2}{p_1}\right) + \left(50 + 25\frac{p_2}{p_1}\right) = 200$$

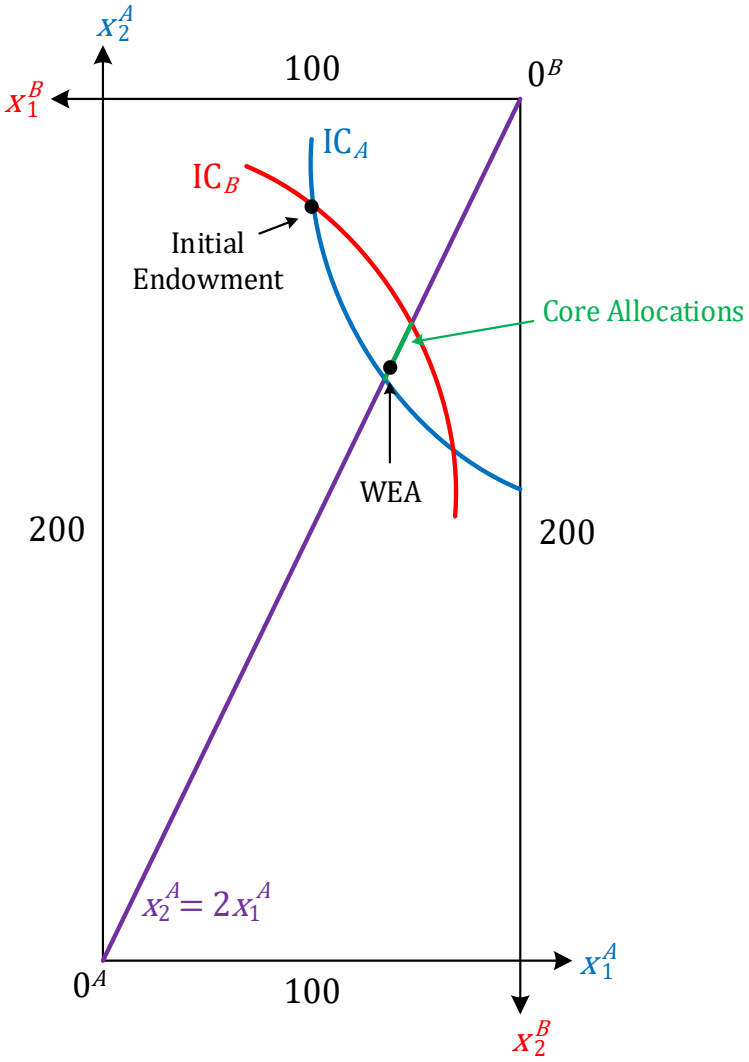
$$\frac{p_2}{p_1} = \frac{1}{2}$$

  - Plugging the relative prices into the Walrasian demands yields Walrasian equilibrium allocation:

$$\left(x_1^{A,*}, x_2^{A,*}, x_1^{B,*}, x_2^{B,*}; \frac{p_1}{p_2}\right) = (137.5, 275, 62.5, 125; 2)$$

General Equilibrium: Competitive Markets

- ***Example 6.6*** (continued):
  - Initial allocation,
  - Core allocation, and
  - Walrasian equilibrium allocations (WEA).



General Equilibrium: Competitive Markets

- ***Equilibrium allocations must be in the Core***:
  - If each consumer's utility function is strictly increasing, then every WEA is in the Core, i.e., $W(\mathbf{e}) \subset C(\mathbf{e})$.
- *Proof* (by contradiction):
  - Take a WEA $\mathbf{x}(\mathbf{p}^*)$ with equilibrium price $\mathbf{p}^*$, but $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$.
  - Since $\mathbf{x}(\mathbf{p}^*)$ is a WEA, it must be feasible.
  - If $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$, we can find a coalition of individuals S and another allocation $\mathbf{y}$ such that

$$u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)) \text{ for all } i \in S$$

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - The above expression:
    - holds with strict inequality for at least one individual in the coalition, i.e., there exists an $i \in S$ that satisfies $u^i(\mathbf{y}^i) > u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i))$
    - is feasible for the coalition, i.e., $\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$.
  - Multiplying both sides of the feasibility condition by $\mathbf{p}^*$ yields

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$$

  - However, the most preferable vector $\mathbf{y}^i$ must be more costly than $\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)$:

$$\mathbf{p}^* \cdot \mathbf{y}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i) = \mathbf{p}^* \cdot \mathbf{e}^i$$

    with strict inequality for at least one individual.

General Equilibrium: Competitive Markets

- *Proof* (continued):
  - Hence, summing over all consumers in the coalition S, we obtain

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i > \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{x}^i\left(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i\right) = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$$

    which contradicts $\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$.

  - Therefore, all WEAs must be part of the Core, i.e.,

$$\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$$

General Equilibrium: Competitive Markets

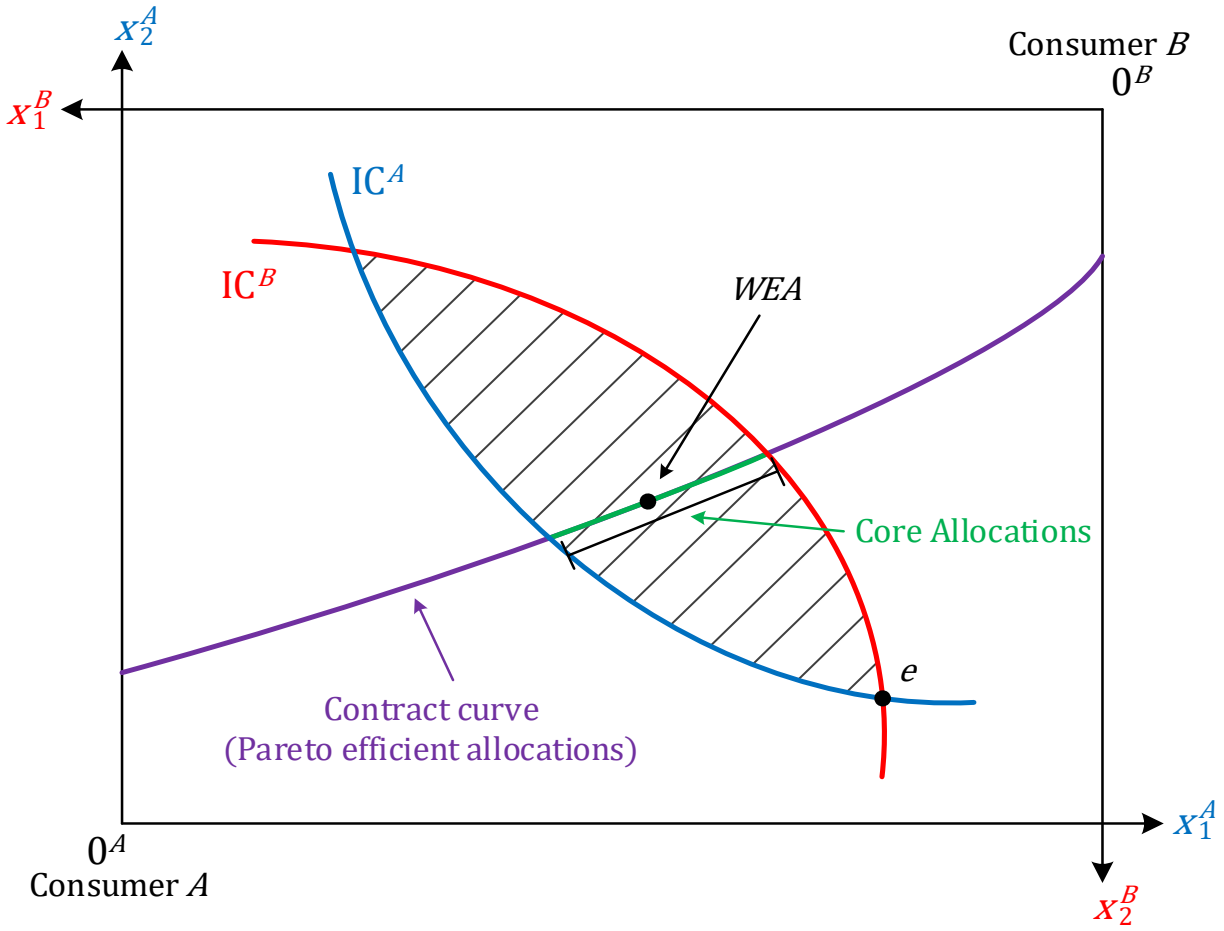
- *Remarks*:
  1) The Core $C(\mathbf{e})$ contains the WEA (or WEAs)
     - That is, the Core is nonempty.
  2) Since all core allocations are Pareto efficient (i.e., we cannot increase the welfare of one consumer without decreasing that of the other consumers), then all WEAs (which are part of the Core) are also Pareto efficient.

General Equilibrium: Competitive Markets

- ***First Welfare Theorem***: Every WEA is Pareto efficient.
  - The WEA lies on the core (the segment of the contract curve within the lens-shaped area),
  - The core is a subset of all Pareto efficient allocations.

General Equilibrium: Competitive Markets

- First Welfare Theorem



General Equilibrium: Competitive Markets

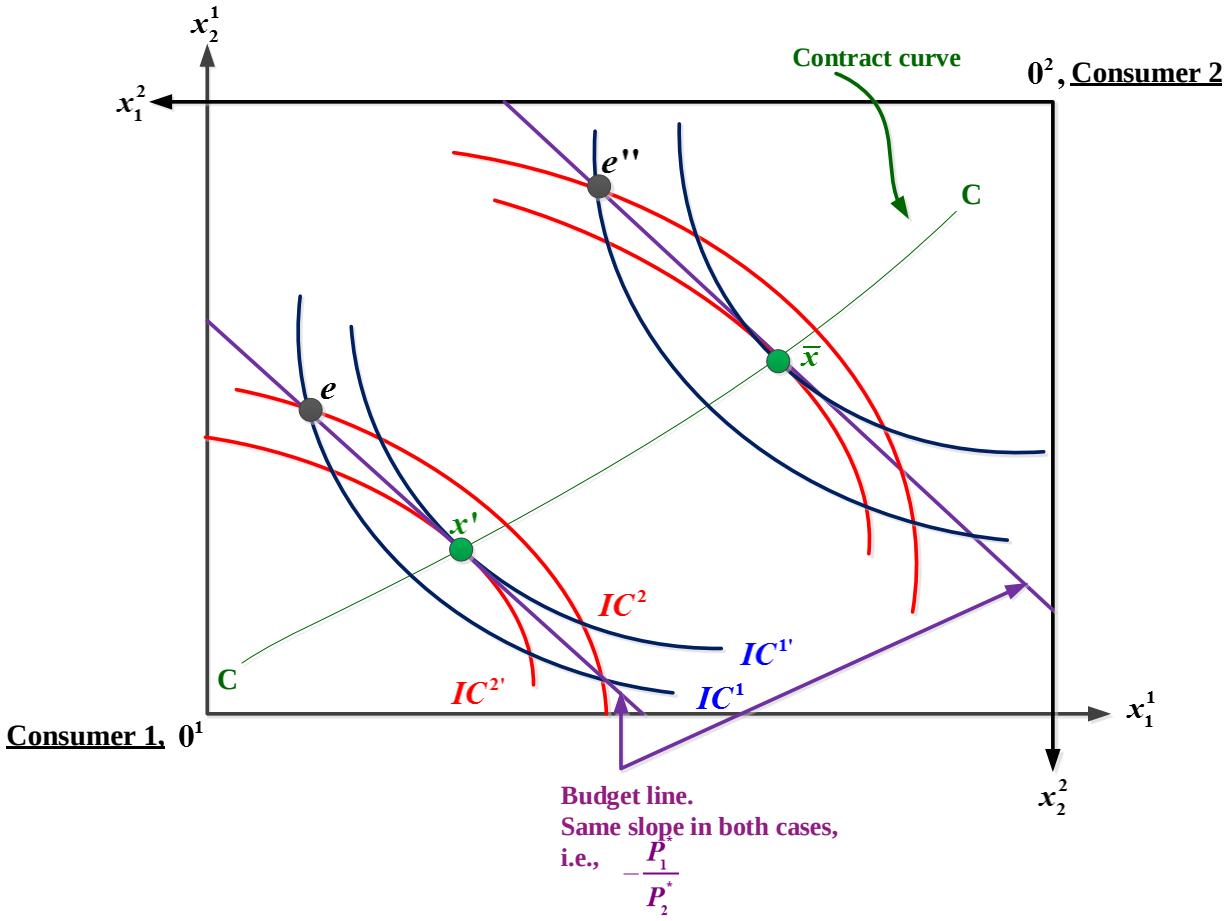
- ***Second Welfare Theorem***:
  - Suppose that $\bar{\mathbf{x}}$ is a Pareto-efficient allocation (i.e., it lies on the contract curve), and that endowments are redistributed so that the new endowment vector $\mathbf{e}^{*i}$ lies on the budget line, thus satisfying

    $$\mathbf{p}^* \cdot \mathbf{e}^{*i} = \mathbf{p}^* \cdot \bar{\mathbf{x}}^i \text{ for every consumer } i$$

  - Then, the Pareto-efficient allocation $\bar{\mathbf{x}}$ is a WEA given the new endowment vector $\mathbf{e}^*$.

General Equilibrium: Competitive Markets

- Second welfare theorem



General Equilibrium: Competitive Markets

- ***Example 6.7*** (WEA and Second welfare theorem):
  - Consider an economy with utility functions
    - $u^A = x_1^A x_2^A$ for consumer A, and
    - $u^B = \min\{x_1^B, x_2^B\}$ for consumer B.
  - The initial endowments are:
    - $\mathbf{e}^A = (3,1)$ and $\mathbf{e}^B = (1,3)$.
  - Good 2 is the numeraire, i.e., $p_2 = 1$.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  1) Pareto Efficient Allocations:

  – Consumer B's preferences are perfect complements. Hence, he consumes at the kink of his indifference curves, i.e.,

  $$x_1^B = x_2^B$$

  – Given feasibility constraints

  $$x_1^A + x_1^B = 4$$
  $$x_2^A + x_2^B = 4$$

  substitute x_2^B for x_1^B in the first constraint to get

  $$x_2^B = 4 - x_1^A$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  1) *Pareto Efficient Allocations*:

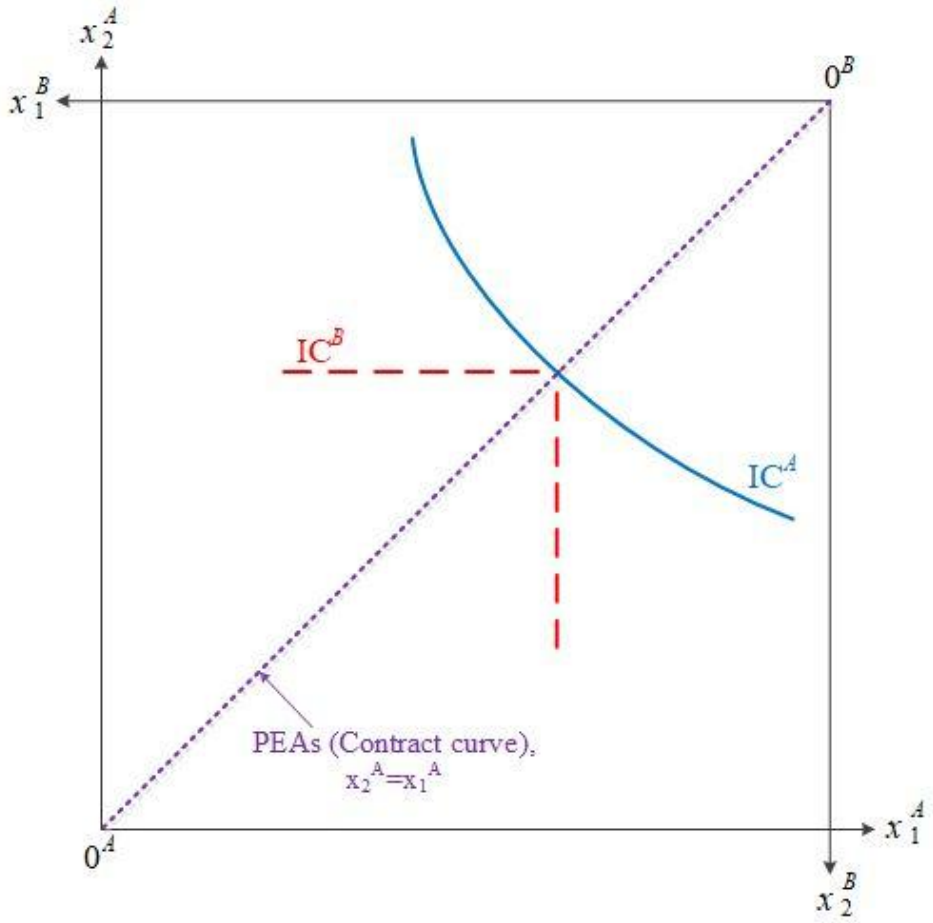
  – Substituting the above expression in the second constraint yields

$$x_2^A + \underbrace{(4 - x_1^A)}_{x_2^B} = 4 \iff x_2^A = x_1^A$$

  – This defines the contract curve, i.e., the set of Pareto efficient allocations.

General Equilibrium: Competitive Markets

- Contract curve



General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  2) WEA:

  – Consumer A's maximization problem is

$$\max_{x_1^A,\, x_2^A} \; x_1^A x_2^A$$

$$\text{s.t. } p_1 x_1^A + x_2^A \leq p_1 \cdot 3 + 1$$

  – FOCs:

$$x_2^A - \lambda p_1 = 0$$

$$x_1^A - \lambda = 0$$

$$p_1 x_1^A + x_2^A = 3p_1 + 1$$

  where λ is the lagrange multiplier.

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):

  2) WEA:

  – Combining the first two equations,

$$\lambda = \frac{x_2^A}{p_1} = x_1^A \quad \text{or} \quad p_1 = \frac{x_2^A}{x_1^A}$$

  – From Pareto efficiency, we know that $x_2^A = x_1^A$. Hence,

$$p_1 = \frac{x_2^A}{x_1^A} = 1$$

General Equilibrium: Competitive Markets

- ***Example 6.7*** (continued):
  - Substituting both the price and Pareto efficient allocation requirement into the budget constraint,

$$1 \cdot x_1^A + x_1^A = 1 \cdot 3 + 1$$

$$\text{or } x_1^{A*} = x_2^{A*} = 2$$

  - Using the feasibility constraint,

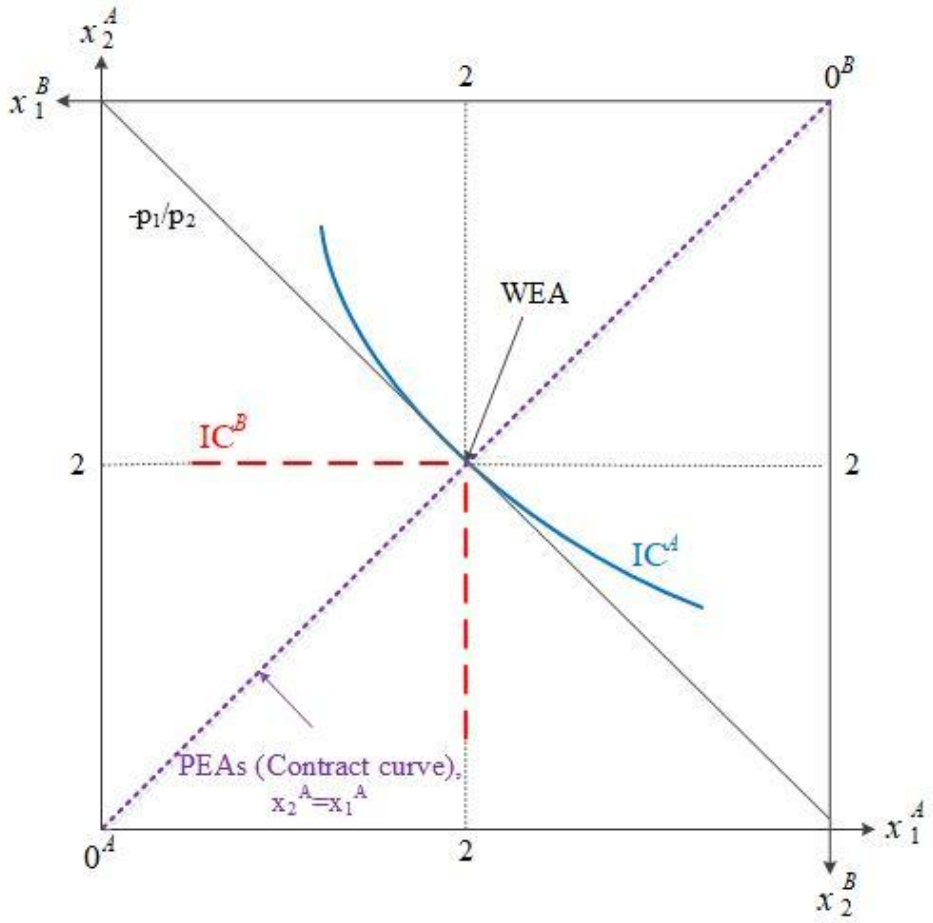
$$\underbrace{2}_{x_1^A} + x_1^B = 4 \text{ or } x_1^{B*} = x_2^{B*} = 2$$

  - Thus, the WEA is

$$\left(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2}\right) = (2,2; 2,2; 1)$$

General Equilibrium: Competitive Markets

- Second welfare theorem: WEA and PEAs

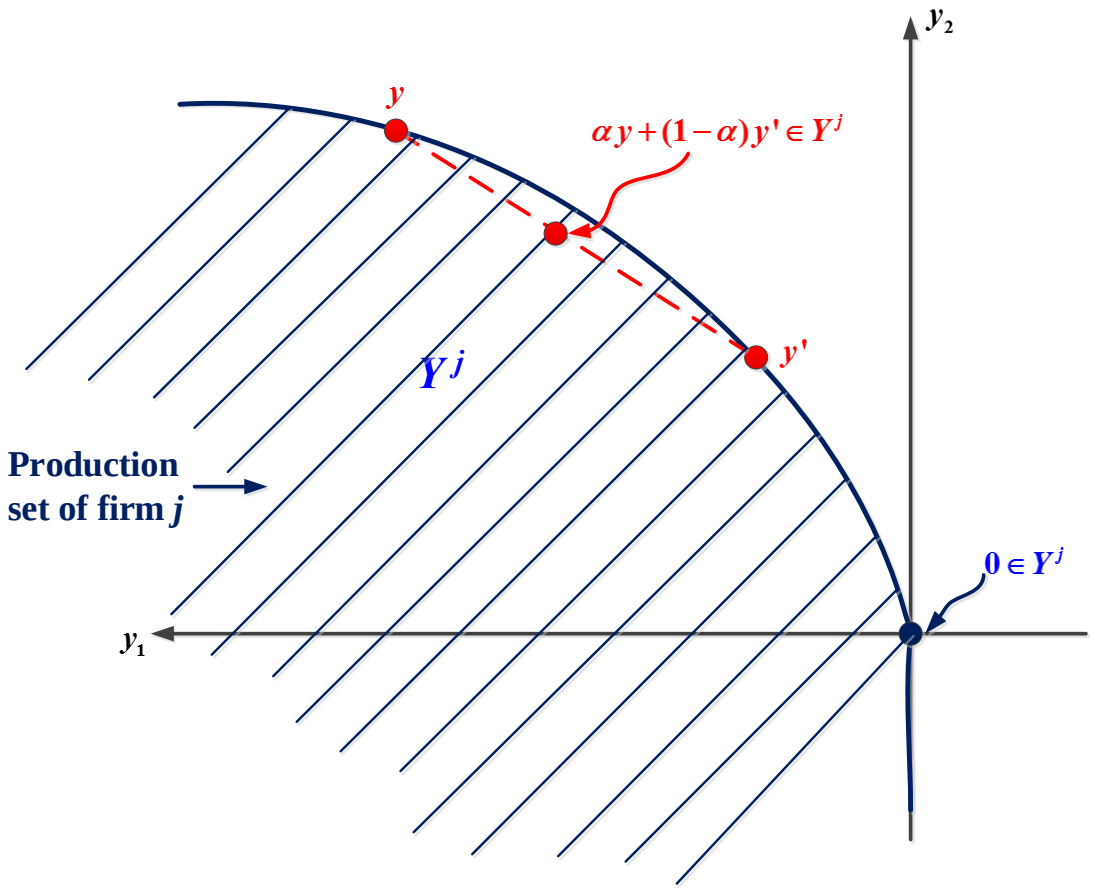


General Equilibrium: Production

- Let us now extend our previous results to setting where firms are also active.
- Assume J firms in the economy, each with production set Y^j, which satisfies:
  - Inaction is possible, i.e., $\mathbf{0} \in Y^j$.
  - Y^j is closed and bounded, so points on the production frontier are part of the production set and thus feasible.
  - Y^j is strictly convex, so linear combinations of two production plans also belong to the production set.

General Equilibrium: Production

- Production set Y^j for a representative firm



General Equilibrium: Production

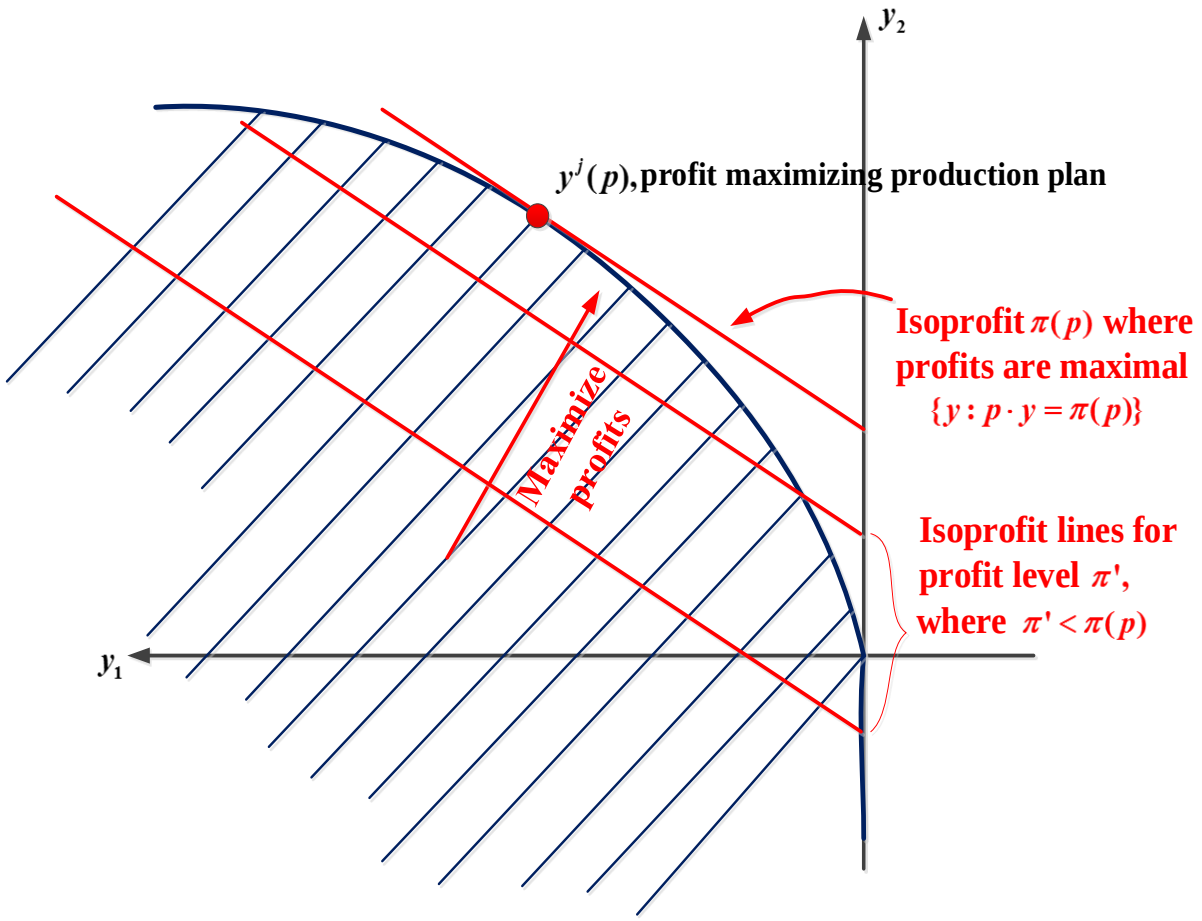
- Every firm j facing a fixed price vector $\mathbf{p} \gg 0$ independently and simultaneously solves

$$\max_{y_j \in Y^j} \quad \mathbf{p} \cdot y_j$$

- A profit-maximizing production plan $y_j(\mathbf{p})$ exists for every firm j, and it is unique.
- By the theorem of the maximum, both the argmax, $y_j(\mathbf{p})$, and the value function, $\pi_j(\mathbf{p}) \equiv \mathbf{p} \cdot y_j(\mathbf{p})$, are continuous in p.

General Equilibrium: Production

- $y^j(p)$ exists and is unique



General Equilibrium: Production

- ***Aggregate production set***:
  - The aggregate production set is the sum of all firms' production plans (either profit maximizing or not):

$$Y = \left\{ \mathbf{y} | \mathbf{y} = \sum_{j=1}^{J} y_j \text{ where } y_j \in Y^j \right\}$$

  - A joint-profit maximizing production plan $\mathbf{y}(\mathbf{p})$ is the sum of each firm's profit-maximizing plan, i.e.,

$$\mathbf{y}(\mathbf{p}) = y_1(\mathbf{p}) + y_2(\mathbf{p}) + \cdots + y_J(\mathbf{p})$$

General Equilibrium: Production

- In an economy with J firms, each of them earning $\pi_j(\mathbf{p})$ profits in equilibrium, how are profits distributed?
  - Assume that each individual i owns a share θ_{ij} of firm j's profits, where $0 \leq \theta_{ij} \leq 1$ and $\sum_{i=1}^{I} \theta_{ij} = 1$.
  - This allows for multiple sharing profiles:
    - $\theta_{ij} = 1$: individual i owns all shares of firm j
    - $\theta_{ij} = 1/I$: every individual's share of firm j coincides

General Equilibrium: Production

– Consumer's budget constraint becomes

$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + \sum_{j=1}^{J} \theta_{ij} \pi_j(\mathbf{p})$$

where $\sum_{j=1}^{J} \theta_{ij} \pi_j(\mathbf{p})$ is new relative to the standard budget constraint.

– Let us express the budget constraint as

$$\mathbf{p} \cdot \mathbf{x}^i \leq \underbrace{\mathbf{p} \cdot \mathbf{e}^i + \sum_{j=1}^{J} \theta_{ij} \pi_j(\mathbf{p})}_{m^i(\mathbf{p})}$$

$$\Longrightarrow \mathbf{p} \cdot \mathbf{x}^i \leq m^i(\mathbf{p})$$

where $m^i(\mathbf{p}) > 0$ (given assumptions on Y^j).

General Equilibrium: Production

- ***Equilibrium with production***:
  - We start defining excess demand functions and use such a definition to identify the set of equilibrium allocations.
  - *Excess demand*: The excess demand function for good k is

$$z_k(\mathbf{p}) \equiv \sum_{i=1}^{I} x_k^i(\mathbf{p}, m^i(\mathbf{p})) - \sum_{i=1}^{I} e_k^i - \sum_{j=1}^{J} y_k^j(\mathbf{p})$$

    where $\sum_{j=1}^{J} y_k^j(\mathbf{p})$ is a new term relative to the analysis of general equilibrium without production.

General Equilibrium: Production

- Hence, the aggregate excess demand vector is

$$\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_L(\mathbf{p}))$$

- *WEA with production*: If the price vector is strictly positive in all of its components, $\mathbf{p}^* \gg 0$, a pair of consumption and production bundles $(\mathbf{x}(\mathbf{p}^*), \mathbf{y}(\mathbf{p}^*))$ is a WEA if:
  1) Each consumer i solves his UMP, which becomes the i^{th} entry of $\mathbf{x}(\mathbf{p}^*)$, i.e., $\mathbf{x}^i(\mathbf{p}^*, m^i(\mathbf{p}^*))$;
  2) Each firm j solves its PMP, which becomes the j^{th} entry of $\mathbf{y}(\mathbf{p}^*)$, i.e., $\mathbf{y}^j(\mathbf{p}^*)$;
  3) Demand equals supply

     $\sum_{i=1}^{I} \mathbf{x}^i(\mathbf{p}^*, m^i(\mathbf{p}^*)) = \sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j(\mathbf{p}^*)$

     which is the market clearing condition.

General Equilibrium: Production

- From every consumer i simultaneously solving his UMP, the marginal rate of substitution between any goods 1 and 2 satisfies $MRS^i_{1,2} = MRS^j_{1,2} = p_1/p_2$.
- Similarly, from every firm j simultaneously solving its PMP, we obtain

  $p_1 F_{1K} = r$ and $p_1 F_{1L} = w$ for firm 1, and

  $p_2 F_{2K} = r$ and $p_2 F_{2L} = w$ for firm 2.
- Dividing these expressions yields

  $MRTS^1_{L,K} \equiv \frac{F_{1L}}{F_{1K}} = \frac{w}{r}$ for firm 1, and

  $MRTS^2_{L,K} \equiv \frac{F_{2L}}{F_{2K}} = \frac{w}{r}$ for firm 2.

General Equilibrium: Production

- Therefore, $MRTS^1_{L,K} = MRTS^2_{L,K}$. Similarly, we divide $p_1 F_{1K} = r$ by $p_2 F_{2K} = r$ to yield $\frac{p_1 F_{1K}}{p_2 F_{2K}} = \frac{r}{r} = 1$, or rearranging, $\frac{p_1}{p_2} = \frac{F_{2K}}{F_{1K}} = MRT^K_{1,2}$.
- A similar result emerges by dividing $p_1 F_{1L} = w$ by $p_2 F_{2L} = w$, which yields $\frac{p_1}{p_2} = \frac{F_{2L}}{F_{1L}} = MRT^L_{1,2}$.
- Combining the equilibrium conditions for every consumer i, $MRS^i_{1,2} = p_1/p_2$, and for every input $m = \{K, L\}$, $p_1/p_2 = MRT^m_{1,2}$, yields

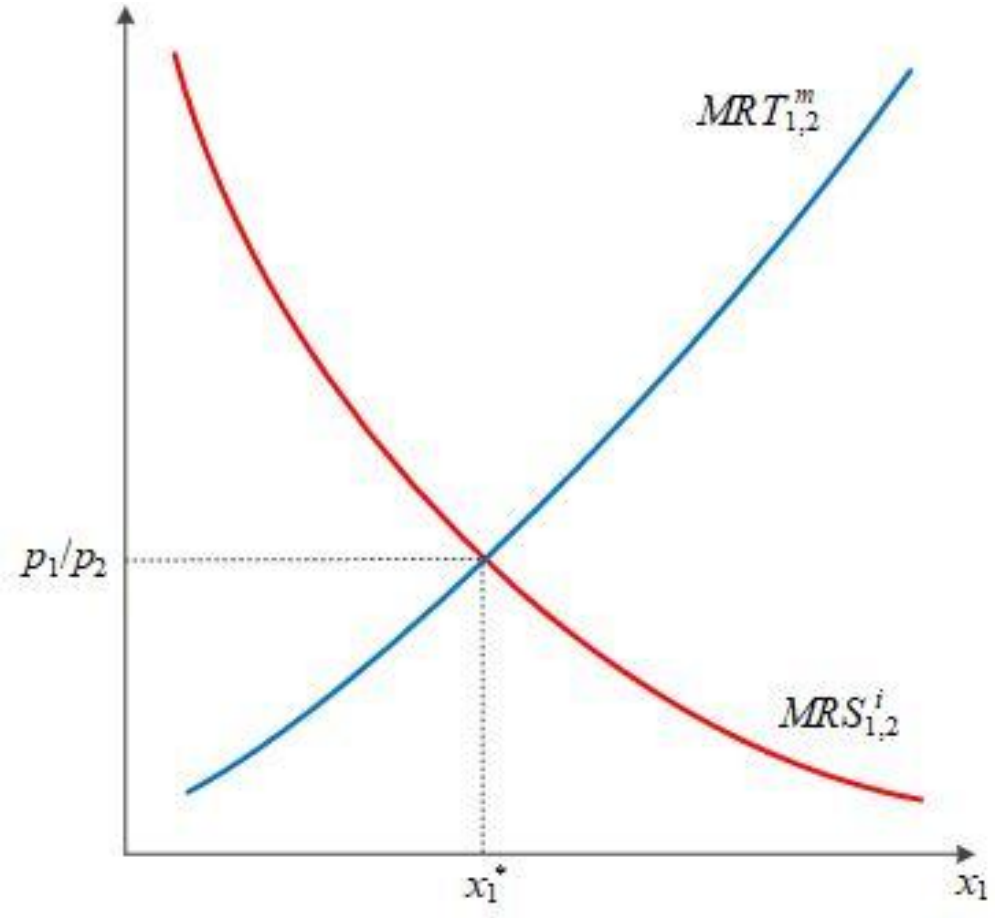
$$MRS^i_{1,2} = MRT^m_{1,2} = \frac{p_1}{p_2}$$

General Equilibrium: Production

- Inspecting $MRS^i_{1,2} = MRT^m_{1,2} = \frac{p_1}{p_2}$, we see that
  - $MRS^i_{1,2} = MU^i_1/MU^i_2$ is decreasing in good 1 (as x_1 increases, MU^i_1 decreases while MU^i_2 increases).
  - In contrast, $MRT^m_{1,2} = F_{2m}/F_{1m}$ is increasing in good 1. Intuitively, in order to increase x_1, we need to move units of input m from firm 2 to firm 1, thus increasing the marginal product of this input for firm 2 and lowering it for firm 1.
- Plotting $MRS^i_{1,2}$ and $MRT^m_{1,2}$ as a function of x_1, they cross each other at the equilibrium level of good 1, x^*_1, at a height of p_1/p_2.

General Equilibrium: Production

- Equilibrium with production



General Equilibrium: Production

– *Existence*: Assume that
  - consumers' utility functions are continuous, strictly increasing and strictly quasiconcave;
  - every firm j's production set Y^j is closed and bounded, strictly convex, and satisfies inaction being possible;
  - every consumer is initially endowed with positive units of at least one good, so the sum $\sum_{i=1}^{I} \mathbf{e}^i \gg 0$.

Then, there is a price vector $\mathbf{p}^* \gg 0$ such that a WEA exists, i.e., $z(\mathbf{p}^*) = 0$.

General Equilibrium: Production

- ***Example 6.8*** (Equilibrium with production):
  - Consider a two-consumer, two-good economy where consumer $i = \{A, B\}$ has utility function $u^i = x_1^i x_2^i$.
  - There are two firms in this economy, and each of them use capital (K) and labor (L) to produce one of the consumption goods each.
  - Firm 1 produces good 1 according to $y_1 = K_1^{0.75} L_1^{0.25}$.
  - Firm 2 produces good 2 according to $y_2 = K_2^{0.25} L_2^{0.75}$.
  - Consumer A is endowed with $(K^A, L^A) = (1,1)$, while consumer B is endowed with $(K^B, L^B) = (2,1)$.
  - Let us find a WEA in this economy with production.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - ***UMPs***: Consumer i's maximization problem is

$$\max_{x_1^i,\, x_2^i} \; x_1^i x_2^i$$

$$\text{s.t. } \; p_1 x_1^i + p_2 x_2^i = rK^i + wL^i$$

where r and w are prices for capital and labor, respectively.

  - FOC:

$$\frac{p_1}{p_2} = MRS_{1,2}^i \implies \frac{p_1}{p_2} = \frac{x_2^i}{x_1^i} \implies p_1 x_1^i = p_2 x_2^i$$

for $i = \{A, B\}$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Taking the above equation for consumers A and B, and adding them together yields

$$p_1(x_1^A + x_1^B) = p_2(x_2^A + x_2^B)$$

    where $x_1^A + x_1^B$ is the left side of the feasibility condition $x_1^A + x_1^B = y_1 = K_1^{0.75} L_1^{0.25}$.
  - Substituting both feasibility conditions into the above expression, and re-arranging, yields

$$\frac{p_1}{p_2} = \frac{K_2^{0.25} L_2^{0.75}}{K_1^{0.75} L_1^{0.25}}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - ***PMPs***: Firm 1's maximization problem is

$$\max_{K_1, L_1} \; p_1 K_1^{0.75} L_1^{0.25} - rK_1 - wL_1$$

  - FOCs:

$$r = 0.75 p_1 K_1^{-0.25} L_1^{0.25}$$
$$w = 0.25 p_1 K_1^{0.75} L_1^{-0.75}$$

  - Combining these conditions gives the tangency condition for profit maximization

$$\frac{r}{w} = MRTS_{K,L}^{1} \implies \frac{r}{w} = 3\frac{L_1}{K_1}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Likewise, firm 2's PMP gives the following FOCs:

$$r = 0.25 p_2 K_2^{-0.75} L_2^{0.75}$$
$$w = 0.75 p_2 K_2^{0.25} L_2^{-0.25}$$

  - Combining these conditions gives the tangency condition for profit maximization

$$\frac{r}{w} = MRTS_{K,L}^2 \implies \frac{r}{w} = \frac{1}{3}\frac{L_2}{K_2}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Combining both $MRTS$ yields,

$$3\frac{L_1}{K_1} = \frac{1}{3}\frac{L_2}{K_2} \Rightarrow \frac{K_1}{L_1} = 9\frac{K_2}{L_2}$$

    - *Intuition*: firm 1 is more capital intensive than firm 2, i.e., its capital-to-labor ratio is higher.
  - Setting both firms' price of capital, r, equal to each other yields

$$0.75p_1K_1^{-0.25}L_1^{0.25} = 0.25p_2K_2^{-0.75}L_2^{0.75}$$

$$\Rightarrow \frac{p_1}{p_2} = \frac{1}{3}\left(\frac{K_1}{L_1}\right)^{0.25}\left(\frac{K_2}{L_2}\right)^{-0.75}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Setting both firms' price of labor, w, equal to each other yields

$$0.25p_1K_1^{0.75}L_1^{-0.75} = 0.75p_2K_2^{0.25}L_2^{-0.25}$$

$$\Rightarrow \frac{p_1}{p_2} = 3\left(\frac{K_1}{L_1}\right)^{-0.75}\left(\frac{K_2}{L_2}\right)^{0.25}$$

  - Setting price ratio from consumers' UMP equal to the first price ratio from firms' PMP yields

$$\frac{K_2^{0.25}L_2^{0.75}}{K_1^{0.75}L_1^{0.25}} = \frac{1}{3}\left(\frac{K_1}{L_1}\right)^{0.25}\left(\frac{K_2}{L_2}\right)^{-0.75} \Rightarrow K_1 = 3K_2$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - By the feasibility conditions, we know that $K_1 + K_2 = K^A + K^B = 3$ or $K_2 = 3 - K_1$.
  - Substituting the above expression into $K_1 = 3K_2$, we find the profit-maximizing demands for capital use by firms 1 and 2:

$$K_1 = 3(3 - K_1) \implies K_1^* = \frac{9}{4}$$

$$K_2^* = \frac{1}{3}K_1^* = \frac{3}{4}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Setting price ratio from consumers' UMP equal to the second price ratio from firms' PMP yields

$$\frac{K_2^{0.25}L_2^{0.75}}{K_1^{0.75}L_1^{0.25}} = 3\left(\frac{K_1}{L_1}\right)^{-0.75}\left(\frac{K_2}{L_2}\right)^{0.25} \Longrightarrow L_1 = \frac{1}{3}L_2$$

  - By the feasibility conditions, we know that $L_1 + L_2 = L^A + L^B = 2$ or $L_2 = 2 - L_1$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Substituting the above expression into $L_1 = \frac{1}{3}L_2$, we find the profit-maximizing demands for labor use by firms 1 and 2:

$$L_1 = \frac{1}{3}(2 - L_1) \implies L_1^* = \frac{1}{2}$$

$$L_2^* = 3L_1^* = \frac{3}{2}$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Substituting the capital and labor demands for firm 1 and 2 into the price ratio from consumers' UMP yields

$$\frac{p_1}{p_2} = \frac{\left(\frac{3}{4}\right)^{0.25}\left(\frac{3}{2}\right)^{0.75}}{\left(\frac{9}{4}\right)^{0.75}\left(\frac{1}{2}\right)^{0.25}} = \sqrt{\frac{2}{3}} \approx 0.82$$

where normalizing the price of good 2, i.e., $p_2 = 1$, gives $p_1 = 0.82$.

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Furthermore, substituting our calculated values into the price of capital and labor yields

$$r^* = 0.75(0.82)\left(\frac{9}{4}\right)^{-0.25}\left(\frac{1}{2}\right)^{0.25} = 0.42$$

$$w^* = 0.25(0.82)\left(\frac{9}{4}\right)^{0.75}\left(\frac{1}{2}\right)^{-0.75} = 0.63$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Using consumer A's tangency condition, we know

$$x_2^A = \frac{p_1}{p_2} x_1^A \implies x_2^A = 0.82 x_1^A$$

  - Substituting this value into consumer A's budget constraint yields

$$p_1 x_1^A + p_2 (0.82 x_1^A) = rK^A + wL^A$$

  - Plugging in our calculated values and solving for x_1^A yields

$$x_1^{A,*} = \frac{0.42 + 0.63}{0.82 + 0.82} = 0.64$$

$$x_2^{A,*} = 0.82 x_1^{A,*} = 0.53$$

General Equilibrium: Production

- ***Example 6.8*** (continued):
  - Performing the same process with the tangency condition of consumer B yields

$$x_1^{B,*} = \frac{(2 \times 0.42) + 0.63}{0.82 + 0.82} = 0.90$$

$$x_2^{B,*} = 0.82 x_1^{B,*} = 0.74$$

  - Thus, our WEA is

$$\left(x_1^{A,*}, x_2^{A,*}; x_1^{B,*}, x_2^{B,*}; \frac{p_1}{p_2}; L_1^*, L_2^*; K_1^*, K_2^*\right) =$$

$$\left(0.64, 0.53; 0.90, 0.74; 0.82; \frac{1}{2}, \frac{3}{2}, \frac{9}{4}, \frac{3}{4}\right)$$

General Equilibrium: Production

- ***Equilibrium with production – Welfare***:
  - We extend the First and Second Welfare Theorems to economies with production, connecting WEA and Pareto efficient allocations.
  - *Pareto efficiency*: A feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto efficient if there is no other feasible allocation $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ such that

$$u^i(\bar{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$$

for every consumer $i \in I$, with $u^i(\bar{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$ for at least one consumer.

General Equilibrium: Production

– In an economy with two goods, two consumers, two firms and two inputs (labor and capital), the set of Pareto efficient allocations solves

$$\max_{x_1^i, x_2^i, x_1^j, x_2^j, L_1, K_1, L_2, K_2 \geq 0} \quad u^i(x_1^i, x_2^i)$$

$$\text{s. t.} \quad u^j\left(x_1^j, x_2^j\right) \geq \bar{u}^j$$

$$\left.\begin{aligned} x_1^i + x_1^j &\leq F_1(L_1, K_1) \\ x_2^i + x_2^j &\leq F_2(L_2, K_2) \end{aligned}\right\} \text{tech. feasibility}$$

$$\left.\begin{aligned} L_1 + L_2 &\leq \bar{L} \\ K_1 + K_2 &\leq \bar{K} \end{aligned}\right\} \text{input feasibility}$$

General Equilibrium: Production

– The Lagrangian is

$$\mathcal{L}$$
$$= u^i\left(x_1^i, x_2^i\right) + \lambda\left[u^j\left(x_1^j, x_2^j\right) - \bar{u}^j\right]$$
$$+ \mu_1\left[F_1(L_1, K_1) - x_1^i - x_1^j\right] + \mu_2\left[F_2(L_2, K_2) - x_2^i - x_2^j\right]$$
$$+ \delta_L[\bar{L} - L_1 - L_2] + \delta_K[\bar{K} - K_1 - K_2]$$

– In the case of interior solutions, the set of FOCs yield a condition for efficiency in consumption similar to barter economics:

$$MRS_{1,2}^i = MRS_{1,2}^j$$

General Equilibrium: Production

- FOCs wrt L_j and K_j, where $j = \{1,2\}$, yield a condition for efficiency that we encountered in production theory

$$\frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

- That is, the $MRTS_{L,K}$ between labor and capital must coincide across firms.
- Otherwise, welfare could be increased by assigning more labor to the firm with the highest $MRTS_{L,K}$.

General Equilibrium: Production

– Combining the above two conditions for efficiency in consumption and production, we obtain

$$\frac{\frac{\partial U^i}{\partial x_1^i}}{\frac{\partial U^i}{\partial x_2^i}} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_1}{\partial L}}$$

– That is, $MRS_{1,2}^i$ must coincide with the rate at which units of good 2 can be transformed into units of good 1, i.e., the marginal rate of transformation $MRT_{1,2}$.

General Equilibrium: Production

- If we move labor from firm 1 to firm 2, the marginal product of good 2 decreases by $\frac{\partial F_2}{\partial L}$ while that of good 1 increases by $\frac{\partial F_1}{\partial L}$. Hence, in order to increase the total output of good 2 by one unit, we need $\frac{\partial F_2}{\partial L} / \frac{\partial F_1}{\partial L}$ units of good 1.
- *Intuition*: for an allocation to be efficient we need that the rate at which consumers are willing to substitute goods 1 and 2 coincides with the rate at which good 1 can be transformed into good 2.

General Equilibrium: Production

- ***First Welfare Theorem with production***: if the utility function of every individual i, u^i, is strictly increasing, then every WEA is Pareto efficient.
- *Proof* (by contradiction):
  - Suppose that $(\mathbf{x}, \mathbf{y})$ is a WEA at prices $\mathbf{p}^*$, but is *not* Pareto efficient.
  - Since $(\mathbf{x}, \mathbf{y})$ is a WEA, then it must be feasible

$$\sum_{i=1}^{I} \mathbf{x}^i = \sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j$$

General Equilibrium: Production

- Because $(\mathbf{x}, \mathbf{y})$ is *not* Pareto efficient, there exists some other feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that

$$u^i(\hat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$$

  for every consumer $i \in I$, with $u^i(\hat{\mathbf{x}}^i) > u^i(\mathbf{x}^i)$ for at least one consumer.

  - That is, the alternative allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ makes at least one consumer strictly better off than WEA .

- But this implies that bundle $\hat{\mathbf{x}}^i$ is more costly than $\mathbf{x}^i$,

$$\mathbf{p}^* \cdot \hat{\mathbf{x}}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i$$

  for every individual i (with at least one strictly inequality).

General Equilibrium: Production

– Summing over all consumers yields

$$\mathbf{p}^* \cdot \sum_{i=1}^{I} \hat{\mathbf{x}}^i > \mathbf{p}^* \cdot \sum_{i=1}^{I} \mathbf{x}^i$$

which can be re-written as

$$\mathbf{p}^* \cdot \left(\sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \hat{\mathbf{y}}^j \right) > \mathbf{p}^* \cdot \left(\sum_{i=1}^{I} \mathbf{e}^i + \sum_{j=1}^{J} \mathbf{y}^j \right)$$

or re-arranging

$$\mathbf{p}^* \cdot \sum_{j=1}^{J} \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \sum_{j=1}^{J} \mathbf{y}^j$$

– However, this result implies that $\mathbf{p}^* \cdot \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \mathbf{y}^j$ for some firm j.

General Equilibrium: Production

- This indicates that production plan $\mathbf{y}^j$ was not profit-maximizing and, as a consequence, it cannot be part of a WEA.
- We therefore reached a contradiction.
- This implies that the original statement was true: if an allocation $(\mathbf{x}, \mathbf{y})$ is a WEA, it must also be Pareto efficient.

General Equilibrium: Production

- ***Example 6.9*** (WEA and PEA with production):
  - Consider the setting described in example 6.8.
  - The set of Pareto efficient allocations must satisfy

$$MRS_{1,2}^{A} = MRS_{1,2}^{B} \text{ and } MRTS_{K,L}^{1} = MRTS_{K,L}^{2}$$

  - We can show that

$$MRS_{1,2}^{A} = \frac{x_2^A}{x_1^A} = \frac{0.53}{0.64} = 0.82$$

$$MRS_{1,2}^{B} = \frac{x_2^B}{x_1^B} = \frac{0.74}{0.90} = 0.82$$

which implies that $MRS_{1,2}^{A} = MRS_{1,2}^{B}$.

General Equilibrium: Production

- ***Example 6.9*** (continued):
  - We can also show that

$$MRTS_{K,L}^{1} = 3\frac{L_1}{K_1} = 3\left(\frac{1}{2}/\frac{9}{4}\right) = \frac{2}{3}$$

$$MRTS_{K,L}^{2} = \frac{1}{3}\frac{L_2}{K_2} = \frac{1}{3}\left(\frac{3}{2}/\frac{3}{4}\right) = \frac{2}{3}$$

    which implies that $MRTS_{K,L}^{1} = MRTS_{K,L}^{2}$.

  - Since both of these conditions hold, our WEA from example 6.8 is Pareto efficient.

General Equilibrium: Production

- ***Second Welfare Theorem with production***:
  - Consider the assumptions on consumers and producers described above.
  - Then, for every Pareto efficient allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ we can find:
    a) a profile of income transfers $(T_1, T_2, \ldots, T_I)$ redistributing income among consumers, i.e., satisfying $\sum_{i=1}^{I} T_i = 0$;
    b) a price vector $\overline{\mathbf{p}}$,

    such that:

General Equilibrium: Production

1) Bundle $\hat{\mathbf{x}}^i$ solves the UMP

$$\max_{\mathbf{x}^i} \quad u^i(\mathbf{x}^i)$$

$$\text{s.t.} \quad \overline{\mathbf{p}} \cdot \mathbf{x}^i \leq m^i(\overline{\mathbf{p}}) + T_i \text{ for every } i \in I$$

where individual i's original income $m^i(\overline{\mathbf{p}})$ is increased (decreased) if the transfer T_i is positive (negative).

2) Production plan $\hat{\mathbf{y}}^j$ solves the PMP

$$\max_{y^j} \quad \overline{\mathbf{p}} \cdot \mathbf{y}^j$$

$$\text{s.t.} \quad \mathbf{y}^j \in Y^j \text{ for every } j \in J$$

General Equilibrium: Production

- ***Example 6.10*** (Second Welfare Theorem with production):
  - Consider an alternative allocation in the set of Pareto efficient allocations identified in example 6.9.
  - Such as $\left(\hat{x}_1^A, \hat{x}_2^A; \hat{x}_1^B, \hat{x}_2^B\right) = (0.75,0.61; 0.79,0.65)$.
  - Consumer A's budget constraint becomes
    $$p_1\hat{x}_1^A + p_2\hat{x}_2^A = rK^A + wL^A + T_A$$
  - Recall that
    $$(p_1, p_2; K^A, L^A; r, w) = (0.82,1; 1,1; 0.42,0.63)$$
    remains unchanged.

General Equilibrium: Production

– Substituting these values into consumer A's budget constraint

$$0.82\hat{x}_1^A + \hat{x}_2^A = 1.05 + T_A$$

– Recall that

$$\frac{p_1}{p_2} = \frac{\hat{x}_2^A}{\hat{x}_1^A} \implies \hat{x}_2^A = 0.82\hat{x}_1^A$$

– Substituting

$$2(0.82)\underbrace{(0.75)}_{\hat{x}_1^A} = 1.05 + T_A \implies T_A = 0.17$$

General Equilibrium: Production

– Likewise for consumer B, his budget constraint becomes

$$p_1\hat{x}_1^B + p_2\hat{x}_2^B = rK^B + wL^B + T_B$$

– Substituting the unchanged values $(p_1, p_2; K^B, L^B; r, w) = (0.82,1; 2,1; 0.42,0.63)$,

$$0.82\hat{x}_1^B + \hat{x}_2^B = 1.47 + T_B$$

– Recall that

$$\frac{p_1}{p_2} = \frac{\hat{x}_2^B}{\hat{x}_1^B} \Longrightarrow \hat{x}_2^B = 0.82\hat{x}_1^B$$

General Equilibrium: Production

- Substituting

$$2(0.82)\underbrace{(0.79)}_{\hat{x}_1^B} = 1.47 + T_B \implies T_B = -0.17$$

- Clearly, $T_A + T_B = 0$
- Thus these transfers will allow for the new allocation to be a WEA.

Comparative Statics

Comparative Statics

- We analyze how equilibrium outcomes are affected by an increase in:
  - the price of one good
  - the endowment of one input
- Consider a setting with two goods, each being produced by two factors 1 and 2 under constant returns to scale (CRS).
- A necessary condition for input prices (w_1^*, w_2^*) to be in equilibrium is

$$c_1(w_1, w_2) = p_1 \text{ and } c_2(w_1, w_2) = p_2$$

  - That is, firms produce until their marginal costs equal the price of the good.

Comparative Statics

- Let $z_{1j}(w)$ denote firm j's demand for factor 1, and $z_{2j}(w)$ be its demand for factor 2.
  - This is equivalent to the factor demand correspondences $z(w, q)$ in production theory.
- The production of good 1 is relatively more intense in factor 1 than is the production of good 2 if

$$\frac{z_{11}(w)}{z_{21}(w)} > \frac{z_{12}(w)}{z_{22}(w)}$$

where $\frac{z_{1j}(w)}{z_{2j}(w)}$ represents firm j's demand for input 1 relative to that of input 2.

Comparative Statics: Price Change

1) ***Changes in the price of one good,*** p_j **(Stolper-Samuelson theorem)**:
   - Consider an economy with two consumers and two firms satisfying the above factor intensity assumption.
   - If the price of good j, p_j, increases, then:
     a) the equilibrium price of the factor more intensively used in the production of good j increases; while
     b) the equilibrium price of the other factor decreases.

Comparative Statics: Price Change

- *Proof*:
  - Let us first take the equilibrium conditions

$$c_1(w_1, w_2) = p_1 \text{ and } c_2(w_1, w_2) = p_2$$

  - Differentiating them yields

$$\frac{\partial c_1(w_1,w_2)}{\partial w_1} dw_1 + \frac{\partial c_1(w_1,w_2)}{\partial w_2} dw_2 = dp_1$$

$$\frac{\partial c_2(w_1,w_2)}{\partial w_1} dw_1 + \frac{\partial c_2(w_1,w_2)}{\partial w_2} dw_2 = dp_2$$

  - Applying Shephard's lemma, we obtain

$$z_{11}(w)dw_1 + z_{21}(w)dw_2 = dp_1$$

$$z_{12}(w)dw_1 + z_{22}(w)dw_2 = dp_2$$

Comparative Statics: Price Change

– If only price p_1 varies, then $dp_2 = 0$.

– Hence, we can rewrite the second expression as

$$z_{12}(w)dw_1 + z_{22}(w)dw_2 = 0$$

$$\Longrightarrow \quad dw_2 = -\frac{z_{12}}{z_{22}}dw_1$$

– Solving for $\frac{dw_1}{dp_1}$ in the first expression yields

$$\frac{dw_1}{dp_1} = \frac{z_{22}}{z_{11}z_{22} - z_{12}z_{21}}$$

– Solving, instead, for $\frac{dw_2}{dp_1}$ yields

$$\frac{dw_2}{dp_1} = -\frac{z_{12}}{z_{11}z_{22} - z_{12}z_{21}}$$

Comparative Statics: Price Change

- From the factor intensity condition, $\frac{z_{11}(w)}{z_{21}(w)} > \frac{z_{12}(w)}{z_{22}(w)}$, we know that $z_{11}z_{22} - z_{12}z_{21} > 0$.
- Hence, the denominator in both $\frac{dw_1}{dp_1}$ and $\frac{dw_2}{dp_1}$ is positive.
- The numerator in both $\frac{dw_1}{dp_1}$ and $\frac{dw_2}{dp_1}$ is also positive (they are just factor demands).
- Thus, $\frac{dw_1}{dp_1} > 0$ and $\frac{dw_2}{dp_1} < 0$.

Comparative Statics: Price Change

- ***Example 6.11***:
  - Let us solve for the input demands in Example 6.8:

$$r = 0.75p_1K_1^{-0.25}L_1^{0.25} \Longrightarrow z_{11} = K_1 = \left(\frac{3p_1}{4r}\right)^4 L_1$$

$$w = 0.25p_1K_1^{0.75}L_1^{-0.75} \Longrightarrow z_{21} = L_1 = \left(\frac{p_1}{4w}\right)^{\frac{4}{3}} K_1$$

$$r = 0.25p_2K_2^{-0.75}L_2^{0.75} \Longrightarrow z_{12} = K_2 = \left(\frac{p_2}{4r}\right)^{\frac{4}{3}} L_2$$

$$w = 0.75p_2K_2^{0.25}L_2^{-0.25} \Longrightarrow z_{22} = L_2 = \left(\frac{3p_2}{4w}\right)^4 K_2$$

Comparative Statics: Price Change

- ***Example 6.11*** (continued):
  - Since firm 1 is more capital intensive than firm 2, then $z_{11}z_{22} - z_{12}z_{21} > 0$ must hold, i.e.,

$$\left(\frac{3p_1}{4r}\right)^4 L_1 \left(\frac{3p_2}{4w}\right)^4 K_2 - \left(\frac{p_2}{4r}\right)^{\frac{4}{3}} L_2 \left(\frac{p_1}{4w}\right)^{\frac{4}{3}} K_1 > 0$$

  - From example 6.8, $\frac{K_1}{L_1} = 9\frac{K_2}{L_2} \Longrightarrow K_1L_2 = 9K_2L_1.$
  - Substituting this value into the above expression,

$$\frac{3^6}{2^{\frac{32}{3}}}\left(\frac{p_1p_2}{rw}\right)^{\frac{8}{3}} - 1 > 0 \Longrightarrow \frac{p_1p_2}{rw} > \frac{2^4}{3^{\frac{9}{4}}} \approx 1.35$$

Comparative Statics: Price Change

- ***Example 6.11*** (continued):
  - In our solution, $\frac{p_1 p_2}{rw} = 3.08$, hence this condition is satisfied.
  - Next, observe that both z_{12} and z_{22} are trivially positive.
  - Applying the ***Stolper-Samuelson theorem*** yields

$$\frac{dw_1}{dp_1} = \frac{z_{22}}{z_{11}z_{22} - z_{12}z_{21}} > 0$$

$$\frac{dw_2}{dp_1} = -\frac{z_{12}}{z_{11}z_{22} - z_{12}z_{21}} < 0$$

Comparative Statics: Endowment Change

2) ***Changes in endowments*** **(Rybczynski theorem)**:
   - Consider an economy with two consumers and two firms satisfying the above factor intensity assumption.
   - If the endowment of a factor increases, then
     a) production of the good that uses this factor more intensively increases; whereas
     b) the production of the other good decreases.

Comparative Statics: Endowment Change

- *Proof*:
  - Consider an economy with two factors, labor and capital, and two goods, 1 and 2.
  - Let $z_{Lj}(w)$ denote firm j's factor demand for labor (when producing one unit of output)
  - Similarly, let $z_{Kj}(w)$ denote firm j's factor demand for capital.
  - Then, factor feasibility requires

$$L = z_{L1}(w) \cdot y_1 + z_{L2}(w) \cdot y_2$$
$$K = z_{K1}(w) \cdot y_1 + z_{K2}(w) \cdot y_2$$

Comparative Statics: Endowment Change

- Differentiating the first condition

$$dL = z_{L1} \cdot \frac{\partial y_1}{\partial L} + z_{L2} \cdot \frac{\partial y_2}{\partial L}$$

- Dividing both sides by L yields

$$\frac{dL}{L} = \frac{z_{L1}}{L} \cdot \frac{\partial y_1}{\partial L} + \frac{z_{L2}}{L} \cdot \frac{\partial y_2}{\partial L}$$

- Multiplying the first term by $\frac{y_1}{y_1}$ and the second term by $\frac{y_2}{y_2}$, we obtain

$$\frac{dL}{L} = \frac{z_{L1} \cdot y_1}{L} \cdot \frac{\frac{\partial y_1}{\partial L}}{y_1} + \frac{z_{L2} \cdot y_2}{L} \cdot \frac{\frac{\partial y_2}{\partial L}}{y_2}$$

Comparative Statics: Endowment Change

– We can express:

a) $\frac{z_{Li}(w)\cdot y_i}{L} \equiv \gamma_{Li}$, i.e., the share of labor used by firm i;

b) $\frac{\frac{\partial y_i}{\partial L}}{y_i} \equiv \%\Delta y_i$, i.e., the percentage increase in the production of firm i brought by the unit increase in the endowment of labor;

c) $\frac{dL}{L} \equiv \%\Delta L$, i.e., the percentage increase in the endowment of labor in the economy.

Comparative Statics: Endowment Change

- Hence, the above expression becomes

$$\%\Delta L = \gamma_{L1} \cdot (\%\Delta y_1) + \gamma_{L2} \cdot (\%\Delta y_2)$$

- A similar expression can be obtained for the endowment of capital:

$$\%\Delta K = \gamma_{K1} \cdot (\%\Delta y_1) + \gamma_{K2} \cdot (\%\Delta y_2)$$

- Note that $\gamma_{L1}, \gamma_{L2} \in (0,1)$
  - Hence, $\%\Delta L$ is a linear combination of $\%\Delta y_1$ and $\%\Delta y_2$.
- Similar argument applied to $\%\Delta K$, where $\gamma_{K1}, \gamma_{K2} \in (0,1)$.

Comparative Statics: Endowment Change

– Capital is assumed to be more intensively used in firm 1, i.e.,

$$\frac{K_1}{L_1} > \frac{K_2}{L_2} \text{ or}$$

$\gamma_{K1} > \gamma_{L1}$ for firm 1 and $\gamma_{K2} < \gamma_{L2}$ for firm 2

– Hence, if capital becomes relatively more abundant than labor, i.e., $\%\Delta K > \%\Delta L$, it must be that $\%\Delta y_1 > \%\Delta y_2$.

Comparative Statics: Endowment Change

– That is

$$\begin{array}{ccccccccc} \%\Delta L & = & \gamma_{L1} & \cdot & (\%\Delta y_1) & + & \gamma_{L2} & \cdot & (\%\Delta y_2) \\ \wedge & & \wedge & & \| & & \vee & & \| \\ \%\Delta K & = & \gamma_{K1} & \cdot & (\%\Delta y_1) & + & \gamma_{K2} & \cdot & (\%\Delta y_2) \end{array}$$

– *Intuition*: the change in the input endowment produces a more-than-proportional increase in the good whose production is intensive in the use of that input.

Comparative Statics: Endowment Change

- ***Example 6.12*** (Rybczynski Theorem):
  - Consider the production decisions of the two firms in Example 6.8, where we found that $K_1 = 3K_2$ and $K_1 + K_2 = \bar{K} = 3$.
  - Assume that total endowment of capital increases to $\bar{K} = 5$, i.e., $K_2 = 5 - K_1$.
  - The profit maximizing demands for capital are

$$K_1 = 3(5 - K_1) \implies K_1^* = \frac{15}{4}$$

$$K_2^* = \frac{1}{3}K_1^* = \frac{5}{4}$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Similarly, for labor we found that $L_1 = \frac{1}{3}L_2$ and $L_1 + L_2 = \bar{L} = 2$.
  - We do not alter the aggregate endowment of labor, $\bar{L} = 2$.
  - Hence, capital use by firm 1 increases from $K_1^* = \frac{9}{4}$ to $\frac{15}{4}$.
  - Firm 1 uses capital more intensively than firm 2 does, i.e., $\frac{K_1}{L_1} > \frac{K_2}{L_2}$, since $\frac{\frac{15}{4}}{\frac{1}{2}} = 7\frac{1}{2} > \frac{5}{6} = \frac{\frac{5}{4}}{\frac{3}{2}}$.

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - From $MRTS_{K,L}^{1} = \frac{r}{w} = 3\frac{L_1}{K_1}$, we have $L_1 = \frac{rK_1}{3w}$ and $K_1 = \frac{3wL_1}{r}$
  - Substituting into $y_1 = {K_1}^{0.75}{L_1}^{0.25}$, we obtain

$$y_1 = {K_1}^{0.75}\left(\frac{rK_1}{3w}\right)^{0.25} = K_1\left(\frac{r}{3w}\right)^{0.25}$$

$$y_1 = \left(\frac{3wL_1}{r}\right)^{0.75}{L_1}^{0.25} = L_1\left(\frac{3w}{r}\right)^{0.75}$$

  - Setting $y_1 = 1$, firm 1's factor demands in producing one unit of good 1 become

$$z_{K1} = \left(\frac{r}{3w}\right)^{-0.25} \text{ and } z_{L1} = \left(\frac{r}{3w}\right)^{0.75}$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - From $MRTS_{K,L}^{2} = \frac{r}{w} = \frac{1}{3}\frac{L_2}{K_2}$, we have $L_2 = \frac{3rK_2}{w}$ and $K_2 = \frac{wL_2}{3r}$
  - Substituting into $y_2 = {K_2}^{0.25}{L_2}^{0.75}$, we obtain

$$y_2 = {K_2}^{0.25}\left(\frac{3rK_2}{w}\right)^{0.75} = K_2\left(\frac{3r}{w}\right)^{0.75}$$

$$y_2 = \left(\frac{wL_2}{3r}\right)^{0.25}{L_2}^{0.75} = L_2\left(\frac{w}{3r}\right)^{0.25}$$

  - Setting $y_2 = 1$, firm 2's factor demands in producing one unit of good 2 become

$$z_{K2} = \left(\frac{w}{3r}\right)^{0.75} \text{ and } z_{L2} = \left(\frac{w}{3r}\right)^{-0.25}$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Since $\frac{r}{w} = 3\frac{L_1}{K_1} = 3\left(\frac{1/2}{15/4}\right) = \frac{2}{5}$, factor demands are

$$z_{K1} = \left(\frac{2}{15}\right)^{-0.25} = 1.65 \text{ and } z_{L1} = \left(\frac{2}{15}\right)^{0.75} = 0.22$$

$$z_{K2} = \left(\frac{5}{6}\right)^{0.75} = 0.87 \quad \text{and} \quad z_{L2} = \left(\frac{5}{6}\right)^{-0.25} = 1.05$$

  - Using the values from example 6.8, we can assign following values:

$$(\gamma_{K1}, \gamma_{L1}, \gamma_{K2}, \gamma_{L2}) = (0.75, 0.25, 0.25, 0.75)$$

Comparative Statics: Endowment Change

- ***Example 6.12*** (continued):
  - Our two equations then become

$$0 = 0.25 \cdot (\%\Delta y_1) + 0.75 \cdot (\%\Delta y_2)$$

$$0.67 = 0.75 \cdot (\%\Delta y_1) + 0.25 \cdot (\%\Delta y_2)$$

  - Solving the above equations simultaneously yields

$$\%\Delta y_1 = 1 = 100\%$$

$$\%\Delta y_2 = -0.333 = -33.3\%$$

  - *Intuition*: an increase in the endowment of capital by $\frac{5-3}{3} = 0.67 = 67\%$ entails an increase in good 1's output by 100% while that of good 2 decreases by 33.3%.

Introducing Taxes

Introducing Taxes: Tax on Goods

- Assume that a sales tax t_k is imposed on good k.
- Then the price paid by consumers increases by $p_k^C = (1 + t_k)p_k^P$, where p_k^P is the price received by producers.
- If the tax on good 1 and 2 coincides, i.e., $t_1 = t_2$, the price ratio consumers and producers face is unaffected:

$$\frac{p_1^C}{p_2^C} = \frac{(1+t_1)p_1^P}{(1+t_2)p_2^P} = \frac{p_1^P}{p_2^P}$$

  - Hence, the after-tax allocation is still Pareto efficient.

Introducing Taxes: Tax on Goods

- However, if only good 1 is affected by the tax, i.e., $t_1 > 0$ while $t_2 = 0$ (i.e., $t_1 \neq t_2$), then the allocation will not be Pareto efficient.
  - In this setting, the $MRTS_{L,K}$ is still the same as before the introduction of the tax:

$$\frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}} = \frac{w_L}{w_K} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

  - Therefore, the allocation of inputs still achieves productive efficiency.

Introducing Taxes: Tax on Goods

– Similarly, the $MRT_{1,2}$ still coincides with the price ratio of goods 1 and 2:

$$\frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_1}{\partial L}} = \frac{p_1^P}{p_2} = \frac{\frac{\partial F_2}{\partial K}}{\frac{\partial F_1}{\partial K}}$$

where the price received by the producer, p_1^P, is the same before and after introducing the tax.

Introducing Taxes: Tax on Goods

- However, while the $MRS_{1,2}$ is equal to the price ratio that consumers face, i.e., $\frac{p_1^C}{p_2} = \frac{(1+t_1)p_1^P}{p_2}$, it now becomes larger than the price ratio that producers face, $\frac{p_1^P}{p_2}$:

$$MRS_{1,2} = \frac{p_1^C}{p_2} = \frac{(1+t_1)p_1^P}{p_2} > \frac{p_1^P}{p_2}$$

- *Intuition*:
  - The rate at which consumers are willing to substitute good 1 for 2 is larger than the rate at which firms can transform good 1 for 2.
  - Thus, the production of good 1 should decrease and that of good 2 should increase.

Introducing Taxes: Tax on Inputs

- Similar arguments extend to the introduction of taxes on inputs
- Price paid by producers is $w_m^P = (1 + t_m)w_m^C$ for input $m = \{L, K\}$.
- If both inputs are subject to the same tax, i.e., $t_L = t_K = t$, the input price ratio consumers and producers face coincides:

$$\frac{w_L^P}{w_K^P} = \frac{(1+t)w_L^C}{(1+t)w_K^C} = \frac{w_L^C}{w_K^C}$$

  - Hence, the efficiency conditions is unaffected.

Introducing Taxes: Tax on Inputs

- However, when taxes differ, $t_L \neq t_K$, productive efficiency no longer holds under such condition:
  - While input consumers satisfy

$$\frac{w_L^C}{w_K^C} = \frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}}$$

and input producers satisfy

$$\frac{w_L^P}{w_K^P} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

Introducing Taxes: Tax on Inputs

the input price ratios they face do not coincide

$$\frac{\frac{\partial F_1}{\partial L}}{\frac{\partial F_1}{\partial K}} = \frac{w_L^C}{w_K^C} \neq \frac{(1+t_L)w_L^C}{(1+t_K)w_K^C} = \frac{w_L^P}{w_K^P} = \frac{\frac{\partial F_2}{\partial L}}{\frac{\partial F_2}{\partial K}}$$

– For instance:

- If $t_L > t_K$, the $MRTS_{L,K}$ is larger for firm 1 than 2.
- Thus, the allocation of inputs is inefficient, i.e., the marginal productivity of additional units of labor (relative to capital) is larger in firm 1 than in firm 2.

Appendix A:
Large Economies and the Core

Large Economies and the Core

- We know that equilibrium allocations (WEAs) are part of the Core.
- We now show that, as the economy becomes larger, the Core shrinks until exactly coinciding with the set of WEAs.

Large Economies and the Core

- Let us first consider an economy with I consumers, each with utility function u^i and endowment vector $\mathbf{e}^i$.
- Consider this economy's replica by doubling the number of consumers to $2I$, each of them still with utility function u^i and endowment vector $\mathbf{e}^i$.
  - There are now two consumers of each type, i.e., "twins," having identical preferences and endowments.
- Define an r-fold replica economy $\mathcal{E}_r$, having consumers of each type, for a total of rI consumers.
  - For any consumer type $i \in I$, all r consumers of that type share the common utility function u^i and have identical endowments $\mathbf{e}^i \gg 0$.

Large Economies and the Core

- When comparing two replica economies, the largest will be that having more of every type of consumers.
- Allocation $\mathbf{x}^{iq}$ indicates the vector of goods for the q^{th} consumer of type i.
- The feasibility condition is

$$\sum_{i=1}^{I}\sum_{q=1}^{r}\mathbf{x}^{iq} = r\sum_{i=1}^{I}\mathbf{e}^{i}$$

Large Economies and the Core

- ***Equal treatment at the Core***: If $\mathbf{x}$ is an allocation in the Core of the r-fold replica economy $\mathcal{E}_r$, then every consumer of type i must have the same bundle, i.e.,

$$\mathbf{x}^{iq} = \mathbf{x}^{iq'}$$

for every two consumers q and q' of type i, $q \neq q' \in \{1,2,\dots,r\}$, and for every type $i \in I$.

  - That is, in the r-fold replica economy, not only similar type of consumers start with the same endowment vector $\mathbf{e}^i$, but they also end up with the same allocation at the Core.

Large Economies and the Core

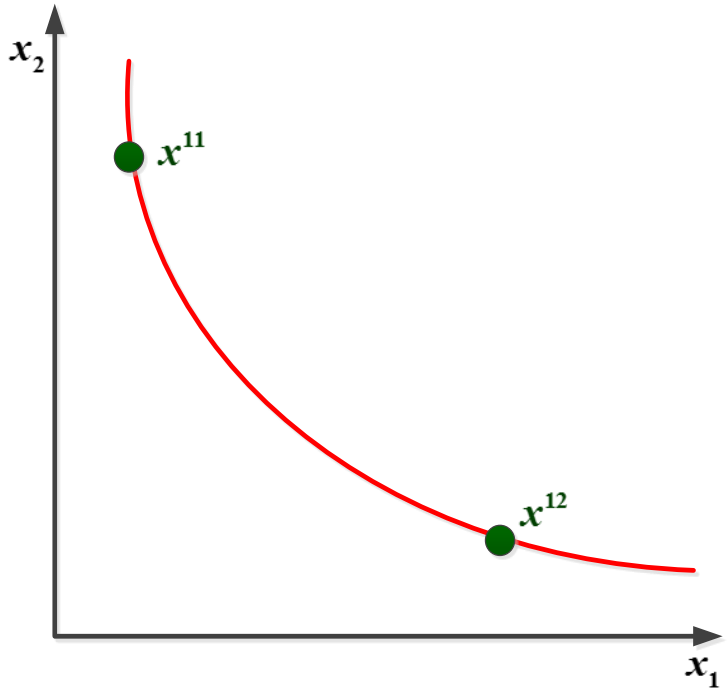
- *Proof* (by contradiction):
  - Consider a two-fold replica economy $\mathcal{E}_2$
    - The results can be generalized to r-fold replicas.
  - Suppose that allocation $\mathbf{x} \equiv \{\mathbf{x}^{11}, \mathbf{x}^{12}, \mathbf{x}^{21}, \mathbf{x}^{22}\}$ is at the core of $\mathcal{E}_2$.
  - Since $\mathbf{x}$ is in the core, then it must be feasible, i.e.,
    $$\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22} = 2\mathbf{e}^1 + 2\mathbf{e}^2$$
  - Assume that allocation $\mathbf{x}$ does not assign the same consumption vectors to the two twins of type-1, i.e., $\mathbf{x}^{11} \neq \mathbf{x}^{12}$.

Large Economies and the Core

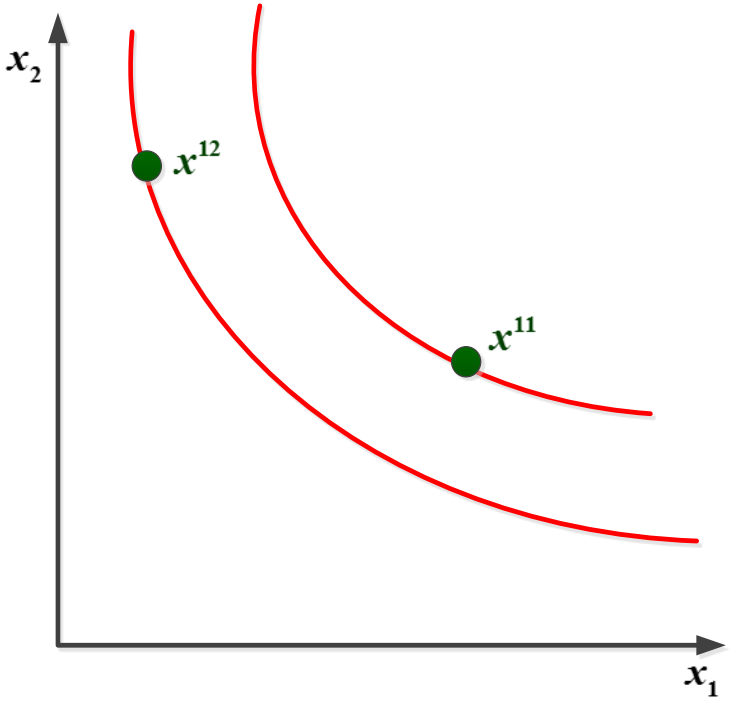
- Assume that type-1 consumer weakly prefers $\mathbf{x}^{11}$ to $\mathbf{x}^{12}$, i.e., $\mathbf{x}^{11} \succsim^{1} \mathbf{x}^{12}$.
  - This is true for both type-1 consumers, since they have the same preferences.
  - A similar result emerges if we instead assume $\mathbf{x}^{12} \succsim^{1} \mathbf{x}^{11}$.

Large Economies and the Core

- Unequal treatment at the core for type-1 consumers



In this case $x^{11} \sim^1 x^{12}$ since both bundles lie on the same indifference curve



In this case $x^{11} \succ^1 x^{12}$

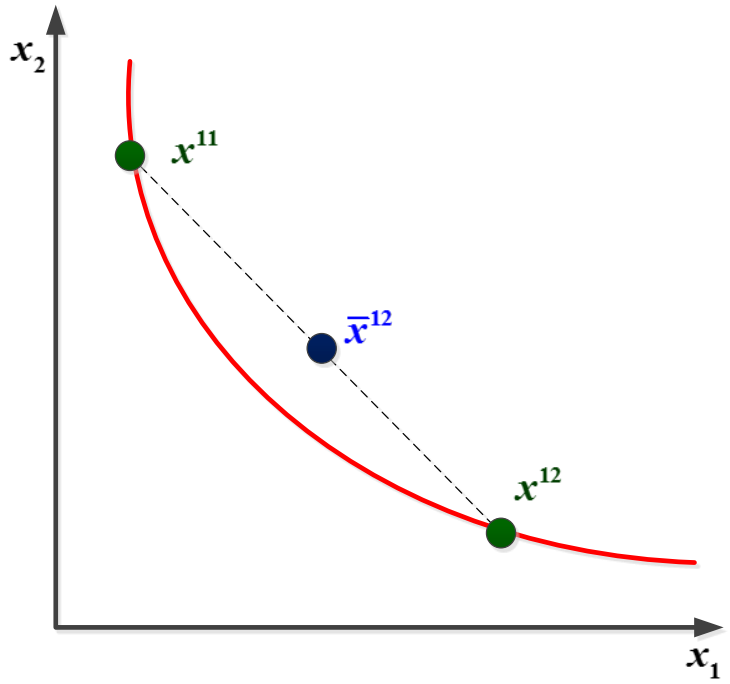
Large Economies and the Core

- Consider that for type-2 consumers we have $\mathbf{x}^{21} \gtrsim^{2} \mathbf{x}^{22}$.
- Hence, consumer 12 is the worst-off type-1 consumer and consumer 22 is the worst-off type 2 consumer.
- Let us take these two "poorly treated" consumers of each type, and check if they can form a blocking coalition to oppose allocation $\mathbf{x}$.
- The average bundles for type-1 and type-2 consumers are

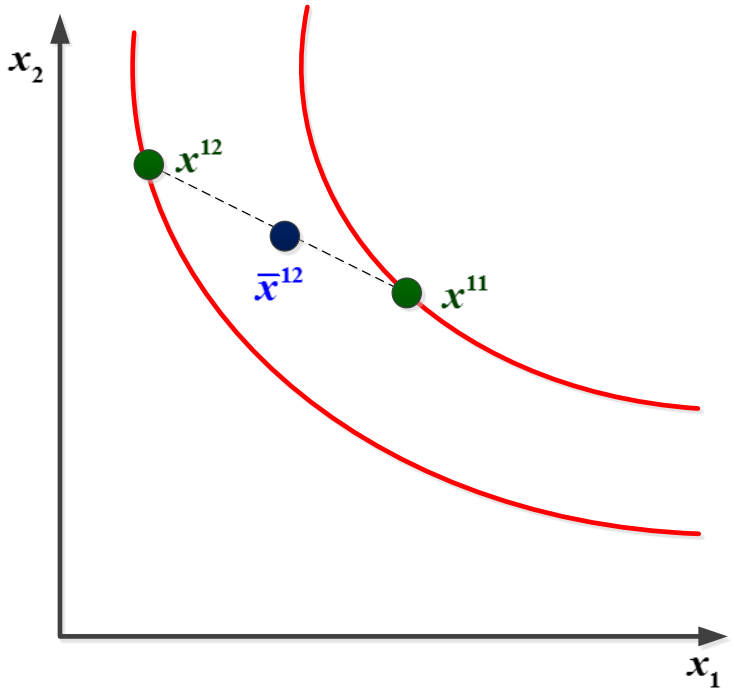
$$\bar{\mathbf{x}}^{12} = \frac{\mathbf{x}^{11}+\mathbf{x}^{12}}{2} \text{ and } \bar{\mathbf{x}}^{22} = \frac{\mathbf{x}^{21}+\mathbf{x}^{22}}{2}$$

Large Economies and the Core

- Average bundles leading to a blocking coalition



In this case $x^{11} \sim^1 x^{12}$ but we can find another bundle, $\bar{x}^{12}$, which satisfies $\bar{x}^{12} \succ^1 x^{12}$



In this case $x^{11} \succ^1 x^{12}$, but we can still find another bundle, $\bar{x}^{12}$, which satisfies $\bar{x}^{12} \succ^1 x^{12}$

Large Economies and the Core

- *Desirability*. Since preferences are strictly convex, the worst-off type-1 consumer prefers $\bar{\mathbf{x}}^{12} \succsim^{1} \mathbf{x}^{12}$,
  - That is, a linear combination between $\mathbf{x}^{11}$ and $\mathbf{x}^{12}$ is preferred to the original bundle $\mathbf{x}^{12}$.
- A similar argument applies to the worst-off type-2 consumer, i.e., $\bar{\mathbf{x}}^{22} \succsim^{2} \mathbf{x}^{22}$.
- Hence, $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$ makes both consumers 12 and 22 better off than at the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$.

Large Economies and the Core

– *Feasibility*. Can consumers 12 and 22 achieve $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$?

– Sum the amount of goods consumers 12 and 22 need to achieve $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$ to obtain

$$\begin{aligned}\bar{\mathbf{x}}^{12} + \bar{\mathbf{x}}^{22} &= \frac{\mathbf{x}^{11}+\mathbf{x}^{12}}{2} + \frac{\mathbf{x}^{21}+\mathbf{x}^{22}}{2} \\ &= \frac{1}{2}(\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22}) \\ &= \frac{1}{2}(2\mathbf{e}^1 + 2\mathbf{e}^2) = \mathbf{e}^1 + \mathbf{e}^2\end{aligned}$$

– Hence, the pair of bundles $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$ is feasible.

Large Economies and the Core

- In summary, pair of bundles $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$:
  - makes consumers 12 and 22 better off than the original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$
  - is feasible
- Thus, these consumers can block $(\mathbf{x}^{12}, \mathbf{x}^{22})$.
  - The original allocation $(\mathbf{x}^{12}, \mathbf{x}^{22})$ cannot be at the Core.
- Therefore, if an allocation is at the Core of the replica economy, it must give consumers of the same type the exact same bundle.

Large Economies and the Core

- If $\mathbf{x}$ is in the core of a r-fold replica economy $\mathcal{E}_r$, i.e., $\mathbf{x} \in C_r$, then (by the equal treatment property) allocation $\mathbf{x}$ must be of the form

$$\mathbf{x} = (\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r \text{ times}}, \underbrace{\mathbf{x}^2, \ldots, \mathbf{x}^2}_{r \text{ times}}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r \text{ times}})$$

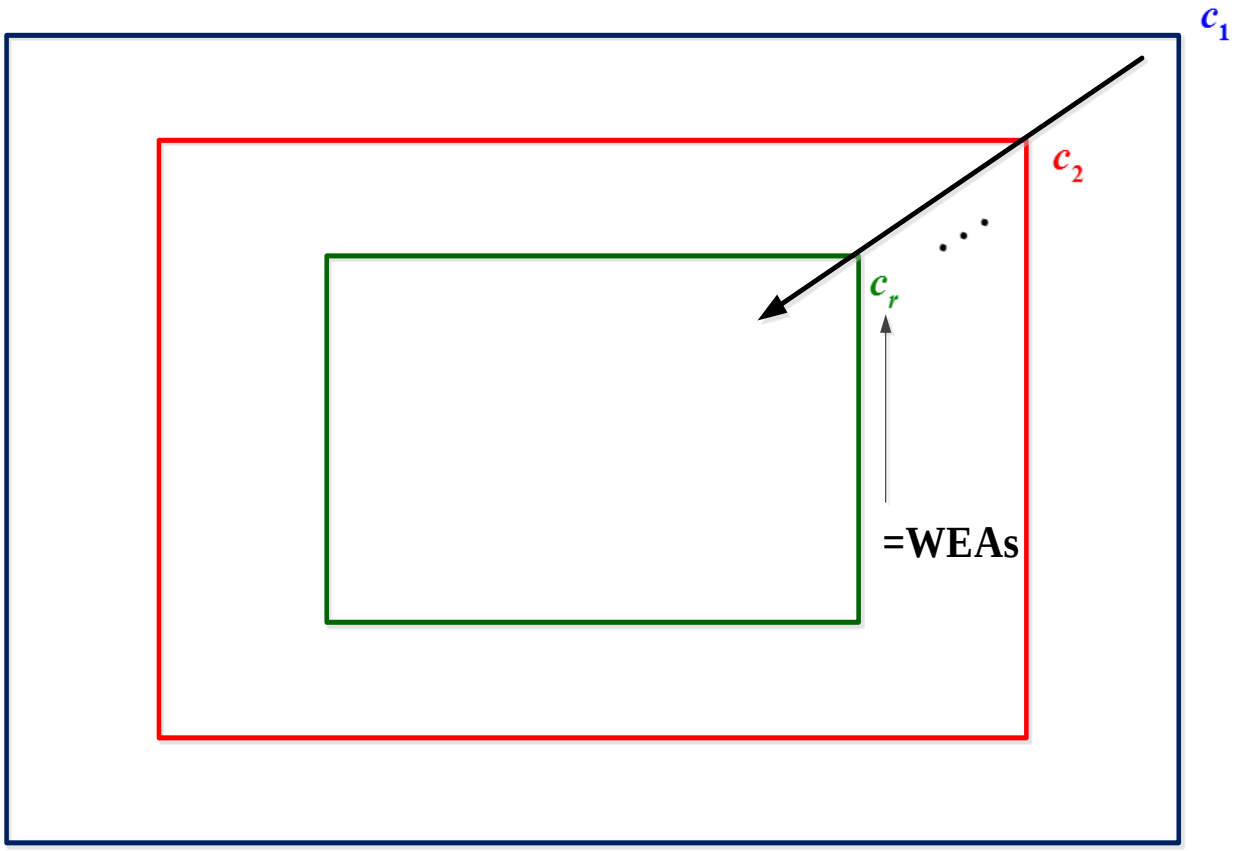
  - All consumers of the same type must receive the same bundle.
  - Core allocations in $\mathcal{E}_r$ are r-fold copies of allocations in $\mathcal{E}_1$, $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$.

Large Economies and the Core

- ***The core shrinks as the economy enlarges***. The sequence of core sets $C_1, C_2, \ldots, C_r$ is decreasing.
- That is,
  - the core of the original (un-replicated) economy, C_1, is a superset of that in the 2-fold replica economy, C_2;
  - the core in the 2-fold replica economy, C_2, is a superset of the 3-fold replica economy, C_3;
  - etc.

Large Economies and the Core

- The Core shrinks as r increases



Large Economies and the Core

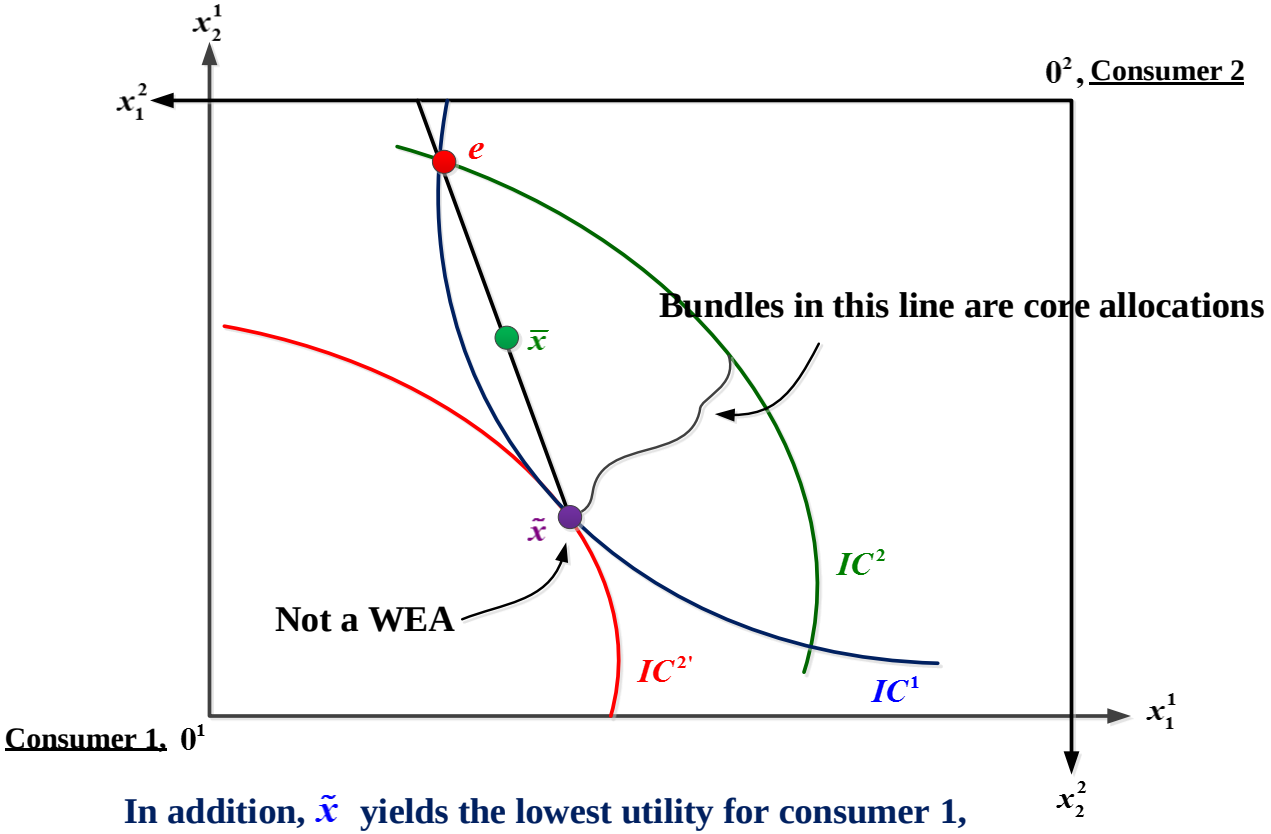
- *Proof*:
  - Since we seek to show that $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_{r-1} \supseteq C_r$, it suffices to show that, for any $r > 1$, $C_{r-1} \supseteq C_r$.
  - Suppose that allocation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I) \in C_r$.
  - There is no blocking coalition to $\mathbf{x}$ in the r-fold replica economy $\mathcal{E}_r$.
  - We then need to show that $\mathbf{x}$ cannot be blocked by any coalition in the $(r-1)$-fold replica economy either.
    - If we could find a blocking coalition to $\mathbf{x}$ in $\mathcal{E}_{r-1}$, then we could also find a blocking coalition in $\mathcal{E}_r$.
    - All members in $\mathcal{E}_{r-1}$ are also present in the larger economy $\mathcal{E}_r$ and their endowments have not changed.

Large Economies and the Core

- Now we need to show that, as r increases, the core shrinks.
- We will do this be demonstrating that allocations at the frontier of C_1 do not belong to the core of the 2-fold replica economy, C_2.

Large Economies and the Core

- Un-replicated economy $\mathcal{E}_1$



In addition, $\tilde{x}$ yields the lowest utility for consumer 1, among all core allocations.

Large Economies and the Core

- The line between $\tilde{\mathbf{x}}$ and $\mathbf{e}$ includes core allocations.
  - All points in the line are part of the core.
  - However, not all points in this line are WEAs.
  - For instance: $\tilde{\mathbf{x}}$ is not a WEA since the price line through $\tilde{\mathbf{x}}$ and $\mathbf{e}$ is not tangent to the consumer's indifference curve at $\tilde{\mathbf{x}}$.
- If the Core shrinks as the economy enlarges, we should be able to show that allocation $\tilde{\mathbf{x}} \notin C_2$.
- Let us build a blocking coalition against $\tilde{\mathbf{x}}$.

Large Economies and the Core

- *Desirability*. Consider the midpoint allocation $\bar{\mathbf{x}}$ and the coalition $S = \{11,12,21\}$. Such a midpoint in the line connecting $\tilde{\mathbf{x}}$ and $\mathbf{e}$ is strictly preferred by both types of consumer 1.
- If the midpoint allocation $\bar{\mathbf{x}}$ is offered to both types of consumer 1 (11 and 12), and the same bundle is offered to type 2 consumer, they will all accept it:

$$\bar{\mathbf{x}}^{11} \equiv \frac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{11}) \succ^1 \tilde{\mathbf{x}}^{11}$$

$$\bar{\mathbf{x}}^{12} \equiv \frac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{12}) \succ^1 \tilde{\mathbf{x}}^{12}$$

$$\tilde{\mathbf{x}}^{21} \sim^2 \tilde{\mathbf{x}}^{21}$$

Large Economies and the Core

– *Feasibility*. Since $\bar{\mathbf{x}}^{11} = \bar{\mathbf{x}}^{12}$, then the sum of the suggested allocations yields

$$\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \tilde{\mathbf{x}}^{21} = 2 \times \frac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{11}) + \tilde{\mathbf{x}}^{21}$$
$$= \mathbf{e}^1 + \tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21}$$

– Recall that $\tilde{\mathbf{x}}$ is part of the un-replicated economy $\mathcal{E}_1$.

- Hence, it must be feasible, i.e., $\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2 = \mathbf{e}^1 + \mathbf{e}^2$.
- Therefore, $\tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21} = \mathbf{e}^1 + \mathbf{e}^2$.

Large Economies and the Core

– We can thus re-write the above equality as

$$\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \tilde{\mathbf{x}}^{21} = \mathbf{e}^1 + \underbrace{\tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21}}_{\mathbf{e}^1 + \mathbf{e}^2}$$

$$= \mathbf{e}^1 + \mathbf{e}^1 + \mathbf{e}^2 = 2\mathbf{e}^1 + \mathbf{e}^2$$

which confirms the feasibility.

– Hence, the frontier allocation $\tilde{\mathbf{x}}$ in the core of the un-replicated economy does not belong to the core of the two-fold economy, $\tilde{\mathbf{x}} \notin C_2$.

- There is a blocking coalition $S = \{11,12,21\}$ and an alternative allocation $\bar{\mathbf{x}} = \{\bar{\mathbf{x}}^{11}, \bar{\mathbf{x}}^{12}, \tilde{\mathbf{x}}^{21}\}$ that they would prefer to $\tilde{\mathbf{x}}$ and that is feasible for the coalition members.

Large Economies and the Core

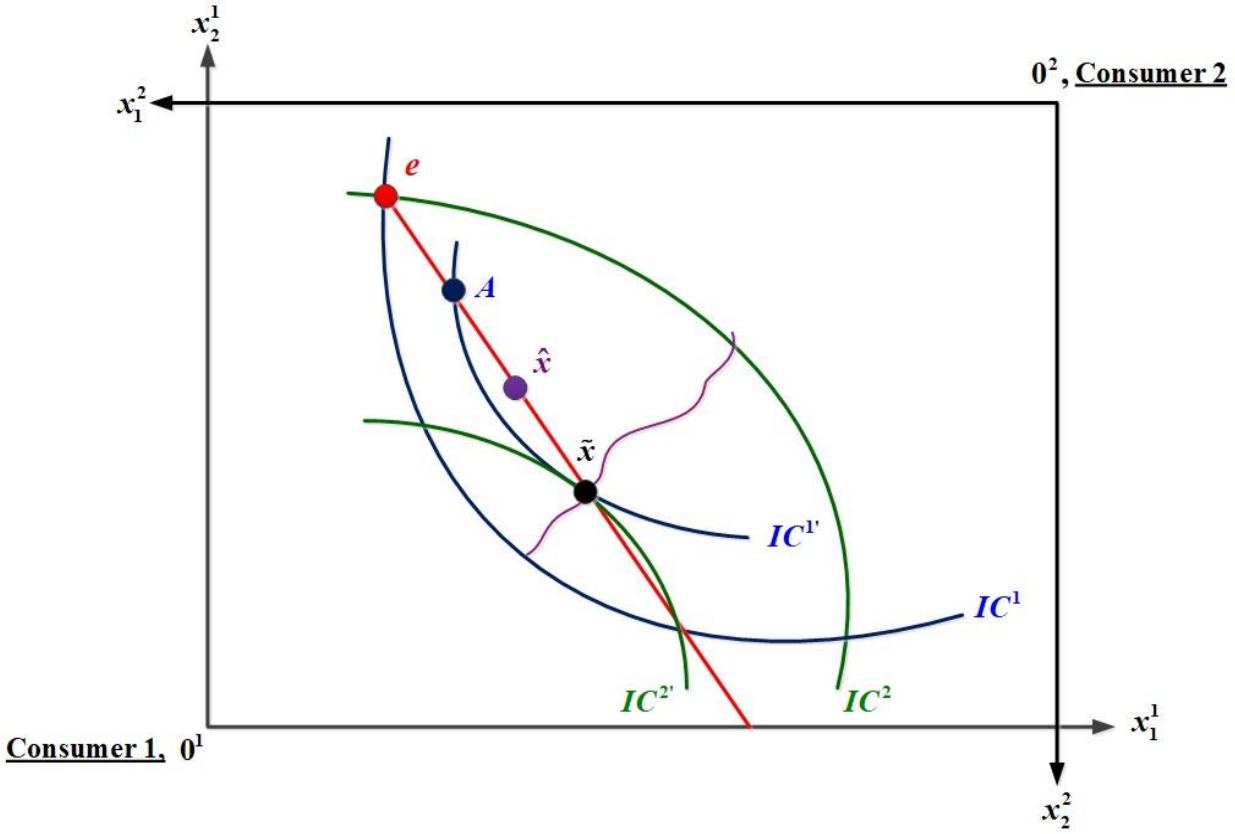
- ***WEA in replicated economies***:
  - Consider a WEA in the un-replicated economy $\mathcal{E}_1$, $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$.
  - An allocation $\mathbf{x}$ is a WEA for the r-fold replica economy $\mathcal{E}_r$ iff it is of the form
    $$\mathbf{x} = (\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r \text{ times}}, \underbrace{\mathbf{x}^2, \ldots, \mathbf{x}^2}_{r \text{ times}}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r \text{ times}})$$
  - If $\mathbf{x}$ is a WEA for $\mathcal{E}_r$, then it also belongs to the core of that economy (by the "equal treatment at the core" property).

Large Economies and the Core

- ***A limit theorem on the Core***: If an allocation $\mathbf{x}$ belongs to the core of all r-fold replica economies then such allocation must be a WEA of the un-replicated economy $\mathcal{E}_1$.
- *Proof* (by contradiction):
  - Consider that an allocation $\tilde{\mathbf{x}}$ belongs to the core of the r-fold replica economy C_r but is *not* a WEA.
  - A core allocation for the un-replicated economy $\mathcal{E}_1$, $\tilde{\mathbf{x}} \in C_1$ satisfies $\tilde{\mathbf{x}} \in C_r$ since $C_1 \supset C_r$.
  - Allocation $\tilde{\mathbf{x}}$ must then be within the lens-shaped area and on the contract curve.

Large Economies and the Core

- A core allocation $\tilde{\mathbf{x}}$ that is not WEA



Large Economies and the Core

- Consider now the line connecting $\tilde{\mathbf{x}}$ and $\mathbf{e}$.
- Since $\tilde{\mathbf{x}}$ is not a WEA, the budget line cannot be tangent to both consumers' indifference curves:

$$\frac{p_1}{p_2} > MRS \text{ or } \frac{p_1}{p_2} < MRS$$

- Can allocation $\tilde{\mathbf{x}}$ be at the Core C_r and yet not be a WEA?
- Let us show that if $\tilde{\mathbf{x}}$ is not a WEA it *cannot* be part of the Core C_r either.
  - To demonstrate that $\tilde{\mathbf{x}} \notin C_r$, let us find a blocking coalition.

Large Economies and the Core

- By the convexity of preferences, we can find a set of bundles (such those between A and $\tilde{\mathbf{x}}$) that consumer 1 prefers to $\tilde{\mathbf{x}}$:

$$\hat{\mathbf{x}} \equiv \frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\tilde{\mathbf{x}}^1$$

  for some $r > 1$, where $\frac{1}{r} + \frac{r-1}{r} = 1$.

- Consider a coalition S with all r type-1 consumers and $r - 1$ type-2 consumers.
- Let us now show that allocation $\hat{\mathbf{x}}$ satisfies the properties of desirability and feasibility for the blocking coalition S.

Large Economies and the Core

- *Desirability*. If we give every type-1 consumer the bundle $\hat{\mathbf{x}}^1$, $\hat{\mathbf{x}}^1 \succ^1 \tilde{\mathbf{x}}^1$. Similarly, if we give every type-2 consumer in the coalition the bundle $\hat{\mathbf{x}}^2 = \tilde{\mathbf{x}}^2$, then $\hat{\mathbf{x}}^2 \sim^2 \tilde{\mathbf{x}}^2$.
- *Feasibility*. Summing over the consumers in coalition S, their aggregate allocation is

$$r\hat{\mathbf{x}}^1 + (r-1)\hat{\mathbf{x}}^2 = r\left[\frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\tilde{\mathbf{x}}^1\right] + (r-1)\tilde{\mathbf{x}}^2$$
$$= \mathbf{e}^1 + (r-1)(\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2)$$

- Since $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2)$ is in the core of the un-replicated economy $\mathcal{E}_1$, then it must be feasible $\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2 = \mathbf{e}^1 + \mathbf{e}^2$.

Large Economies and the Core

– Combining the above two results, we find that

$$r\hat{\mathbf{x}}^1 + (r-1)\hat{\mathbf{x}}^2 = \mathbf{e}^1 + (r-1)\underbrace{(\mathbf{e}^1 + \mathbf{e}^2)}_{\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2}$$

$$= \mathbf{e}^1 + r(\mathbf{e}^1 + \mathbf{e}^2) - (\mathbf{e}^1 + \mathbf{e}^2)$$

$$= r\mathbf{e}^1 + (r-1)\mathbf{e}^2$$

which confirms feasibility.

– Hence, r type-1 consumers and $r-1$ type-2 consumers can get together in coalition and block allocation $\tilde{\mathbf{x}}$.

Large Economies and the Core

- Thus if $\tilde{\mathbf{x}}$ is not a WEA, then, $\tilde{\mathbf{x}}$ cannot be in the Core of the r-fold replica economy $\mathcal{E}_r$.
- As a consequence, if $\tilde{\mathbf{x}} \in C_r$ for all $r > 1$, then $\tilde{\mathbf{x}}$ must be a WEA.

Appendix B: Marshall–Hicks Four Laws of Derived Demand

Marshall–Hicks Four Laws

- Consider a production function $q = f(K, L)$, with positive marginal products, $f_L, f_K > 0$.
- Assume that the supply of each input $(w(L), r(K))$ is positively sloped, $w'(L) > 0$ and $r'(K) > 0$.
- Demand for output is given by $q = g(p)$, which satisfies $g'(p) < 0$.
- The total cost is $w(L)L + r(K)K$.
- Assume that the capital market is perfectly competitive, but the labor and output markets are not necessarily competitive.

Marshall–Hicks Four Laws

- Define:
  - $\varepsilon_{q,p} = (\partial q/\partial p)(p/q)$ as the price elasticity of output
  - $s_{K,r} = (\partial K/\partial r)(r/K)$ as the elasticity of capital supply to a change in its price
  - $s_{L,r} = (\partial L/\partial r)(r/L)$ as the elasticity of labor supply to a change in the price of capital
  - $s_{L,w} = (\partial L/\partial w)(w/L)$ as the elasticity of labor supply to a change in its price
  - σ as the elasticity of substitution between inputs
- We use superscript i to refer to the elasticity that an individual firm faces ($\varepsilon^{i}_{q,p}$).
- The industry elasticities do not include superscripts ($\varepsilon_{q,p}$).

Marshall–Hicks Four Laws

- Let $\theta_L \equiv wL/pq$ and $\theta_K \equiv rK/pq$ be the cost of labor and capital, respectively, relative to total sales.
- This implies that $\theta_L = 1 - \theta_K$.
- For compactness, let us define

$$A \equiv 1 - (1/\varepsilon^i_{q,p})$$
$$B \equiv 1 + (1/s^i_{L,w})$$

Marshall–Hicks Four Laws

- Marshall, Hicks, and Allen analyze how the input demand of a perfectly competitive input, such as capital, is affected by a marginal change in the price of capital:

$$s_{K,r} = \frac{-A\left[\theta_K \varepsilon_{q,p} + \sigma\left(\varepsilon_{q,p}/s_{L,w}\right)A + \sigma\theta_L B\right]}{\left(\theta_K + \theta_L B\right)^2 + \theta_K\left(\sigma/s_{L,w}\right)A + \theta_L\left(\varepsilon_{q,p}/s_{L,w}\right)AB}$$

Marshall–Hicks Four Laws

- Marshall–Hicks's four laws of input demand ("**derived demand**") state that an input demand becomes more elastic, whereby $s_{K,r}$ decreases, in
  1. the elasticity of substitution between inputs σ
  2. the price-elasticity of output demand $\varepsilon_{q,p}$
  3. the cost of the input relative to total sales θ_K
  4. the elasticity of the other input's supply to a change in its price $s_{L,w}$
- We analyze these four comparative statics under two market structures:
  1. **The Marshall's presentation**: $\varepsilon^i_{q,p} = s^i_{L,w} = \infty$, $\sigma = 0$
  2. **The Hicks' presentation**: $\varepsilon^i_{q,p} = s^i_{L,w} = \infty$ (no assumptions on σ)

Marshall's Presentation

- Assumptions:
  - Output and input markets are perfectly competitive, $\varepsilon_{q,p}^{i} = s_{L,w}^{i} = \infty$, for every firm i, which implies $A = B = 1$
  - Inputs cannot be substituted in the production process, $\sigma = 0$
- The expression for $s_{K,r}$ can be simplified to

$$s_{K,r} = -\frac{\theta_K \varepsilon_{q,p} s_{L,w}}{s_{L,w} + \theta_L \varepsilon_{q,p}}$$

Marshall's Presentation

- The derivatives testing the laws are:

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{\theta_K (s_{L,w})^2}{(s_{L,w} + \theta_L \varepsilon_{q,p})^2}$$

$$\frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{s_{L,w} \cdot \varepsilon_{q,p}(s_{L,w} + \varepsilon_{q,p})}{(s_{L,w} + \theta_L \varepsilon_{q,p})^2}$$

$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{\theta_K \theta_L (\varepsilon_{q,p})^2}{(s_{L,w} + \theta_L \varepsilon_{q,p})^2}$$

- If labor is a "normal" input, $s_{L,w} > 0$, the three derivatives are all negative (the three laws hold).
- If labor is inferior, $s_{L,w} < 0$, $s_{K,r}$ is still decreasing in $\varepsilon_{q,p}$ and in $s_{L,w}$, but not necessarily in θ_K.

Hicks' Presentation

- Assumptions:
  - Output and input markets are perfectly competitive, $\varepsilon_{q,p}^{i} = s_{L,w}^{i} = \infty$, for every firm i, so $A = B = 1$
  - No condition imposed on the substitution of inputs (σ)
- The expression for $s_{K,r}$ can be simplified to

$$s_{K,r} = \frac{-\left[\theta_K \varepsilon_{q,p} s_{L,w} + \sigma\left(\varepsilon_{q,p} + \theta_L s_{L,w}\right)\right]}{s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}}$$

Hicks' Presentation

- The derivatives testing the laws are

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{\theta_K (s_{L,w} + \sigma)^2}{(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p})^2}$$

$$\frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{\varepsilon_{q,p}(s_{L,w}^2 - \sigma^2) + \theta_L s_{L,w} (\varepsilon_{q,p}^2 - \sigma^2)}{(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p})^2}$$

$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{\theta_K \theta_L (\varepsilon_{q,p} - \sigma)^2}{(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p})^2}$$

$$\frac{\partial s_{K,r}}{\partial \sigma} = -\frac{\theta_L (\varepsilon_{q,p} + s_{L,w})^2}{(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p})^2}$$

- Hence, $s_{K,r}$ decreases in $\varepsilon_{q,p}$, $s_{L,w}$, and σ (the three laws hold).
- $s_{K,r}$ also decreases in θ_K if the input is "normal", $s_{L,w} > 0$, and inputs are not extremely easy to substitute, $\varepsilon_{q,p} > \sigma$.

Advanced Microeconomic Theory

Chapter 6: Partial Equilibrium

Outline

- Partial equilibrium analysis
- Comparative statics
- Welfare analysis
- Appendix: Marshall–Hicks four laws of derived demand

Partial Equilibrium Analysis

- In a competitive equilibrium (CE), all agents must select an optimal allocation given their resources:
  - Firms choose profit-maximizing production plans given their technology;
  - Consumers choose utility-maximizing bundles given their budget constraint.
- A competitive equilibrium allocation will emerge at a price that makes consumers' purchasing plans coincide with the firms' production decisions.

Partial Equilibrium Analysis

- ***Firm***:
  - Assume that firm j's cost function is strictly convex, $c_j''(q_j) > 0$.
  - Given the market price p^*, firm j's equilibrium output level q_j^* must solve

$$\max_{q_j \geq 0} \; p^* q_j - c_j(q_j)$$

Partial Equilibrium Analysis

- ***Firm*** (continued):
  - Since c_j is strictly convex, this problem yields the necessary and sufficient condition

$$p^* \leq c_j'(q_j^*), \qquad \text{with equality if } \; q_j^* > 0$$

  - That is, every firm j produces up to the point at which its marginal cost, $c_j'(q_j^*)$, coincides with the current market price.

Partial Equilibrium Analysis

- ***Consumers***:
  - Consider a quasilinear utility function

$$u_i(m_i, x_i) = m_i + \phi_i(x_i)$$

where m_i denotes the numeraire, and $\phi_i'(x_i) > 0$, $\phi_i''(x_i) < 0$ for all $x_i > 0$.

  - Normalizing, $\phi_i(0) = 0$. Recall that with quasilinear utility functions, the wealth effects for all non-numeraire commodities are zero.

Partial Equilibrium Analysis

– Consumer i's UMP is

$$\max_{m_i \in \mathbb{R}, x_i \in \mathbb{R}_+} \quad m_i + \phi_i(x_i)$$

$$\text{s.t.} \quad \underbrace{m_i + p^* x_i}_{\text{Total expend.}} \leq \underbrace{w_{m_i} + \sum_{j=1}^{J} \theta_{ij} \left(\underbrace{p^* q_j^* - c_j\left(q_j^*\right)}_{\text{Profits}} \right)}_{\text{Total resources (endowment + profits)}}$$

– where w_{m_i} is consumer i's endowment of the numeraire and θ_{ij} is his ownership share of firm j.

Partial Equilibrium Analysis

- The budget constraint binds because u_i is strictly increasing in m_i. Hence,

$$m_i = -p^* x_i + \left[w_{m_i} + \sum_{j=1}^{J} \theta_{ij} \left(p^* q_j^* - c_j\left(q_j^*\right) \right) \right]$$

- Since $m_i \in \mathbb{R}$, we do not need to impose $m_i \geq 0$ when we substitute this expression into the objective function.

Partial Equilibrium Analysis

– Substituting the budget constraint into the objective function,

$$\max_{x_i \in \mathbb{R}_+} \quad \phi_i(x_i) - p^* x_i + w_{m_i} + \sum_{j=1}^{J} \theta_{ij} \left(p^* q_j^* - c_j(q_j^*) \right)$$

– The FOC with respect to x_i yields

$$\phi_i'(x_i^*) \leq p^*, \qquad \text{with equality if } x_i^* > 0$$

– That is, the consumer increases the amount he buys of good x up to the point at which the marginal utility he obtains coincides with the market price he has to pay for it.

Partial Equilibrium Analysis

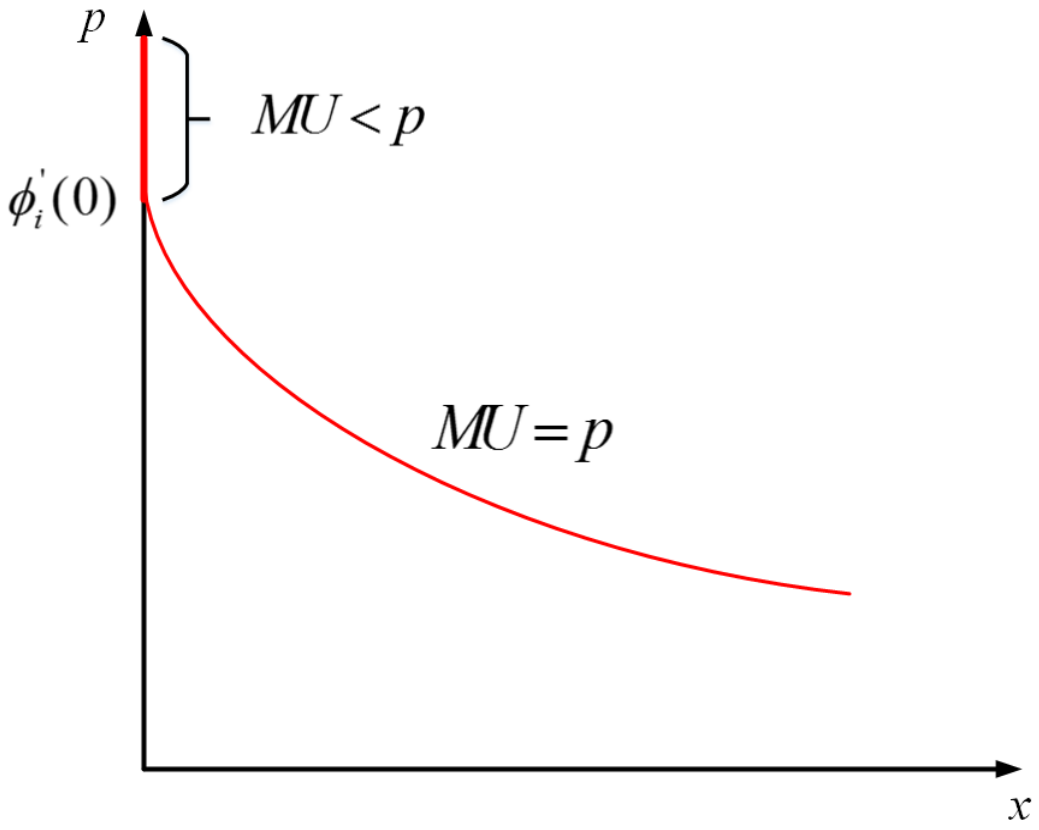
- Hence, an allocation $\left(x_1^*, x_2^*, \ldots, x_I^*, q_1^*, q_2^*, \ldots, q_J^*\right)$ and a price $p^* \in \mathbb{R}_+$ constitute a CE if:
  - $p^* \leq c_j'\left(q_j^*\right)$, with equality if $q_j^* > 0$, for every firm j;
  - $\phi_i'(x_i^*) \leq p^*$, with equality if $x_i^* > 0$, for every consumer i; and

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$$

- Note that these conditions do not depend on consumers' endowments w_{m_i} or ownership shares θ_{ij}.

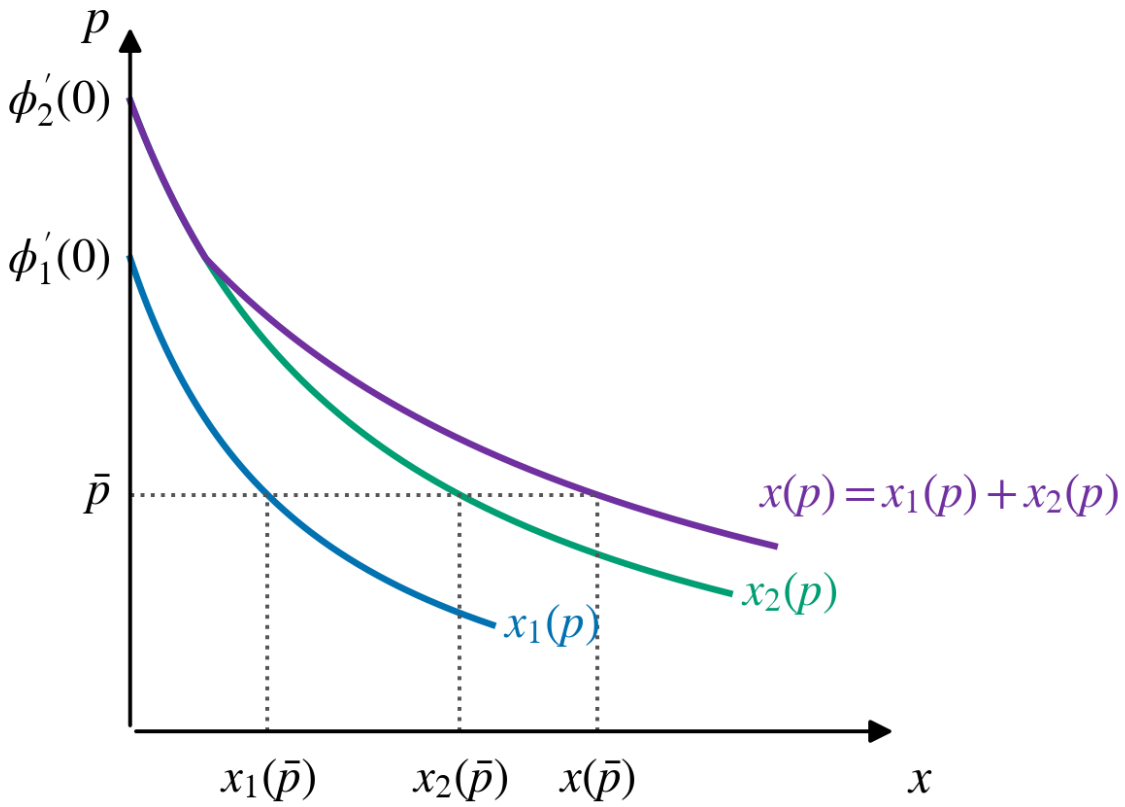
Partial Equilibrium Analysis

- The individual demand curve, where $\phi_i'(x_i^*) \leq p^*$



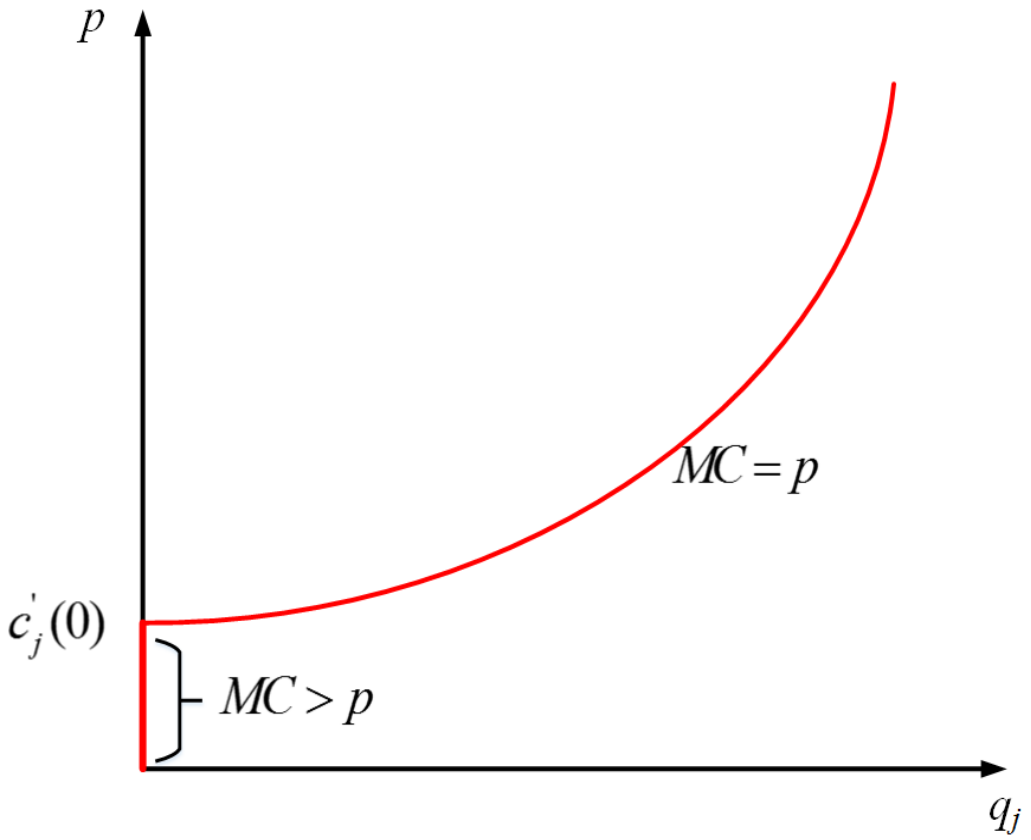
Partial Equilibrium Analysis

- Horizontally summing individual demand curves yields the aggregate demand curve.



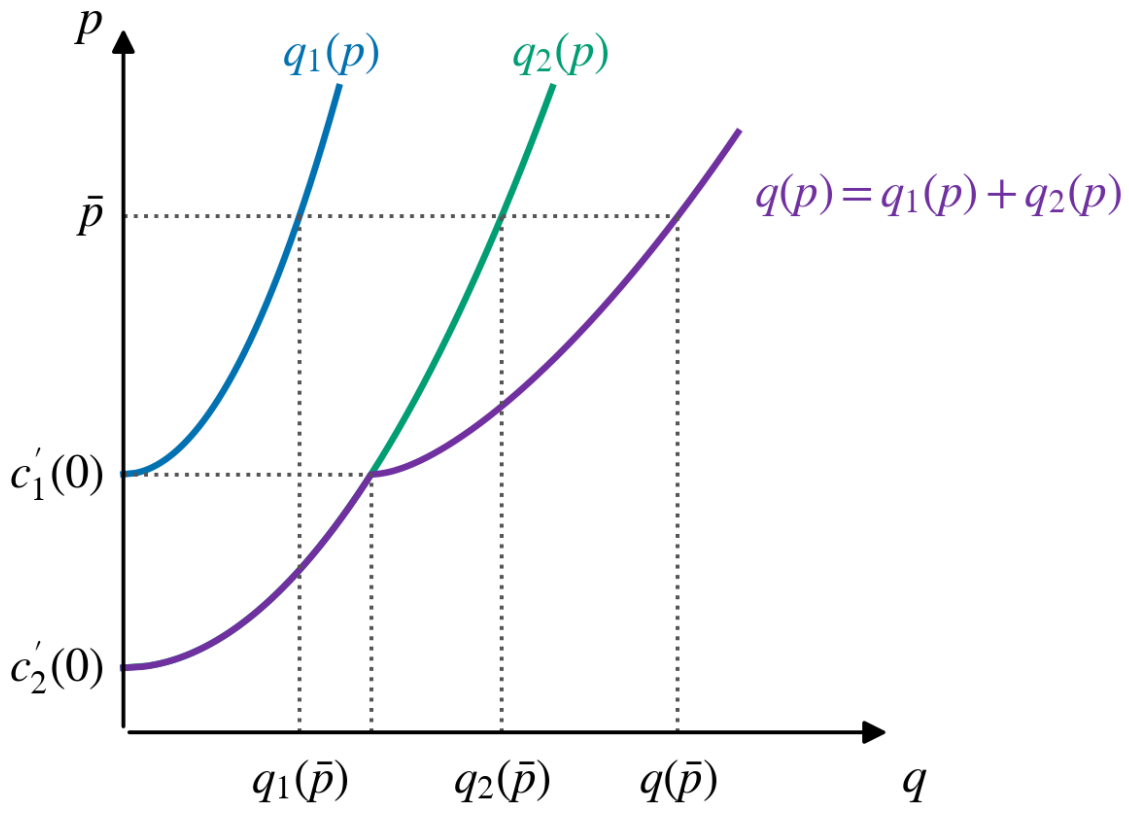
Partial Equilibrium Analysis

- The individual supply curve, where $p^* \leq c_j'(q_j^*)$



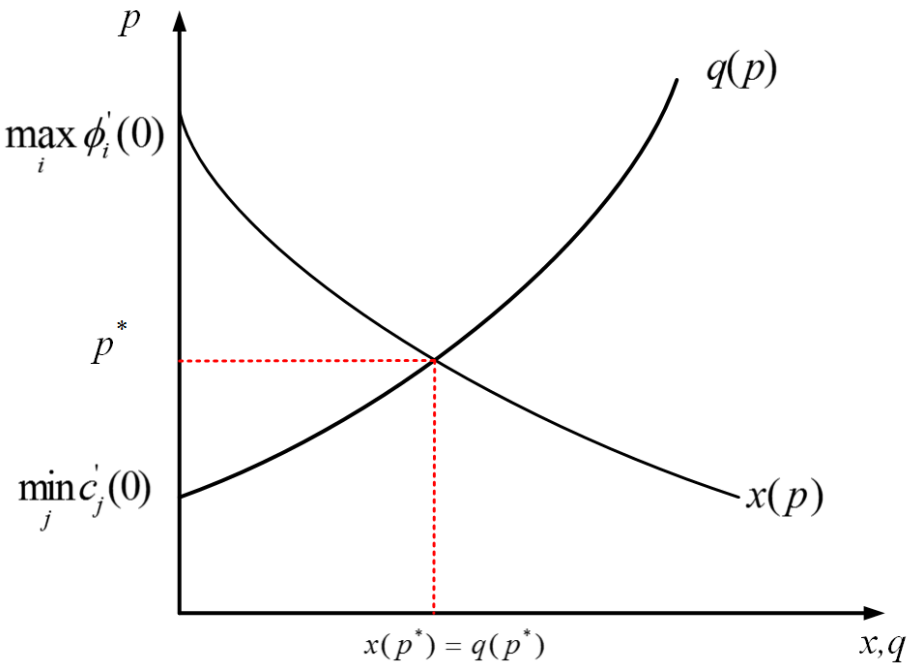
Partial Equilibrium Analysis

- Horizontally summing individual supply curves yields the aggregate supply curve.



Partial Equilibrium Analysis

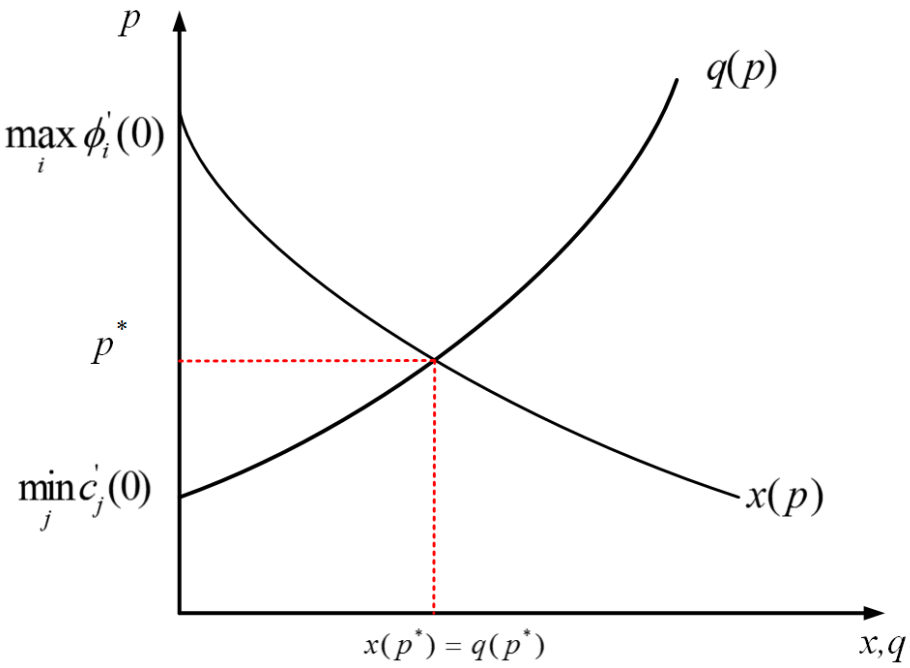
- Superimposing aggregate demand and aggregate supply curves, we obtain the CE allocation of good x.



Partial Equilibrium Analysis

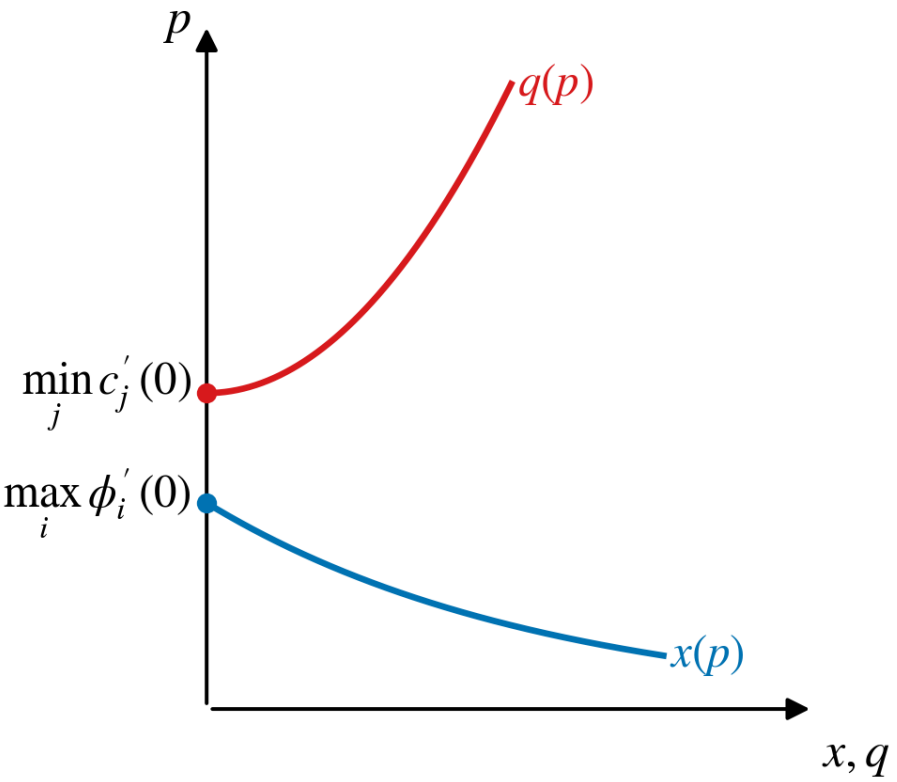
- For the CE to involve positive production and consumption, we need
$$\max_i \phi_i'(0) > \min_j c_j'(0)$$
- The equilibrium price then satisfies

$$\max_i \phi_i'(0) > p^* > \min_j c_j'(0)$$



Partial Equilibrium Analysis

- If we have
  $\max_i \phi_i'(0) < \min_j c_j'(0)$
  then there is no positive production or consumption of good x.
- The CE then has
  $x^* = q^* = 0$



Partial Equilibrium Analysis

- Also, since $\phi_i'(x_i)$ is downward sloping in x_i, and $c_j'(q_j)$ is upward sloping in q_j (since c_j is strictly convex), aggregate demand and supply cross at a unique point, given $\max_i \phi_i'(0) > \min_j c_j'(0)$.
  - Hence, the CE allocation is unique.

Partial Equilibrium Analysis

- ***Example 6.1***:
  - Assume a perfectly competitive industry consisting of two types of firms: 100 firms of type *A* and 30 firms of type *B.*
  - Short-run supply curve of type *A* firm is
$$s_A(p) = 2p$$
  - Short-run supply curve of type *B* firm is
$$s_B(p) = 10p$$
  - The Walrasian market demand curve is
$$x(p) = 5000 - 500p$$

Partial Equilibrium Analysis

- ***Example 6.1*** (continued):
  - Summing the individual supply curves of the 100 type-A firms and the 30 type-B firms,

$$S(p) = 100 \cdot 2p + 30 \cdot 10p = 500p$$

  - The short-run equilibrium occurs at the price at which quantity demanded equals quantity supplied,

$$5000 - 500p = 500p, \qquad \text{or} \quad p = 5$$

Partial Equilibrium Analysis

- ***Example 6.1*** (continued):
  - Each type-A firm supplies
$$s_A(5) = 2 \cdot 5 = 10$$
  - Each type-B firm supplies
$$s_B(5) = 10 \cdot 5 = 50$$
  - The equilibrium quantity is

$$x(5) = 5000 - 2500 = 2500 = 100 \cdot 10 + 30 \cdot 50$$

Comparative Statics

Comparative Statics

- Parameters:
  - Let us assume that the consumer's preferences are affected by a vector of parameters $\alpha \in \mathbb{R}^M$.
    - Then, consumer i's utility from good x is $\phi_i(x_i, \alpha)$.
  - Similarly, firms' technology is affected by a vector of parameters $\beta \in \mathbb{R}^S$.
    - Then, firm j's cost function is $c_j(q_j, \beta)$.

Comparative Statics

- Tax notation:
  - $\hat{p}_C(p,t)$ is the effective price paid by consumers.
  - $\hat{p}_F(p,t) = p$ is the effective price received by firms.
  - Per-unit tax: $\hat{p}_C(p,t) = p + t$.
    - Example: $t = \$2$, regardless of the price p.
  - Ad valorem tax: $\hat{p}_C(p,t) = p + pt = p(1+t)$.
    - Example: $t = 0.1$ (10%).

Comparative Statics

- If consumption and production are strictly positive in the CE, then
  - $\phi_i'(x_i^*, \alpha) = \hat{p}_C(p^*, t)$ for every consumer i;
  - $c_j'(q_j^*, \beta) = \hat{p}_F(p^*, t)$ for every firm j; and

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$$

- Then we have $I + J + 1$ equations, which depend on parameter values α, β and t, in the $I + J + 1$ unknowns (x_i^*, q_j^*, p^*).

Comparative Statics

- In order to understand how x_i^* or q_j^* depends on parameters α, β and t, we can use the Implicit Function Theorem.
  - The above functions have to be differentiable.

Comparative Statics

- Implicit Function Theorem:
  - Let $u(x, y)$ be a utility function, where x and y are amounts of two goods.
  - If $\frac{\partial u(\overline{x},\overline{y})}{\partial y} \neq 0$, then the equation $u(x, y) = u(\overline{x}, \overline{y})$ implicitly defines $y(x)$ near $(\overline{x}, \overline{y})$, and along it

$$\frac{\partial u(x, y)}{\partial x} dx + \frac{\partial u(x, y)}{\partial y} dy = 0$$

Comparative Statics

– Rearranging, we obtain

$$\frac{dy(x)}{dx} = -\frac{\frac{\partial u(x,y)}{\partial x}}{\frac{\partial u(x,y)}{\partial y}}$$

– Similarly, at every $(\overline{x}, \overline{y})$ where $\frac{\partial u(\overline{x},\overline{y})}{\partial x} \neq 0$, the equation $u(x,y) = u(\overline{x}, \overline{y})$ implicitly defines $x(y)$, with

$$\frac{dx(y)}{dy} = -\frac{\frac{\partial u(x,y)}{\partial y}}{\frac{\partial u(x,y)}{\partial x}}$$

Comparative Statics

- In comparative statics, we apply the IFT to the equilibrium condition rather than to the utility function.
- If $F(x, \alpha) = 0$ (e.g., the FOC $\phi'(x, \alpha) - p = 0$) and $\frac{\partial F}{\partial x} \neq 0$, then

$$\frac{dx(\alpha)}{d\alpha} = -\frac{\partial F/\partial \alpha}{\partial F/\partial x} = -\frac{\phi''_{x\alpha}(x, \alpha)}{\phi''_{xx}(x, \alpha)}$$

- The left-hand side is unknown.
- The right-hand side is, however, easier to find.

Comparative Statics

- ***Per-unit tax: general case*** (Example 6.2):
  - The aggregate demand is now $x(p + t)$, because the effective price that the consumer pays is $p + t$.
  - In equilibrium, the market price after imposing the tax, $p^*(t)$, must hence satisfy

$$x(p^*(t) + t) = q(p^*(t))$$

Comparative Statics

- If the per-unit tax is marginally increased, and functions are differentiable at $p = p^*(t)$,
  $x'(p^*(t) + t) \cdot (p^{*\prime}(t) + 1) = q'(p^*(t)) \cdot p^{*\prime}(t)$
- Rearranging, we obtain

$$p^{*\prime}(t) \cdot [x'(p^*(t) + t) - q'(p^*(t))] = -x'(p^*(t) + t)$$

Comparative Statics

– Hence,

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t) - q'(p^*(t))}$$

– Since $x(p)$ is decreasing in prices, $x'(p^*(t)+t) < 0$, and $q(p)$ is increasing in prices, $q'(p^*(t)) > 0$,

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t) - q'(p^*(t))} = -\frac{-}{-} = -$$

$$\qquad\qquad\qquad\qquad\qquad\quad -\qquad\qquad\quad +$$

Comparative Statics

– Let

$$|x'| = -x'(p^*(t) + t) > 0 \quad \text{and} \quad q' = q'(p^*(t)) > 0$$

– Then

$$p^{*\prime}(t) = -\frac{|x'|}{|x'| + q'} \in (-1,0)$$

– and

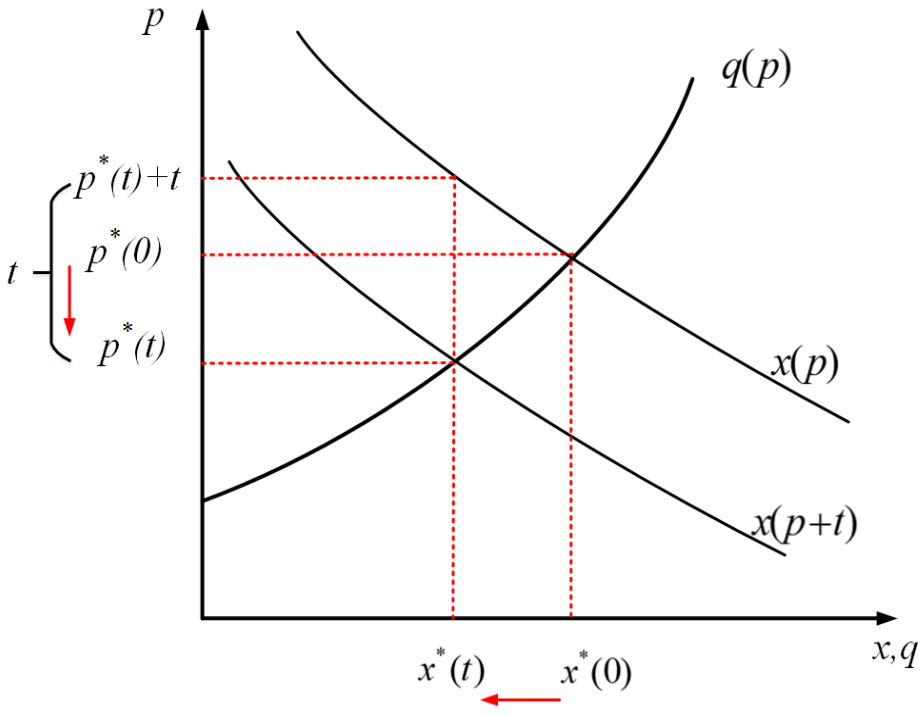
$$p^{*\prime}(t) + 1 = \frac{q'}{q' + |x'|} \in (0,1)$$

Comparative Statics

- Hence, $p^{*\prime}(t) < 0$.
- Moreover, $p^{*\prime}(t) \in (-1,0)$.
  - That is, the price received by producers falls with the tax, but by less than the tax increase.
- Additionally, since $p^*(t) + t$ is the price paid by consumers, $p^{*\prime}(t) + 1$ is the marginal increase in the price paid by consumers when the tax marginally increases.
  - Since $p^{*\prime}(t) > -1$, $p^{*\prime}(t) + 1 < 1$, and the price paid by consumers also rises by less than the tax increase.

Comparative Statics

- *No tax:*
  - CE occurs at $p^*(0)$ and $x^*(0)$
- *Tax:*
  - x^* decreases from $x^*(0)$ to $x^*(t)$
  - Consumers now pay $p^*(t) + t$
  - Producers now receive $p^*(t)$ for the $x^*(t)$ units they sell.



Comparative Statics

- ***Per-unit tax*** (extreme cases):
  - The supply is very responsive to price changes, i.e., $q'(p^*(t))$ is large.

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t) - q'(p^*(t))} \to 0$$

  - Therefore, $p^{*\prime}(t) \to 0$, and the price received by producers barely falls (in the limit, does not fall).

Comparative Statics

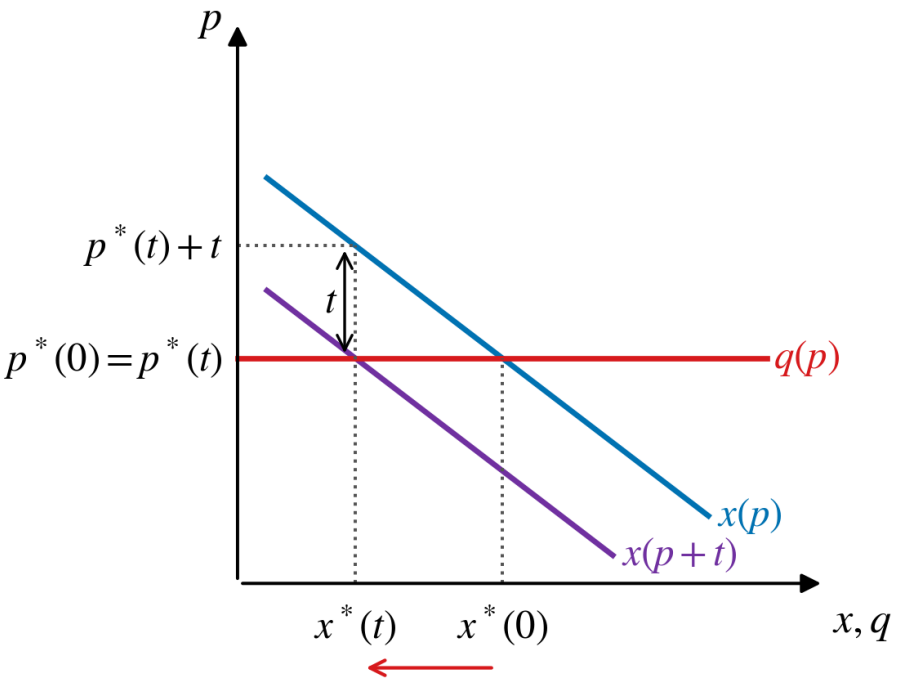
- ***Per-unit tax*** (extreme cases, continued):
  - However, consumers still have to pay $p^*(t) + t$.
    - A marginal increase in taxes therefore provides an increase in the consumer's price of

$$p^{*\prime}(t) + 1 \to 0 + 1 = 1$$

    - In the limit, the tax is borne entirely by consumers.

Comparative Statics

- Perfectly elastic supply (limit case):
  - The price received by producers does not fall.
  - But the price paid by consumers increases by the amount of the tax.



Comparative Statics

- ***Per-unit tax*** (extreme cases, continued):
  - The supply is not responsive to price changes, i.e., $q'(p^*(t))$ is close to zero.

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t) - q'(p^*(t))} \to -1$$

  - Therefore, $p^{*\prime}(t) \to -1$, and the price received by producers falls by \$1 for every extra dollar in taxes.
    - Producers bear the entire tax burden (in the limit).

Comparative Statics

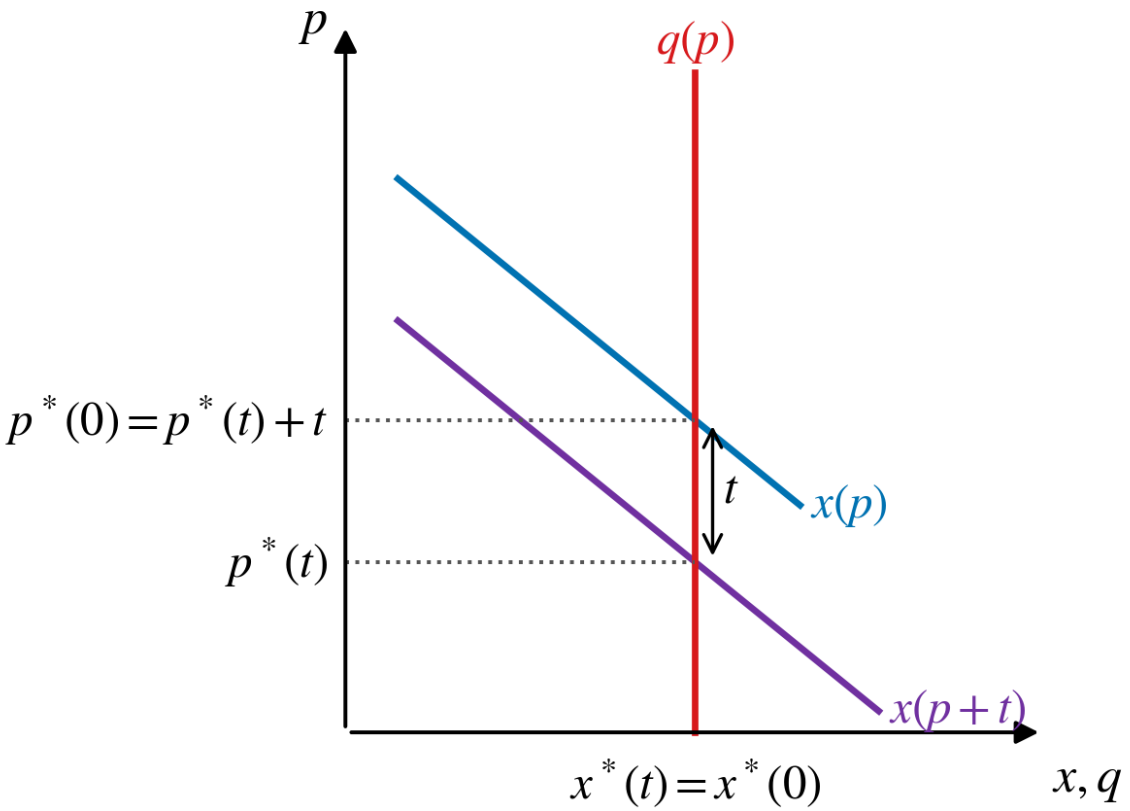
- ***Per-unit tax*** (extreme cases, continued):
  - In contrast, consumers pay $p^*(t) + t$.
    - A marginal increase in taxes produces an increase in the consumer's price of

$$p^{*\prime}(t) + 1 \rightarrow -1 + 1 = 0$$

    - Consumers do not bear the tax burden at all.

Comparative Statics

- Perfectly inelastic supply curve (limit case)



Comparative Statics

- ***Example 6.3***:
  - Consider a competitive market in which the government will be imposing a per-unit tax t.
  - Aggregate demand curve is $x(p) = Ap^{\varepsilon}$, where $A > 0$ and $\varepsilon < 0$, and aggregate supply curve is $q(p) = ap^{\gamma}$, where $a > 0$ and $\gamma > 0$.
  - Let us evaluate how the equilibrium price is affected by a marginal increase in the tax.

Comparative Statics

- ***Example 6.3*** (continued):
  - At $t = 0$, the change in the price received by producers is

$$p^{*\prime}(0) = -\frac{x'(p^*)}{x'(p^*) - q'(p^*)}$$

$$= -\frac{A\varepsilon(p^*)^{\varepsilon-1}}{A\varepsilon(p^*)^{\varepsilon-1} - a\gamma(p^*)^{\gamma-1}} = -\frac{A\varepsilon(p^*)^{\varepsilon}}{A\varepsilon(p^*)^{\varepsilon} - a\gamma(p^*)^{\gamma}}$$

$$= -\frac{\varepsilon x(p^*)}{\varepsilon x(p^*) - \gamma q(p^*)} = -\frac{\varepsilon}{\varepsilon - \gamma}$$

  - since $x(p^*) = q(p^*)$ at $t = 0$.

Comparative Statics

- ***Example 6.3*** (continued):
  - At $t = 0$, the change in the price paid by consumers is

$$p^{*\prime}(0) + 1 = -\frac{\varepsilon}{\varepsilon - \gamma} + 1 = -\frac{\gamma}{\varepsilon - \gamma}$$

Comparative Statics

- ***Example 6.3*** (continued):
  - In the limit $\gamma \to 0$ (i.e., supply is perfectly inelastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax since

$$p^{*\prime}(0) \to -\frac{\varepsilon}{\varepsilon - 0} = -1$$

    - That is, producers bear the full burden of the tax.

Comparative Statics

- ***Example 6.3*** (continued):
  - In the limit $\varepsilon \to 0$ (i.e., demand is perfectly inelastic), the price received by producers is unchanged, and the price paid by consumers increases by the amount of the tax because

$$p^{*\prime}(0) + 1 \to -\frac{\gamma}{0 - \gamma} = 1$$

    - That is, consumers bear the full burden of the tax.

Comparative Statics

- ***Example 6.3*** (continued):
  - When $\varepsilon \to -\infty$ (i.e., demand is perfectly elastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax since

$$\lim_{\varepsilon \to -\infty} p^{*\prime}(0) = -\frac{1}{1 - \lim\limits_{\varepsilon \to -\infty} \frac{\gamma}{\varepsilon}} = -1$$

Comparative Statics

- ***Example 6.3*** (continued):
  - When $\gamma \to +\infty$ (i.e., supply is perfectly elastic), the price received by producers is unchanged, and the price paid by consumers increases by the amount of the tax because

$$\lim_{\gamma \to +\infty} p^{*\prime}(0) + 1 = -\frac{1}{\lim_{\gamma \to +\infty} \frac{\varepsilon}{\gamma} - 1} = 1$$

Welfare Analysis

Welfare Analysis

- Let us now measure the changes in the aggregate social welfare due to a change in the allocation.
- Consider the aggregate surplus

$$S = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$$

Welfare Analysis

- Take a differential change in the quantity of good x that individuals consume and that firms produce such that $\sum_{i=1}^{I} dx_i = \sum_{j=1}^{J} dq_j$.
- The change in the aggregate surplus is

$$dS = \sum_{i=1}^{I} \phi_i'(x_i)dx_i - \sum_{j=1}^{J} c_j'(q_j)dq_j$$

Welfare Analysis

- Let $P(x)$ denote the inverse aggregate demand and $C(q)$ the aggregate (industry) cost function.
- Since
  - $\phi_i'(x_i) = P(x)$ for all consumers with $x_i > 0$ and
    - That is, every individual consumes until $\mathrm{MB} = P(x)$.
  - $c_j'(q_j) = C'(q)$ for all firms,
    - That is, every firm's MC coincides with aggregate MC,

Welfare Analysis

- then the change in surplus can be rewritten as

$$dS = \sum_{i=1}^{I} P(x)dx_i - \sum_{j=1}^{J} C'(q)dq_j$$

$$= P(x)\sum_{i=1}^{I} dx_i - C'(q)\sum_{j=1}^{J} dq_j$$

Welfare Analysis

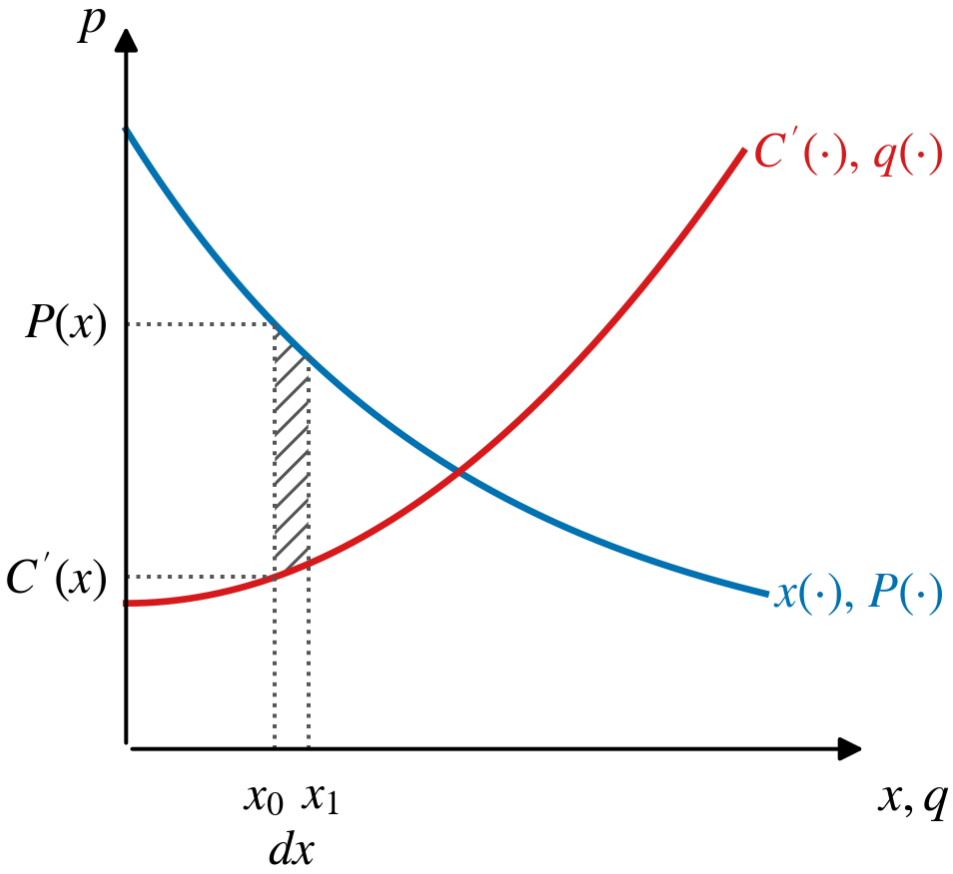
- But since $\sum_{i=1}^{I} dx_i = \sum_{j=1}^{J} dq_j = dx$, and $x = q$ by market feasibility, then

$$dS = [P(x) - C'(q)]dx$$

- *Intuition*:
  - The change in surplus of a marginal increase in consumption (and production) reflects the difference between the consumers' additional utility and the firms' additional cost of production.

Welfare Analysis

- Differential change in surplus



Welfare Analysis

- We can also integrate the above expression, eliminating the differentials, in order to obtain the total surplus for an aggregate consumption level of x:

$$S(x) = S_0 + \int_0^x [P(s) - C'(s)]ds$$

- where $S_0 = S(0)$ is the constant of integration, and represents the aggregate surplus when aggregate consumption is zero, $x = 0$.

Welfare Analysis

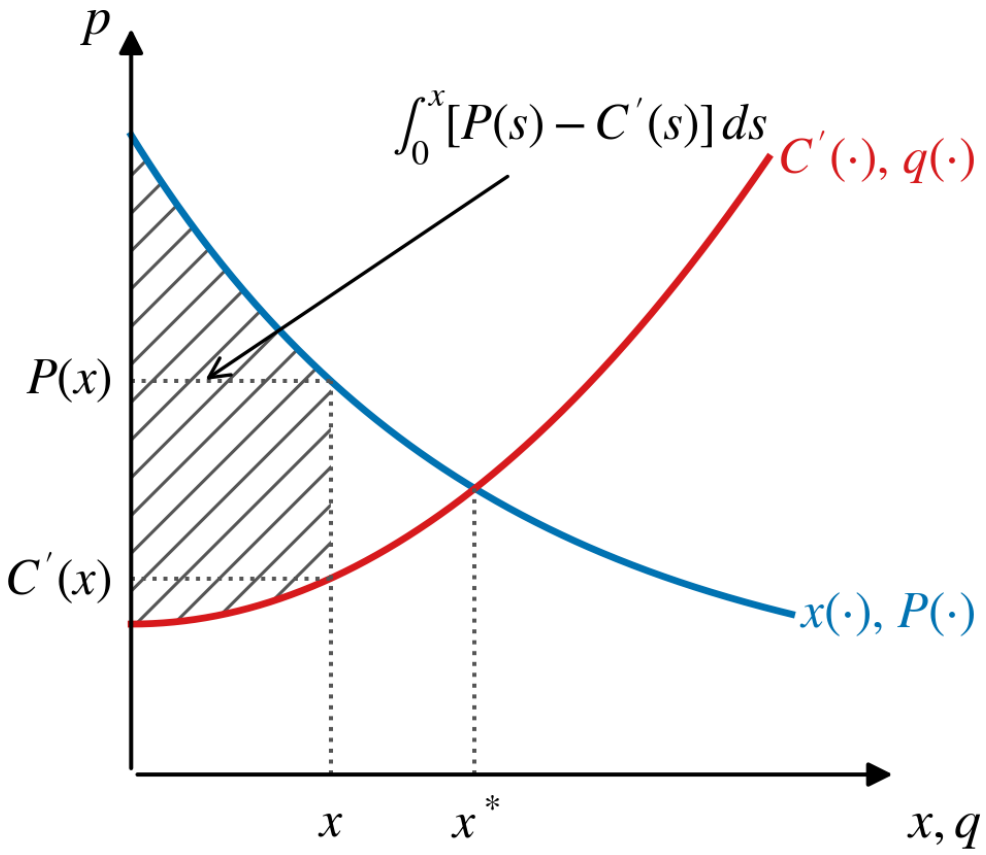
- Since $\phi_i(0) = 0$ for every consumer,

$$S_0 = -\sum_{j=1}^{J} c_j(0)$$

  - $S_0 = 0$ if there are no fixed costs, $c_j(0) = 0$ for all J firms.

Welfare Analysis

- Surplus at aggregate consumption x



Welfare Analysis

- For which consumption level is aggregate surplus $S(x)$ maximized?
  - Differentiating $S(x)$ with respect to x,
    $$S'(x^*) = P(x^*) - C'(x^*) \leq 0$$
  - with equality if $x^* > 0$; or, $P(x^*) \leq C'(x^*)$.
  - The second-order (sufficient) condition is
    $$S''(x) = \underset{-}{P'(x)} - \underset{+}{C''(x)} < 0, \qquad \text{for all} \quad x$$
    - Hence, $S(x)$ is concave.
    - Then, when $x^* > 0$, aggregate surplus $S(x)$ is maximized at $P(x^*) = C'(x^*)$.

Welfare Analysis

- In the CE, $P(x^*) = p^* = C'(x^*)$, so the CE satisfies the condition on the previous slide.
- Therefore, the CE allocation maximizes aggregate surplus.
- With quasilinear utility, maximizing aggregate surplus is equivalent to Pareto optimality.
- This is the First Welfare Theorem:
  - Every CE is Pareto optimal (PO).

Welfare Analysis

- ***Example 6.4***:
  - Consider an aggregate demand $x(p) = a - bp$ and aggregate supply $q(p) = J \cdot \frac{p}{2}$, where J is the number of firms in the industry.
  - The CE price solves

$$a - bp = J \cdot \frac{p}{2}, \qquad \text{or} \quad p^* = \frac{2a}{2b + J}$$

Welfare Analysis

- ***Example 6.4*** (continued):
  - Intuitively, as demand increases (as the number of firms increases), the equilibrium price increases (decreases):

$$\frac{\partial p^*}{\partial a} = \frac{2}{2b + J} > 0$$

$$\frac{\partial p^*}{\partial J} = -\frac{2a}{(2b + J)^2} < 0$$

Welfare Analysis

- ***Example 6.4*** (continued):
  - Therefore, equilibrium output is

$$x^* = a - b\frac{2a}{2b+J} = \frac{aJ}{2b+J}$$

  - Surplus is

$$S(x^*) = \int_0^{x^*} \big(P(x) - C'(x)\big)dx$$

Welfare Analysis

- ***Example 6.4*** (continued):
  - where $P(x) = \frac{a-x}{b}$ and, since $q = \frac{pJ}{2}$, solving for p and rearranging yields the marginal cost function

$$p = C'(x) = \frac{2x}{J}$$

  - so that

$$S(x^*) = \int_0^{x^*} \left(\frac{a-x}{b} - \frac{2x}{J} \right) dx$$

Welfare Analysis

- ***Example 6.4*** (continued):
  - Integrating,

$$S(x^*) = \left[\frac{ax}{b} - \frac{x^2}{2b} - \frac{x^2}{J}\right]_0^{x^*}$$

$$= \frac{a^2 J}{4b^2 + 2bJ} = \frac{a^2}{2b\left(1 + \frac{2b}{J}\right)}$$

  - which is increasing in the number of firms J,

$$\frac{\partial S(x^*)}{\partial J} = \frac{a^2}{(2b + J)^2} > 0$$

Appendix:
Marshall–Hicks Four Laws of Derived Demand

Marshall–Hicks Four Laws

- Consider a production function $q = f(K, L)$, with positive marginal products, $f_L, f_K > 0$.
- Assume that the supply of capital is positively sloped, $r'(K) > 0$, while the supply of labor, $w(L)$, can be upward sloping, $w'(L) > 0$, or backward bending, $w'(L) < 0$.
- Demand for output is given by $q = g(p)$, which satisfies $g'(p) < 0$.

Marshall–Hicks Four Laws

- The total cost is $w(L)L + r(K)K$.
- Assume that the capital market is perfectly competitive, but the labor and output markets are not necessarily competitive.

Marshall–Hicks Four Laws

- Define:
  - $\varepsilon_{q,p} = -\left(\frac{\partial q}{\partial p}\right)\left(\frac{p}{q}\right) > 0$ as the absolute price elasticity of output demand
  - $s_{K,r} = \left(\frac{\partial K}{\partial r}\right)\left(\frac{r}{K}\right)$ as the elasticity of the (derived) demand for capital with respect to its price r
  - $s_{L,w} = \left(\frac{\partial L}{\partial w}\right)\left(\frac{w}{L}\right)$ as the elasticity of labor supply to a change in its price
  - σ as the elasticity of substitution between inputs

Marshall–Hicks Four Laws

- We use superscript i to refer to the elasticity that an individual firm faces ($\varepsilon_{q,p}^{i}$).
- The industry elasticities do not include superscripts ($\varepsilon_{q,p}$).

Marshall–Hicks Four Laws

- Let $\theta_L \equiv \frac{wL}{pq}$ and $\theta_K \equiv \frac{rK}{pq}$ be the cost of labor and capital, respectively, relative to total sales.
- Under constant returns and zero profits,

$$pq = wL + rK, \qquad \text{so} \quad \theta_L = 1 - \theta_K$$

- For compactness, let us define

$$A \equiv 1 - \frac{1}{\varepsilon^i_{q,p}}$$

$$B \equiv 1 + \frac{1}{s^i_{L,w}}$$

Marshall–Hicks Four Laws

- Marshall, Hicks, and Allen analyze how the demand for an input bought in a competitive market, such as capital, is affected by a marginal change in the price of capital:

$$s_{K,r} = \frac{-A\left[\theta_K \varepsilon_{q,p} + \sigma\left(\varepsilon_{q,p}/s_{L,w}\right)A + \sigma\theta_L B\right]}{\left(\theta_K + \theta_L B\right)^2 + \theta_K\left(\sigma/s_{L,w}\right)A + \theta_L\left(\varepsilon_{q,p}/s_{L,w}\right)AB}$$

Sources: Marshall (1920), *Principles of Economics*, 8th ed.; Hicks (1932), *The Theory of Wages*; Allen (1938), *Mathematical Analysis for Economists*.

Marshall–Hicks Four Laws

- Marshall–Hicks's four laws of input demand ("derived demand") state that an input demand becomes more elastic ($s_{K,r}$ decreases) as each of the following increases:
  - the elasticity of substitution between inputs, σ
  - the price elasticity of output demand, $\varepsilon_{q,p}$
  - the cost of the input relative to total sales, θ_K (if $\varepsilon_{q,p} > \sigma$ and $s_{L,w} > 0$)
  - the elasticity of the other input's supply to a change in its price, $s_{L,w}$

Marshall–Hicks Four Laws

- We analyze these four comparative statics in two cases:
  - Marshall's presentation: $\varepsilon_{q,p}^{i} = s_{L,w}^{i} = \infty, \sigma = 0$
  - Hicks's presentation: $\varepsilon_{q,p}^{i} = s_{L,w}^{i} = \infty$ (no assumptions on σ)

Marshall's Presentation

- Assumptions:
  - Output and input markets are perfectly competitive, $\varepsilon_{q,p}^{i} = s_{L,w}^{i} = \infty$, for every firm i, so $A = B = 1$.
  - Inputs cannot be substituted in the production process, $\sigma = 0$.
- The expression for $s_{K,r}$ can be simplified to

$$s_{K,r} = -\frac{\theta_K \varepsilon_{q,p} s_{L,w}}{s_{L,w} + \theta_L \varepsilon_{q,p}}$$

Marshall's Presentation

- The derivatives testing the laws are:

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{\theta_K \left(s_{L,w}\right)^2}{\left(s_{L,w} + \theta_L \varepsilon_{q,p}\right)^2}$$

$$\frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{s_{L,w} \cdot \varepsilon_{q,p}\left(s_{L,w} + \varepsilon_{q,p}\right)}{\left(s_{L,w} + \theta_L \varepsilon_{q,p}\right)^2}$$

$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{\theta_K \theta_L \left(\varepsilon_{q,p}\right)^2}{\left(s_{L,w} + \theta_L \varepsilon_{q,p}\right)^2}$$

Marshall's Presentation

- If labor supply is upward sloping ($s_{L,w} > 0$), the three derivatives are all negative (the three laws hold).
- If labor supply is backward bending ($s_{L,w} < 0$), $s_{K,r}$ is still decreasing in $\varepsilon_{q,p}$ and in $s_{L,w}$, but not necessarily in θ_K.

Hicks's Presentation

- Assumptions:
  - Output and input markets are perfectly competitive, $\varepsilon_{q,p}^{i} = s_{L,w}^{i} = \infty$, for every firm i, so $A = B = 1$.
  - No condition imposed on the substitution of inputs (σ).
- The expression for $s_{K,r}$ can be simplified to

$$s_{K,r} = -\frac{\theta_K \varepsilon_{q,p} s_{L,w} + \sigma\left(\varepsilon_{q,p} + \theta_L s_{L,w}\right)}{s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}}$$

Hicks's Presentation

- The derivatives testing the laws are

$$\frac{\partial s_{K,r}}{\partial \varepsilon_{q,p}} = -\frac{\theta_K \left(s_{L,w} + \sigma\right)^2}{\left(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}\right)^2}$$

$$\frac{\partial s_{K,r}}{\partial \theta_K} = -\frac{\left(\varepsilon_{q,p} + s_{L,w}\right)\left(\varepsilon_{q,p} - \sigma\right)\left(s_{L,w} + \sigma\right)}{\left(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}\right)^2}$$

$$\frac{\partial s_{K,r}}{\partial s_{L,w}} = -\frac{\theta_K \theta_L \left(\varepsilon_{q,p} - \sigma\right)^2}{\left(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}\right)^2}$$

$$\frac{\partial s_{K,r}}{\partial \sigma} = -\frac{\theta_L \left(\varepsilon_{q,p} + s_{L,w}\right)^2}{\left(s_{L,w} + \theta_K \sigma + \theta_L \varepsilon_{q,p}\right)^2}$$

Hicks's Presentation

- Hence, $s_{K,r}$ decreases in $\varepsilon_{q,p}$, $s_{L,w}$, and σ (the three laws hold).
- If labor supply is upward sloping ($s_{L,w} > 0$), $s_{K,r}$ also decreases in θ_K when the elasticity of substitution is below the output-demand elasticity, $\sigma < \varepsilon_{q,p}$.
- If labor supply is backward bending ($s_{L,w} < 0$), $s_{K,r}$ does not necessarily decrease in θ_K.

Advanced Microeconomic Theory

Chapter 7: Monopoly

Outline

- Barriers to Entry
- Profit Maximization under Monopoly
- Welfare Loss of Monopoly
- Multiplant Monopolist
- Price Discrimination
- Advertising in Monopoly
- Regulation of Natural Monopolies
- Monopsony

Barriers to Entry

Barriers to Entry

- ***Entry barriers***: elements that make the entry of potential competitors either impossible or very costly.

- Three main categories:

  1) ***Legal***: the incumbent firm in an industry has the legal right to charge monopoly prices during the life of the patent

     – *Example*: newly discovered drugs

Barriers to Entry

2) ***Structural***: the incumbent firm has a cost or demand advantage relative to potential entrants.
   - superior technology
   - a loyal group of customers
     - positive network externalities (Facebook, eBay)

3) ***Strategic***: the incumbent monopolist has a reputation of fighting off newcomers, ultimately driving them off the market.
   - price wars

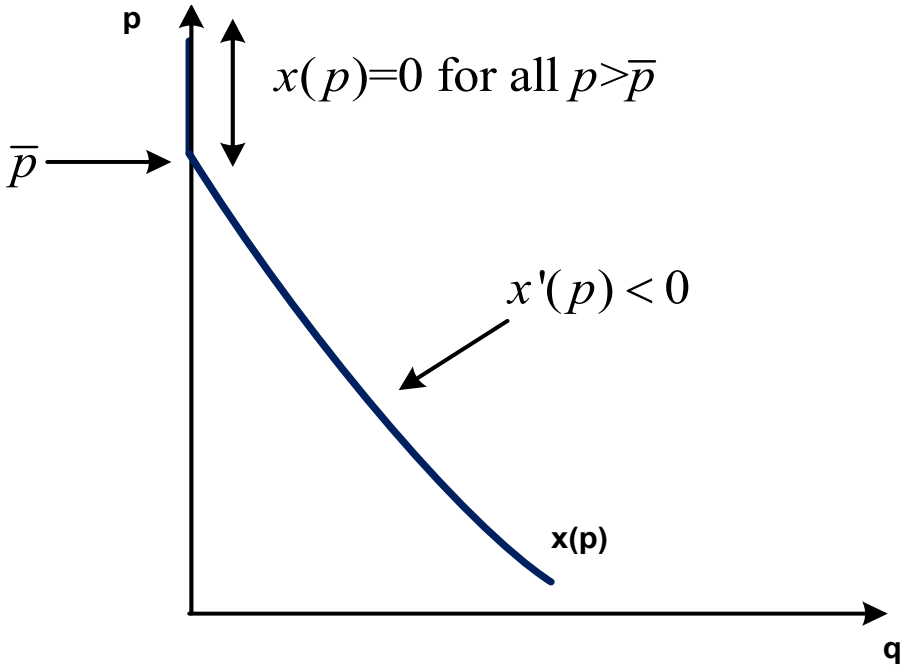
Profit Maximization under Monopoly

Profit Maximization

- Consider a demand function $x(p)$, which is continuous and strictly decreasing in p, i.e., $x'(p) < 0$.
- We assume that there is price $\bar{p} < \infty$ such that $x(p) = 0$ for all $p > \bar{p}$.
- Also, consider a general cost function $c(q)$, which is increasing and convex in q.

Profit Maximization

- $\bar{p}$ is a "choke price"
- No consumers buy positive amounts of the good for $p > \bar{p}$.



Profit Maximization

- Monopolist's decision problem is

$$\max_{p} \; px(p) - c(x(p))$$

- Alternatively, using $x(p) = q$, and taking the inverse demand function $p(q) = x^{-1}(q)$, we can rewrite the monopolist's problem as

$$\max_{q \geq 0} \; p(q)q \; - c(q)$$

Profit Maximization

- Differentiating with respect to q,

$$p(q^m) + p'(q^m)q^m - c'(q^m) \leq 0$$

- Rearranging,

$$\underbrace{p(q^m) + p'(q^m)q^m}_{MR=\frac{d[p(q)q]}{dq}} \leq \underbrace{c'(q^m)}_{MC}$$

with equality if $q^m > 0$.

- Recall that total revenue is $TR(q) = p(q)q$

Profit Maximization

- In addition, we assume that $p(0) \geq c'(0)$.
  - That is, the inverse demand curve originates above the marginal cost curve.
  - Hence, the consumer with the highest willingness to pay for the good is willing to pay more than the variable costs of producing the first unit.
- Then, we must be at an interior solution $q^m > 0$, implying

$$\underbrace{p(q^m) + p'(q^m)q^m}_{MR} = \underbrace{c'(q^m)}_{MC}$$

Profit Maximization

- Note that

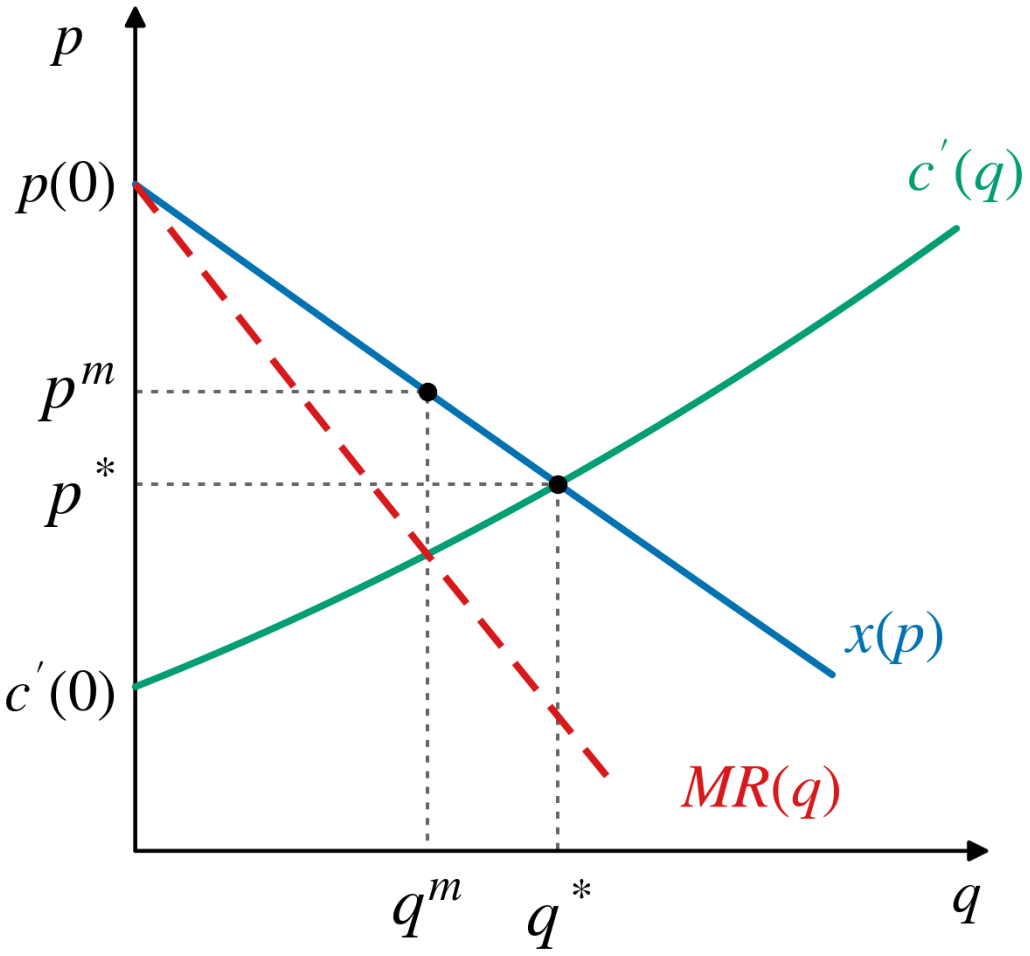
$$p(q^m) + \underbrace{p'(q^m)q^m}_{-} = c'(q^m)$$

- Then, $p(q^m) > c'(q^m)$, i.e.,

monopoly price $> MC$

- Moreover, we know that in competitive equilibrium $p(q^*) = c'(q^*)$.
- Then, $p^m > p^*$ and $q^m < q^*$.

Profit Maximization



Profit Maximization

- Marginal revenue in monopoly

$$MR = p(q^m) + p'(q^m)q^m$$

MR describes two effects:

  - **A *direct* (positive) effect**: an additional unit can be sold at $p(q^m)$, thus increasing revenue by $p(q^m)$.
  - **An *indirect* (negative) effect**: selling an additional unit can only be done by reducing the market price of all units (the new and all previous units), ultimately reducing revenue by $p'(q^m)q^m$.
    - *Inframarginal units* – initial units before the marginal increase in output.

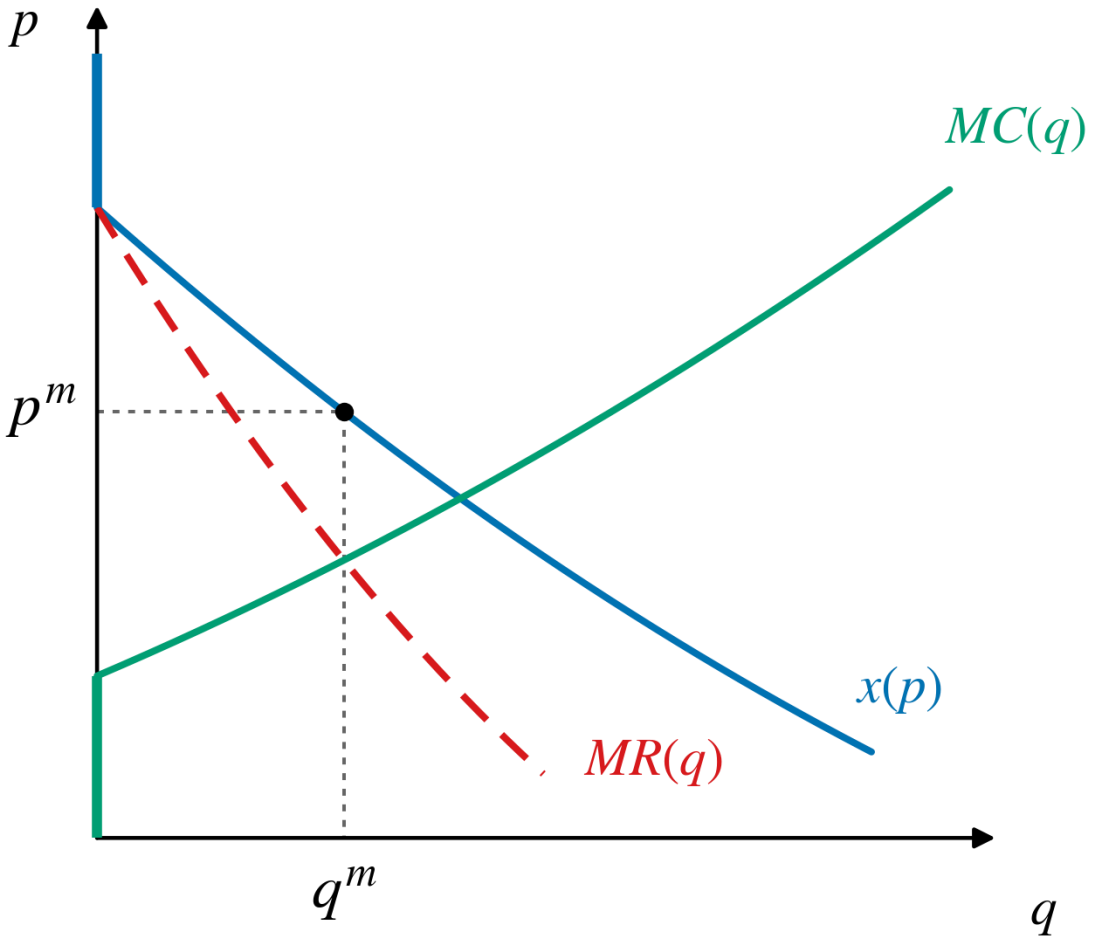
Profit Maximization

- Is the above FOC also sufficient?
  - Let's take the FOC $p(q^m) + p'(q^m)q^m - c'(q^m)$, and differentiate it wrt q,

$$\underbrace{p'(q) + p'(q) + p''(q)q}_{\frac{dMR}{dq}} - \underbrace{c''(q)}_{\frac{dMC}{dq}} \leq 0$$

  - That is, $\frac{dMR}{dq} \leq \frac{dMC}{dq}$.
  - Since MR curve is decreasing and MC curve is weakly increasing, the second-order condition is satisfied for all q.

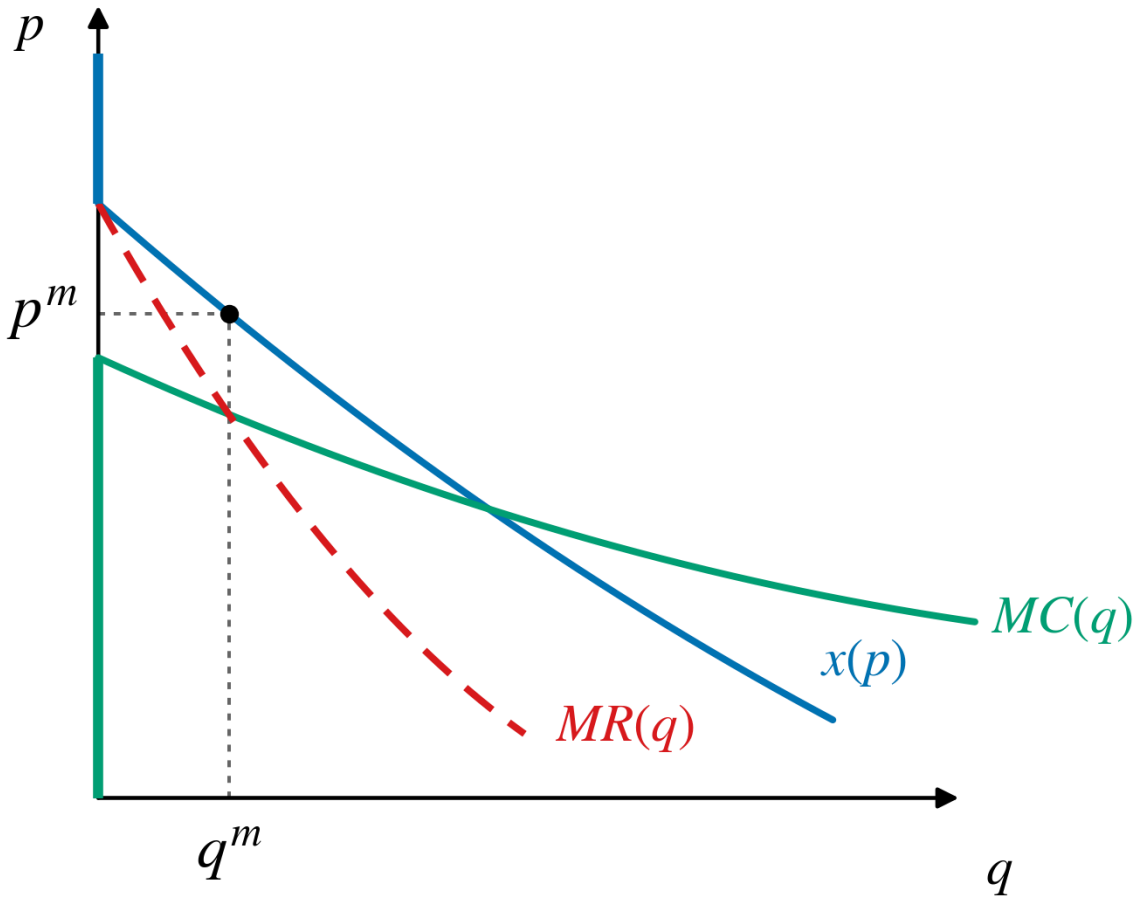
Profit Maximization



Profit Maximization

- What would happen if MC curve was decreasing in q (e.g., concave technology given the presence of increasing returns to scale)?
  - Then, the slopes of MR and MC curves are both decreasing.
  - At the optimum, MR curve must be steeper than MC curve.

Profit Maximization



Profit Maximization: Lerner Index

- Can we re-write the FOC in a more intuitive way? Yes.
  - Just take $MR = p(q) + p'(q)q = p + \frac{\partial p}{\partial q} q$ and multiply by $\frac{p}{p}$,

$$MR = p\frac{p}{p} + \underbrace{\frac{\partial p}{\partial q}\frac{q}{p}}_{1/\varepsilon_d} p = p + \frac{1}{\varepsilon_d} p$$

  - In equilibrium, $MR(q) = MC(q)$. Hence, we can replace MR with MC in the above expression.

Profit Maximization: Lerner Index

- Rearranging yields

$$\frac{p - MC(q)}{p} = -\frac{1}{\varepsilon_d}$$

- This is the ***Lerner index*** of market power
  - The price mark-up over marginal cost that a monopolist can charge is a function of the elasticity of demand.
- Note:
  - If $\varepsilon_d \to \infty$, then $\frac{p-MC(q)}{p} \to 0 \implies p = MC(q)$
  - If $\varepsilon_d \to 0$, then $\frac{p-MC(q)}{p} \to \infty \implies$ substantial mark-up

Profit Maximization: Lerner Index

- The Lerner index can also be written as

$$p = \frac{MC(q)}{1+\frac{1}{\varepsilon_d}}$$

  which is referred to as the ***Inverse Elasticity Pricing Rule*** (IEPR).
- *Example* (Perloff, 2012):
  - Prilosec OTC: $\varepsilon_d = -1.2$. Then price should be $p = \frac{MC(q)}{1+\frac{1}{-1.2}} = 6MC$
  - Designer jeans: $\varepsilon_d = -2$. Then price should be $p = \frac{MC(q)}{1+\frac{1}{-2}} = 2MC$

Profit Maximization: Lerner Index

- ***Example 1*** (linear demand):
  - Market inverse demand function is
    $$p(q) = a - bq$$
    where $b > 0$
  - Monopolist's cost function is $c(q) = cq$
  - We usually assume that $a > c \geq 0$
    - To guarantee $p(0) > c'(0)$
    - That is, $p(0) = a - b0 = a$ and $c'(q) = c$, thus implying $c'(0) = c$

Profit Maximization: Lerner Index

- ***Example 1*** (continued):
  - Monopolist's objective function

$$\pi(q) = (a - bq)q - cq$$

  - FOC: $a - 2bq - c = 0$
  - SOC: $-2b < 0$ (concave)
    - Note that as long as $b > 0$, i.e., negatively sloped demand function, profits will be concave in output.
    - Otherwise (i.e., Giffen good, with positively sloped demand function) profits will be convex in output.

Profit Maximization: Lerner Index

- ***Example 1*** (continued):
  - Solving for the optimal q^m in the FOC, we find monopoly output

$$q^m = \frac{a-c}{2b}$$

  - Inserting $q^m = \frac{a-c}{2b}$ in the demand function, we obtain monopoly price

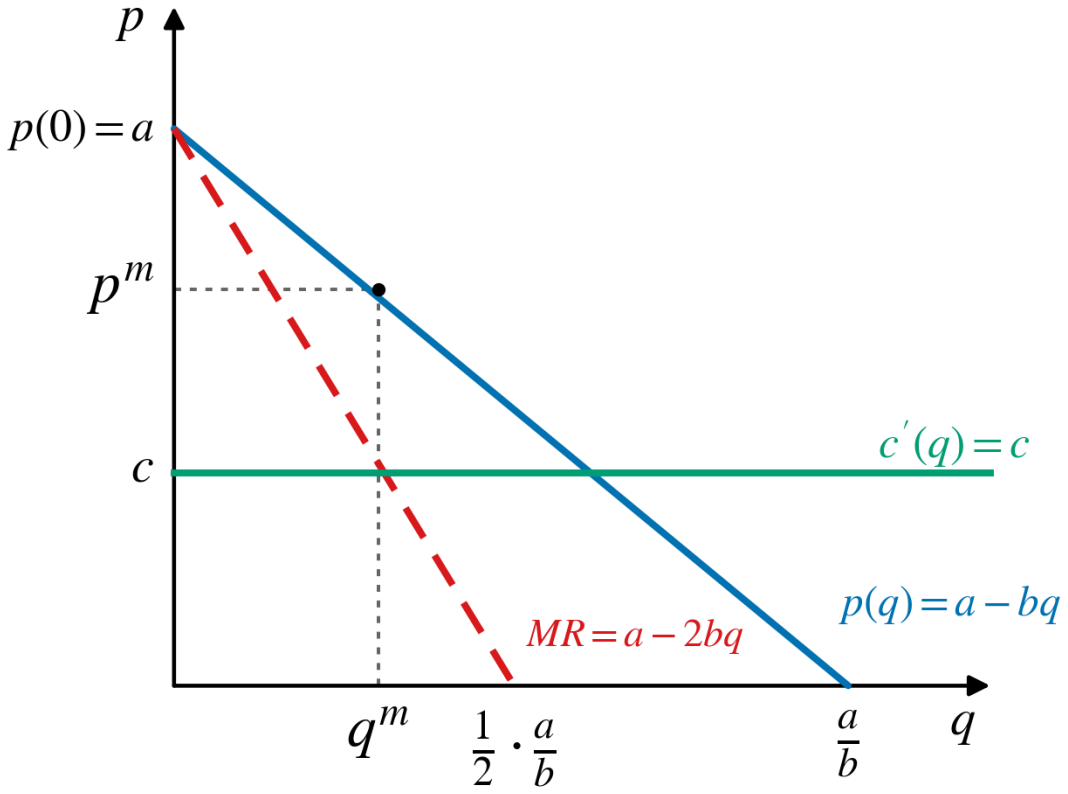
$$p^m = a - b\left(\frac{a-c}{2b}\right) = \frac{a+c}{2}$$

  - Hence, monopoly profits are

$$\pi^m = p^m q^m - cq^m = \frac{(a-c)^2}{4b}$$

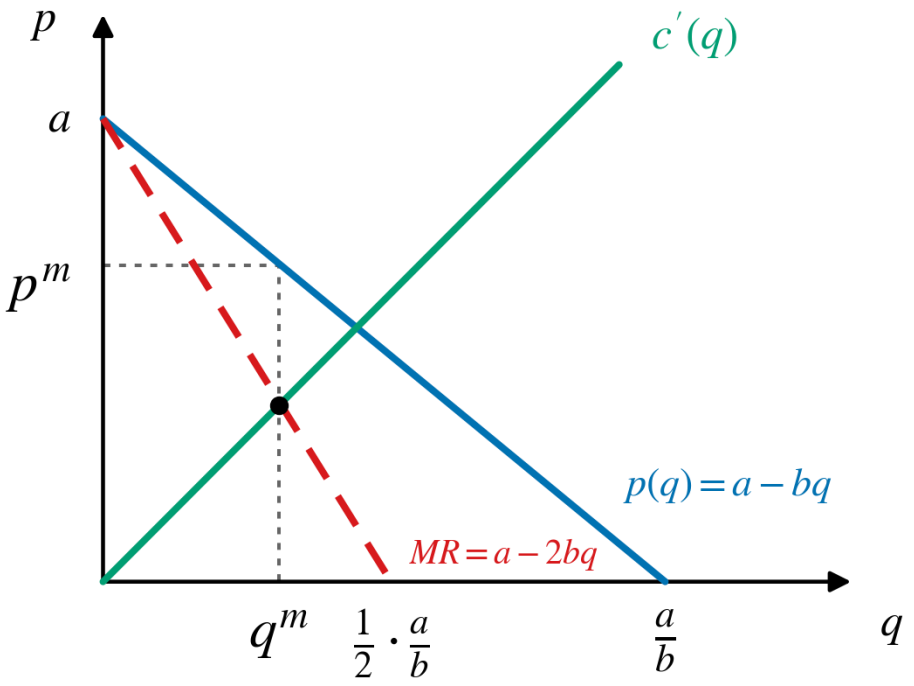
Profit Maximization: Lerner Index

- ***Example 1*** (continued):



Profit Maximization: Lerner Index

- ***Example 1*** (continued):
  - Non-constant marginal cost
  - The cost function is convex in output
    $$c(q) = cq^2$$
  - Marginal cost is
    $$c'(q) = 2cq$$



Profit Maximization: Lerner Index

- ***Example 2*** (Constant elasticity demand):
  - The demand function is

$$q(p) = Ap^{-b}$$

  - We can show that $\varepsilon(q) = -b$ for all q, i.e.,

$$\varepsilon(q) = \frac{\partial q(p)}{\partial p}\frac{p}{q} = \underbrace{(-b)Ap^{-b-1}}_{\frac{\partial q(p)}{\partial p}}\underbrace{\frac{p}{Ap^{-b}}}_{\frac{p}{q}}$$

$$= -b\frac{p^{-b}}{p}\frac{p}{p^{-b}} = -b$$

Profit Maximization: Lerner Index

- ***Example 2*** (continued):
  - We can now plug $\varepsilon(q) = -b$ into the Lerner index,

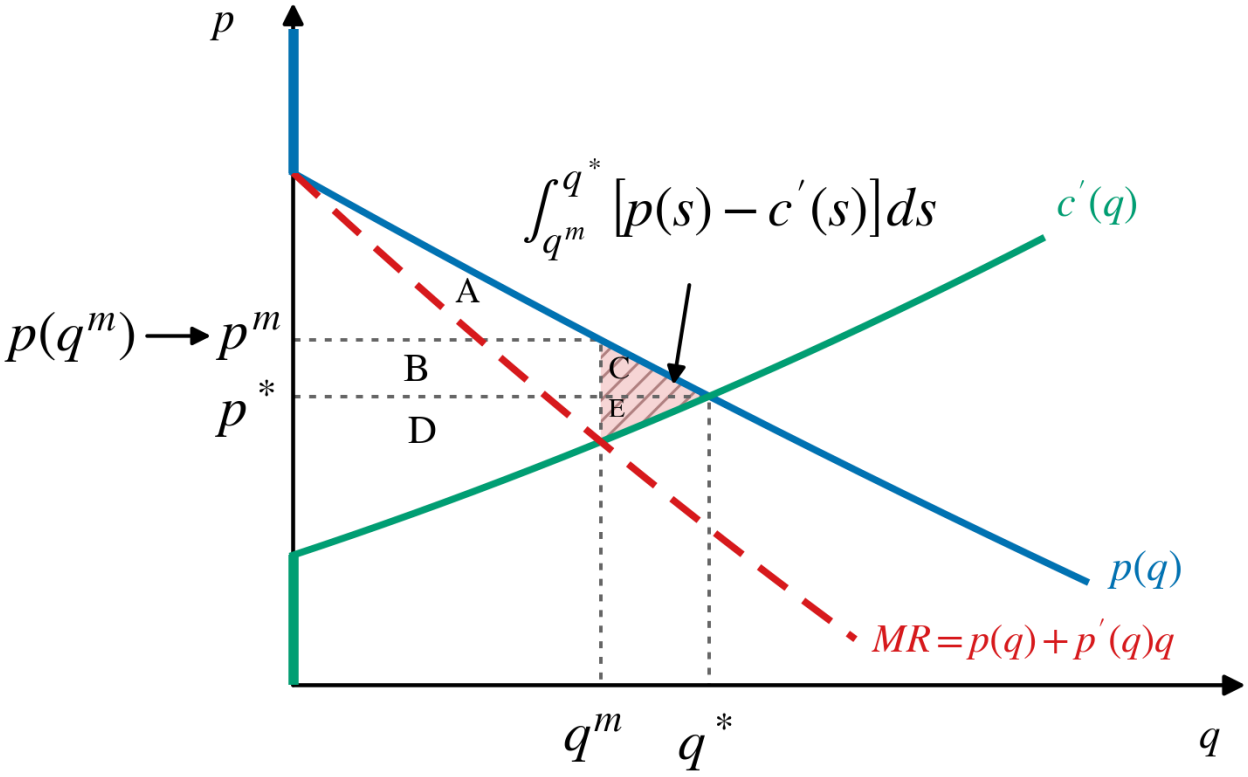
$$p^m = \frac{c}{1 + \frac{1}{\varepsilon(q)}} = \frac{c}{1 - \frac{1}{b}} = \frac{bc}{b-1}$$

  - That is, price is a constant mark-up over marginal cost.

Welfare Loss of Monopoly

Welfare Loss of Monopoly

- Welfare comparison for perfect competition and monopoly.



Welfare Loss of Monopoly

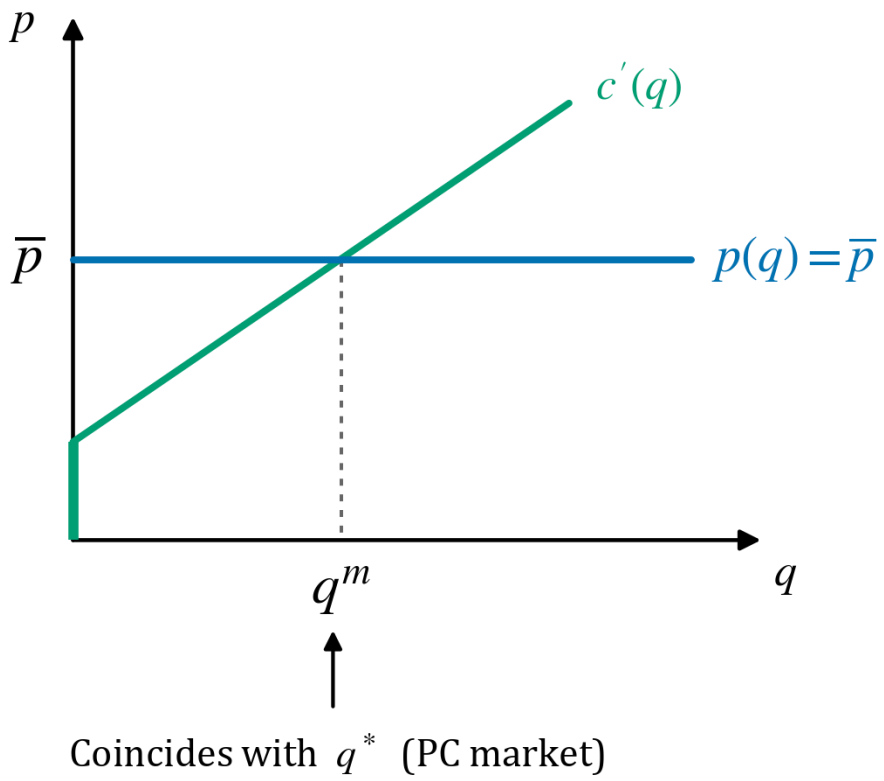
- Consumer surplus
  - Perfect competition: *A*+*B*+*C*
  - Monopoly: *A*
- Producer surplus:
  - Perfect competition: *D*+*E*
  - Monopoly: *B*+*D*
- Deadweight loss of monopoly: *C*+*E*

$$DWL = \int_{q^m}^{q^*} [p(s) - c'(s)]ds$$

- DWL decreases as demand and/or supply become more elastic.

Welfare Loss of Monopoly

- Infinitely elastic demand
$$p'(q) = 0$$
- The inverse demand curve becomes totally flat.
- Marginal revenue coincides with inverse demand:
$$MR(q) = p(q) + 0 \cdot q$$
$$= p(q)$$
- Profit-maximizing q
$$MR(q) = MC(q) \Longrightarrow$$
$$p(q) = MC(q)$$
- Hence, $q^m = q^*$ and $DWL = 0$.



Welfare Loss of Monopoly

- ***Example*** (Welfare losses and elasticity):
  - Consider a monopolist with constant marginal and average costs, $c'(q) = c$, who faces a market demand with constant elasticity

    $$q(p) = p^e$$

    where e is the price elasticity of demand ($e < -1$)
  - Perfect competition: $p_c = c$
  - Monopoly: using the IEPR

    $$p^m = \frac{c}{1 + \frac{1}{e}}$$

Welfare Loss of Monopoly

- ***Example*** (continued):
  - The consumer surplus associated with any price (p_0) can be computed as

$$CS = \int_{p_0}^{\infty} Q(P)dp = \int_{p_0}^{\infty} p^e dp = \frac{p^{e+1}}{e+1}\Bigg|_{p_0}^{\infty} = -\frac{p_0^{e+1}}{\underbrace{e+1}_{<0}} > 0$$

  - Under perfect competition, $p_c = c$,

$$CS = -\frac{c^{e+1}}{e+1}$$

  - Under monopoly, $p^m = \frac{c}{1+1/e}$,

$$CS_m = -\frac{\left(\frac{c}{1+1/e}\right)^{e+1}}{e+1}$$

Welfare Loss of Monopoly

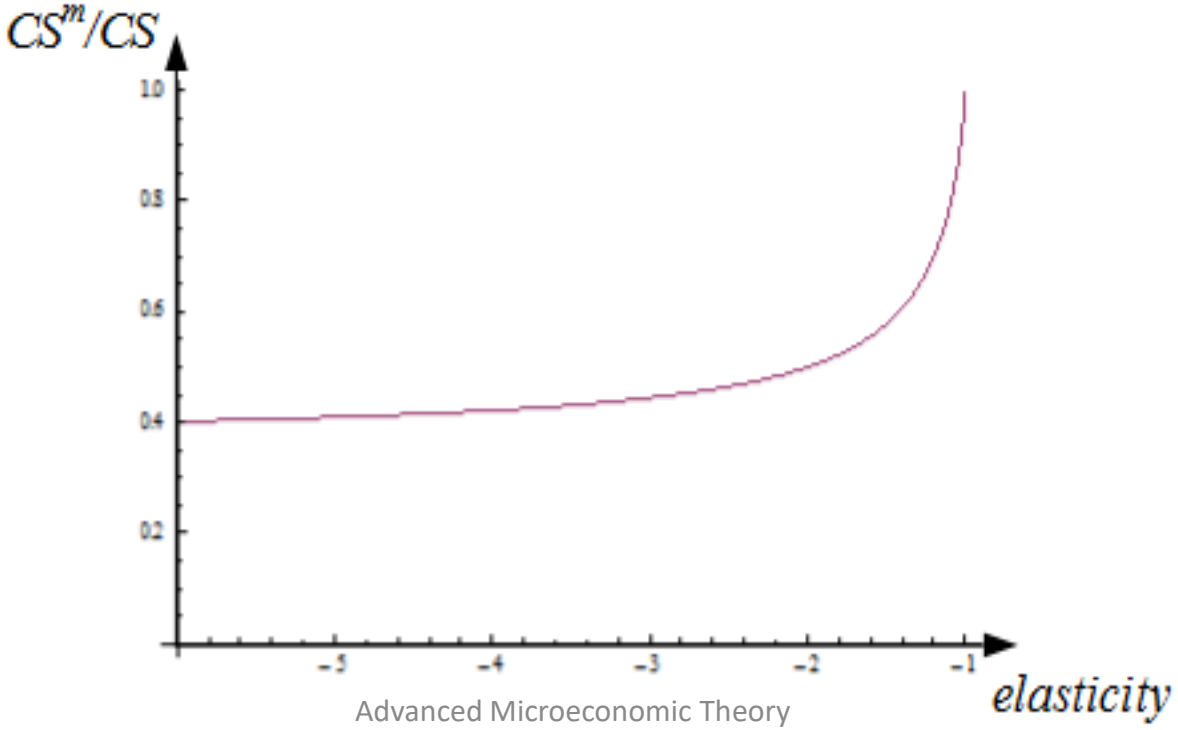
- ***Example*** (continued):
  - Taking the ratio of these two surpluses

$$\frac{CS_m}{CS} = \left(\frac{1}{1+1/e}\right)^{e+1}$$

  - If $e = -2$, this ratio is ½
    - CS under monopoly is half of that under perfectly competitive markets

Welfare Loss of Monopoly

- ***Example*** (continued):
  - The ratio $\frac{CS_m}{CS} = \left(\frac{1}{1+1/e}\right)^{e+1}$ decreases as demand becomes more elastic (e increases in absolute value).



Welfare Loss of Monopoly

- ***Example*** (continued):
  - Monopoly profits are given by

$$\pi^m = p^m q^m - cq^m = \left(\frac{c}{1+1/e} - c\right) q^m$$

where $q^m(p) = p^e = \left(\frac{c}{1+1/e}\right)^e$.

  - Rearranging,

$$\pi^m = \left(\frac{-c/e}{1+1/e}\right)\left(\frac{c}{1+1/e}\right)^e$$

$$= -\left(\frac{c}{1+1/e}\right)^{e+1} \cdot \frac{1}{e}$$

Welfare Loss of Monopoly

- ***Example*** (continued):
  - To find the transfer from CS into monopoly profits that consumers experience when moving from perfect competition to a monopoly, divide monopoly profits by the competitive CS

$$\frac{\pi^m}{CS} = \left(\frac{e+1}{e}\right)\left(\frac{1}{1+1/e}\right)^{e+1} = \left(\frac{e}{1+e}\right)^{e}$$

  - If $e = -2$, this ratio is ¼
    - One-fourth of the consumer surplus under perfectly competitive markets is transferred to monopoly profits

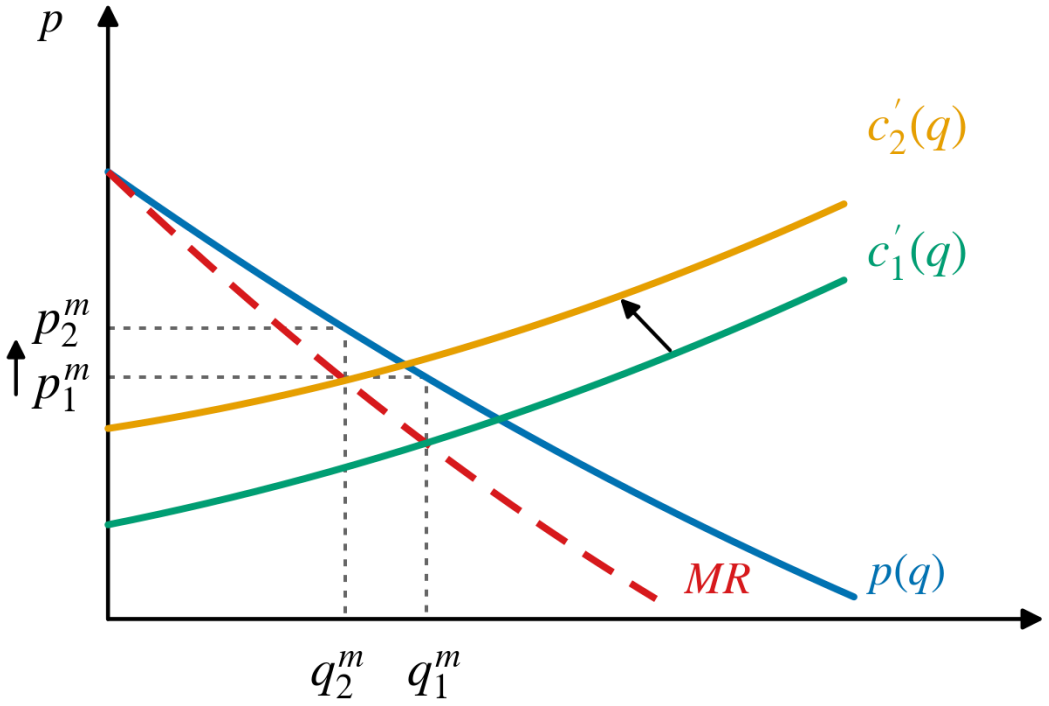
Welfare Loss of Monopoly

- More social costs of monopoly:
  - Excessive R&D expenditure (patent race)
  - Persuasive (not informative) advertising
  - Lobbying costs (different from bribes)
  - Resources to avoid entry of potential firms in the industry

Comparative Statics

Comparative Statics

- We want to understand how q^m varies as a function of the monopolist's marginal cost



Comparative Statics

- Formally, we know that at the optimum, $q^m(c)$, the monopolist maximizes its profits

$$\frac{\partial \pi(q^m(c), c)}{\partial q^m} = 0$$

- Differentiating wrt c, and using the chain rule,

$$\frac{\partial^2 \pi(q^m(c), c)}{\partial q^2} \frac{dq^m(c)}{dc} + \frac{\partial^2 \pi(q^m(c), c)}{\partial q \partial c} = 0$$

- Solving for $\frac{dq^m(c)}{dc}$, we have

$$\frac{dq^m(c)}{dc} = -\frac{\frac{\partial^2 \pi(q^m(c), c)}{\partial q \partial c}}{\frac{\partial^2 \pi(q^m(c), c)}{\partial q^2}}$$

Comparative Statics

- ***Example***:
  - Assume linear demand curve $p(q) = a - bq$
  - Then, the cross-derivative is

$$\frac{\partial^2 \pi(q^m(c), c)}{\partial q \partial c} = \frac{\partial \left(\frac{\partial [(a - bq)q - cq]}{\partial q} \right)}{\partial c}$$

$$= \frac{\partial [a - 2bq - c]}{\partial c} = -1$$

and

$$\frac{dq^m(c)}{dc} = -\frac{\frac{\partial^2 \pi(q^m(c), c)}{\partial q \partial c}}{\frac{\partial^2 \pi(q^m(c), c)}{\partial q^2}} = -\frac{-1}{-2b} < 0$$

Comparative Statics

- ***Example*** (continued):
  - That is, an increase in marginal cost, c, decreases monopoly output, q^m.
  - Similarly for any other demand.
  - Even if we don't know the accurate demand function, but know the sign of

$$\frac{\partial^2 \pi(q^m(c), c)}{\partial q \partial c}$$

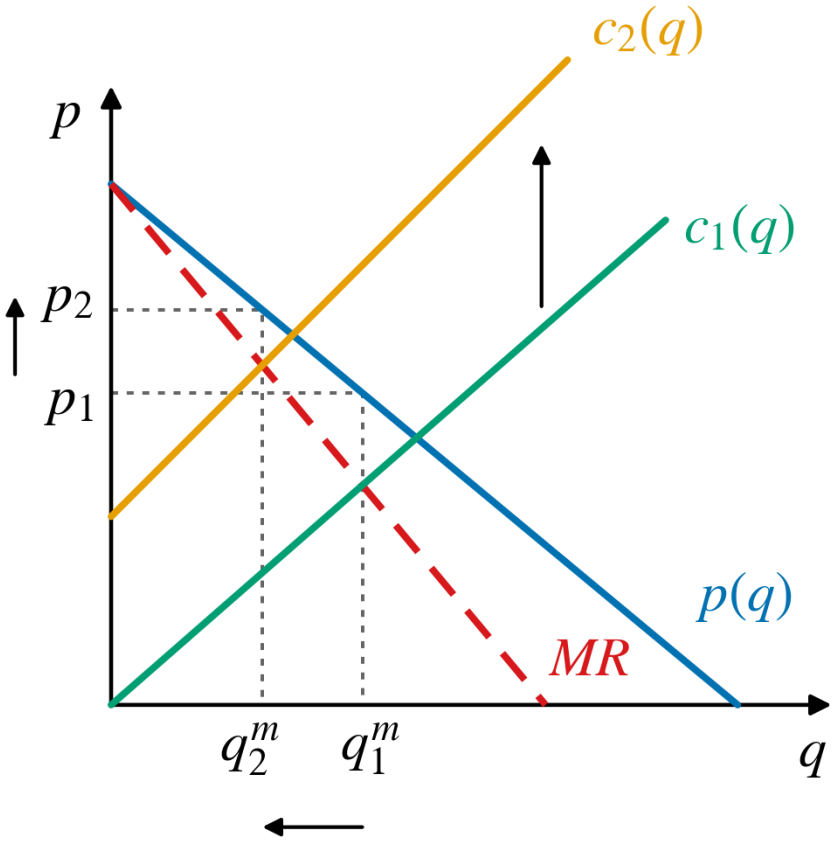
Comparative Statics

- ***Example*** (continued):
  - Marginal costs are increasing in q
  - For convex cost curve $c(q) = cq^2$, monopoly output is

$$q^m(c) = \frac{a}{2(b+c)}$$

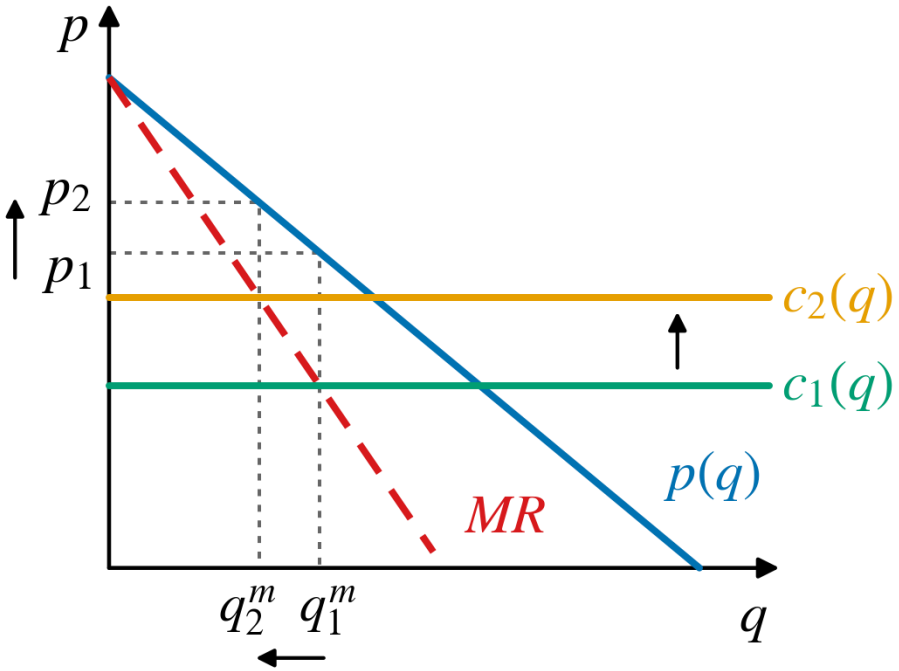
  - Here,

$$\frac{dq^m(c)}{dc} = -\frac{a}{2(b+c)^2} < 0$$



Comparative Statics

- ***Example*** (continued):
  - Constant marginal cost
  - For the constant-elasticity demand curve $q(p) = p^e$, we have $p^m = \frac{c}{1+1/e}$ and $q^m(c) = \left(\frac{ec}{1+e}\right)^e$
  - Here,
$$\frac{dq^m(c)}{dc} = \frac{e}{c}\left(\frac{ec}{1+e}\right)^e = \frac{e}{c}q^m < 0$$



Multiplant Monopolist

Multiplant Monopolist

- Monopolist produces output $q_1, q_2, \dots, q_N$ across N plants it operates, with total costs $TC_i(q_i)$ at each plant $i = \{1,2, \dots, N\}$.
- Profits-maximization problem

$$\max_{q_1,\dots,q_N} \left[a - b \sum_{i=1}^{N} q_i\right] \sum_{i=1}^{N} q_i - \sum_{i=1}^{N} TC_i(q_i)$$

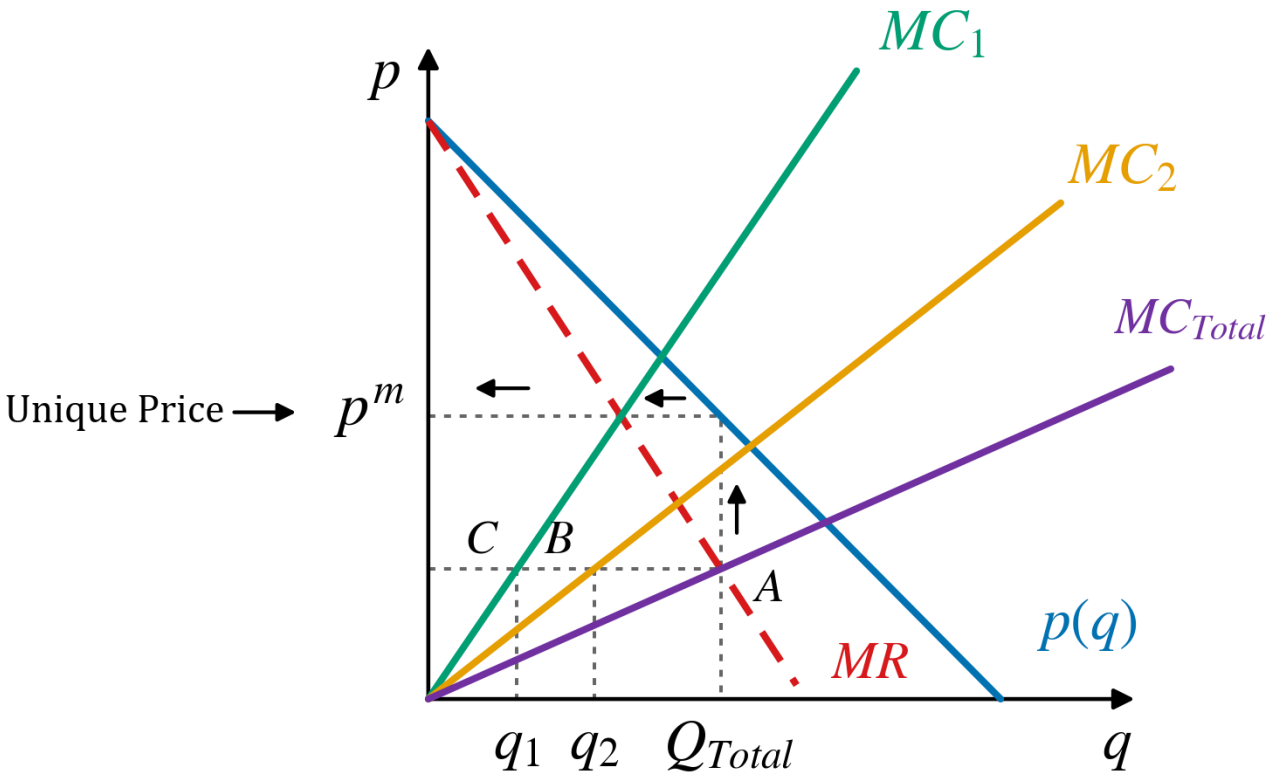
- FOCs wrt production level at every plant j

$$a - 2b \sum_{i=1}^{N} q_i - MC_j(q_j) = 0$$
$$\Leftrightarrow MR(Q) = MC_j(q_j)$$

for all j.

Multiplant Monopolist

- Multiplant monopolist operating two plants with marginal costs MC_1 and MC_2.



Multiplant Monopolist

- Total marginal cost is $MC_{total} = MC_1 + MC_2$ (i.e., horizontal sum)
- Q_{total} is determined by $MR = MC_{total}$ (i.e., point A)
- Mapping Q_{total} in the demand curve, we obtain price p^m (both plants selling at the same price)
- At the MC level for which $MR = MC_{total}$ (i.e., point A), extend a line to the left crossing MC_1 and MC_2.
- This will give us output levels q_1 and q_2 that plants 1 and 2 produce, respectively.

Multiplant Monopolist

- ***Example 1*** (symmetric plants):
  - Consider a monopolist operating N plants, where all plants have the *same* cost function $TC_i(q_i) = F + cq_i^2$. Hence, all plants produce the same output level $q_1 = q_2 = \cdots = q_N = q$ and $Q = Nq_j$. The linear demand function is given by $p = a - bQ$.
  - FOCs:

$$a - 2b\sum_{j=1}^{N} q_j = 2cq_j \quad \text{or} \quad a - 2bNq_j = 2cq_j$$

$$q_j = \frac{a}{2(bN + c)}$$

Multiplant Monopolist

- ***Example 1*** (continued):
  - Total output produced by the monopolist is

$$Q = Nq_j = \frac{Na}{2(bN + c)}$$

and market price is

$$p = a - bQ = a - b\frac{Na}{2(bN + c)} = \frac{a(bN + 2c)}{2(bN + c)}$$

  - Hence, the profits of every plant j are

$$\pi_j = \left(\frac{a(bN + 2c)}{2(bN + c)}\right)\frac{a}{2(bN + c)} - c\left(\frac{a}{2(bN + c)}\right)^2 = \frac{a^2}{4(bN + c)} - F$$

  - Total profits become

$$\pi_{total} = \frac{Na^2}{4(bN + c)} - NF$$

Multiplant Monopolist

- ***Example 1*** (continued):
  - The optimal number of plants N^* is determined by

$$\frac{d\pi_{total}}{dN} = \frac{a^2}{4}\frac{c}{(bN+c)^2} - F = 0$$

and solving for N

$$N^* = \frac{1}{b}\left(\frac{a}{2}\sqrt{\frac{c}{F}} - c\right)$$

  - N^* is decreasing in the fixed costs F
  - N^* is decreasing in c as long as $a < 4\sqrt{cF}$, since

$$\frac{dN^*}{dc} = \frac{1}{b}\left(\frac{a - 4\sqrt{CF}}{4\sqrt{CF}}\right)$$

Multiplant Monopolist

- ***Example 1*** (continued):
  - Note that when $N = 1$, $Q = q^m$ and $p = p^m$.
  - Note that an increase in N decreases q_j and π_j, as

$$\frac{dq_j}{dN} = -\frac{ab}{2(bN + c)^2} < 0$$

$$\frac{d\pi_j}{dN} = -\frac{a^2 b}{4(bN + c)^2} < 0$$

Multiplant Monopolist

- ***Example 2*** (asymmetric plants):
  - Consider a monopolist operating two plants with marginal costs $MC_1(q_1) = 10 + 20q_1$ and $MC_2(q_2) = 60 + 5q_2$, respectively. A linear demand function is given by $p(Q) = 120 - 3Q$.
  - Note that $MC_{total} \neq MC_1(q_1) + MC_2(q_2)$
    - This is a vertical (not a horizontal) sum.
  - Instead, first invert the marginal cost functions

$$MC_1(q_1) = 10 + 20q_1 \Leftrightarrow q_1 = \frac{MC_1}{20} - \frac{1}{2}$$

$$MC_2(q_2) = 60 + 5q_2 \Leftrightarrow q_2 = \frac{MC_2}{5} - 12$$

Multiplant Monopolist

- ***Example 2*** (continued):
  - Second,

$$Q_{total} = q_1 + q_2 = \frac{MC_{total}}{20} - \frac{1}{2} + \frac{MC_{total}}{5} - 12$$
$$= \frac{1}{4} MC_{total} - 12.5$$

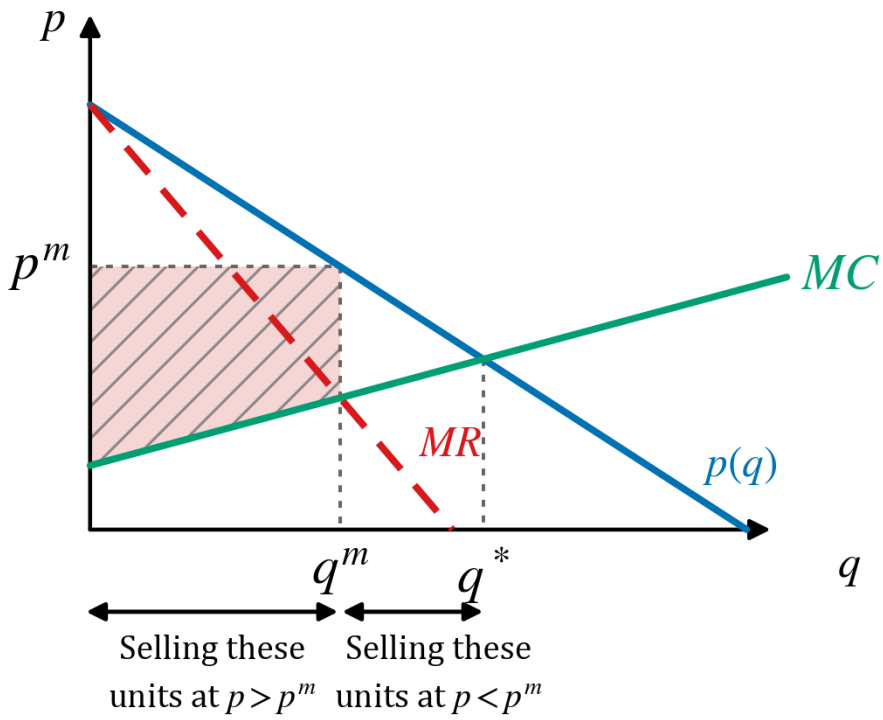
  - Hence, $MC_{total} = 50 + 4Q_{total}$
  - Setting $MR(Q) = 120 - 6Q = 50 + 4Q = MC_{total}$, we obtain $Q_{total} = 7$ and $p = 120 - 3 \cdot 7 = 99$.
  - Since $MR(Q_{total}) = 120 - 6 \cdot 7 = 78$, then

$$MR(Q_{total}) = MC_1(q_1) \Longrightarrow 78 = 10 + 20q_1 \Longrightarrow q_1 = 3.4$$
$$MR(Q_{total}) = MC_2(q_2) \Longrightarrow 78 = 60 + 5q_2 \Longrightarrow q_2 = 3.6$$

Price Discrimination

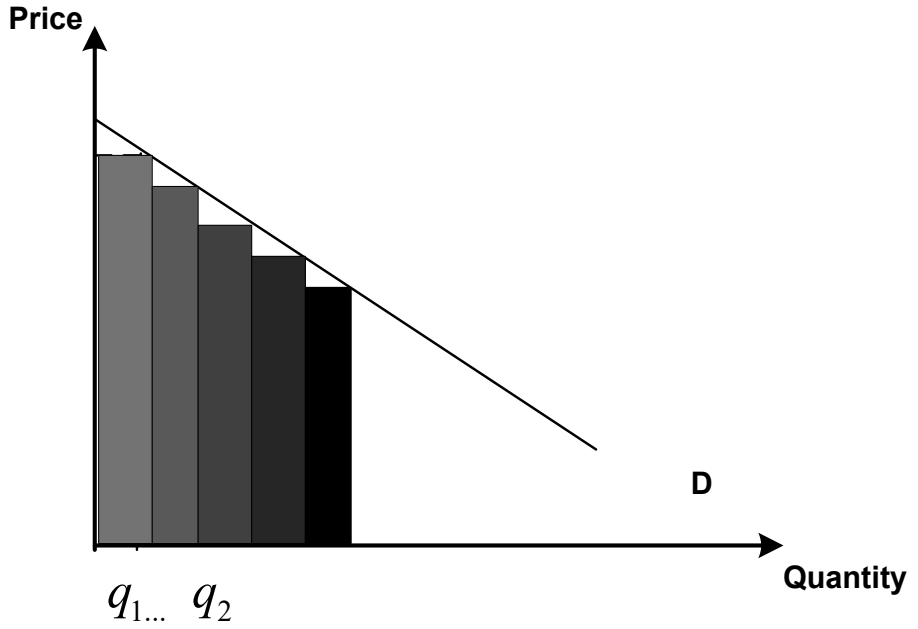
Price Discrimination

- Can the monopolist capture an even larger surplus?
  - Charge $p > p^m$ to those who buy the product at p^m and are willing to pay more
  - Charge $c < p < p^m$ to those who do not buy the product at p^m, but whose willingness to pay for the good is still higher than the marginal cost of production, c.
  - With p^m for all units, the monopolist does not capture the surplus of neither of these segments.



Price Discrimination: First-degree

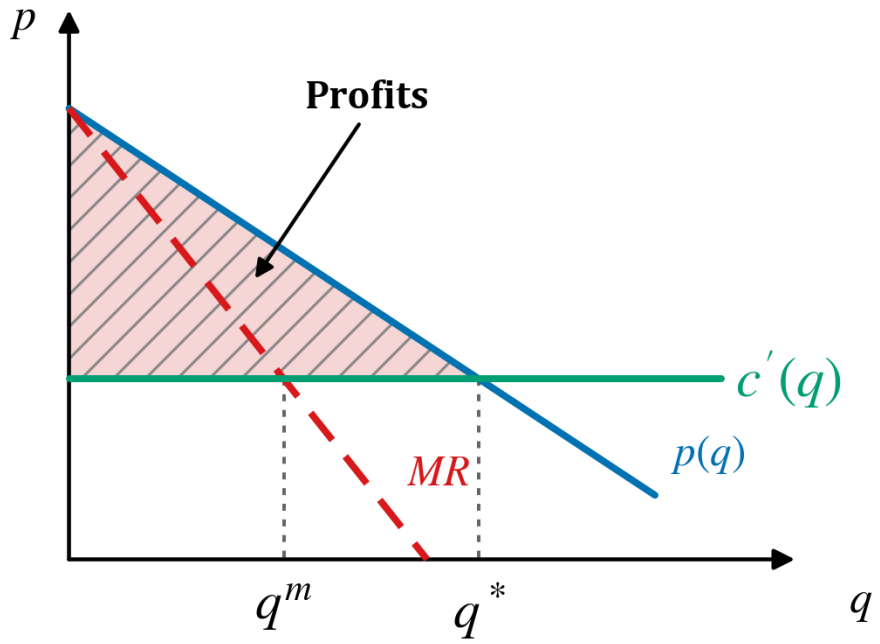
- ***First-degree (perfect) price discrimination***:
  - The monopolist charges to every customer his/her maximum willingness to pay for the object.
  - *Personalized price*: The first buyer pays p_1 for the q_1 units, the second buyer pays p_2 for $q_2 - q_1$ units, etc.



Price Discrimination: First-degree

– The monopolist continues doing so until the last buyer is willing to pay the marginal cost of production.

– In the limit, the monopolist captures all the area below the demand curve and above the marginal cost (i.e., consumer surplus).



Price Discrimination: First-degree

- Suppose that the monopolist can charge a fixed fee, r^*, and an amount of the good, q^*, that maximizes profits.
- PMP:

$$\max_{r,q} \; r - cq$$
$$s.t. \; u(q) \geq r$$

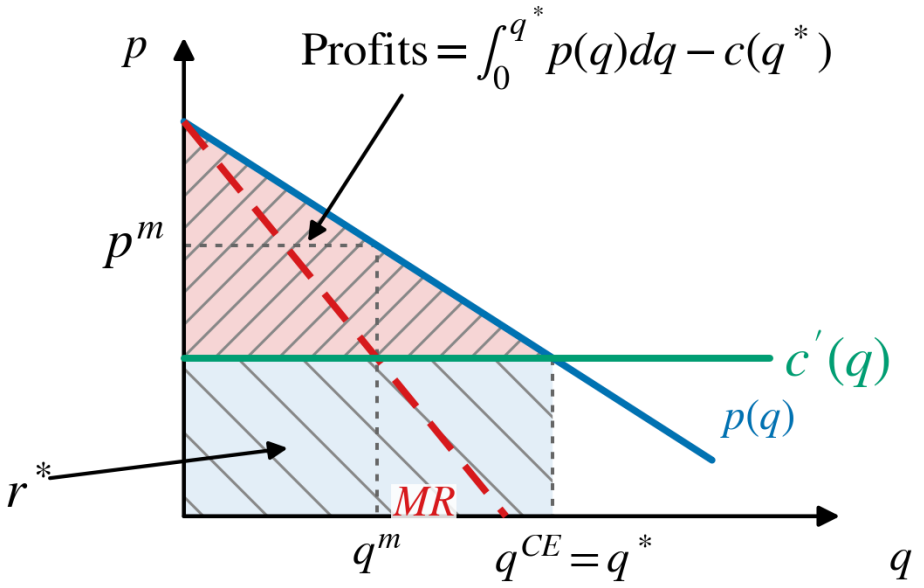
- Note that the monopolist raises the fee r until $u(q) = r$. Hence we can reduce the set of choice variables

$$\max_{q} \; u(q) - cq$$

- FOC: $u'(q^*) - c = 0$ or $u'(q^*) = c$.
  - *Intuition*: the monopolist increases output until the marginal utility that consumers obtain from additional units coincides with the marginal cost of production

Price Discrimination: First-degree

- Given the level of production q^*, the optimal fee is
$$r^* = u(q^*)$$
- *Intuition*: the monopolist charges a fee r^* that coincides with the utility that the consumer obtains from consuming q^*.



Price Discrimination: First-degree

- ***Example***:
  - A monopolist faces inverse demand curve $p(q) = 20 - q$ and constant marginal costs $c = \$2$.
  - No price discrimination:

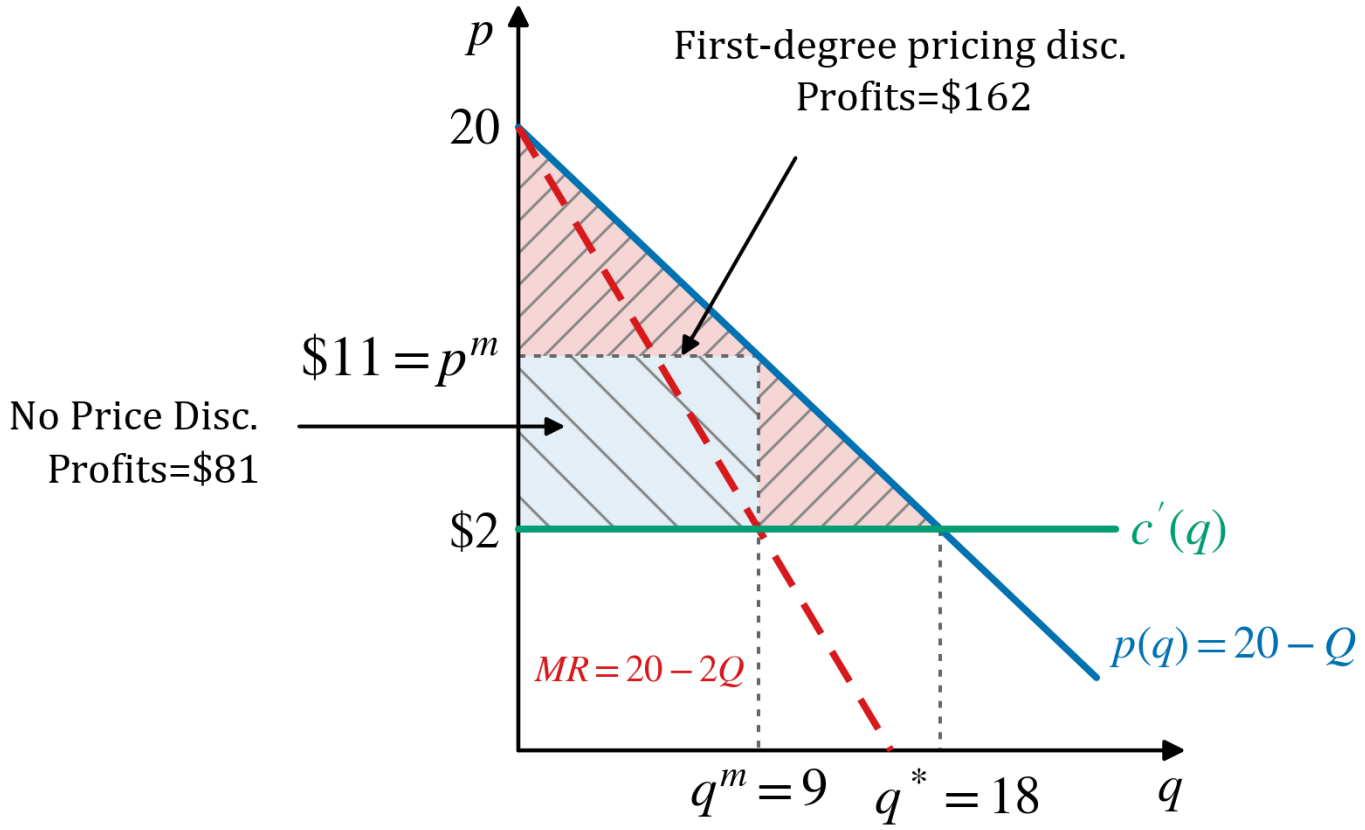
$$MR = MC \implies 20 - 2q = 2 \implies q^m = 9$$
$$p^m = \$11, \qquad \pi^m = \$81$$

  - Price discrimination:

$$p(Q) = MC \implies 20 - Q = 2 \implies Q = 18$$
$$\pi = \frac{18 \times (20 - 2)}{2} = \$162$$

Price Discrimination: First-degree

- ***Example*** (continued):

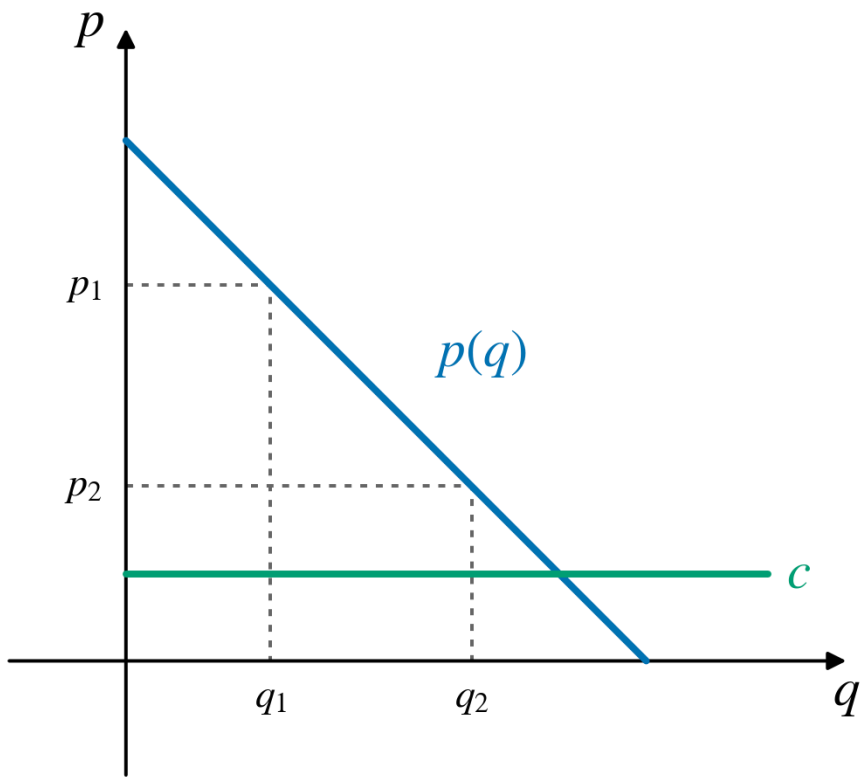


Price Discrimination: First-degree

- Summary:
  - Total output coincides with that in perfect competition
  - Unlike in perfect competition, the consumer does not capture any surplus
  - The producer captures all the surplus
  - Due to information requirements, we do not see many examples of it in real applications
    - Financial aid in undergraduate education ("tuition discrimination")

Price Discrimination: First-degree

- ***Example*** (two-block pricing):
  - A monopolist faces a inverse demand curve $p(q) = a - bq$, with constant marginal costs $c < a$.
  - Under two-block pricing, the monopolist sells the first q_1 units at a price $p(q_1) = p_1$ and the remaining $q_2 - q_1$ units at a price $p(q_2) = p_2$.



Price Discrimination: First-degree

- ***Example*** (continued):
  - Profits from the first q_1 units

$$\pi_1 = p(q_1) \cdot q_1 - cq_1 = (a - bq_1 - c)q_1$$

while from the remaining $q_2 - q_1$ units

$$\pi_2 = p(q_2) \cdot (q_2 - q_1) - c \cdot (q_2 - q_1)$$
$$= (a - bq_2 - c)(q_2 - q_1)$$

  - Hence total profits are

$$\pi = \pi_1 + \pi_2$$
$$= (a - bq_1 - c)q_1 + (a - bq_2 - c)(q_2 - q_1)$$

Price Discrimination: First-degree

- ***Example*** (continued):
  - FOCs:

$$\frac{\partial \pi}{\partial q_1} = a - 2bq_1 - c - a + bq_2 + c = 0$$

$$\frac{\partial \pi}{\partial q_2} = -b(q_2 - q_1) + a - bq_2 - c = 0$$

  - Solving for q_1 and q_2

$$q_1 = \frac{a-c}{3b} \qquad q_2 = \frac{2(a-c)}{3b}$$

which entails prices of

$$p(q_1) = a - b \cdot \frac{a-c}{3b} = \frac{2a+c}{3} \qquad p(q_2) = \frac{a+2c}{3}$$

where $p(q_1) > p(q_2)$ since $a > c$.

Price Discrimination: First-degree

- ***Example*** (continued):
  - The monopolist's profits from each block are

$$\pi_1 = (p(q_1) - c) \cdot q_1$$
$$= \left(\frac{2a + c}{3} - c\right) \cdot \frac{a - c}{3b} = \frac{2}{b}\left(\frac{a - c}{3}\right)^2$$

$$\pi_2 = (p(q_2) - c)(q_2 - q_1)$$
$$= \left(\frac{a + 2c}{3} - c\right) \cdot \left(\frac{2(a - c)}{3b} - \frac{a - c}{3b}\right) = \frac{1}{b}\left(\frac{a - c}{3}\right)^2$$

  - Thus, $\pi = \pi_1 + \pi_2 = \frac{(a-c)^2}{3b}$, which is larger than those arising from uniform pricing , $\pi^u = \frac{(a-c)^2}{4b}$.

Price Discrimination: Third-degree

- ***Third degree price discrimination***:
  - The monopolist charges different prices to two or more groups of customers (each group must be easily recognized by the seller).
    - *Example*: youth vs. adult at the movies, airline tickets
  - Firm's PMP:

$$\max_{x_1,x_2} \; p_1(x_1)x_1 + p_2(x_2)x_2 - cx_1 - cx_2$$

  - FOCs:

$$p_1(x_1) + p_1'(x_1)x_1 - c = 0 \implies MR_1 = MC$$
$$p_2(x_2) + p_2'(x_2)x_2 - c = 0 \implies MR_2 = MC$$

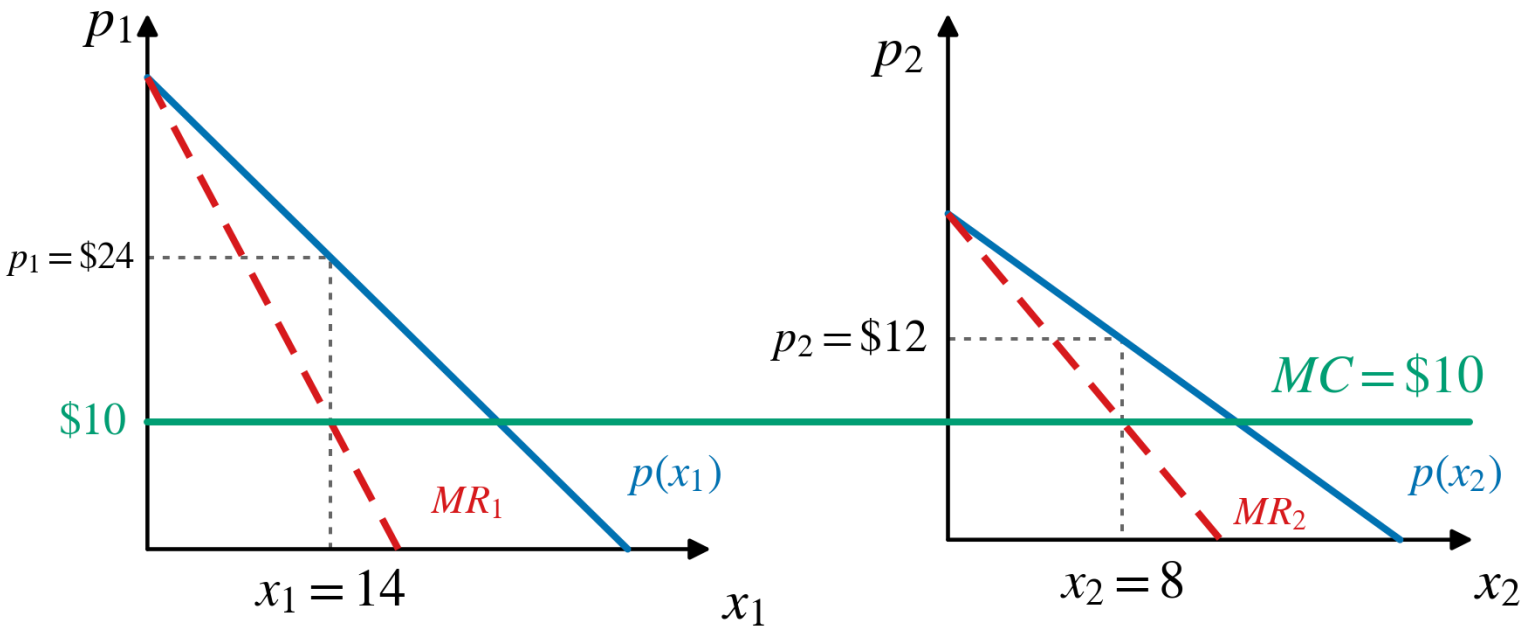
  - FOCs coincides with those of a regular monopolist who serves two completely separated markets practicing uniform pricing .

Price Discrimination: Third-degree

– ***Example***: $p_1(x_1) = 38 - x_1$ for adults and $p_2(x_2) = 14 - 1/4x_2$ for seniors, with $MC = \$10$ for both markets.

$MR_1(x_1) = MC \implies 38 - 2x_1 = 10 \implies x_1 = 14 \quad p_1 = \$24$

$MR_2(x_2) = MC \implies 14 - 1/2x_2 = 10 \implies x_2 = 8 \quad p_2 = \$12$



Market 1
Adults at the movies

Market 2
Seniors at the movies

Price Discrimination: Third-degree

- Using the Inverse Elasticity Pricing Rule (IERP), we can obtain the prices

$$p_1(x_1) = \frac{c}{1+1/\varepsilon_1} \text{ and } p_2(x_2) = \frac{c}{1+1/\varepsilon_2}$$

where c is the common marginal cost

- Then, $p_1(x_1) > p_2(x_2)$ if and only if

$$\frac{c}{1+1/\varepsilon_1} > \frac{c}{1+1/\varepsilon_2} \Rightarrow 1+\frac{1}{\varepsilon_2} < 1+\frac{1}{\varepsilon_1}$$

$$\Rightarrow \frac{1}{\varepsilon_2} > \frac{1}{\varepsilon_1} \Rightarrow \varepsilon_2 < \varepsilon_1$$

- *Intuition*: the monopolist charges lower price in the market with more elastic demand.

Price Discrimination: Third-degree

- ***Example*** (Pullman-Seattle route):
  - The price-elasticity of demand for business-class seats is -1.15, while that for economy seats is -1.52
  - From the IEPR,

$$p_B = \frac{MC}{1 - 1/1.15} \implies 0.13p_B = MC$$

$$p_E = \frac{MC}{1 - 1/1.52} \implies 0.34p_E = MC$$

  - Hence, $0.13p_B = 0.34p_E$ or $p_B = 2.62p_E$
    - Airline maximizes its profits by charging business-class seats a price 2.62 times higher than that of economy-class seats

Price Discrimination: Second-degree

- ***Second-degree price discrimination***:
  - The monopolist cannot observe the type of each consumer (e.g., his willingness to pay).
  - Hence the monopolist offers a menu of two-part tariffs, (F_L, q_L) and (F_H, q_H), with the property that the consumer with type $i = \{L, H\}$ has the incentive to self-select the two-part tariff (F_i, q_i) meant for him.

Price Discrimination: Second-degree

- Assume the utility function of type i consumer

$$U_i(q_i, F_i) = \theta_i u(q_i) - F_i$$

where

  - q_i is the quantity of a good consumed
  - F_i is the fixed fee paid to the monopolist for q_i
  - θ_i measures the valuation consumer assigns to the good, where $\theta_H > \theta_L$, with corresponding probabilities p and $1 - p$.

- The monopolist's constant marginal cost c satisfies $\theta_i > c$ for all $i = \{L, H\}$.

Price Discrimination: Second-degree

- The monopolist must guarantee that
  1) both types of customers are willing to participate ("***participation constraint***")
     - the two-part tariff meant for each type of customer provides him with a weakly positive utility level
  2) customers do not have incentives to choose the two-part tariff meant for the other type of customer ("***incentive compatibility***")
     - type i customer prefers (F_i, q_i) over (F_j, q_j) where $j \neq i$

Price Discrimination: Second-degree

- The participation constraints (PC) are

$$\theta_L u(q_L) - F_L \geq 0 \qquad PC_L$$

$$\theta_H u(q_H) - F_H \geq 0 \qquad PC_H$$

- The incentive compatibility conditions are

$$\theta_L u(q_L) - F_L \geq \theta_L u(q_H) - F_H \qquad IC_L$$

$$\theta_H u(q_H) - F_H \geq \theta_H u(q_L) - F_L \qquad IC_H$$

Price Discrimination: Second-degree

- Rearranging the four inequalities, the monopolist's profit maximization problem becomes:

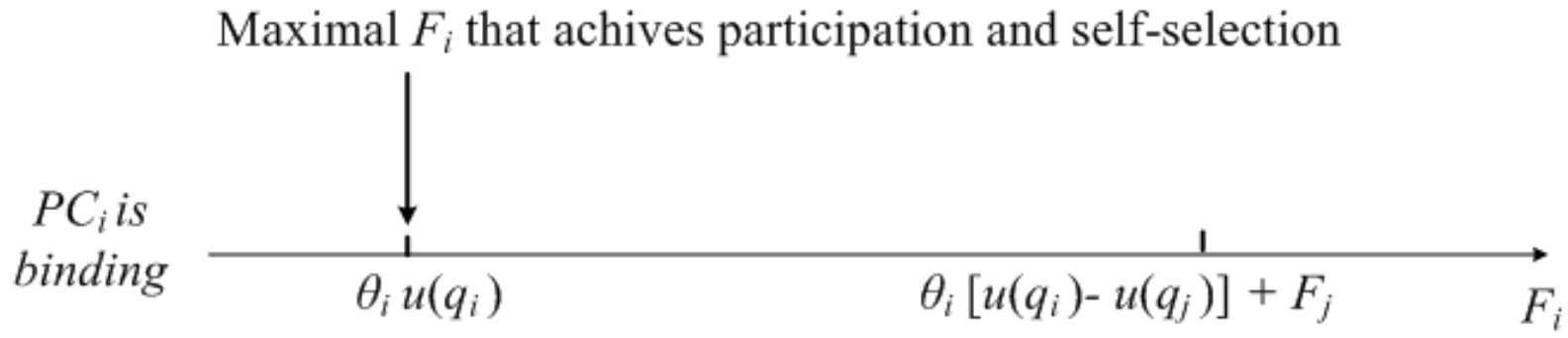
$$\max_{F_L, q_L, F_H, q_H} \quad p[F_H - cq_H] + (1-p)[F_L - cq_L]$$

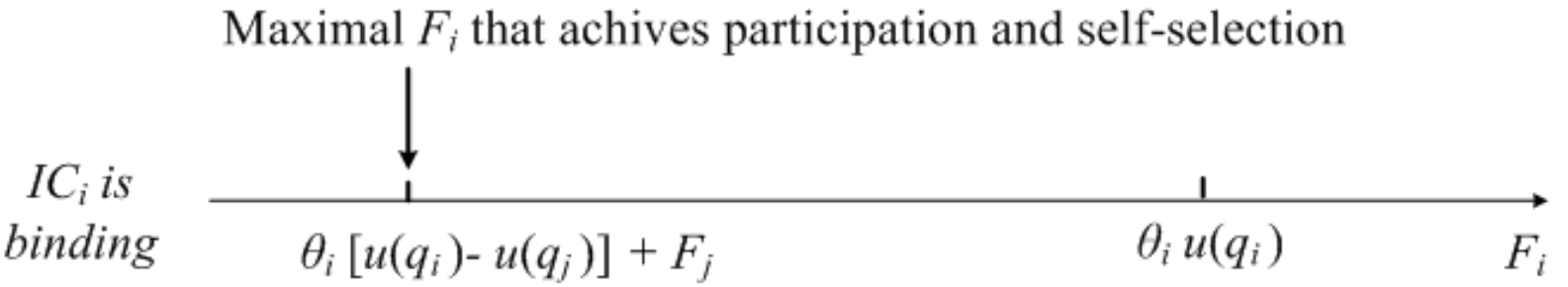
$$\theta_L u(q_L) \geq F_L$$
$$\theta_H u(q_H) \geq F_H$$
$$\theta_L[u(q_L) - u(q_H)] + F_H \geq F_L$$
$$\theta_H[u(q_H) - u(q_L)] + F_L \geq F_H$$

Price Discrimination: Second-degree

- Both PC_H and IC_H are expressed in terms of the fee F_H
  - The monopolist increases F_H until such fee coincides with the lowest of $\theta_H u(q_H)$ and $\theta_H[u(q_H) - u(q_L)] + F_L$.
  - Otherwise, one (or both) constraints will be violated, leading the high-demand customer to not participate

Price Discrimination: Second-degree





Price Discrimination: Second-degree

- ***High-demand customer***:
  - Let us show that IC_H is binding
  - An indirect way to show that

$$F_H = \theta_H[u(q_H) - u(q_L)] + F_L$$

    is to demonstrate that $F_H < \theta_H u(q_H)$

  - Proving this by contradiction, assume that

$$F_H = \theta_H u(q_H)$$

Price Discrimination: Second-degree

– Then, IC_H can be written as

$$F_H - \theta_H u(q_L) + F_L \geq F_H$$
$$\Longrightarrow F_L \geq \theta_H u(q_L)$$

– Combining this result with the fact that $\theta_H > \theta_L$,

$$F_L \geq \theta_H u(q_L) > \theta_L u(q_L)$$

which implies $F_L > \theta_L u(q_L)$

– However, this violates PC_L

- We then reached a contradiction
- Thus, $F_H < \theta_H u(q_H)$
- IC_H is binding but PC_H is not.

Price Discrimination: Second-degree

- ***Low-demand customer***:
  - Let us show that PC_L binding
  - Similarly as for the high-demand customer, an indirect way to show that

$$F_L = \theta_L u(q_L)$$

    is to demonstrate that $F_L < \theta_L[u(q_L) - u(q_H)] + F_H$
  - Proving this by contradiction, assume that

$$F_L = \theta_L[u(q_L) - u(q_H)] + F_H$$

Price Discrimination: Second-degree

– Then, IC_H can be written as

$$\theta_H[u(q_H) - u(q_L)] + \theta_L[u(q_L) - u(q_H)] + F_H = F_H$$
$$\Longrightarrow \theta_H[u(q_H) - u(q_L)] = \theta_L[u(q_H) - u(q_L)]$$
$$\Longrightarrow \theta_H = \theta_L$$

which violates the initial assumption $\theta_H > \theta_L$

- We reached a contradiction
- Thus, $F_L < \theta_L[u(q_L) - u(q_H)] + F_H$
- PC_L is binding but IC_L is not

Price Discrimination: Second-degree

- In summary:
  - From PC_L binding we have
$$\theta_L u(q_L) = F_L$$
  - From IC_H binding we have
$$\theta_H[u(q_H) - u(q_L)] + F_L = F_H$$
  - In addition,
    - PC_L binding implies that IC_L holds, and
    - IC_H binding entails that PC_H is also satisfied,
    - That is, all four constraints hold.

Price Discrimination: Second-degree

- The monopolist's expected PMP can then be written as unconstrained problem, as follows,

$$\begin{aligned}
\max_{q_L, q_H \geq 0} \; & p\,[F_H - cq_H] + (1-p)[F_L - cq_L] \\
&= p\left\{\underbrace{\theta_H[u(q_H) - u(q_L)] + F_L}_{F_H} - cq_H\right\} \\
&\qquad + (1-p)\left\{\underbrace{\theta_L u(q_L)}_{F_L} - cq_L\right\} \\
&= p\left\{\theta_H[u(q_H) - u(q_L)] + \underbrace{\theta_L u(q_L)}_{F_L} - cq_H\right\} \\
&\qquad + (1-p)\{\theta_L u(q_L) - cq_L\} \\
&= p[\theta_H u(q_H) - (\theta_H - \theta_L)u(q_L) - cq_H] \\
&\qquad + (1-p)[\theta_L u(q_L) - cq_L]
\end{aligned}$$

Price Discrimination: Second-degree

- FOC with respect to q_H:

$$p[\theta_H u'(q_H) - c] = 0 \quad \Longrightarrow \quad \theta_H u'(q_H) = c$$

  - which coincides with that under complete information.
  - That is, there is not output distortion for high-demand buyer
  - Informally, we say that there is "**no distortion at the top**".

- FOC with respect to q_L:

$$p[-(\theta_H - \theta_L)u'(q_L)] + (1-p)[\theta_L u'(q_L) - c] = 0$$

which can be re-written as

$$u'(q_L)[\theta_L - p\theta_H] = (1-p)c$$

Price Discrimination: Second-degree

- Dividing both sides by $(1-p)$, we obtain

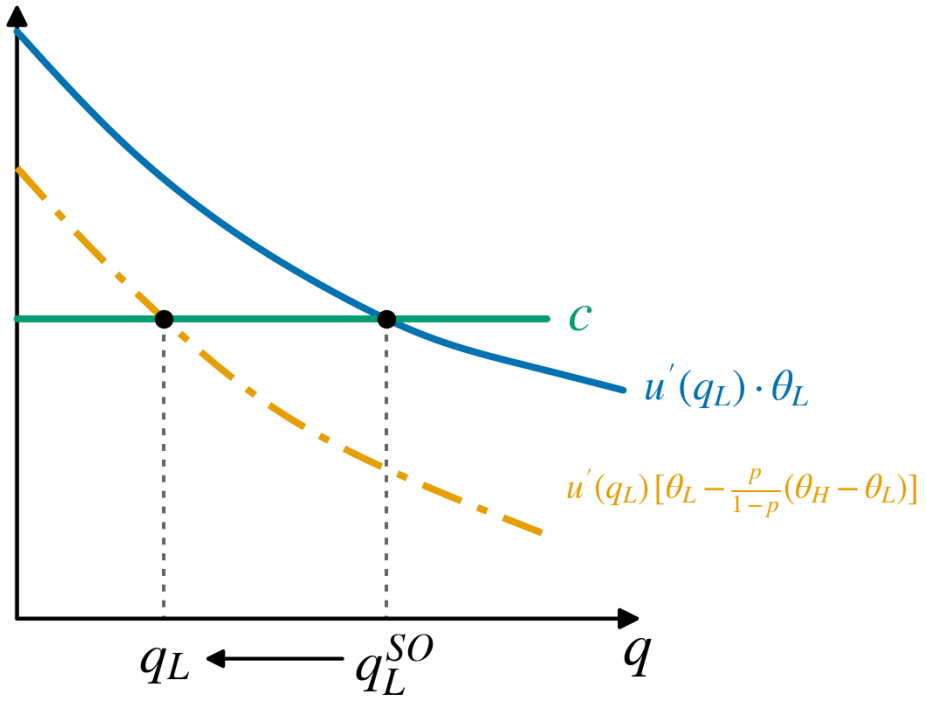
$$u'(q_L)\left[\frac{\theta_L - p\theta_H}{1-p}\right] = c$$

- The above expression can alternatively be written as

$$u'(q_L)\left[\theta_L - \frac{p}{1-p}(\theta_H - \theta_L)\right] = c$$

Price Discrimination: Second-degree

- $u'(q_L) \cdot \theta_L$ depicts the socially optimal output q_L^{so}, i.e., that arising under complete information
- The output offered to high-demand customers is socially efficient due to the absence of output distortion for high-type agents
- The output offered to low-demand customers entails a distortion, i.e., $q_L < q_L^{so}$
- Per-unit price for high-type and low-type differs, i.e., $F_H \neq F_L$
  - Monopolist practices price discrimination among the two types of customers.



Price Discrimination: Second-degree

- Since constraint PC_L binds while PC_H does not, then only the high-demand customer retains a positive utility level, i.e., $\theta_H u(q_H) - F_H > 0$.
- The firm's lack of information provides the high-demand customer with an "information rent."
  - Intuitively, the information rent emerges from the seller's attempt to reduce the incentives of the high-type customer to select the contract meant for the low type.
  - The seller also achieves self-selection by setting an attractive output for the low-type buyer, i.e., q_L is lower than under complete information.

Price Discrimination: Second-degree

- ***Example***:
  - Consider a monopolist selling a textbook to two types of graduate students, low- and high-demand, with utility function

$$U_i(q_i, F_i) = \frac{q_i^2}{2} - \theta_i q_i - F_i$$

where $i = \{L, H\}$ and $\theta_H > \theta_L$.

  - Hence, the UMP of type i student is

$$\max_{q_i} \quad \frac{q_i^2}{2} - \theta_i q_i - F_i \quad \text{s.t.} \quad pq_i + F_i \leq w_i$$

where $w_i > 0$ denotes the student's wealth.

Price Discrimination: Second-degree

- ***Example*** (continued):
  - By Walras' law, the constraint binds

$$F_i = w_i - pq_i$$

  - Then, the UMP can be expressed as

$$\max_{q_i} \quad \frac{{q_i}^2}{2} - \theta_i q_i - (w_i - pq_i)$$

  - FOCs wrt q_i yields the direct demand function:

$$q_i - \theta_i + p = 0 \quad \text{or} \quad q_i = \theta_i - p$$

Price Discrimination: Second-degree

- ***Example*** (continued):
  - Assume that the proportion of high-demand (low-demand) students is γ ($1-\gamma$, respectively).
  - The monopolist's constant marginal cost is $c > 0$, which satisfies $\theta_i > c$ for all $i = \{L, H\}$.
  - Consider for simplicity that $\theta_L > \frac{\theta_H + c}{2}$.
  - This implies that each type of student would buy the textbook, both when the firm practices uniform pricing and when it sets two-part tariffs
    - Exercise.

Advertising in Monopoly

Advertising in Monopoly

- ***Advertising***: non-price strategy to capture surplus
- The monopolist must balance the additional demand that advertising entails and its associated costs (A dollars)
- The monopolist solves

$$\max_{A} \quad p \cdot q(p, A) - TC\big(q(p, A)\big) - A$$

where the demand function $q(p, A)$ depends on price and advertising.

Advertising in Monopoly

- Taking FOCs with respect to A,

$$p \cdot \frac{\partial q(p,A)}{\partial A} - \underbrace{\frac{\partial TC}{\partial q}}_{MC} \cdot \frac{\partial q(p,A)}{\partial A} - 1 = 0$$

Rearranging, we obtain

$$(p - MC)\frac{\partial q(p,A)}{\partial A} = 1$$

- Let us define the advertising elasticity of demand

$$\varepsilon_{q,A} = \frac{\%\ \text{increse in}\ q}{\%\ \text{increse in}\ A} = \frac{\partial q(p,A)}{\partial A} \cdot \frac{A}{q}$$

Or, rearranging,

$$\varepsilon_{q,A} \cdot \frac{q}{A} = \frac{\partial q(p,A)}{\partial A}$$

Advertising in Monopoly

- We can then rewrite the above FOC as

$$(p - MC)\,\underbrace{\varepsilon_{q,A} \cdot \frac{q}{A}}_{\frac{\partial q(p,A)}{\partial A}} = 1$$

- Dividing both sides by $\varepsilon_{q,A}$ and rearranging

$$p - MC = \frac{1}{\varepsilon_{q,A}} \cdot \frac{A}{q}$$

- Dividing both sides by p

$$\frac{p - MC}{p} = \frac{1}{\varepsilon_{q,A}} \cdot \frac{A}{p \cdot q}$$

Advertising in Monopoly

- From the Lerner index, we know that $\frac{p-MC}{p} = -\frac{1}{\varepsilon_{q,p}}$.

  Hence,

$$-\frac{1}{\varepsilon_{q,p}} = \frac{1}{\varepsilon_{q,A}} \cdot \frac{A}{p \cdot q}$$

- And rearranging

$$-\frac{\varepsilon_{q,A}}{\varepsilon_{q,p}} = \frac{A}{p \cdot q}$$

  - The right-hand side represents the *advertising-to-sales ratio*.
  - For two markets with the same $\varepsilon_{q,p}$, the advertising-to-sales ratio must be larger in the market where demand is more sensitive to advertising (higher $\varepsilon_{q,A}$).

Advertising in Monopoly

- ***Example***:
  - If the price-elasticity in a given monopoly market is $\varepsilon_{q,p} = -1.5$ and the advertising-elasticity is $\varepsilon_{q,A} = 0.1$, the advertising-to-sales ratio should be

$$\frac{A}{p \cdot q} = -\frac{0.1}{-1.5} = 0.067$$

  - Advertising should account for 6.7% of this monopolist's revenue.

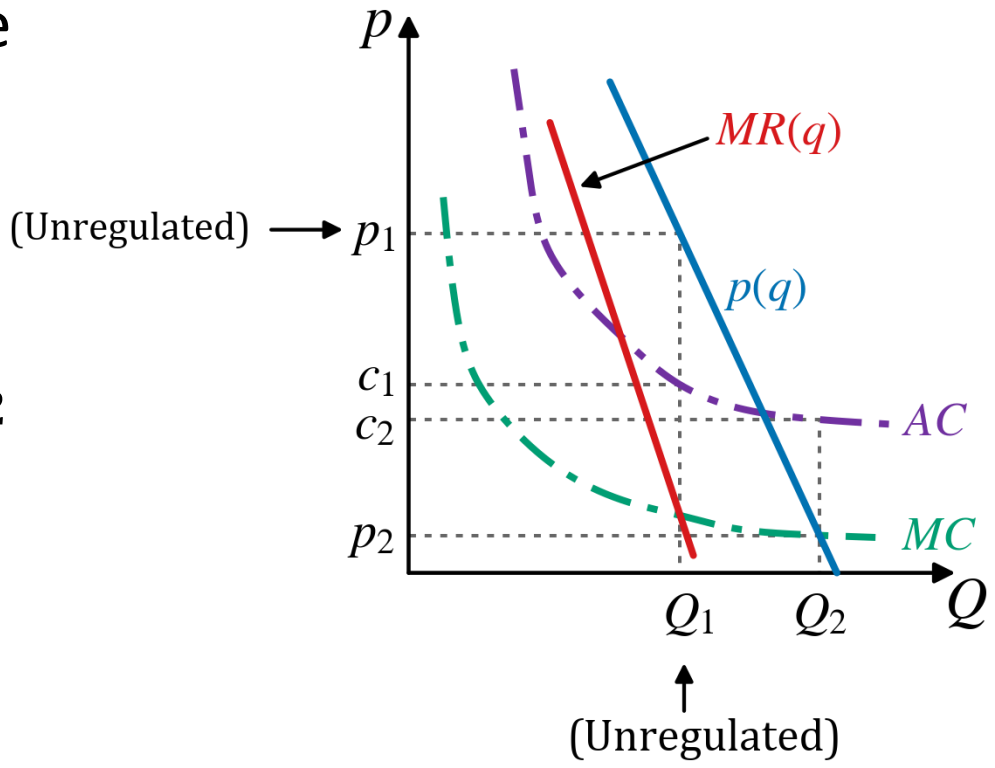
Regulation of Natural Monopolies

Regulation of Natural Monopolies

- ***Natural monopolies***: Monopolies that exhibit *decreasing* cost structures, with the MC curve lying below the AC curve.
- Hence, having a single firm serving the entire market is cheaper than having multiple firms, as aggregate average costs for the entire industry would be lower.

Regulation of Natural Monopolies

- Unregulated natural monopolist maximizes profits at the point where MR=MC, producing Q_1 units and selling them at a price p_1.
- Regulated natural monopolist will charge p_2 (where demand crosses MC) and produce Q_2 units.
- The production level Q_2 implies a loss of $p_2 - c_2$ per unit.



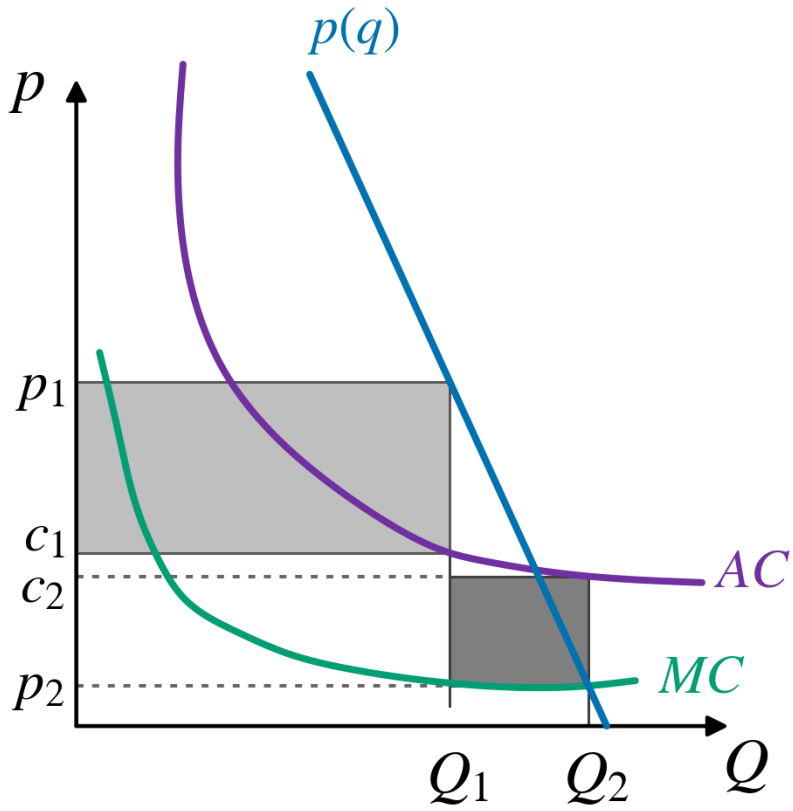
Regulation of Natural Monopolies

- Dilemma with natural monopolies:
  - abandon the policy of setting prices equal to marginal cost, OR
  - continue applying marginal cost pricing but subsidize the monopolist for his losses
- Solution to the dilemma:
  - A multi-price system that allows for price discrimination
  - Charging some users a high price while maintaining a low price to other users

Regulation of Natural Monopolies

- Multi-price system:
  - a high price p_1
  - a low price p_2
- *Benefit*: $(p_1 - c_1)$ per unit in the interval from 0 to Q_1
- *Loss*: $(c_2 - p_2)$ per unit in the interval $(Q_2 - Q_1)$
- The monopolist price discriminates iff

$$(p_1 - c_1)Q_1 > (c_2 - p_2)(Q_2 - Q_1)$$



Ramsey Pricing

- ***Average-cost pricing***: the regulator sets the price where demand crosses AC.
  - The firm breaks even, so no subsidy is needed.
  - But p > MC, so a deadweight loss remains.
- If the firm sells several products, which prices cover the fixed cost at the lowest efficiency loss?
  - Answer: Ramsey pricing (Ramsey, 1927; Boiteux, 1956).

Ramsey Pricing

- ***Setup***: a natural monopoly sells n products, with independent demands $q_i(p_i)$, and cost

$$C(q_1, \dots, q_n) = F + \sum_{i=1}^{n} c_i q_i$$

  - where $F > 0$ is the fixed cost.
- Examples: a railway carrying passengers and freight; a utility serving homes and factories.

Ramsey Pricing

- The regulator chooses prices to maximize total surplus,

$$W = \sum_i [CS_i(p_i) + (p_i - c_i)q_i(p_i)] - F$$

- subject to the firm breaking even without a subsidy,

$$\pi = \sum_i (p_i - c_i)q_i(p_i) - F \geq 0$$

Ramsey Pricing

- The Lagrangian is

$$\mathcal{L} = W + \lambda\pi$$

- Since a higher price reduces consumer surplus by the quantity bought,

$$\frac{\partial CS_i}{\partial p_i} = -q_i$$

Ramsey Pricing

- Hence, the derivatives of total surplus and profit with respect to p_i are

$$\frac{\partial W}{\partial p_i} = -q_i + q_i + (p_i - c_i)q_i' = (p_i - c_i)q_i'$$

$$\frac{\partial \pi}{\partial p_i} = q_i + (p_i - c_i)q_i'$$

Ramsey Pricing

- The FOC with respect to p_i is

$$(p_i - c_i)q_i' + \lambda[q_i + (p_i - c_i)q_i'] = 0$$

- Rearranging,

$$(1 + \lambda)(p_i - c_i)q_i' = -\lambda q_i$$

Ramsey Pricing

- Dividing both sides by the price, we obtain the ***inverse-elasticity rule*** (Ramsey rule),

$$\frac{p_i - c_i}{p_i} = \frac{\lambda}{1+\lambda}\frac{1}{\varepsilon_i}$$

  - where

$$\varepsilon_i \equiv -q_i' \frac{p_i}{q_i}$$

  - is the price elasticity of demand in market i, in absolute value.

Ramsey Pricing

- The markup is higher in markets with less elastic demand.
- Two limit cases:

$$\lambda = 0: \quad p_i = c_i$$

  - the break-even constraint does not bind (e.g., $F = 0$): marginal-cost pricing;

$$\lambda \to \infty: \quad \frac{p_i - c_i}{p_i} = \frac{1}{\varepsilon_i}$$

  - each market is priced as an unregulated monopolist would (Lerner rule).

Ramsey Pricing

- Intuitively, each markup raises revenue to cover F, but it also creates a deadweight loss.
  - A markup in a market with inelastic demand reduces quantity by little, so it raises revenue at a low efficiency cost.
- At the optimum, the last dollar of revenue raised in every market costs the same deadweight loss, λ.

Ramsey Pricing

- ***Example*** (Ramsey pricing): a natural monopoly sells two products with demands

$$q_1 = 22 - 2p_1 \quad \text{and} \quad q_2 = 20 - p_2$$

  - and cost

$$C = 108 + 2(q_1 + q_2)$$

- Marginal-cost pricing:

$$p_1 = p_2 = 2: \quad q_1 = q_2 = 18, \qquad \text{so} \quad \pi = -108$$

  - so the firm needs a subsidy of 108.

Ramsey Pricing

- ***Example*** (Ramsey pricing, continued):
  - With linear demands, the slope is the constant $q_i' = -b_i$, so the Ramsey rule becomes

$$p_i - 2 = k\frac{q_i}{b_i}, \qquad \text{with} \quad b_1 = 2, \qquad b_2 = 1$$

  - where

$$k \equiv \frac{\lambda}{1+\lambda}$$

  - Solving for quantities and markups,

$$q_i = \frac{18}{1+k} \quad \text{and} \quad p_i - 2 = \frac{18k}{b_i(1+k)}$$

Ramsey Pricing

- ***Example*** (Ramsey pricing, continued):
  - The firm must break even:

$$\frac{k}{(1+k)^2}\left(\frac{324}{2}+\frac{324}{1}\right)=\frac{486k}{(1+k)^2}=108$$

  - which simplifies to

$$2k^2-5k+2=0, \qquad \text{so} \quad k=\frac{1}{2} \quad \text{and} \quad \lambda=1$$

  - The other root, k = 2, would require a negative λ.

Ramsey Pricing

- ***Example*** (Ramsey pricing, continued):
  - The Ramsey prices and quantities are
    $$p_1 = 5, \qquad q_1 = 12; \qquad p_2 = 8, \qquad q_2 = 12$$
  - and profit is zero,
    $$\pi = 3(12) + 6(12) - 108 = 0$$
  - Market 1 has the more elastic demand, so it bears the lower markup: 3 versus 6.

Ramsey Pricing

- ***Example*** (Ramsey pricing, continued):
  - Elasticities and Lerner indices at the Ramsey prices:

$$\varepsilon_1 = 2 \cdot \frac{5}{12} = \frac{5}{6}, \qquad L_1 = \frac{3}{5}$$

$$\varepsilon_2 = \frac{8}{12} = \frac{2}{3}, \qquad L_2 = \frac{6}{8} = \frac{3}{4}$$

  - so the inverse-elasticity rule holds:

$$L_1\varepsilon_1 = L_2\varepsilon_2 = \frac{1}{2} = \frac{\lambda}{1+\lambda}$$

Ramsey Pricing

- ***Example*** (Ramsey pricing, continued):
  - Uniform average-cost pricing, with the same price in both markets:

$$(p-2)(42-3p)=108,$$
$$\text{so}\quad (p-8)^2=0\quad\text{and}\quad p=8$$

  - so that $q_1=6\quad\text{and}\quad q_2=12.$
  - With linear demands, the deadweight loss in market i is

$$DWL_i=\frac{1}{2}b_i(p_i-c_i)^2$$

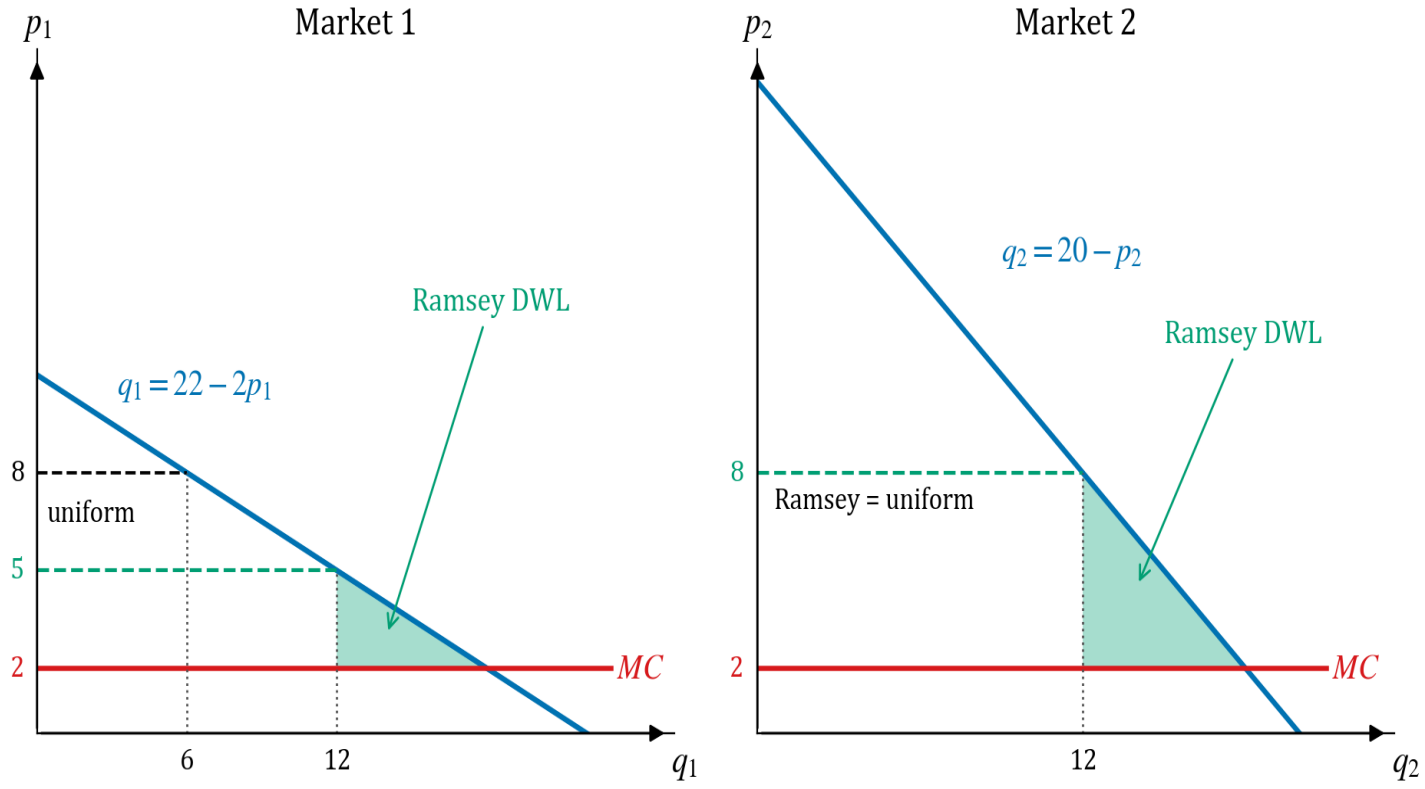
Ramsey Pricing

- ***Example*** (Ramsey pricing, continued):
  - Comparing the four regimes:

Regime	p1	p2	Profit	DWL
Marginal-cost pricing	2	2	−108	0
Ramsey pricing	5	8	0	27
Uniform average-cost pricing	8	8	0	54
Unregulated monopoly	6.5	11	13.5	60.75

Ramsey Pricing

- ***Example*** (Ramsey pricing, continued):
  - Ramsey pricing lowers the price in the elastic market.



Ramsey Pricing

- ***Remarks***:
  - Ramsey pricing requires the regulator to know demand elasticities.
  - It charges more to customers with fewer alternatives, which can raise equity concerns.
  - With interdependent demands, the rule uses elasticities that account for cross-price effects.

Regulation of Natural Monopolies

- An alternative regulation:
  - allow the monopolist to charge a price above marginal cost that is sufficient to earn a “fair” rate of return on capital investments
- Two difficulties:
  - what is a “fair” rate of return
  - overcapitalization

Regulation of Natural Monopolies

- ***Overcapitalization of natural monopolies***:
  - Suppose a production function of the form $q = f(k, l)$. An unregulated monopoly with profit function $pf(k, l) - wl - rk$ has a rate of return on capital, r. Suppose furthermore that the rate of return on capital investments, r, is constrained by a regulatory agency to be equal to r_0.

Regulation of Natural Monopolies

- PMP:

$$L = pf(k,l) - wl - rk + \lambda[wl + r_0 k - pf(k,l)]$$

where $0 < \lambda < 1$.

- FOCs:

$$\frac{\partial L}{\partial l} = pf_l - w + \lambda(w - pf_l) = 0$$

$$\frac{\partial L}{\partial k} = pf_k - r + \lambda(r_0 - pf_k) = 0$$

$$\frac{\partial L}{\partial \lambda} = wl + r_0 k - pf(k,l) = 0$$

Regulation of Natural Monopolies

- From the first FOC:

$$pf_l = w$$

- From the second FOC:

$$(1-\lambda)pf_k = r - \lambda r_0$$

and rearranging

$$pf_k = \frac{r-\lambda r_0}{1-\lambda} = r - \frac{\lambda(r_0-r)}{1-\lambda}$$

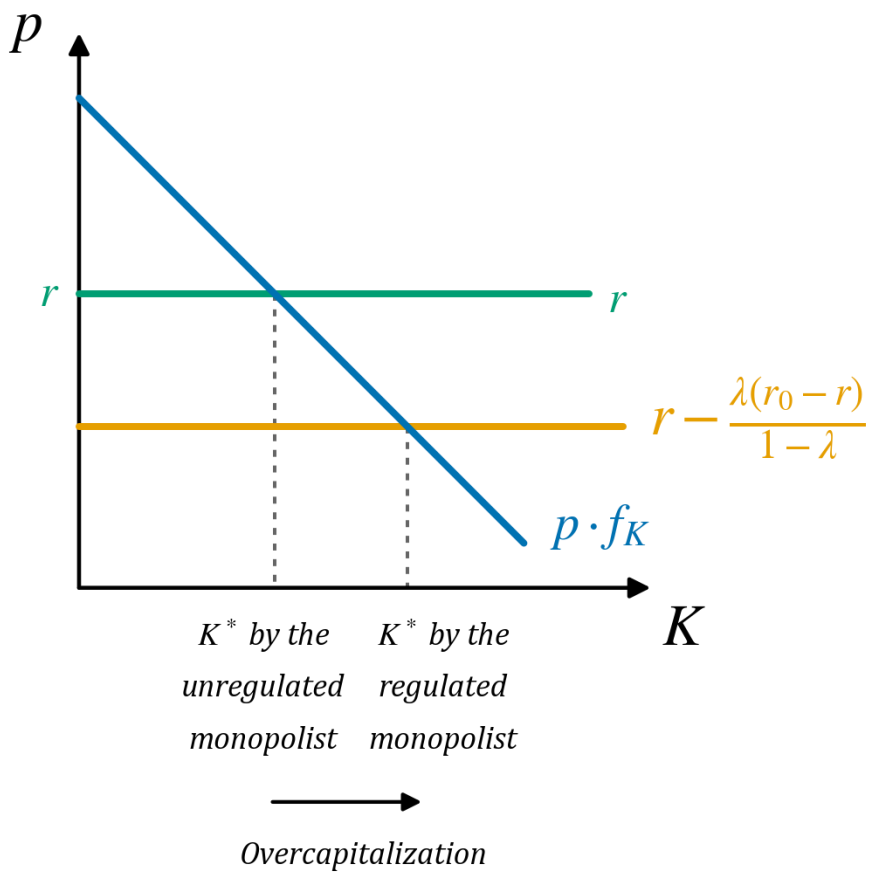
  - Since $r_0 > r$ and $0 < \lambda < 1$, then $pf_k < r$.
  - Hence, the firm would hire *more capital* than under unregulated condition, where $pf_k = r$.

Regulation of Natural Monopolies

- pf_k is the value of the marginal product of capital
  - It is decreasing in k (due to diminishing marginal return, i.e., $f_{kk} < 0$)
- r and $r - \frac{\lambda(r_0 - r)}{1-\lambda}$ are the marginal cost of additional units of capital in the unregulated and regulated monopoly, respectively

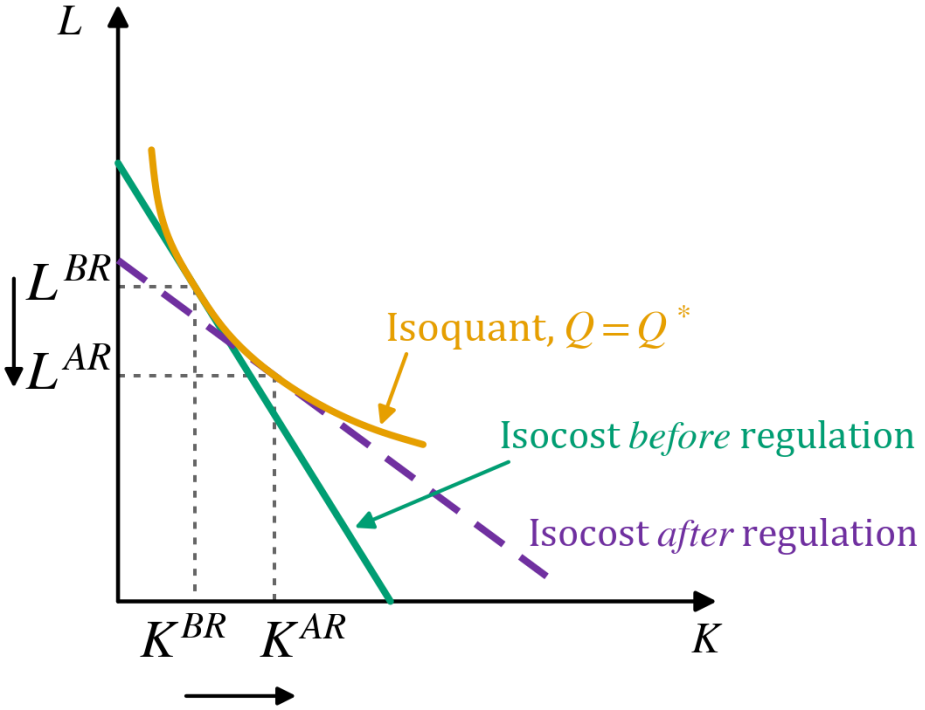
$$r > r - \frac{\lambda(r_0 - r)}{1-\lambda}$$

- *Example*: electricity and water suppliers



Regulation of Natural Monopolies

- An alternative illustration of the overcapitalization (***Averch-Johnson effect***)
- Before regulation, the firm selects (L^{BR}, K^{BR})
- After regulation, the firm selects (L^{AR}, K^{AR}), where $K^{AR} > K^{BR}$ but $L^{AR} < L^{BR}$
- The overcapitalization result only captures the substitution effect of a cheaper input.
  - Output effect?



Monopsony

Monopsony

- ***Monopsony***: A single buyer of goods and services exercises "buying power" by paying prices below those that would prevail in a perfectly competitive context.
- Monopsony (single buyer) is analogous to that of a monopoly (single seller).
- *Examples*: a coal mine, Walmart Superstore in a small town, etc.

Monopsony

- Consider that the monopsony faces competition in the product market, where prices are given at $p > 0$, but is a monopsony in the input market (e.g., labor services).
- Assume an increasing and concave production function, i.e., $f'(x) > 0$ and $f''(x) \leq 0$.
  - This yields a total revenue of $pf(x)$.
- Consider a cost function $w(x) \cdot x$, where $w(x)$ denotes the inverse supply function of labor x.
  - Assume that $w'(x) > 0$ for all x.
  - This indicates that, as the firm hires more workers, labor becomes scarce, thus increasing the wages of additional workers.

Monopsony

- The monopsony PMP is

$$\max_{x} \quad pf(x) - w(x)x$$

- FOC wrt the amount of labor services (x) yields

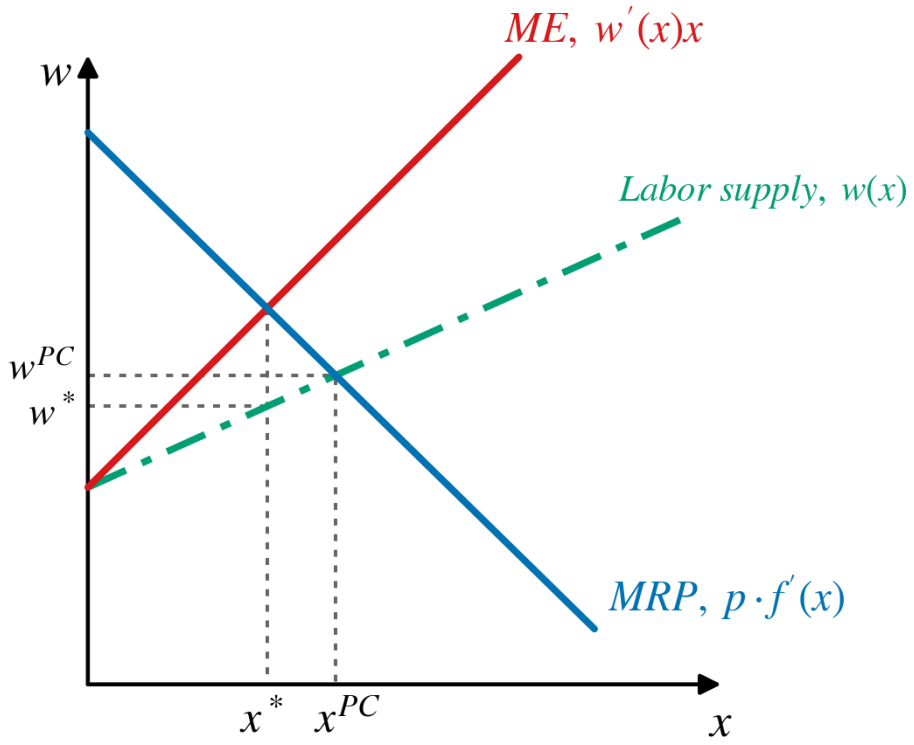
$$pf'(x^*) - w(x^*) - w'(x^*)x^* = 0$$

$$\Longrightarrow \underbrace{pf'(x^*)}_{A} = \underbrace{w(x^*) + w'(x^*)x^*}_{B}$$

  - A: "marginal revenue product" of labor.
  - B: "marginal expenditure" (ME) on labor.
    - The additional worker entails a monetary outlay of $w(x^*)$.
    - Hiring more workers make labor become more scarce, ultimately forcing the monopsony to raise the prevailing wage on all inframarginal workers, as captured by $w'(x^*)x^*$.

Monopsony

- Monopsonist hiring and salary decisions.
  - The marginal revenue product of labor, $pf'(x)$, is decreasing in x given that $f''(x) \leq 0$.
  - The labor supply, $w(x)$, is increasing in x since $w'(x) > 0$.
  - The marginal expenditure (ME) on labor lies above the supply function $w(x)$ since $w'(x) > 0$.
  - The monopsony hires x^* workers at a salary of $w(x^*)$.



Monopsony

- A deadweight loss from monopsony is

$$DWL = \int_{x^*}^{x^{PC}} [pf'(x) - w(x)]dx$$

- That is, the area below the marginal revenue product and above the supply curve, between x^* and x^{PC} workers.

Monopsony

- We can write the monopsony profit-maximizing condition, i.e., $pf'(x^*) = w(x^*) + w'(x^*)x^*$, in terms of labor supply elasticity, using the following steps:

$$pf'(x^*) = w(x^*) + \frac{\partial w(x^*)}{\partial x^*} x^*$$
$$= w(x^*)\left(1 + \frac{\partial w(x^*)}{\partial x^*} \frac{x^*}{w(x^*)}\right)$$

- And rearranging,

$$pf'(x^*) = w(x^*)\left(1 + \frac{1}{\frac{\partial x^* w(x^*)}{\partial w \quad x^*}}\right)$$

Monopsony

- Since $\frac{\partial x^*}{\partial w}\frac{w(x^*)}{x^*}$ represents the elasticity of labor supply ε, then

$$pf'(x^*) = w(x^*)\left(1 + \frac{1}{\varepsilon}\right)$$

- Intuitively, as $\varepsilon \to \infty$ (labor supply becoming perfectly elastic), the behavior of the monopsonist approaches that of a pure competitor.

Monopsony

- The equilibrium condition above is also sufficient as long as

$$pf''(x^*) - 2w'(x^*) - w''(x^*)x^* < 0$$

- Since $f''(x^*) < 0$, $w'(x^*) > 0$ (by assumption), we only need that either:
  a) the supply function is convex, i.e., $w''(x^*) > 0$; or
  b) if it is concave, i.e., $w''(x^*) < 0$, its concavity is not very strong, that is

$$pf''(x^*) - 2w'(x^*) < w''(x^*)x^*$$

Monopsony

- ***Example***:
  - Consider a monopsonist with production function $f(x) = ax$, where $a > 0$, and facing a given market price $p > 0$ per unit of output.
  - Labor supply is $w(x) = bx$, where $b > 0$.
  - The marginal revenue product of hiring an additional worker is
$$pf'(x) = pa$$
  - The marginal expenditure on labor is
$$w(x) + w'(x)x = bx + bx = 2bx$$

Monopsony

- ***Example*** (continued):
  - Setting them equal to each other, $ap = 2bx^*$, yields a profit-maximizing amount of labor:

$$x^* = \frac{ap}{2b}$$

  - x^* increases in the price of output, p, and in the marginal productivity of labor, a; but decreases in the slope of labor supply, b.
  - Sufficiency holds since

$$pf''(x^*) - 2w'(x^*) = p0 - 2b < 0 = w''(x^*)x^*$$

Advanced Microeconomic Theory

Chapter 8: Game Theory and Imperfect Competition

Outline

- Game Theory Tools
- Bertrand Model of Price Competition
- Cournot Model of Quantity Competition
- Product Differentiation
- Dynamic Competition
- Capacity Constraints
- Endogenous Entry
- Repeated Interaction

Introduction

- Monopoly: a single firm
- Oligopoly: a limited number of firms
  - When allowing for N firms, the equilibrium predictions embody the results in perfectly competitive and monopoly markets as special cases.

Game Theory Tools

Game Theory Tools

- Consider a setting with I players (e.g., firms, individuals, or countries) each choosing a strategy s_i from a strategy set S_i, where $s_i \in S_i$ and $i \in I$.
  - An output level, a price, or an advertising expenditure
- Let (s_i, s_{-i}) denote a strategy profile where s_{-i} represents the strategies selected by all firms $i \neq j$, i.e., $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I)$.
- **Dominated strategy**: Strategy s_i^* strictly dominates another strategy $s_i' \neq s_i^*$ for player i if

$$\pi_i(s_i^*, s_{-i}) > \pi_i(s_i', s_{-i}) \text{ for all } s_{-i}$$

  - That is, s_i^* yields a strictly higher payoff than s_i' does, regardless of the strategy s_{-i} selected by all of player i's rivals.

Game Theory Tools

- Payoff matrix (Normal Form Game)

		Firm B: Low prices	Firm B: High prices
Firm A	Low prices	5, 5	9, 1
	High prices	1, 9	7, 7

- "Low prices" yields a higher payoff than "high prices" both when a firm's rival chooses "low prices" and when it selects "high prices."
  - "Low prices" is a strictly ***dominant strategy*** for both firms (i.e., s_i^*).
  - "High prices" is referred to as a strictly ***dominated strategy*** (i.e., s_i').

Game Theory Tools

- A strictly dominated strategy can be deleted from the set of strategies a rational player would use.
- This helps to reduce the number of strategies to consider as optimal for each player.
- In the above payoff matrix, both firms will select “low prices” in the unique equilibrium of the game.
- However, games do not always have a strictly dominated strategy.

Game Theory Tools

		Firm B: Adopt	Firm B: Not adopt
Firm A	Adopt	3, 1	0, 0
	Not adopt	0, 0	1, 3

- $Adopt$ is better than $Not\ adopt$ for firm A if its opponent selects $Adopt$, but becomes worse otherwise.
- Similar argument applies for firm B.
- Hence, no strictly dominated strategies for either player.
- What is the equilibrium of the game then?

Game Theory Tools

- A strategy profile (s_i^*, s_{-i}^*) is a ***Nash equilibrium (NE)*** if, for every player i,

$$\pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i, s_{-i}^*) \text{ for every } s_i \neq s_i^*$$

  - That is, s_i^* is player i's best response to his opponents choosing s_{-i}^* as s_i^* yields a better payoff than any of his available strategies $s_i \neq s_i^*$.

Game Theory Tools

- In the previous game:
  - Firm A's best response to firm B's playing $Adopt$ is $BR_A(Adopt) = Adopt$, while to firm B playing $Not\ adopt$ is $BR_A(Not\ adopt) = Not\ adopt$.
  - Similarly, firm B's best response to firm A choosing U is $BR_B(Adopt) = Adopt$, whereas best response to firm A selecting D is $BR_B(Not\ adopt) = Not\ adopt$.
  - Hence, strategy profiles $(Adopt, Adopt)$ and $(Not\ adopt, Not\ adopt)$ are mutual best responses (i.e., the two Nash equilibria).

Mixed-Strategy Nash Equilibrium

- Insofar we restricted players to use one of their available strategies 100% of the time (commonly known as "**pure strategies**")
- Generally, players could randomize (mix) their choices.
  - *Example*: Choose strategy A with probability p and strategy B with probability $1 - p$.

Mixed-Strategy Nash Equilibrium

- **Mixed-strategy Nash equilibrium** (msNE): Consider a strategy profile $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$, where σ_i is a mixed strategy for player i. Strategy profile σ_i is a msNE if and only if

$$\pi_i(\sigma_i, \sigma_{-i}) \geq \pi_i(s_i', \sigma_{-i}) \text{ for all } s_i' \in S_i$$

  - That is, σ_i is a best response of player i, i.e., $\sigma_i \in BR_i(\sigma_i)$, to the strategy profile σ_{-i} of the other $N - 1$ players.

Mixed-Strategy Nash Equilibrium

- Three points about msNE:
  1. Players must be indifferent among all (or at least some) of their pure strategies.
  2. Since players never use strictly dominated strategies, a msNE assigns a zero probability to dominated strategies.
  3. In games with a finite set of players and a finite set of available actions, there is generally an odd number of equilibria.

Mixed-Strategy Nash Equilibrium

- ***Example*** (no NE in pure strategies):

		Firm B: *Adopt*	Firm B: *Not adopt*
Firm A: *Adopt*		3, -3	-4, 0
Firm A: *Not adopt*		-3, 1	2, -2

  - There is no cell of the matrix in which players select mutual best responses.
  - Thus we cannot find a NE in pure strategies.
  - Firm A (B) seeks to coordinate (miscoordinate) its decision with that of firm B (A, respectively).

Mixed-Strategy Nash Equilibrium

- ***Example*** (continued):
  - Given their opposed incentives, firm A would like to make its choice difficult to anticipate.
  - If firm A chooses a specific action with certainty, firm B will be driven to select the opposite action.
  - An analogous argument applies to firm B.
  - As a consequence, players have incentives to randomize their actions.

Mixed-Strategy Nash Equilibrium

- ***Example*** (continued):
  - Let p (q) denote the probability with which firm A (B, respectively) adopts the technology.
  - If firm A is indifferent between adopting and not adopting the technology, its expected utility must satisfy

$$EU_A(Adopt) = EU_A(Not\ adopt)$$
$$3q + (-4)(1-q) = -3q + 2(1-q)$$
$$6q = 6(1-q) \Rightarrow q = 1/2$$

  - Hence firm B adopts the technology with probability $q = 1/2$.

Mixed-Strategy Nash Equilibrium

- ***Example*** (continued):
  - Similarly, firm B must be indifferent between adopting and not adopting the technology:

$$EU_B(Adopt) = EU_B(Not\ adopt)$$
$$-3p + 1(1-p) = 0p + (-2)(1-p)$$
$$1 - p = p \Rightarrow p = 1/2$$

  - Hence firm A adopts the technology with probability $p = 1/2$.
  - Combining our results, we obtain the msNE

$$\left(\frac{1}{2}Adopt\,\frac{1}{2}Not\ adopt, \frac{1}{2}Adopt\,\frac{1}{2}Not\ adopt\right)$$

Mixed-Strategy Nash Equilibrium

- ***Example*** (Technology adoption game):

		Firm B: *Adopt*	Firm B: *Not adopt*
Firm A	*Adopt*	3, 1	0, 0
	Not adopt	0, 0	1, 3

  - The game has two Nash equilibria in pure strategies: $(Adopt, Adopt)$ and $(Not\ Adopt, Not\ Adopt)$.
  - There is, however, a third Nash equilibria in which both firms use a mixed strategy.

Mixed-Strategy Nash Equilibrium

- ***Example*** (continued):
  - Let p (q) denote the probability with which firm A (B, respectively) adopts the technology.
  - If firm A is indifferent between adopting and not adopting the technology, its expected utility must satisfy

$$EU_A(Adopt) = EU_A(Not\ adopt)$$
$$3q + 0(1-q) = 0q + 1(1-q)$$
$$3q = 1 - q \Rightarrow q = 1/4$$

  - Hence firm B adopts the technology with probability $q = 1/4$.

Mixed-Strategy Nash Equilibrium

- ***Example*** (continued):
  - Similarly, firm B must be indifferent between adopting and not adopting the technology:

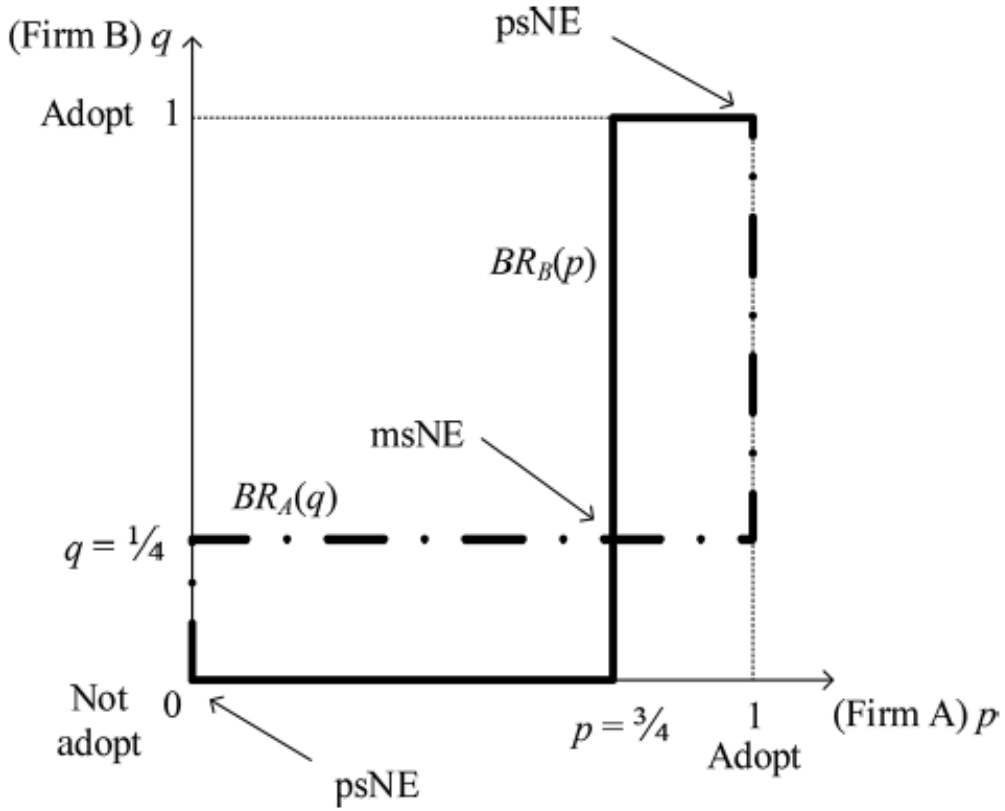
$$EU_B(Adopt) = EU_B(Not\ adopt)$$
$$1p + 0(1-p) = 0p + 3(1-p)$$
$$p = 3 - 3p \Rightarrow p = 3/4$$

  - Hence firm A adopts the technology with probability $p = 3/4$.
  - Combining our results, we obtain the msNE

$$\left(\frac{3}{4}Adopt\,\frac{1}{4}Not\ adopt, \frac{1}{4}Adopt\,\frac{3}{4}Not\ adopt\right)$$

Mixed-Strategy Nash Equilibrium

- ***Example*** (continued): best-response



Sequential-Move Games

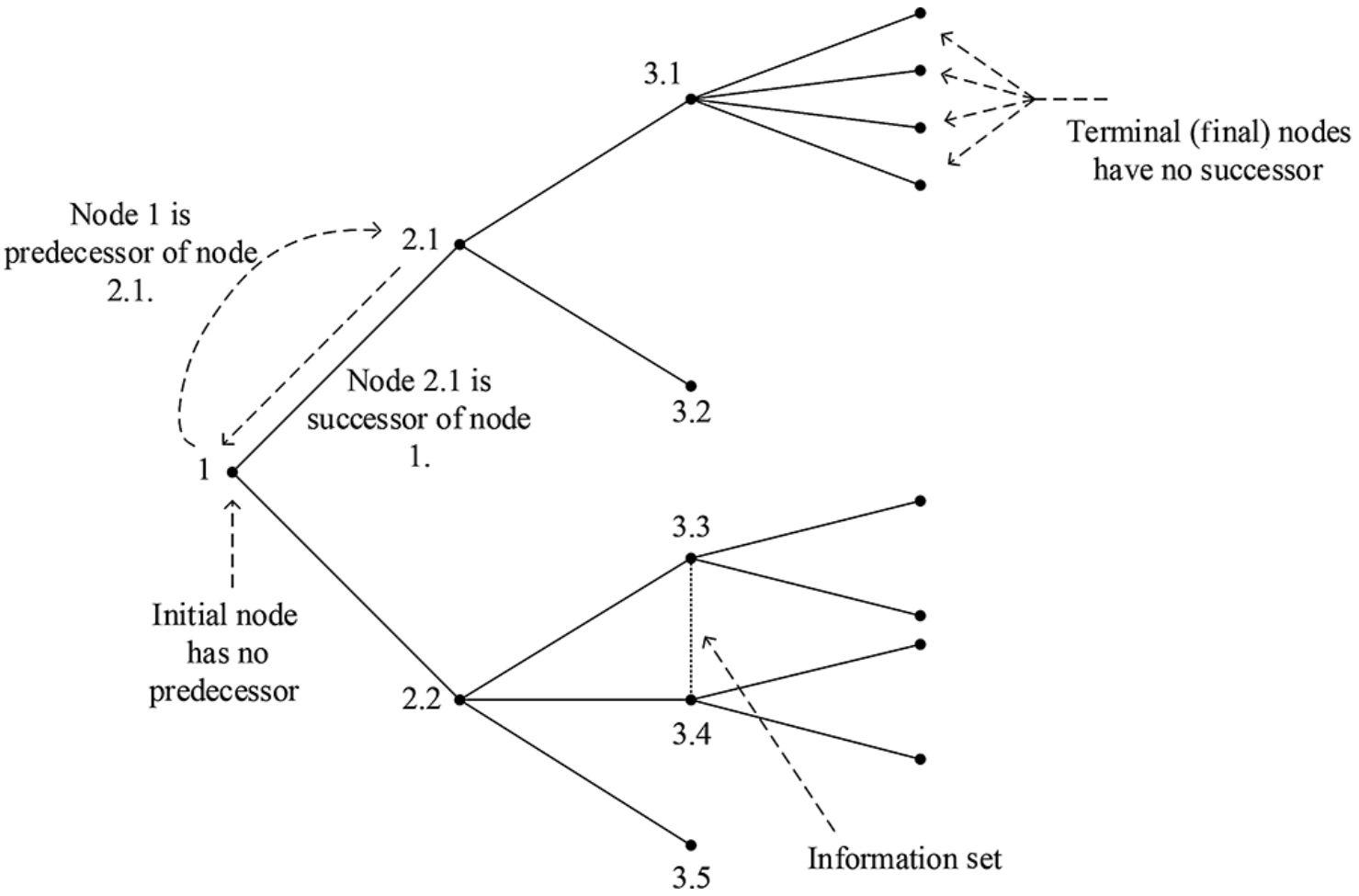
Sequential-Move Games

- When players choose their strategies *sequentially*, rather than *simultaneously*, the definition of strategy becomes more involved.
- Strategy is now a complete contingent plan describing what action player i chooses at each point at which he is called on to move, given the previous history of play.
- Such history can be observable or not observable by player i.

Sequential-Move Games

- Sequential-move games are represented using **game trees** rather than with matrices.
  - The “root” of the tree, where the game starts, is referred to as the **initial node**.
  - The last nodes of the tree, where no more branches originate, are the **terminal nodes**.

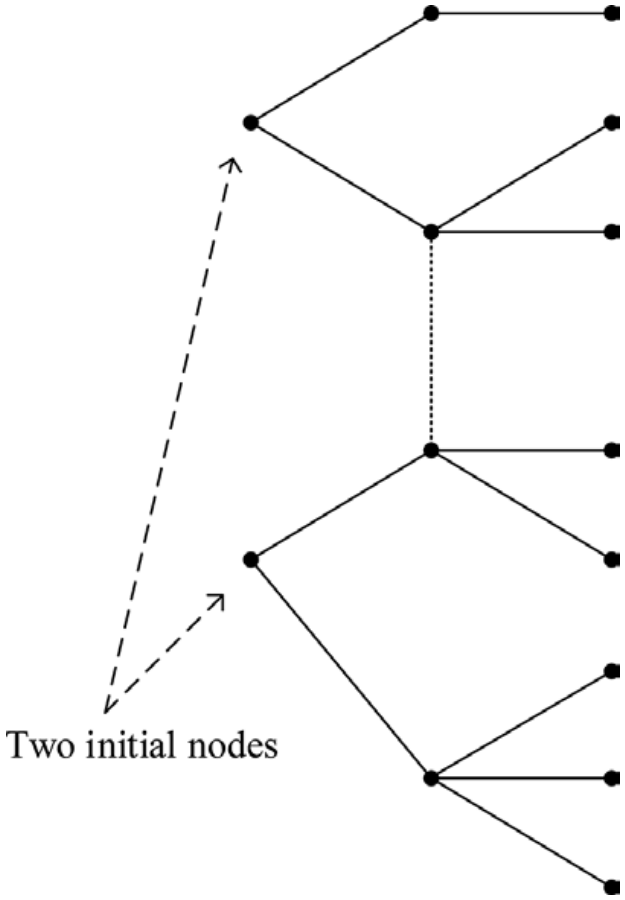
Sequential-Move Games



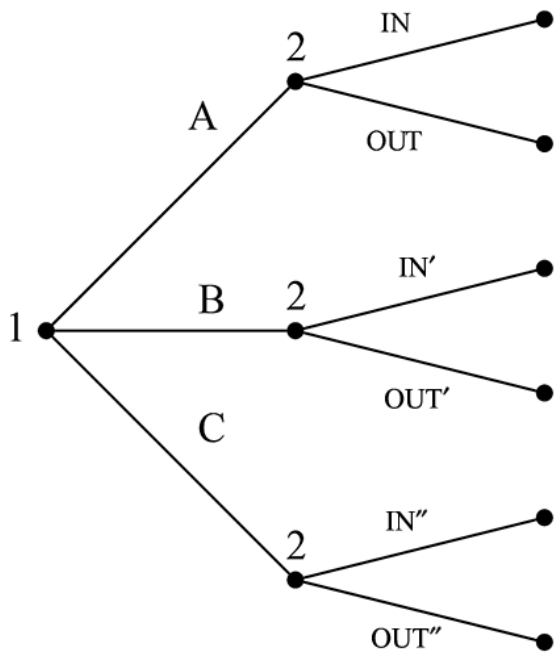
Sequential-Move Games

- Basic rules:
  1. A tree must have only one initial node.
  2. Every node of the tree has exactly one immediate predecessor except the initial node, which has no predecessor.
  3. Multiple branches extending from the same node must have different action labels.
  4. Every information set contains decision nodes for only one of the players in the game.
  5. All nodes in a given information set have the same immediate successors.

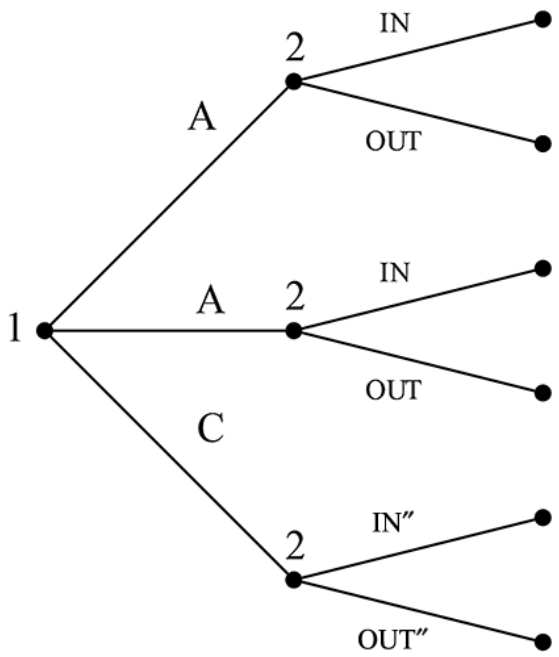
Sequential-Move Games



Sequential-Move Games



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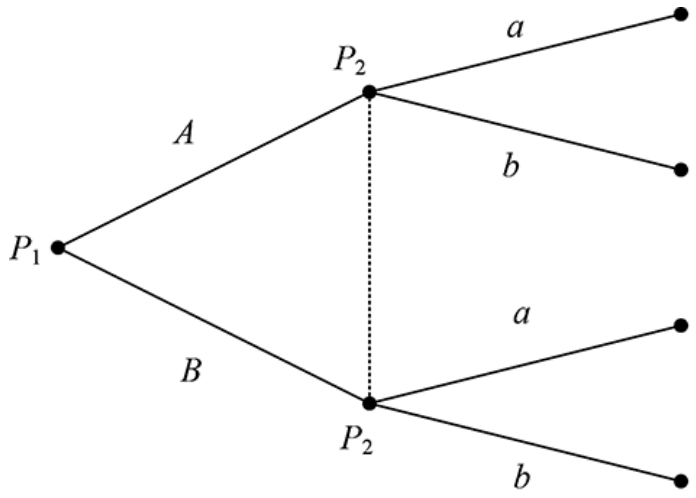


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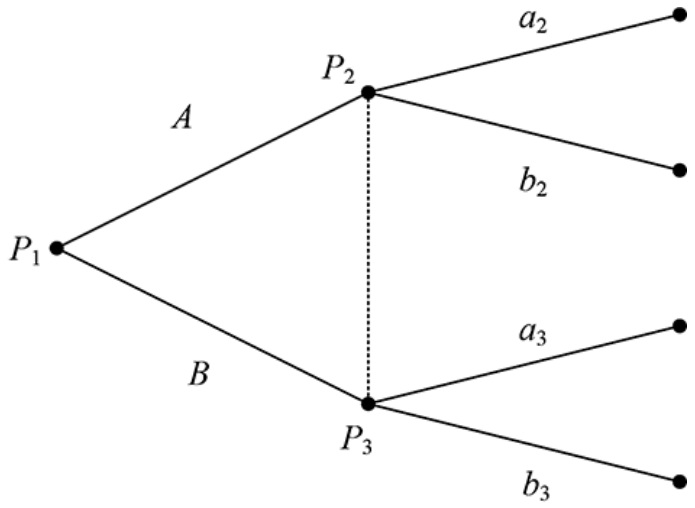
Sequential-Move Games

- **Information sets** are used to denote a group of nodes among which a player cannot distinguish.
- A common feature of trees representing games of **incomplete information**.
- Information sets arise when a player does not observe the action that his predecessor chose.

Sequential-Move Games

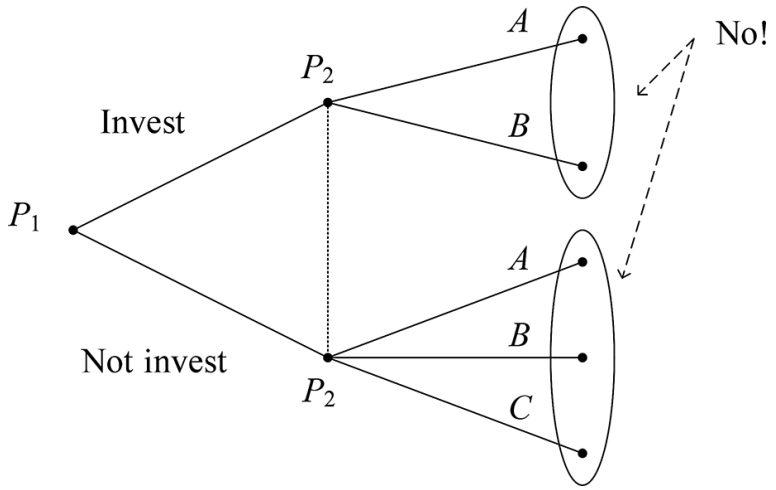


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Sequential-Move Games

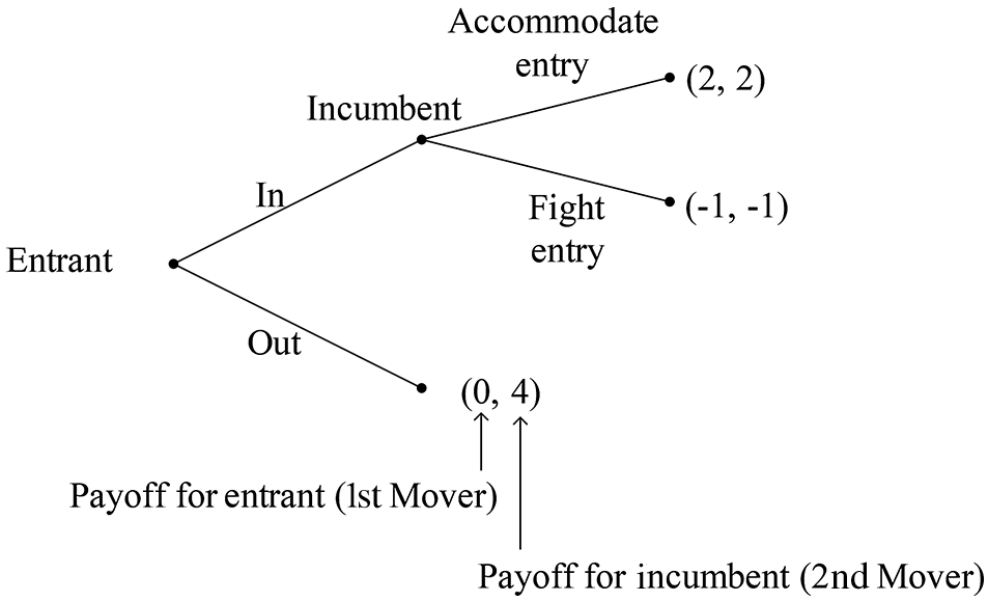


Sequential-Move Games

- Can we simply use the NE solution concept in order to find equilibrium predictions in sequential-move games?
- We can, but some of the NE predictions are not very sensible (credible).

Sequential-Move Games

- ***Example*** (Entry and predation game):
  - Consider an entrant's decision on whether to enter into an industry where an incumbent firm operates or to stay out.



Sequential-Move Games

- ***Example*** (continued):
  - In order to find the NE of this game, it is useful to represent the game in matrix form.

		Incumbent	
		Accommodate	*Fight*
Entrant	*In*	2, 2	-1, -1
	Out	0, 4	0, 4

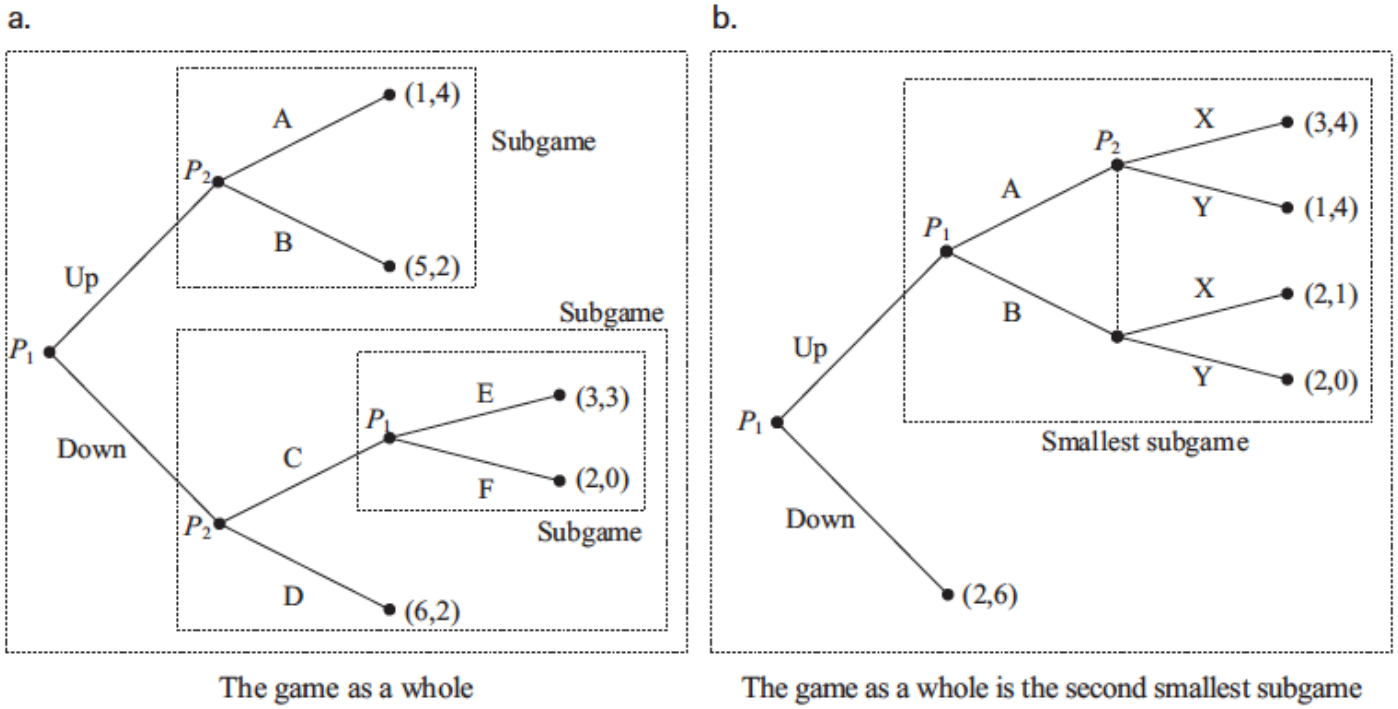
  - Two NEs: $(In, Accommodate)$ and $(Out, Fight)$
  - The first equilibrium seem credible, while the second equilibrium does not look credible at all.

Sequential-Move Games

- The preceding example indicates the need to require a notion of credibility in sequential-move games that did not exist in the NE solution concept
  - A requirement commonly known as "**sequential rationality**"
- Player i's strategy is sequentially rational if it specifies an optimal action for player i at any node (or information set) of the game, even those information sets that player i does not believe will be reached in the equilibrium of the game.
  - That is, player i behaves optimally at every node (or information set), both nodes that belong to the equilibrium path of the game tree and those that lie off-the-equilibrium path.
- How can we guarantee that it holds when finding equilibria in sequential-move games?
  - **Backward induction**: starting from every terminal node, each player uses optimal actions at every *subgame* of the game tree.

Sequential-Move Games

- A **subgame** can be identified by drawing a rectangle around a section of the game tree without "breaking" any information set

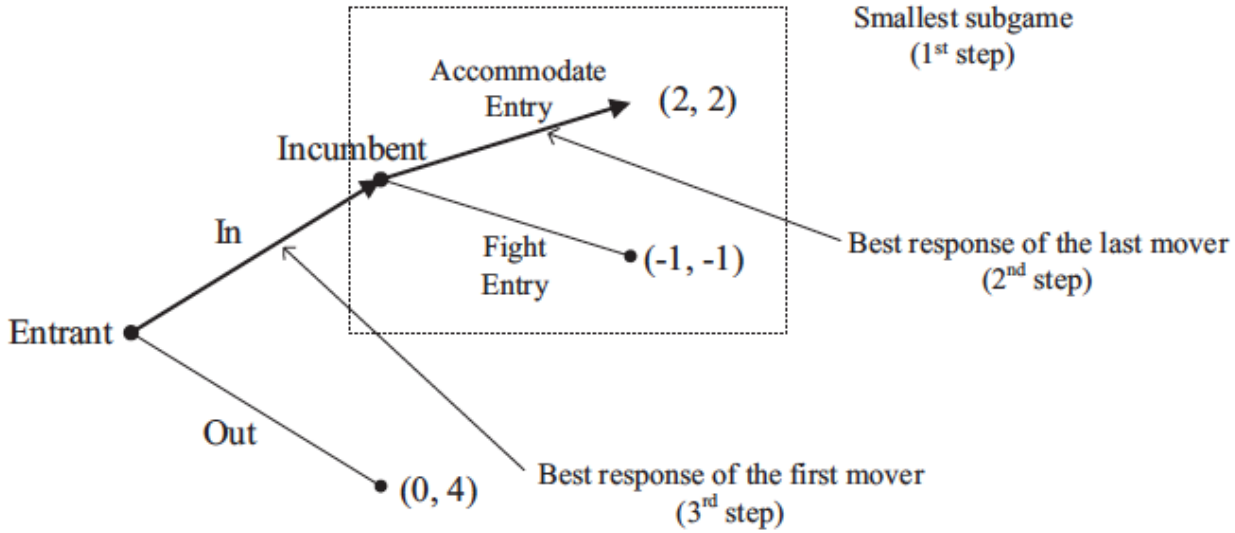


Sequential-Move Games

- The backward induction requires us to find the strategy that every player i finds optimal at every subgame along the game tree.
  - Start by identifying the optimal behavior of the player who acts last (in the last subgame of the tree).
  - Taking the optimal action of this player into account, move to the previous to the last player and identify his optimal behavior.
  - Repeat this process until the initial node.
- **Subgame perfect Nash equilibrium** (SPNE): A strategy profile $(s_1^*, s_2^*, \ldots, s_N^*)$ is a SPNE if it specifies a NE for each subgame.

Sequential-Move Games

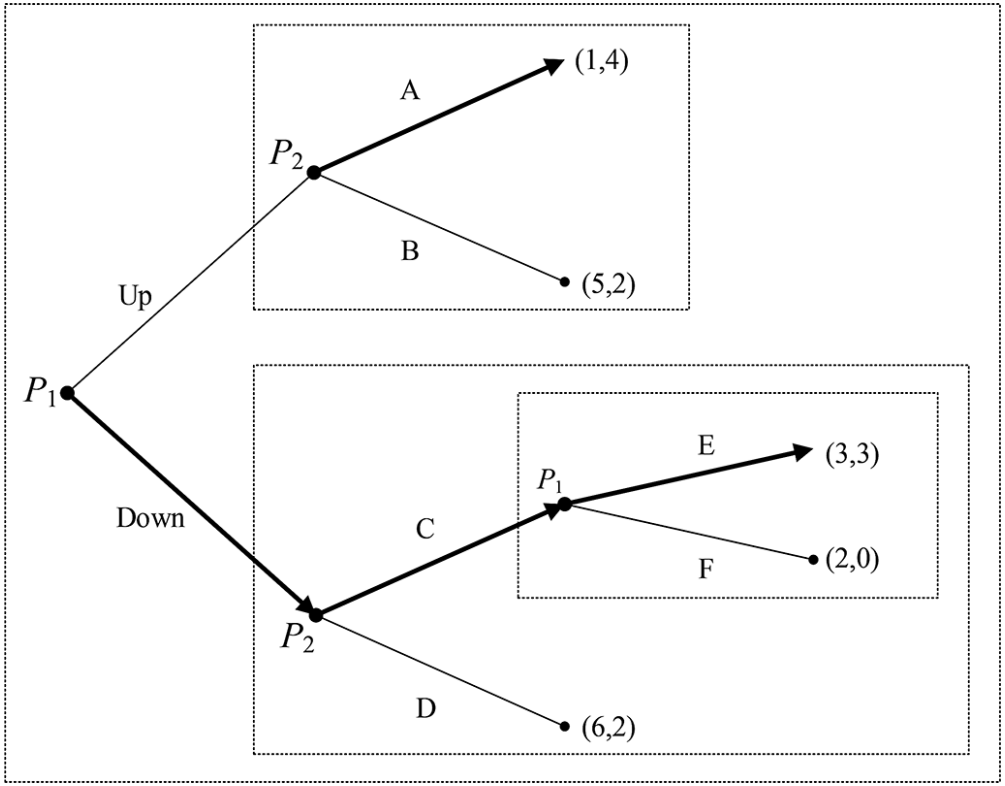
- ***Example*** (Entry and predation game):
  - Identify the subgames of the game tree



  - The SPNE is $(In, Accommodate)$, which coincides with one of the NE of this game.

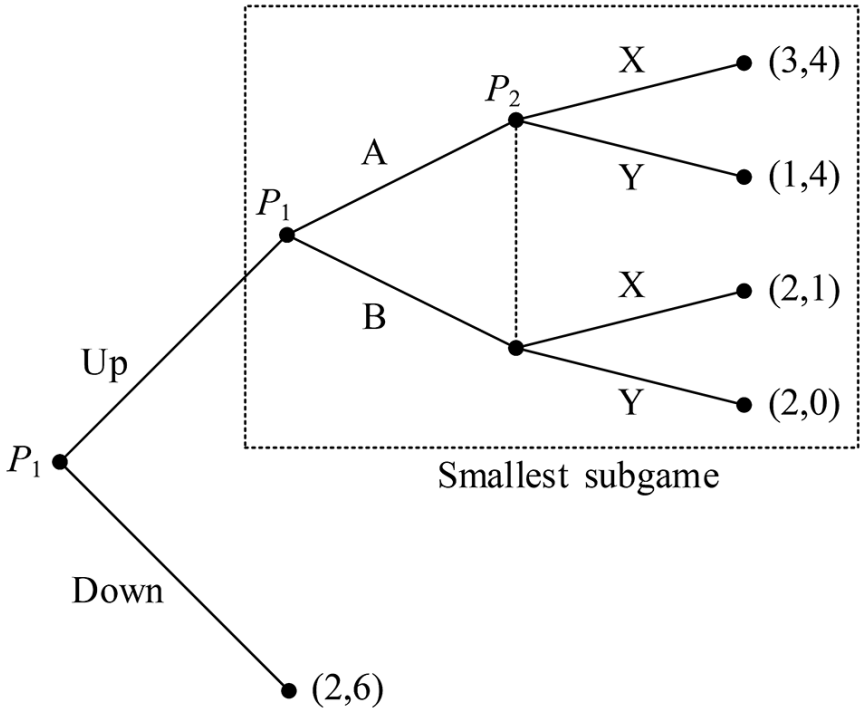
Sequential-Move Games

- ***Example*** (backward induction in three steps):



Sequential-Move Games

- ***Example*** (backward induction in information sets):



Sequential-Move Games

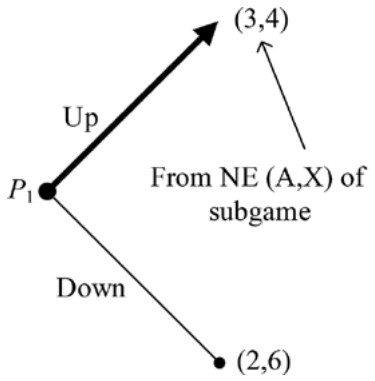
- ***Example*** (continued):
  - The smallest subgame is is strategically equivalent to one in which player 1 and 2 choose their actions simultaneously.

		Player 2	
		X	Y
Player 1	A	3, 4	1, 4
	B	2, 1	2, 0

  - The NE of the subgame is (A, X).

Sequential-Move Games

- ***Example*** (continued):
  - Once we have a reduced-form game tree, we can move one step backward (the initial node)



  - The SPNE of this game is $(Up|A, X)$.
  - Player 1's strategy: play Up in the first node and A afterwards
  - Player 2's strategy: play X

Sequential-Move Games

- ***Example*** (continued):
  - Normal-form representation of the sequential game

		Player 2	
		X	Y
Player 1	Up/A	3, 4	1, 4
	Up/B	2, 1	2, 0
	Down/A	2, 6	2, 6
	Down/B	2, 6	2, 6

  - Three NEs: $(Up|A, X)$, $(Down|A, Y)$, $(Down|B, Y)$.
  - Only the first equilibrium is sequentially rational.

Simultaneous-Move Games of Incomplete Information

Simultaneous-Move Games of Incomplete Information

- The strategic settings previously analyzed assume that all players are perfectly informed about all relevant details of the game.
- There are often real-life situations where players operate without such information.
- Players act under "**incomplete information**" if at least one player cannot observe a piece of information.
  - *Example*: marginal costs of rival firms

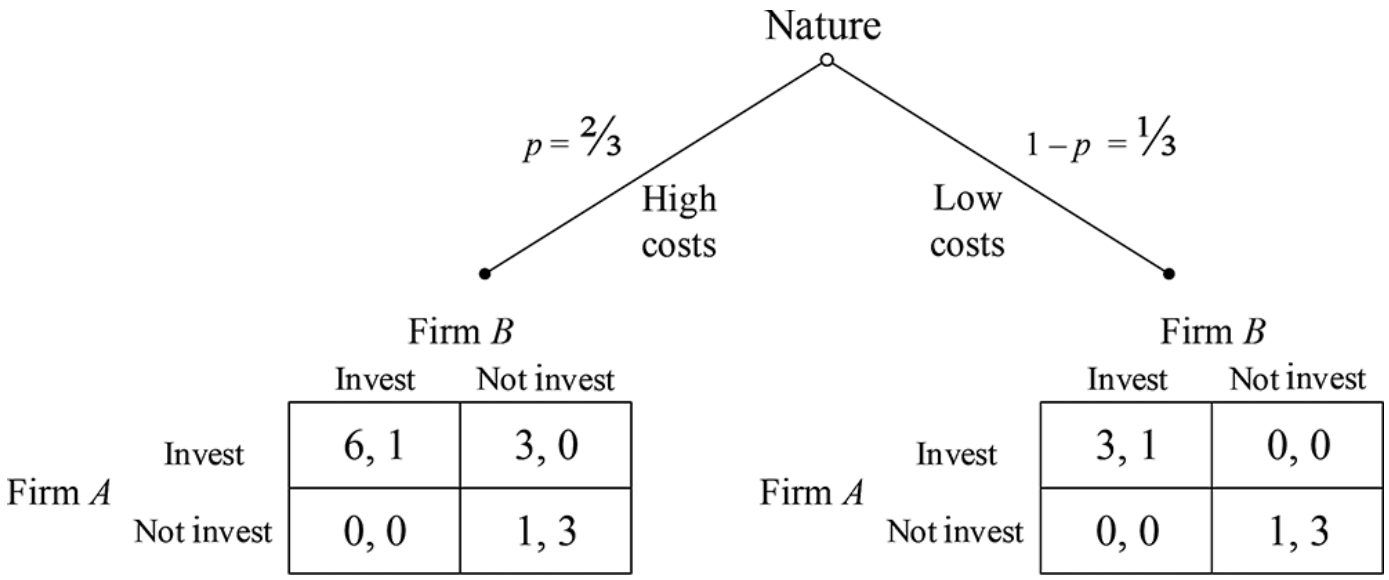
Simultaneous-Move Games of Incomplete Information

- For compactness, we refer to private information as player i's "type" and denote it as θ_i.
- While player j might not observe player i's type, he knows the probability distribution of each type.
- *Example*:
  - Marginal costs can be either high or low, whereby $\Theta_i = \{H, L\}$.
  - The probability of firm i's costs being high is $Prob(\theta_i = H) = p$ and the probability of its costs being low is $Prob(\theta_i = L) = 1 - p$, where $p \in (0,1)$.

Simultaneous-Move Games of Incomplete Information

- ***Example*** (technology adoption):
  - A first move of nature determines the precise type of θ_i.
  - Firm A has two possible types, either high or low costs, with associated probabilities 2/3 and 1/3.
  - Firm A observes its own type, but firm B cannot observe it.
  - Graphically, firm A knows which payoff matrix firms are playing, while firm B can only assign a probability 2/3 (1/3) to playing the left-hand (right-hand) matrix.

Simultaneous-Move Games of Incomplete Information



Simultaneous-Move Games of Incomplete Information

- Every player i's strategy in an incomplete information context needs to be a function of its privately observed type θ_i

$$s_i(\theta_i)$$

- Player i's strategy is not conditioned on other players' types

$$\theta_{-i} = (\theta_1, \theta_2, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_n)$$

- That is, we do not write $s_i(\theta_i, \theta_{-i})$ because player i cannot observe the types of all other players.
  - If all players could observe the types of all of their rivals, we would be describing a complete information game.
- For simplicity, types are independently distributed, which entails that every player i cannot infer his rivals' types θ_{-i} after observing his own type θ_i.

Simultaneous-Move Games of Incomplete Information

- **Bayesian Nash equilibrium** (BNE): A strategy profile $(s_1^*(\theta_1), s_2^*(\theta_2), \ldots, s_N^*(\theta_N))$ is a BNE of a game of incomplete information if

$$EU_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i}) \geq EU_i(s_i(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i, \theta_{-i})$$

  for every strategy $s_i^*(\theta_i) \in S_i$, every type $\theta_i \in \Theta_i$, and every player i.
- When all other players select equilibrium strategies, the expected utility that player i obtains from selecting $s_i^*(\theta_i)$ when his type is θ_i is larger than that of deviating to any other strategy $s_i(\theta_i)$.

Simultaneous-Move Games of Incomplete Information

- **Approach 1**: Four steps to find all BNEs in simultaneous-move games of incomplete information.
- ***Example*** (technology adoption):
  1. *Strategy sets*: Identify the strategy set for each player, which can be a function of his privately observed type

$$S_1 = \{I_H I_L, I_H NI_L, NI_H I_L, NI_H NI_L\}$$
$$S_2 = \{I, NI\}$$

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):
  2. *Bayesian normal-form representation*: Use the strategy sets identified in step 1 to construct the "Bayesian normal-form" representation of the incomplete information game.

		Firm B	
		I	NI
Firm A	$I_H I_L$		
	$I_H NI_L$		
	$NI_H I_L$		
	$NI_H NI_L$		

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):
  3. *Expected payoffs*: Find the expected payoffs that would go in every cell.

		Firm B	
		I	NI
Firm A	$I_H I_L$	5, 1	2, 0
	$I_H NI_L$	4, 2/3	2 1/3, 1
	$NI_H I_L$	1, 1/3	2/3, 2
	$NI_H NI_L$	0, 0	1, 3

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):
  4. *Find best responses for each player*: Follow an approach similar to that in simultaneous-move games of complete information to find best-response payoffs.

     – The BNEs are $(I_H I_L, I)$ and $(I_H N I_L, NI)$.

Simultaneous-Move Games of Incomplete Information

- **Approach 2**: Find the set of BNEs by first analyzing best responses for the privately informed player, and then use those in our identification of best responses for the uninformed player.

Simultaneous-Move Games of Incomplete Information

- ***Example*** (technology adoption):
  - The two possible games that firms could be playing.

		Firm *B*: Invest (β)	Firm *B*: Not invest ($1-\beta$)
Firm *A*: Invest	α	6, 1	3, 0
Firm *A*: Not invest	$1-\alpha$	0, 0	1, 3

Firm A is high type with probability p

		Firm *B*: Invest (β)	Firm *B*: Not invest ($1-\beta$)
Firm *A*: Invest	γ	3, 1	0, 0
Firm *A*: Not invest	$1-\gamma$	0, 0	1, 3

Firm A is low type with probability $1-p$

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):
  - First, we look at the privately informed firm A.
  - If firm A is of the high type, $Invest$ strictly dominates $Not\ invest$.
  - If firm A is of the low type, neither strategy strictly dominates the other.
  - Need to compare the expected utilities
    $EU_A(Invest|Low) = 3 \cdot \beta + 0 \cdot (1-\beta) = 3\beta$
    $EU_A(Not\ invest|Low) = 0 \cdot \beta + 1 \cdot (1-\beta) = 1-\beta$
  - Firm A invests if $3\beta \geq 1-\beta$ or $\beta \geq 1/4$.

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):
  - Next, we look at the uninformed firm B.
  - Since firm B does not know firm A's type, we have to model in the probability (p) that firm A is of the high type.

$$EU_B(Invest)$$

$$= \overbrace{1 \cdot p}^{\text{If firm A is high type, it invests}} + (1-p) \cdot \overbrace{\left[\underbrace{1 \cdot \gamma}_{\substack{Firm\ A\ invests \\ when\ low\ type}} + \underbrace{0 \cdot (1-\gamma)}_{\substack{Firm\ A\ does\ not \\ invest\ when\ low \\ type}} \right]}^{\text{If firm A is low type}}$$

$$= p + (1-p)\gamma$$

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):

$$EU_B(Invest)$$

$$= \underbrace{1 \cdot p}_{\text{If firm A is high type, it invests}} + (1-p) \cdot \overbrace{\left[\underbrace{1 \cdot \gamma}_{\substack{Firm\ A\ invests \\ when\ low\ type}} + \underbrace{0 \cdot (1-\gamma)}_{\substack{Firm\ A\ does\ not \\ invest\ when\ low \\ type}} \right]}^{\text{If firm A is low type}}$$

$$= p + (1-p)\gamma$$

$$EU_B(Not\ invest)$$

$$= \underbrace{0 \cdot p}_{\text{If firm A is high type, it invests}} + (1-p) \cdot \overbrace{\left[\underbrace{0 \cdot \gamma}_{\substack{Firm\ A\ invests \\ when\ low\ type}} + \underbrace{3 \cdot (1-\gamma)}_{\substack{Firm\ A\ does\ not \\ investwhen\ low \\ type}} \right]}^{\text{If firm A is low type}}$$

$$= 3(1-p)(1-\gamma)$$

Simultaneous-Move Games of Incomplete Information

- ***Example*** (continued):
  - Therefore, firm B invests if
$$p + (1-p)\gamma \geq 3(1-p)(1-\gamma)$$
  - Since $p = 2/3$, the above inequality reduces to
$$2 \geq 3 - 4\gamma$$
$$\gamma \geq 1/4$$
  - Two BNEs:
    1. If $\gamma \geq 1/4$, $(I_H I_L, I)$.
    2. If $\gamma < 1/4$, $(I_H N I_L, NI)$.

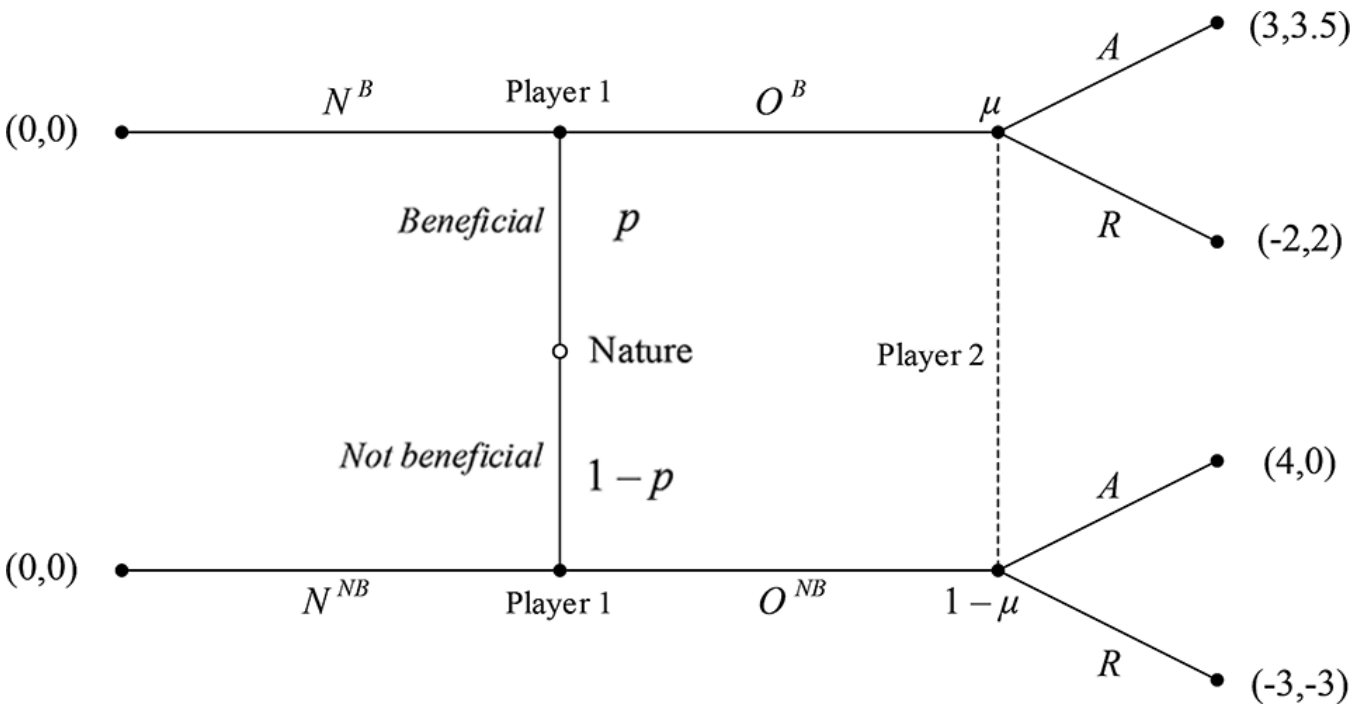
Sequential-Move Games under Incomplete Information

Sequential-Move Games under Incomplete Information

- The BNE solution concept helps us find equilibrium outcomes in settings where players interact under *incomplete information*.
- While the applications in the previous section considered that players act *simultaneously*, we can also find the BNEs of incomplete information games in which players act *sequentially*.

Sequential-Move Games under Incomplete Information

- Investment game:



Sequential-Move Games under Incomplete Information

- In order to find the set of BNEs, we first represent the Bayesian *normal-form* representation of the game tree.
- The matrix includes expected payoffs for each player.

		Player 2	
		A	R
Player 1	$O^B O^{NB}$	4-p, 3.5p	-3+p, -3+5p
	$O^B N^{NB}$	3p, 3.5p	-2p, 2p
	$N^B O^{NB}$	4-4p, 0	-3+3p, -3+3p
	$N^B N^{NB}$	0, 0	0, 0

Sequential-Move Games under Incomplete Information

- There are two BNEs in this game:

$$(O^B O^{NB}, A)$$
$$(N^B N^{NB}, R)$$

- The first BNE is rather sensible
  - Player 1 makes the offer regardless of his type, and thus the uninformed player 2 chooses to accept the offer if he receives one.

Sequential-Move Games under Incomplete Information

- The second BNE is difficult to rationalize
  - No type of sender makes an offer in equilibrium, and the responder rejects any offer presented to him.
  - If an offer was ever observed, the receiver should compare the expected utility of accepting and rejecting the offer, based on the off-the equilibrium belief μ.

$$EU_2(A) = 3.5 \cdot \mu + 0 \cdot (1 - \mu) = 3.5\mu$$
$$EU_2(R) = 2 \cdot \mu + (-3) \cdot (1 - \mu) = -3 + 5\mu$$

  - Player 2 accepts the offer, since $3.5\mu > -3 + 5\mu \Rightarrow 1.5\mu < 3$, which holds for all $\mu \in (0,1)$.
  - Therefore, the offer rejection that $(N^B N^{NB}, R)$ prescribes cannot be sequentially rational.

Sequential-Move Games under Incomplete Information

- In order to avoid identifying equilibrium predictions that are not sequentially rational, we apply the Perfect Bayesian Equilibrium (PBE) that can deal with sequential move games with incomplete information.
- **The Perfect Bayesian Equilibrium** (PBE): A strategy profit $(s_1, s_2, \ldots, s_N)$ and beliefs μ over the nodes at all information sets are a PBE if:
  1. each player's strategies specify optimal actions, given the strategies of the other players, and given his beliefs, and
  2. beliefs are consistent with Bayes's rule, whenever possible.

Sequential-Move Games under Incomplete Information

- The first condition resembles the definition of BNE.
- The second condition was not present in the definition of BNE.
  - It states that beliefs must be consistent with Bayes's rule whenever possible
- Applying Bayes's rule in the investment game, player 2's probability that the investment is beneficial after receiving an offer is

$$p(B|\text{Offer}) = \frac{p(B) \cdot p(\text{Offer}|B)}{p(\text{Offer})}$$

$$= \frac{p(B) \cdot p(\text{Offer}|B)}{p(B) \cdot p(\text{Offer}|B) + p(NB) \cdot p(\text{Offer}|NB)}$$

Sequential-Move Games under Incomplete Information

- Denoting $\mu = p(B|\text{Offer})$, $\alpha_i = p(\text{Offer}|i)$, where $i = \{B, NB\}$, $p = p(B)$, and $1 - p = p(NB)$, player 2's belief can be expressed as

$$\mu = \frac{p \cdot \alpha_B}{p \cdot \alpha_B + (1-p) \cdot \alpha_{NB}}$$

- If player 2 assigns probabilities $\alpha_B = 1/8$ and $\alpha_{NB} = 1/16$, then

$$\mu = \frac{1/2 \cdot 1/8}{1/2 \cdot 1/8 + 1/2 \cdot 1/16} = \frac{2}{3}$$

- We refer to μ as off-the-equilibrium beliefs
  - The probability of being in a node of an information set that is actually not reached in equilibrium.

Sequential-Move Games under Incomplete Information

- **Procedure to Find PBEs:**
  1. Specify a strategy profile for the privately informed player.
     - In the investment example, there are four possible strategy profiles for the privately informed player 1.
     - Two separating strategy profiles: $O^B N^{NB}$, $N^B O^{NB}$.
     - Two pooling strategy profiles: $O^B O^{NB}$, $N^B N^{NB}$.
  2. Update the **uninformed** player's beliefs using Bayes's rule at all information sets, whenever possible.

Sequential-Move Games under Incomplete Information

- **Procedure to Find PBEs:** (continued)
  3. Given the **uninformed** player's updated beliefs, find his optimal response
     - In the investment example, we need to determine the optimal response of player 2 upon receiving an offer from player 1 given his updated belief.
  4. Given the optimal response of the uninformed player obtained in step 3, find the optimal action (message) for each type of **informed** player.
     - In the investment example, first check if player 1 makes an offer when the investment is beneficial.
     - Then check whether player 1 prefers to make an offer, when the investment is not beneficial.

Sequential-Move Games under Incomplete Information

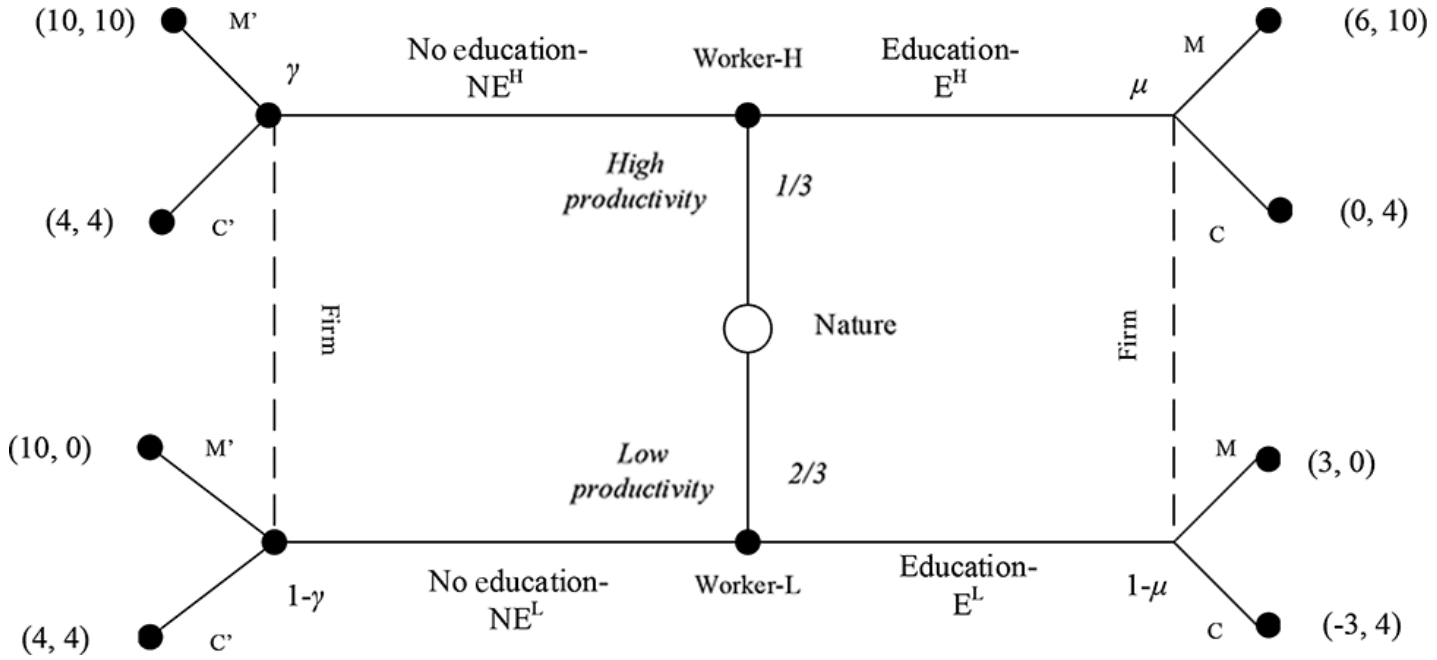
- **Procedure to Find PBEs:** (continued)

5. Check if the strategy profile for the informed player found in step 4 coincides with the profile suggested in step 1.
   - If it coincides, then this strategy profile, updated beliefs, and optimal responses can be supported as a PBE of the incomplete information game.
   - Otherwise, we say that this strategy profile cannot be sustained as a PBE of the game.

Sequential-Move Games under Incomplete Information

- ***Example*** (Labor market signaling game):
  - The sequential game with incomplete information.
  - A worker privately observes whether he has a high productivity or a low productivity.
  - The worker then decides whether to pursue more education (e.g., an MBA) that he might use as a signal about his productivity.
  - The firm can either hire him as a manager (M) or as a cashier (C).

Sequential-Move Games under Incomplete Information



Sequential-Move Games under Incomplete Information

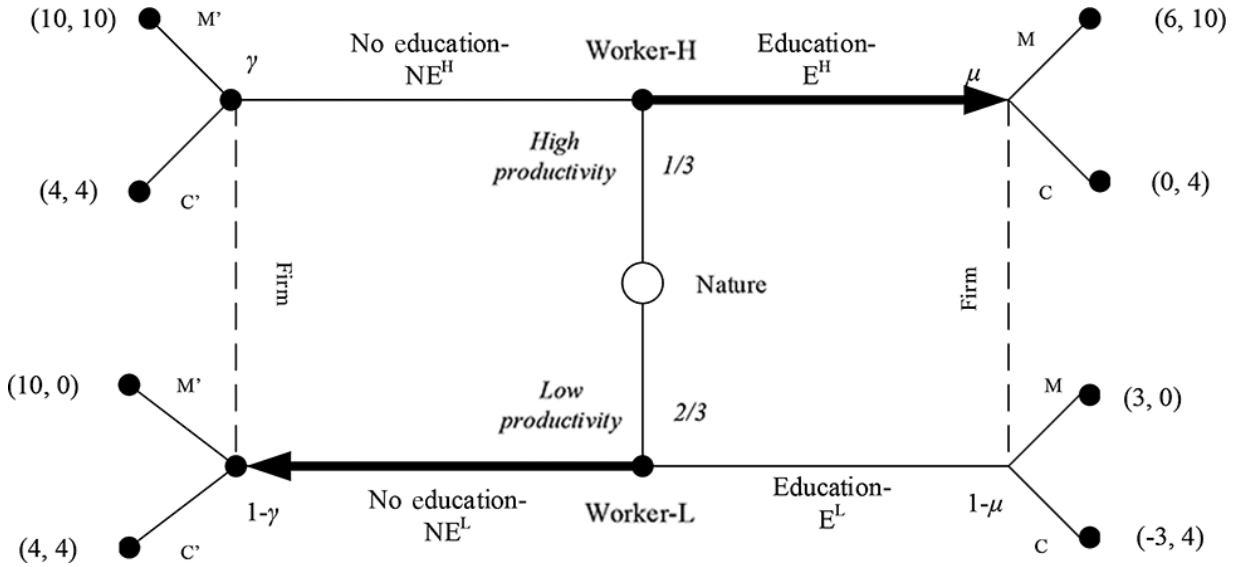
- ***Example*** (continued):
  - We focus on:
    - Separating strategy profiles: (E^H, NE^L)
    - Pooling strategy profile: (NE^H, NE^L)
  - Exercise:
    - Separating strategy profiles: (NE^H, E^L)
    - Pooling strategy profile: (E^H, E^L)

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):

1. Separating PBE $\left(E^H, NE^L\right)$:

– *Step 1*: Specify the separating strategy profile E^H, NE^L for the informed player.



Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 2*: Use Bayes' rule to update the uninformed player's (firm) beliefs.
    - Taking into account that $\alpha_H = 1$ while $\alpha_L = 0$, the firm updates its beliefs for an educated applicant as

$$\mu = \frac{1/3 \cdot \alpha_H}{1/3 \cdot \alpha_H + 2/3 \cdot \alpha_L} = 1$$

    - Intuitively, after observing that the applicant acquired education, the firm assigns full probability to the applicant being of high productivity.

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - Now, taking into account that $\alpha_E = 0$ while $\alpha_{NE} = 1$, the firm updates its beliefs for a less educated applicant as

$$\gamma = \frac{1/3 \cdot \alpha_H}{1/3 \cdot \alpha_H + 2/3 \cdot \alpha_L} = 0$$

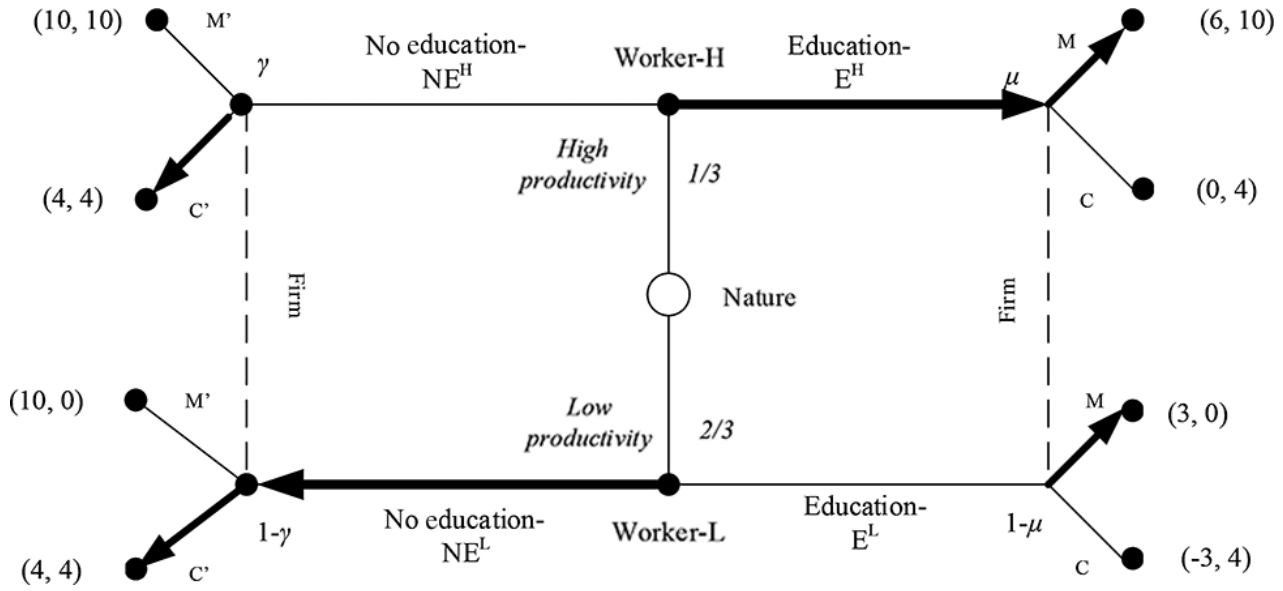
    or that

$$1 - \gamma = 1$$

  - Intuitively, the firm that observes the less educated applicant believes that such an applicant must be of low productivity.

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 3*: Given the firm's beliefs, determine the firm's optimal response, after observing the education level of the worker.



Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 4*: Given these strategy profiles, examine the worker's optimal action.
    - *High-productivity type*: Does not have an incentive to deviate from the strategy profile (acquiring more education).
    - *Low-productivity type*: The cost of acquiring education is too high for the low-productivity worker; and thus that worker chooses not to pursue it.

Sequential-Move Games under Incomplete Information

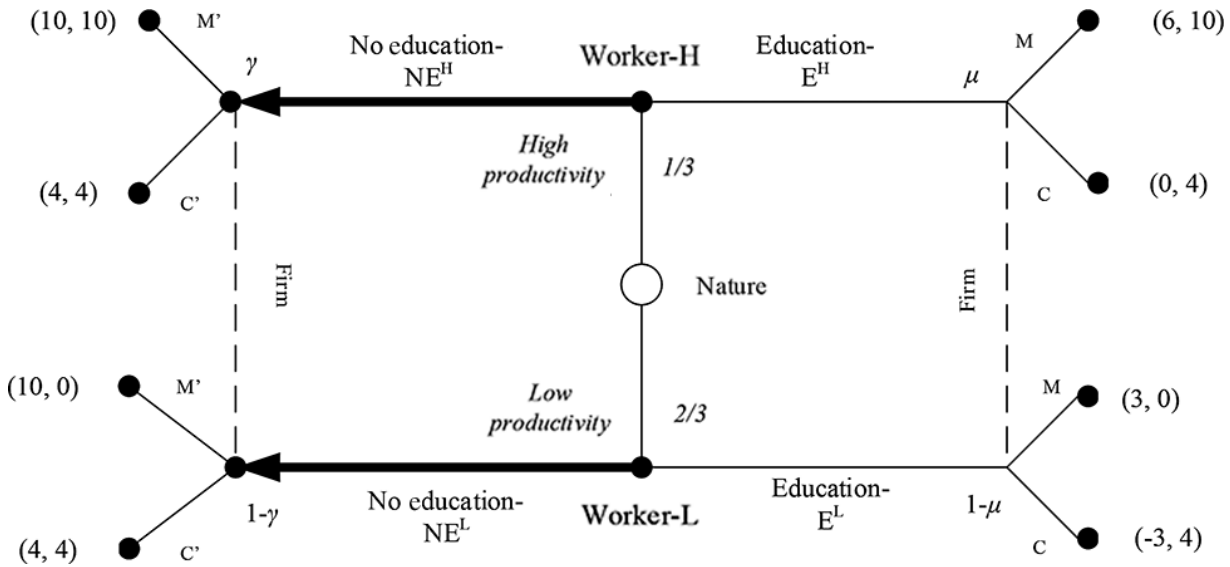
- ***Example*** (continued):
  - *Step 5*: The separating strategy profile (E^H, NE^L) can be sustained as the PBE of this incomplete information game.
    - Neither type of worker has the incentive to deviate from the prescribed separating strategy profile (E^H, NE^L).

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):

2. Pooling PBE $\left(NE^H, NE^L\right)$:

– *Step 1*: Specify the separating strategy profile NE^H, NE^L for the informed player.



Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 2*: Use Bayes's rule to update the uninformed player's (firm) beliefs.
    - Taking into account that $\alpha_H = 1$ while $\alpha_L = 1$, the firm updates its beliefs for a less educated applicant as

$$\gamma = \frac{1/3 \cdot \alpha_H}{1/3 \cdot \alpha_H + 2/3 \cdot \alpha_L} = 1/3$$

    - Intuitively, since neither type of applicant obtains education in this strategy profile, the firm's observation of an uneducated applicant does not allow the firm to further restrict its posterior beliefs about the applicant's type.

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - Taking into account that $\alpha_H = 0$ while $\alpha_L = 0$, the firm updates its beliefs for a more educated applicant as

$$\mu = \frac{1/3 \cdot \alpha_H}{1/3 \cdot \alpha_H + 2/3 \cdot \alpha_L} = 0$$

  - This player's off-the-equilibrium beliefs are left unrestricted at $\mu \in [0,1]$.

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 3*: Given the firm's beliefs, determine the firm's optimal response, after observing the education level of the worker.
    - Upon observing a less educated applicant:

$$EU_{\text{firm}}(M'|\text{No education}) = \frac{1}{3}\cdot 10 + \frac{2}{3}\cdot 0 = \frac{10}{3}$$

$$EU_{\text{firm}}(C'|\text{No education}) = \frac{1}{3}\cdot 4 + \frac{2}{3}\cdot 4 = 4$$

    - Hence, the firm optimally responds by offering the applicant a cashier position.

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - Upon observing a more educated applicant:

$$EU_{\text{firm}}(M|\text{Education}) = \mu \cdot 10 + (1-\mu) \cdot 0 = 10\mu$$
$$EU_{\text{firm}}(C|\text{Education}) = \mu \cdot 4 + (1-\mu) \cdot 4 = 4$$

  - The firm responds by offering the applicant a manager position if and only if

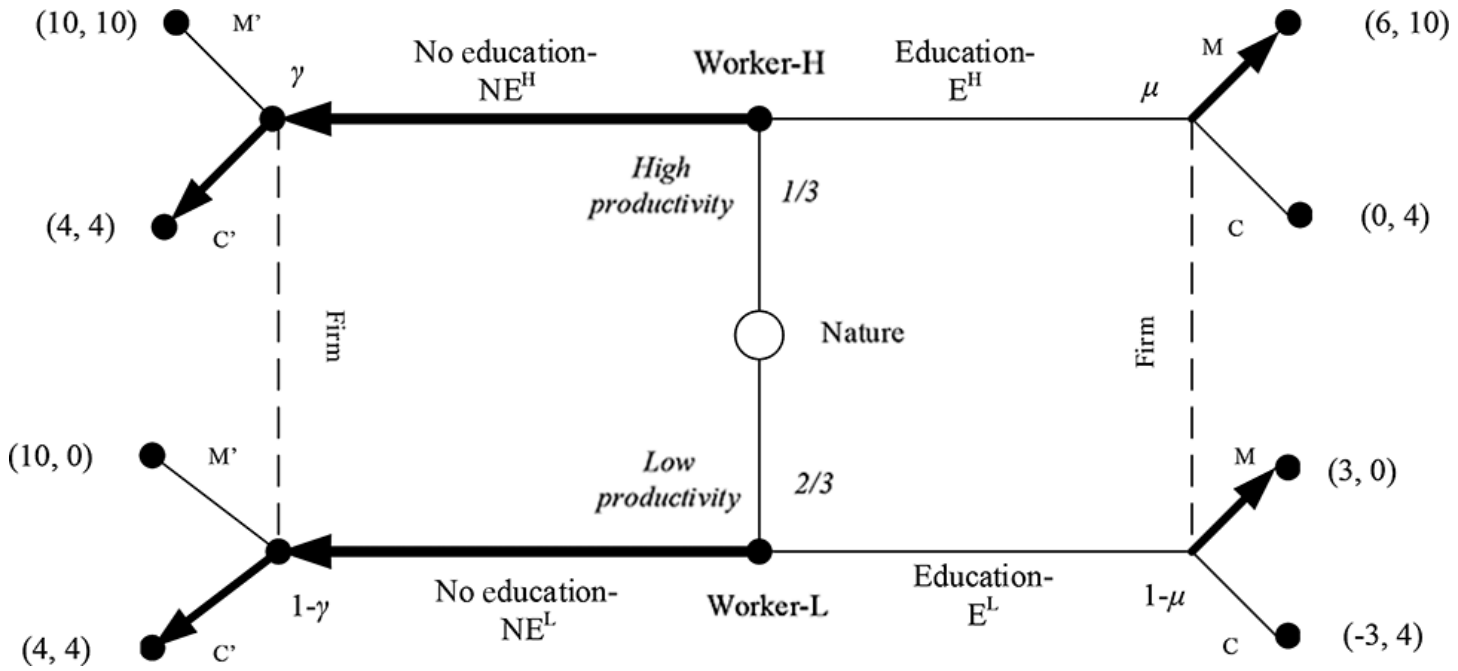
$$10\mu > 4 \Rightarrow \mu > 2/5$$

  - We thus need to divide the fifth step (the optimal actions of the worker) into two cases:
    1. $\mu > 2/5$, where the firm responds with M
    2. $\mu \leq 2/5$, where the firm responds with C

Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 4*: Given these strategy profiles, examine the worker's optimal action.
  - Case 1: $\mu > 2/5$
    - *High-productivity type*: Has an incentive to deviate from the prescribed strategy profile. Thus it cannot be supported as a PBE.
    - *Low-productivity type*: Does not have incentives to deviate from the prescribed strategy profile.

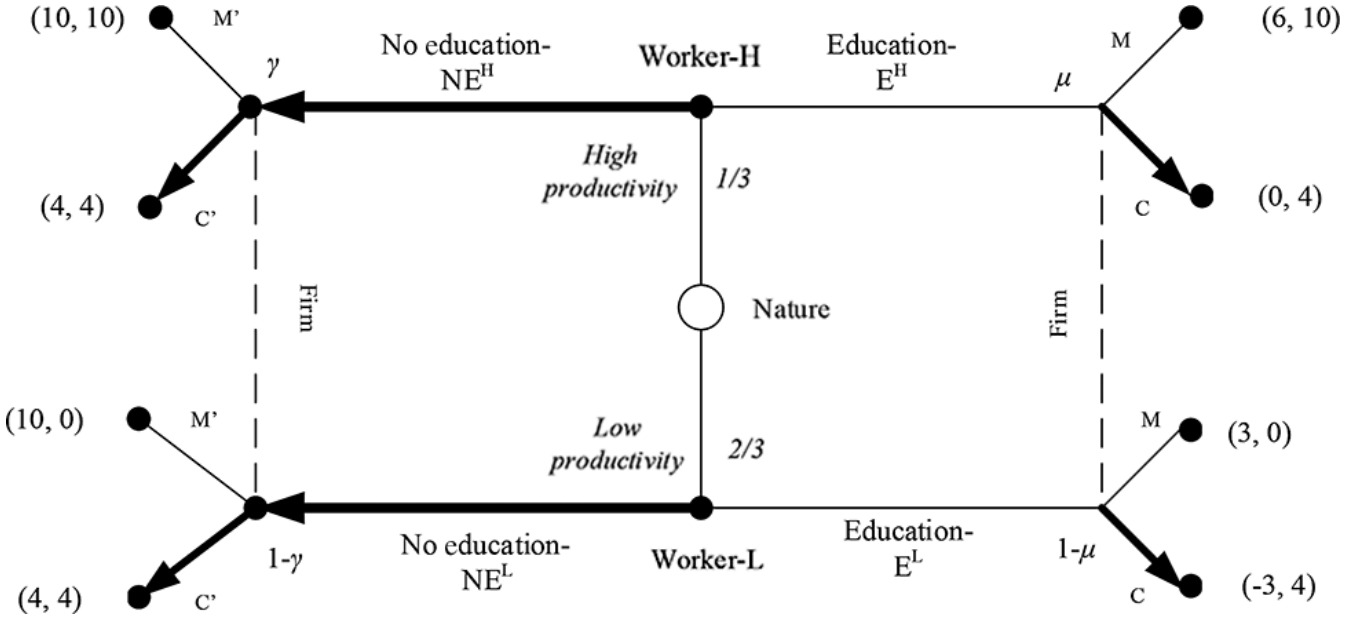
Sequential-Move Games under Incomplete Information



Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - Case 2: $\mu \leq 2/5$
    - *High-productivity type*: Does not have incentives to deviate from the prescribed strategy profile.
    - *Low-productivity type*: Does not have incentives to deviate from the prescribed strategy profile.

Sequential-Move Games under Incomplete Information



Sequential-Move Games under Incomplete Information

- ***Example*** (continued):
  - *Step 5*: The pooling strategy profile (NE^H, NE^L) can be supported as the PBE when off-the-equilibrium beliefs satisfy $\mu \leq 2/5$.

Bertrand Model of Price Competition

Bertrand Model of Price Competition

- Consider:
  - An industry with two firms, 1 and 2, selling a homogeneous product
  - Firms face market demand $x(p)$, where $x(p)$ is continuous and strictly decreasing in p
  - There exists a high enough price (choke price) $\bar{p} < \infty$ such that $x(p) = 0$ for all $p > \bar{p}$
  - Both firms are symmetric in their constant marginal cost $c > 0$, where $x(c) \in (0, \infty)$
  - Every firm j *simultaneously* sets a price p_j

Bertrand Model of Price Competition

- Firm j's demand is

$$x_j(p_j, p_k) = \begin{cases} x(p_j) & \text{if } p_j < p_k \\ \frac{1}{2}x(p_j) & \text{if } p_j = p_k \\ 0 & \text{if } p_j > p_k \end{cases}$$

- *Intuition*: Firm j captures
  - all market if its price is the lowest, $p_j < p_k$
  - no market if its price is the highest, $p_j > p_k$
  - shares the market with firm k if the price of both firms coincides, $p_j = p_k$

Bertrand Model of Price Competition

- Given prices p_j and p_k, firm j's profits are therefore

$$(p_j - c) \cdot x_j\,(p_j, p_k)$$

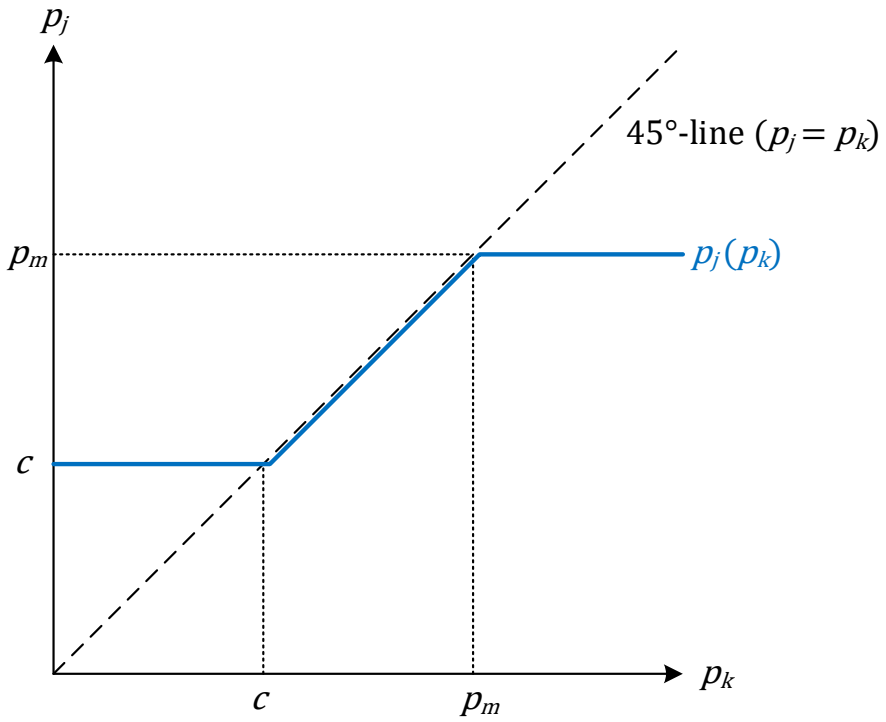
- We are now ready to find equilibrium prices in the Bertrand duopoly model.
  - There is a unique NE (p_j^*, p_k^*) in the Bertrand duopoly model. In this equilibrium, both firms set prices equal to marginal cost, $p_j^* = p_k^* = c$.

Bertrand Model of Price Competition

- Let us describe the best response function of firm j.
- If $p_k < c$, firm j sets its price at $p_j = c$.
  - Firm j does not undercut firm k since that would entail negative profits.
- If $c < p_k < p_m$, firm j slightly undercuts firm k, i.e., $p_j = p_k - \varepsilon$.
  - This allows firm j to capture all sales and still make a positive margin on each unit.
- If $p_k > p_m$, where p_m is a monopoly price, firm j does not need to charge more than p_m, i.e., $p_j = p_m$.
  - p_m allows firm j to capture all sales and maximize profits as the only firm selling a positive output.

Bertrand Model of Price Competition

- Firm j's best response has:
  - a flat segment for all $p_k < c$, where $p_j(p_k) = c$
  - a positive slope for all $c < p_k < p_m$, where firm j charges a price slightly below firm k
  - a flat segment for all $p_k > p_m$, where $p_j(p_k) = p_m$

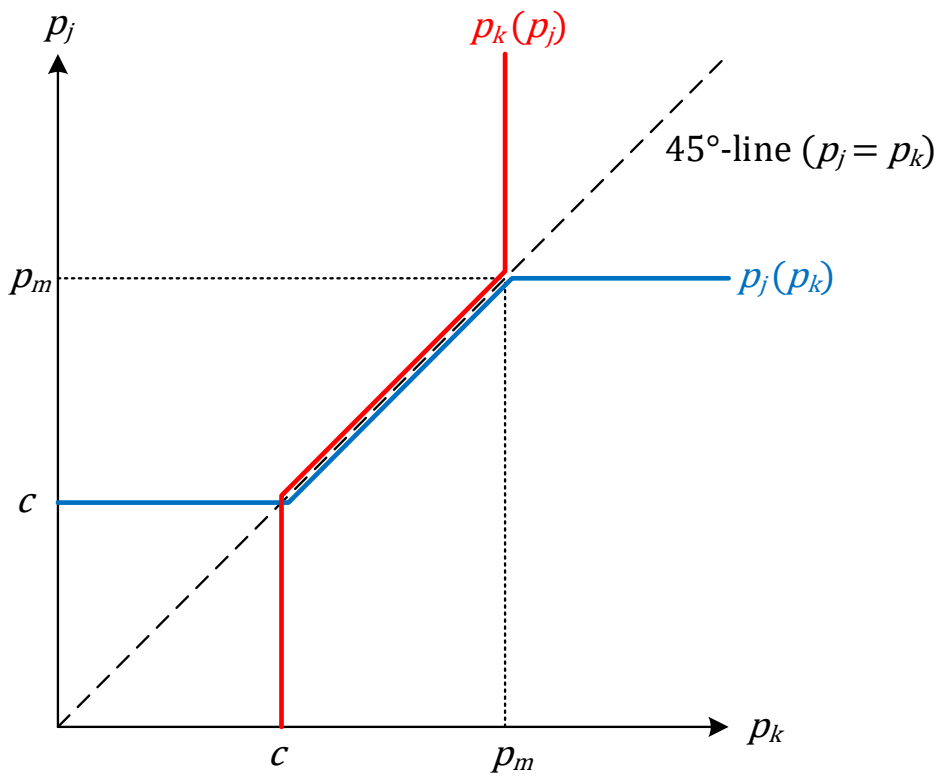


Bertrand Model of Price Competition

- A symmetric argument applies to the construction of the best response function of firm k.
- A mutual best response for both firms is

$$(p_1^*, p_2^*) = (c, c)$$

  where the two best response functions cross each other.
- This is the NE of the Bertrand model
  - Firms make no economic profits.



Bertrand Model of Price Competition

- With only two firms competing in prices we obtain the perfectly competitive outcome, where firms set prices equal to marginal cost.
- Price competition makes each firm j face an infinitely elastic demand curve at its rival's price, p_k.
  - Any increase (decrease) from p_k infinitely reduces (increases, respectively) firm j's demand.

Bertrand Model of Price Competition

- How much does Bertrand equilibrium hinge into our assumptions?
  - Quite a lot
- The competitive pressure in the Bertrand model with homogeneous products is ameliorated if we instead consider:
  - Price competition (but allowing for heterogeneous products)
  - Quantity competition (still with homogeneous products)
  - Capacity constraints

Bertrand Model of Price Competition

- ***Remark***:
  - How would our results be affected if firms face different production costs, i.e., $0 < c_1 < c_2$?
  - The most efficient firm sets a price equal to the marginal cost of the least efficient firm, $p_1 = c_2$.
  - Other firms will set a random price in the uniform interval

$$[c_1, c_1 + \eta]$$

where $\eta > 0$ is some small random increment with probability distribution $f(p, \eta) > 0$ for all p.

Cournot Model of Quantity Competition

Cournot Model of Quantity Competition

- Let us now consider that firms compete in quantities.
- Assume that:
  - Firms bring their output q_1 and q_2 to a market, the market clears, and the price is determined from the inverse demand function $p(q)$, where $q = q_1 + q_2$.
  - $p(q)$ satisfies $p'(q) < 0$ at all output levels $q \geq 0$,
  - Both firms face a common marginal cost $c > 0$
  - $p(0) > c$ in order to guarantee that the inverse demand curve crosses the constant marginal cost curve at an interior point.

Cournot Model of Quantity Competition

- Let us first identify every firm's best response function
- Firm 1's PMP, for a given output level of its rival, $\bar{q}_2$,

$$\max_{q_1 \geq 0} \quad \underbrace{p(q_1 + \bar{q}_2)}_{\text{Price}} q_1 - cq_1$$

- When solving this PMP, firm 1 treats firm 2's production, $\bar{q}_2$, as a parameter, since firm 1 cannot vary its level.

Cournot Model of Quantity Competition

- FOCs:

$$p'(q_1 + \bar{q}_2)q_1 + p(q_1 + \bar{q}_2) - c \leq 0$$

with equality if $q_1 > 0$

- Solving this expression for q_1, we obtain firm 1's best response function (BRF), $q_1(\bar{q}_2)$.
- A similar argument applies to firm 2's PMP and its best response function $q_2(\bar{q}_1)$.
- Therefore, a pair of output levels (q_1^*, q_2^*) is NE of the Cournot model if and only if

$$q_1^* \in q_1(\bar{q}_2) \text{ for firm 1's output}$$
$$q_2^* \in q_2(\bar{q}_1) \text{ for firm 2's output}$$

Cournot Model of Quantity Competition

- To show that $q_1^*, q_2^* > 0$, let us work by contradiction, assuming $q_1^* = 0$.
  - Firm 2 becomes a monopolist since it is the only firm producing a positive output.
- Using the FOC for firm 1, we obtain

$$p'(0 + q_2^*)0 + p(0 + q_2^*) \leq c$$

$$\text{or } p(q_2^*) \leq c$$

- And using the FOC for firm 2, we have

$$p'(q_2^* + 0)q_2^* + p(q_2^* + 0) \leq c$$

$$\text{or } p'(q_2^*)q_2^* + p(q_2^*) \leq c$$

- This implies firm 2's MR under monopoly is lower than its MC. Thus, $q_2^* = 0$.

Cournot Model of Quantity Competition

- Hence, if $q_1^* = 0$, firm 2's output would also be zero, $q_2^* = 0$.
- But this implies that $p(0) < c$ since no firm produces a positive output, thus violating our initial assumption $p(0) > c$.
  - Contradiction!
- As a result, we must have that both $q_1^* > 0$ and $q_2^* > 0$.
- *Note*: This result does not necessarily hold when both firms are asymmetric in their production costs.

Cournot Model of Quantity Competition

- ***Example*** (symmetric costs):
  - Consider an inverse demand curve $p(q) = a - bq$, and two firms competing à la Cournot both facing a marginal cost $c > 0$.
  - Firm 1's PMP is
$$[a - b(q_1 + \bar{q}_2)]q_1 - cq_1$$
  - FOC wrt q_1:
$$a - 2bq_1 - b\bar{q}_2 - c \leq 0$$
with equality if $q_1 > 0$

Cournot Model of Quantity Competition

- ***Example*** (continue):
  - Solving for q_1, we obtain firm 1's BRF

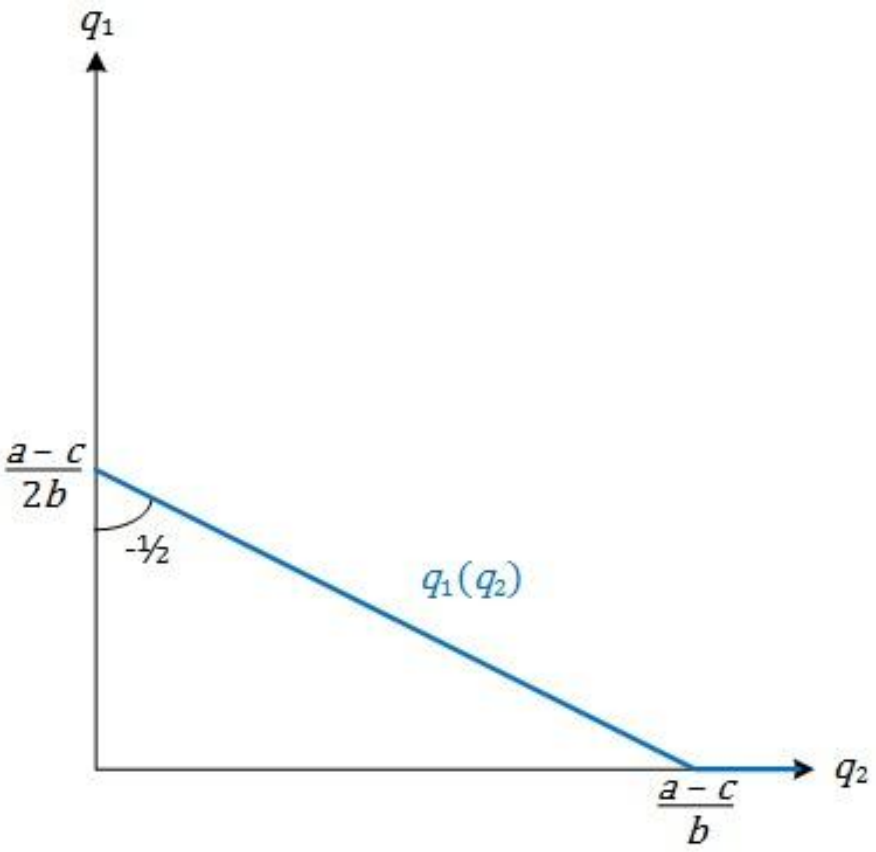
$$q_1(\bar{q}_2) = \frac{a-c}{2b} - \frac{\bar{q}_2}{2}$$

  - Analogously, firm 2's BRF

$$q_2(\bar{q}_1) = \frac{a-c}{2b} - \frac{\bar{q}_1}{2}$$

Cournot Model of Quantity Competition

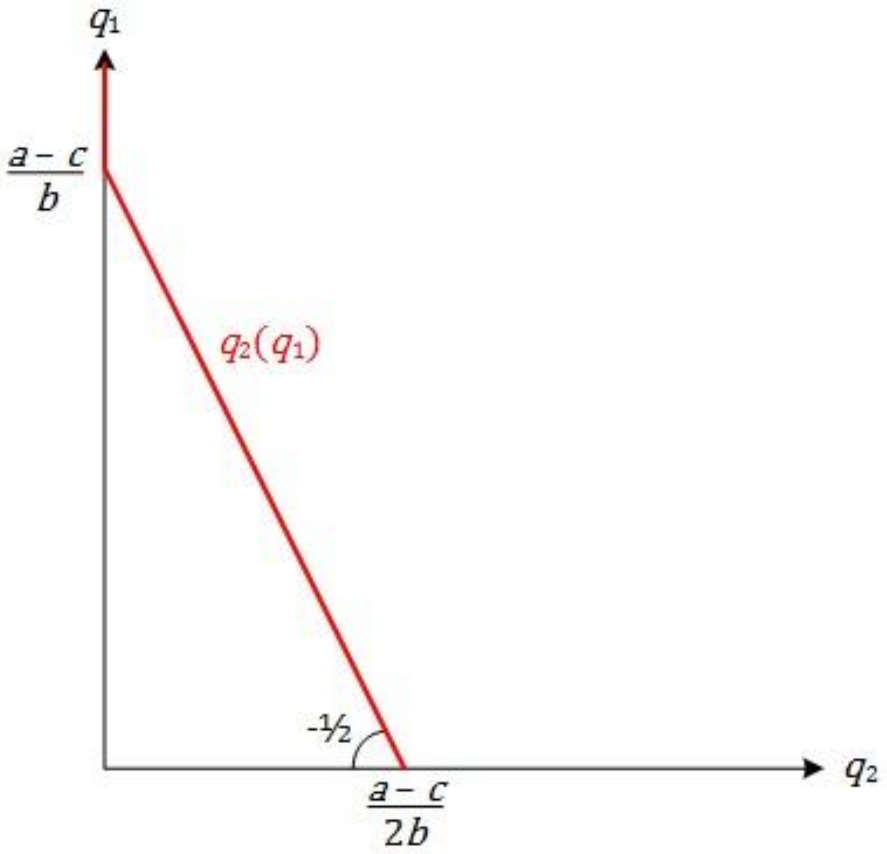
- Firm 1's BRF:
  - When $q_2 = 0$, then $q_1 = \frac{a-c}{2b}$, which coincides with its output under monopoly.
  - As q_2 increases, q_1 decreases (i.e., firm 1's and 2's output are strategic substitutes)
  - When $q_2 = \frac{a-c}{b}$, then $q_1 = 0$.



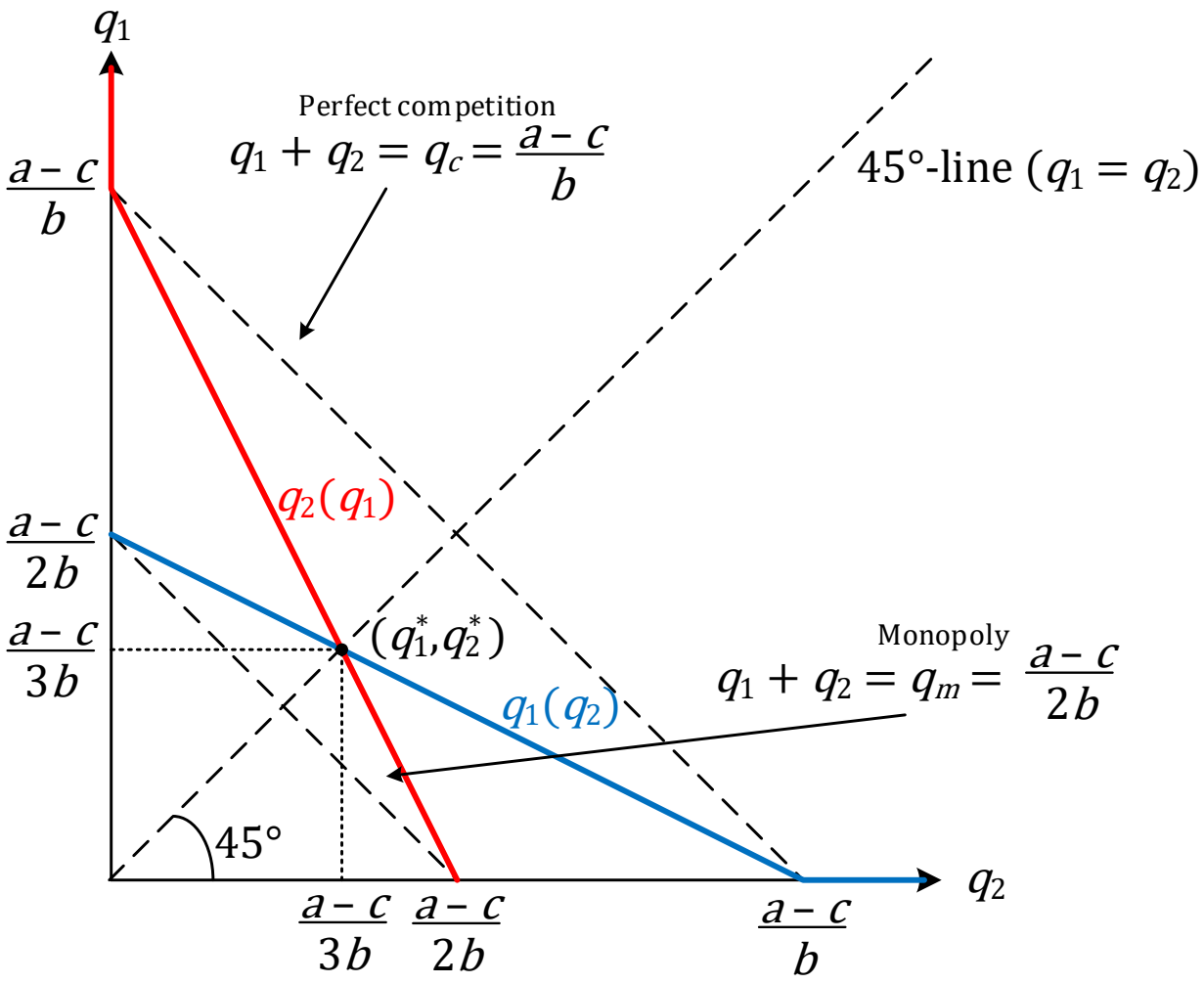
Cournot Model of Quantity Competition

- A similar argument applies to firm 2's BRF.

- Superimposing both firms' BRFs, we obtain the Cournot equilibrium output pair (q_1^*, q_2^*).



Cournot Model of Quantity Competition



Cournot Model of Quantity Competition

- Cournot equilibrium output pair (q_1^*, q_2^*) occurs at the intersection of the two BRFs, i.e.,

$$(q_1^*, q_2^*) = \left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)$$

- Aggregate output becomes

$$q^* = q_1^* + q_2^* = \frac{a-c}{3b} + \frac{a-c}{3b} = \frac{2(a-c)}{3b}$$

which is larger than under monopoly, $q_m = \frac{a-c}{2b}$, but smaller than under perfect competition, $q_c = \frac{a-c}{b}$.

Cournot Model of Quantity Competition

- The equilibrium price becomes

$$p(q^*) = a - bq^* = a - b\left(\frac{2(a-c)}{3b}\right) = \frac{a+2c}{3}$$

which is lower than under monopoly, $p_m = \frac{a+c}{2}$, but higher than under perfect competition, $p_c = c$.

- Finally, the equilibrium profits of every firm j

$$\pi_j^* = p(q^*)q_j^* - cq_j^* = \left(\frac{a+2c}{3}\right)\left(\frac{a-c}{3b}\right) - c\left(\frac{a-c}{3b}\right) = \frac{(a-c)^2}{9b}$$

which are lower than under monopoly, $\pi_m = \frac{(a-c)^2}{4b}$, but higher than under perfect competition, $\pi_c = 0$.

Cournot Model of Quantity Competition

- Quantity competition (Cournot model) yields less competitive outcomes than price competition (Bertrand model), whereby firms' behavior mimics that in perfectly competitive markets
  - That's because, the demand that every firm faces in the Cournot game is not infinitely elastic.
  - A reduction in output does not produce an infinite increase in market price, but instead an increase of $-p'(q_1 + q_2)$.
  - Hence, if firms produce the same output as under marginal cost pricing, i.e., half of $\frac{a-c}{b}$, each firm would have incentives to deviate from such a high output level by marginally reducing its output.

Cournot Model of Quantity Competition

- Equilibrium output under Cournot does not coincide with the monopoly output either.
  - That's because, every firm i, individually increasing its output level q_i, takes into account how the reduction in market price affects its own profits, but ignores the profit loss (i.e., a negative external effect) that its rival suffers from such a lower price.
  - Since every firm does not take into account this external effect, aggregate output is too large, relative to the output that would maximize firms' joint profits.

Cournot Model of Quantity Competition

- ***Example*** (Cournot vs. Cartel):
  - Let us demonstrate that firms' Cournot output is larger than that under the cartel.
  - PMP of the cartel is

$$\max_{q_1,q_2} [(a - b(q_1+q_2))q_1 - cq_1] + [(a - b(q_1+q_2))q_2 - cq_2]$$

  - Since $Q = q_1 + q_2$, the PMP can be written as

$$\max_{q_1,q_2} (a - b(q_1+q_2))(q_1+q_2) - c(q_1+q_2) = \max_{Q} (a - bQ)Q - cQ = aQ - bQ^2 - cQ$$

Cournot Model of Quantity Competition

- ***Example*** (continued):
  - FOC with respect to Q

$$a - 2bQ - c \leq 0$$

  - Solving for Q, we obtain the aggregate output

$$Q^* = \frac{a-c}{2b}$$

    which is positive since $a > c$, i.e., $p(0) = a > c$.
  - Since firms are symmetric in costs, each produces

$$q_i = \frac{Q}{2} = \frac{a-c}{4b}$$

Cournot Model of Quantity Competition

- ***Example*** (continued):
  - The equilibrium price is

$$p = a - bQ = a - b\frac{a-c}{2b} = \frac{a+c}{2}$$

  - Finally, the equilibrium profits are

$$\pi_i = p \cdot q_i - cq_i$$

$$= \frac{a+c}{2} \cdot \frac{a-c}{4b} - c\frac{a-c}{4b} = \frac{(a-c)^2}{8b}$$

which is larger than firms would obtain under Cournot competition, $\frac{(a-c)^2}{9b}$.

Cournot Model of Quantity Competition: Cournot Pricing Rule

- Firms' market power can be expressed using a variation of the Lerner index.
  - Consider firm j's profit maximization problem

$$\pi_j = p(q)q_j - c_j(q_j)$$

  - FOC for every firm j

$$p'(q)q_j + p(q) - c_j = 0$$

$$\text{or } \; p(q) - c_j = -p'(q)q_j$$

  - Multiplying both sides by q and dividing them by $p(q)$ yield

$$q\frac{p(q) - c_j}{p(q)} = \frac{-p'(q)q_j}{p(q)}q$$

Cournot Model of Quantity Competition: Cournot Pricing Rule

– Recalling $\frac{1}{\varepsilon} = -p'(q) \cdot \frac{q}{p(q)}$, we have

$$q\frac{p(q)-c_j}{p(q)} = \frac{1}{\varepsilon}q_j$$

$$\text{or } \frac{p(q)-c_j}{p(q)} = \frac{1}{\varepsilon}\frac{q_j}{q}$$

– Defining $\alpha_j \equiv \frac{q_j}{q}$ as firm j's market share, we obtain

$$\frac{p(q)-c_j}{p(q)} = \frac{\alpha_j}{\varepsilon}$$

which is referred to as the ***Cournot pricing rule***.

Cournot Model of Quantity Competition: Cournot Pricing Rule

– *Note*:

- When $\alpha_j = 1$, implying that firm j is a monopoly, the IEPR becomes a special case of the Cournot pricing rule.
- The larger the market share α_j of a given firm, the larger the price markup of firm j.
- The more inelastic is the demand, the smaller is the value of ε, and the larger the price markup of firm j.

Cournot Model of Quantity Competition: Cournot Pricing Rule

- ***Example*** (Merger effects on Cournot Prices):
  - Consider an industry with n firms and a constant-elasticity demand function $q(p) = ap^{-1}$, where $a > 0$ and $\varepsilon = -1$.
  - Before merger, we have

$$\frac{p^B - c}{p^B} = \frac{1}{n} \implies p^B = \frac{nc}{n-1}$$

  - After the merger of $k < n$ firms $n - k + 1$ firms remain in the industry, and thus

$$\frac{p^A - c}{p^A} = \frac{1}{n-k+1} \implies p^A = \frac{(n-k+1)c}{n-k}$$

Cournot Model of Quantity Competition: Cournot Pricing Rule

- ***Example*** (continued):
  - The percentage change in prices is

$$\%\Delta p = \frac{p^A - p^B}{p^B} = \frac{\frac{(n-k+1)c}{n-k} - \frac{nc}{n-1}}{\frac{nc}{n-1}}$$

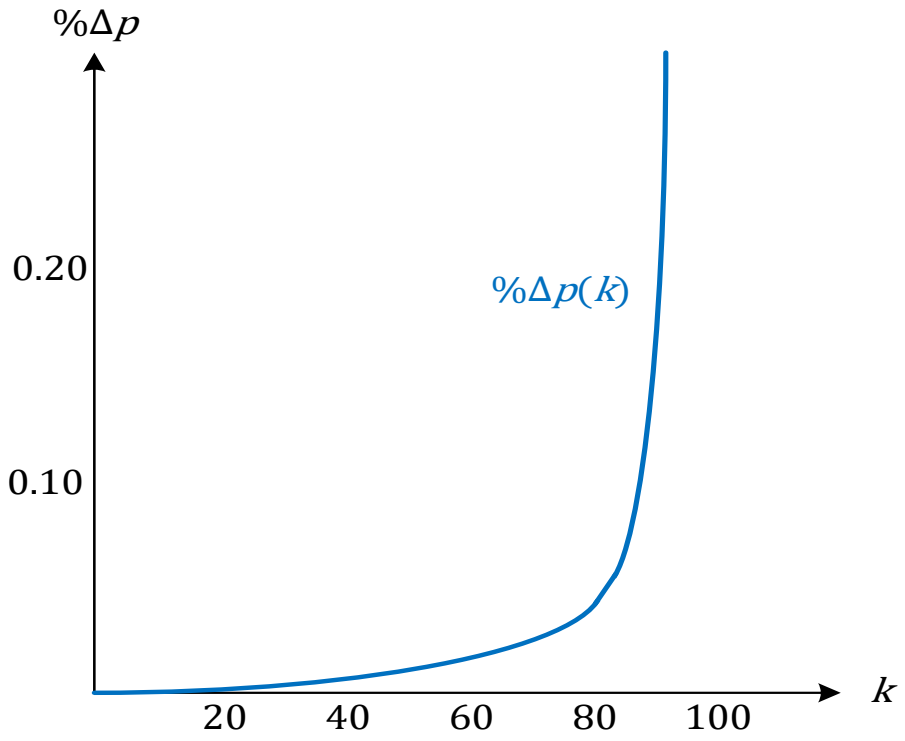
$$= \frac{k-1}{n(n-k)} > 0$$

  - Hence, prices increase after the merger.
  - Also, $\%\Delta p$ increases as the number of merging firms k increases

$$\frac{\partial \%\Delta p}{\partial k} = \frac{n-1}{n(n-k)^2} > 0$$

Cournot Model of Quantity Competition: Cournot Pricing Rule

- ***Example*** (continued):
  - The percentage increase in price after the merger, $\%\Delta p$, as a function of the number of merging firms, k.
  - For simplicity, $n = 100$.



Cournot Model of Quantity Competition: SOC

- Let us check if the first order (necessary) conditions are also sufficient.
- Recall that FOCs are

$$p'(q)q_j + p(q) - c_j'(q_j) \leq 0$$

- Differentiating FOCs wrt q_j yields

$$p''(q)q_j + 2p'(q) - c_j''(q_j) \leq 0$$

  - $p'(q) < 0$: by definition (negatively sloped inverse demand curve)
  - $c_j''(q_j) \geq 0$: by assumption (constant or increasing marginal costs)
  - $p''(q)q_j \leq 0$: as long as the demand curve decreases at a constant or decreasing rate

Cournot Model of Quantity Competition: SOC

- ***Example*** (linear demand):
  - The linear inverse demand curve is $p(q) = a - bq$ and constant marginal cost is $c > 0$.
  - Since $p'(q) = -b < 0$, $p''(q) = 0$, $c'(q) = c$ and $c''(q) = 0$, the SOC reduces to

$$0 - 2b - 0 = -2b < 0$$

    where $b > 0$ by definition.
  - Hence the equilibrium output is indeed profit maximizing.

Cournot Model of Quantity Competition: SOC

- Note that SOCs coincide with the cross-derivative

$$\frac{\partial^2 \pi_j}{\partial q_j \partial q_k} = \frac{\partial}{\partial q_k}\left[p'(q)q_j + p(q) - c'(q_j)\right]$$
$$= p''(q)q_j + p'(q) \text{ for all } k \neq j.$$

- Hence, the firm j's BRF decreases in q_k as long as $p''(q)q_j + p'(q) < 0$
  - That is, firm j's BRF is negatively sloped.

Cournot Model of Quantity Competition: Asymmetric Costs

- Assume that firm 1 and 2's constant marginal costs of production differ, i.e., $c_1 > c_2$, so firm 2 is more efficient than firm 1. Assume also that the inverse demand function is $p(Q) = a - bQ$, and $Q = q_1 + q_2$.
- Firm i's PMP is

$$\max_{q_i} \left(a - b(q_i + q_j)\right)q_i - c_i q_i$$

- FOC:

$$a - 2bq_i - bq_j - c_i = 0$$

Cournot Model of Quantity Competition: Asymmetric Costs

- Solving for q_i (assuming an interior solution) yields firm i's BRF

$$q_i(q_j) = \frac{a - c_i}{2b} - \frac{q_j}{2}$$

- Firm 1's optimal output level can be found by plugging firm 2's BRF into firm 1's

$$q_1^* = \frac{a - c_1}{2b} - \frac{1}{2}\left(\frac{a - c_2}{2b} - \frac{q_1^*}{2}\right) \Leftrightarrow q_1^* = \frac{a - 2c_1 + c_2}{3b}$$

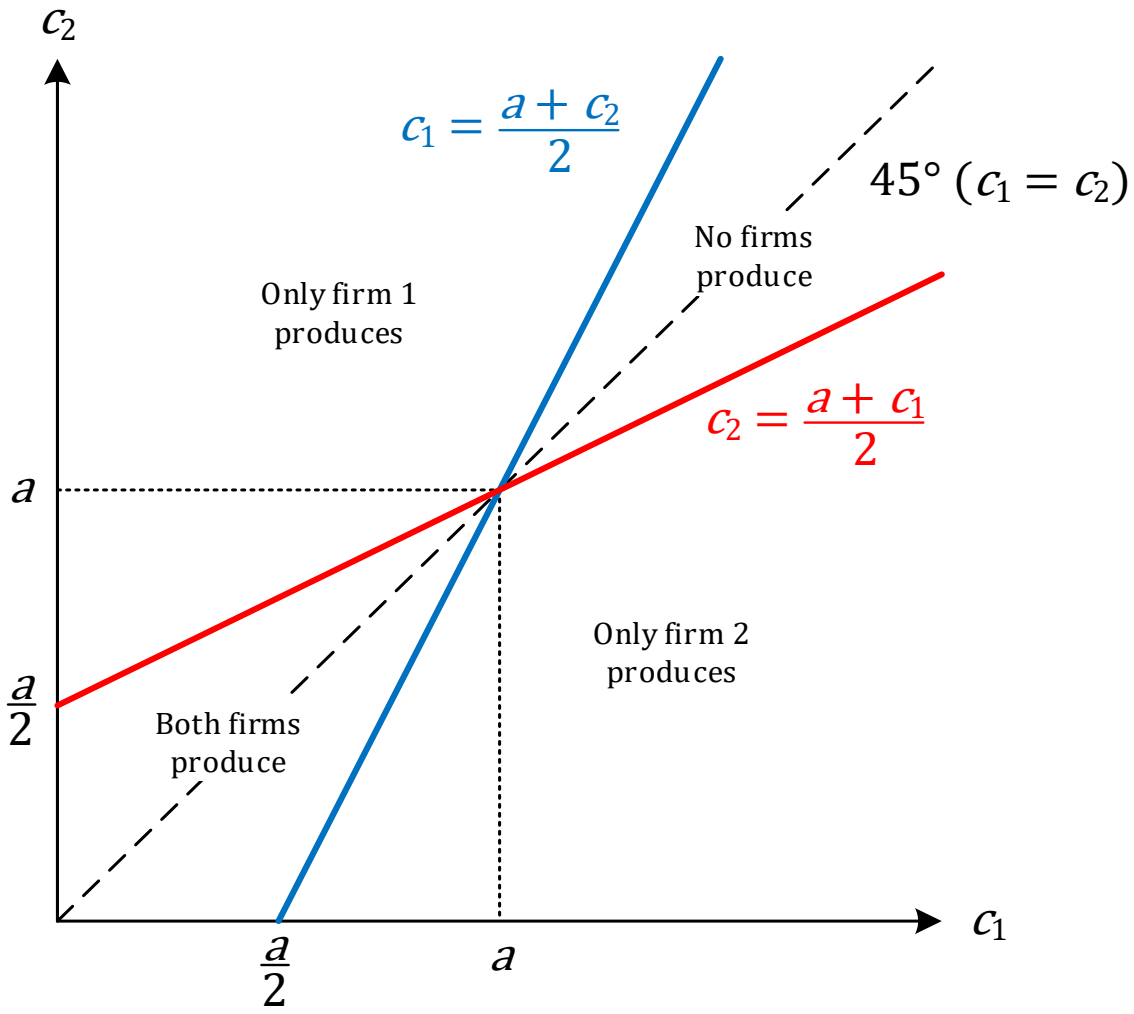
- Similarly, firm 2's optimal output level is

$$q_2^* = \frac{a - c_2}{2b} - \frac{q_1^*}{2} = \frac{a + c_1 - 2c_2}{3b}$$

Cournot Model of Quantity Competition: Asymmetric Costs

- If firm i's costs are sufficiently high it will not produce at all.
  - Firm 1: $q_1^* \leq 0$ if $\frac{a+c_2}{2} \leq c_1$
  - Firm 2: $q_2^* \leq 0$ if $\frac{a+c_1}{2} \leq c_2$
- Thus, we can identify three different cases:
  - If $c_i \geq \frac{a+c_j}{2}$ for all firms $i = \{1,2\}$, no firm produces a positive output
  - If $c_i \geq \frac{a+c_j}{2}$ but $c_j < \frac{a+c_i}{2}$, then only firm j produces a positive output
  - If $c_i < \frac{a+c_j}{2}$ for all firms $i = \{1,2\}$, both firms produce a positive output

Cournot Model of Quantity Competition: Asymmetric Costs



Cournot Model of Quantity Competition: Asymmetric Costs

- The output levels (q_1^*, q_2^*) also vary when (c_1, c_2) changes

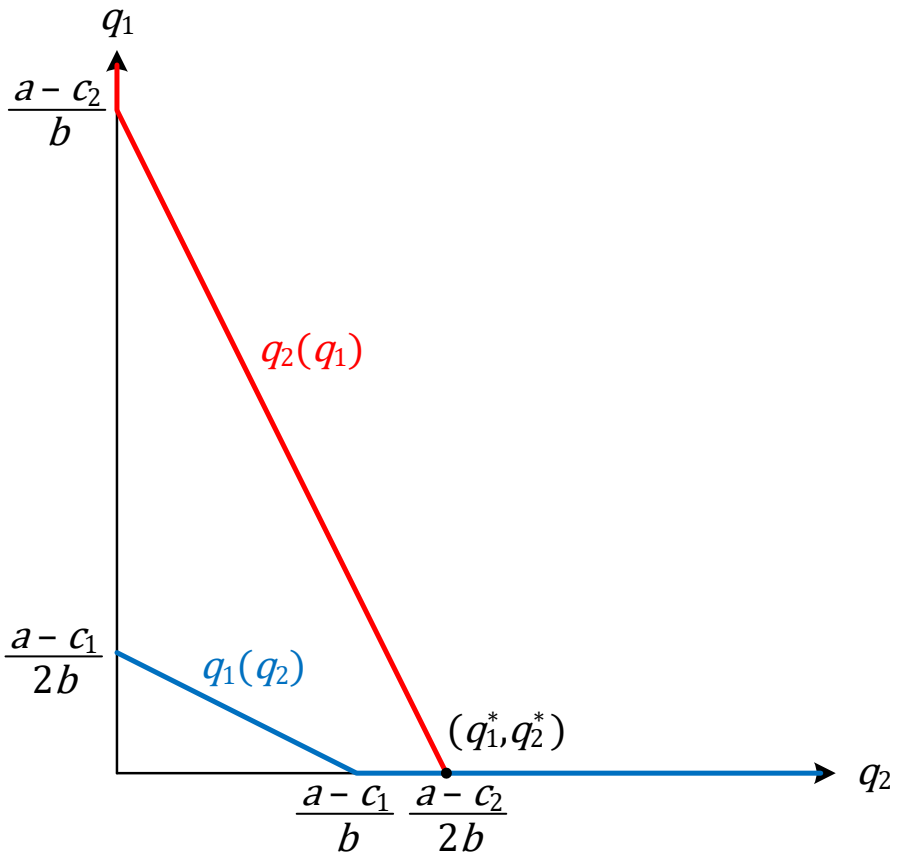
$$\frac{\partial q_1^*}{\partial c_1} = -\frac{2}{3b} < 0 \text{ and } \frac{\partial q_1^*}{\partial c_2} = \frac{1}{3b} > 0$$

$$\frac{\partial q_2^*}{\partial c_1} = \frac{1}{3b} > 0 \text{ and } \frac{\partial q_2^*}{\partial c_2} = -\frac{2}{3b} < 0$$

- *Intuition*: Each firm's output decreases in its own costs, but increases in its rival's costs.

Cournot Model of Quantity Competition: Asymmetric Costs

- BRFs for firms 1 and 2 when $c_1 > \frac{a+c_2}{2}$ (i.e., only firm 2 produces).
- BRFs cross at the horizontal axis where $q_1^* = 0$ and $q_2^* > 0$ (i.e., a corner solution)



Cournot Model of Quantity Competition: $J > 2$ firms

- Consider $J > 2$ firms, all facing the same constant marginal cost $c > 0$.
- The linear inverse demand curve is $p(Q) = a - bQ$, where $Q = \sum_k q_k$.
- Alternatively, we can write $p(Q) = a - b(q_i + Q_{-i})$, where $Q_{-i} = \sum_{k \neq i} q_k$ denotes the output from firm i's rivals.

- Firm i's PMP is

$$\max_{q_i} \; [a - b(q_i + Q_{-i})]q_i - cq_i$$

- FOC:

$$a - 2bq_i^* - bQ_{-i}^* - c \leq 0$$

Cournot Model of Quantity Competition: $J > 2$ firms

- Solving for q_i^*, we obtain firm i's BRF

$$q_i^* = \frac{a-c}{2b} - \frac{1}{2}Q_{-i}^*$$

- Since all firms are symmetric, their BRFs are also symmetric, implying $q_1^* = q_2^* = \cdots = q_J^*$. This implies that $Q_{-i}^* = Jq_i^* - q_i^* = (J-1)q_i^*$.
- Hence, the BRF becomes

$$q_i^* = \frac{a-c}{2b} - \frac{1}{2}(J-1)q_i^*$$

Cournot Model of Quantity Competition: $J > 2$ firms

- Solving for q_i^*

$$q_i^* = \frac{a - c}{(J + 1)b}$$

which is also the equilibrium output for other $J - 1$ firms.

- Therefore, aggregate output is

$$Q^* = Jq_i^* = \frac{J}{J + 1}\frac{a - c}{b}$$

and the corresponding equilibrium price is

$$p^* = a - bQ^* = \frac{a + Jc}{J + 1}$$

Cournot Model of Quantity Competition: $J > 2$ firms

- Firm i's equilibrium profits are

$$\pi_i^* = (a - bQ^*)q_i^* - cq_i^*$$

$$= \left(a - b\left(\frac{J}{J+1}\frac{a-c}{b}\right)\right)\left(\frac{a-c}{(J+1)b}\right) - c\left(\frac{a-c}{(J+1)b}\right)$$

$$= \frac{(a-c)^2}{b(J+1)^2}$$

Cournot Model of Quantity Competition: $J > 2$ firms

- We can show that

$$q_i^*(2) = \frac{a-c}{(2+1)b} = \frac{a-c}{3b}$$

$$Q^*(2) = \frac{2(a-c)}{(2+1)b} = \frac{2(a-c)}{3b}$$

$$p^*(2) = \frac{a+2c}{2+1} = \frac{a+2c}{3}$$

  which exactly coincide with our results in the Cournot duopoly model.

Cournot Model of Quantity Competition: $J > 2$ firms

- We can show that

$$q_i^*(1) = \frac{a-c}{(1+1)b} = \frac{a-c}{2b}$$

$$Q^*(1) = \frac{1(a-c)}{(1+1)b} = \frac{a-c}{2b}$$

$$p^*(1) = \frac{a+1c}{(1+1)} = \frac{a+c}{2}$$

which exactly coincide with our findings in the monopolist's model.

Cournot Model of Quantity Competition: $J > 2$ firms

- We can show that when there are extremely large number of firms, that is, $J \to \infty$,

$$\lim_{J\to\infty} q_i^* = 0$$

$$\lim_{J\to\infty} Q^* = \frac{a-c}{b}$$

$$\lim_{J\to\infty} p^* = c$$

which coincides with the solution in a perfectly competitive market.

Product Differentiation

Product Differentiation

- So far we assumed that firms sell homogenous (undifferentiated) products.
- What if the goods firms sell are differentiated?
  - For simplicity, we will assume that product attributes are exogenous (not chosen by the firm), and production costs are zero.

Product Differentiation: Bertrand Model

- Consider the case where every firm i, for $i = \{1,2\}$, faces demand curve

$$q_i(p_i, p_j) = a - bp_i + cp_j$$

where $a, b, c > 0$ and $j \neq i$.

- Hence, an increase in p_j increases firm i's sales.
- Firm i's PMP:

$$\max_{p_i \geq 0} \ (a - bp_i + cp_j)p_i$$

- FOC:

$$a - 2bp_i + cp_j = 0$$

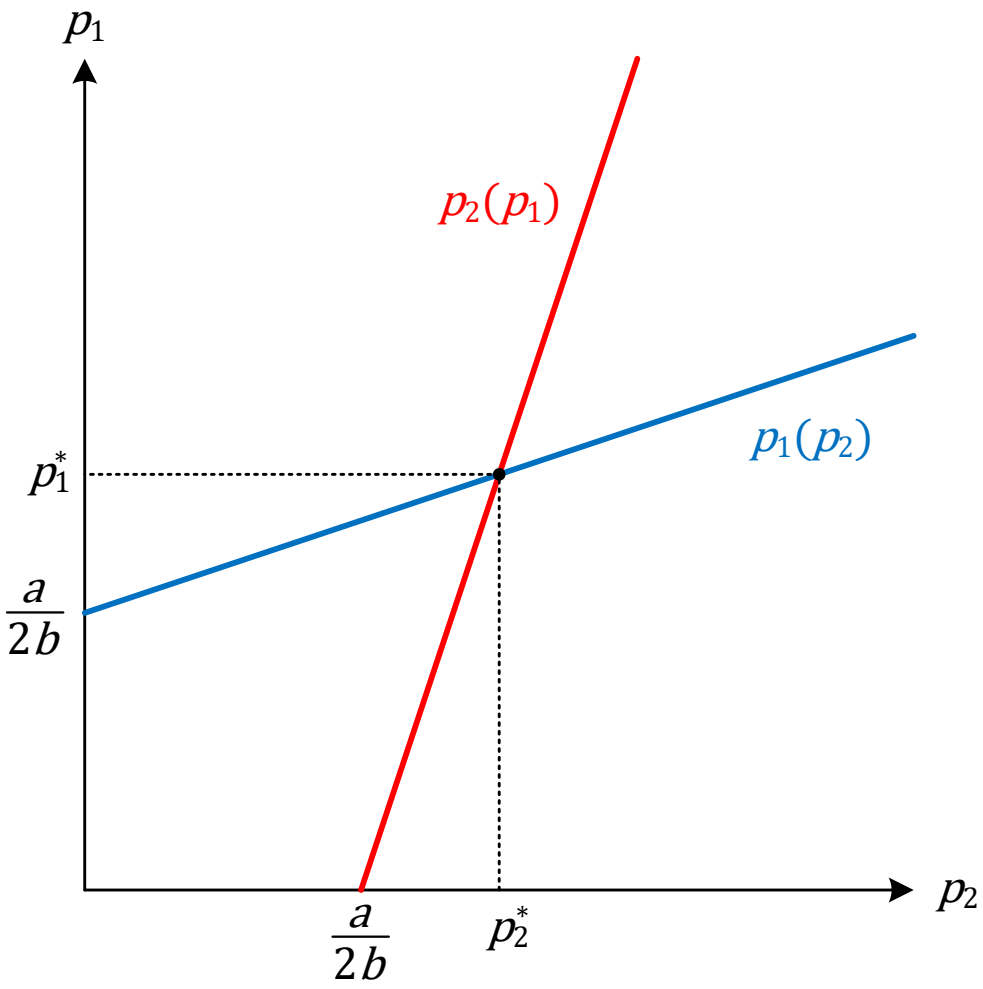
Product Differentiation: Bertrand Model

- Solving for p_i, we find firm i's BRF

$$p_i(p_j) = \frac{a + cp_j}{2b}$$

- Firm j also has a symmetric BRF.
- *Note*:
  - BRFs are now positively sloped
  - An increase in firm j's price leads firm i to increase his, and vice versa
  - In this case, firms' choices (i.e., prices) are strategic complements

Product Differentiation: Bertrand Model



Product Differentiation: Bertrand Model

- Simultaneously solving the two BRFs yields

$$p_i^* = \frac{a}{2b - c}$$

with corresponding equilibrium sales of

$$q_i^*(p_i^*, p_j^*) = a - bp_i^* + cp_j^* = \frac{ab}{2b - c}$$

and equilibrium profits of

$$\pi_i^* = p_i^* \cdot q_i^*(p_i^*, p_j^*) = \left(\frac{a}{2b - c}\right)\left(\frac{ab}{2b - c}\right)$$

$$= \frac{a^2 b}{(2b - c)^2}$$

Product Differentiation: Cournot Model

- Consider two firms with the following linear inverse demand curves

$$p_1(q_1, q_2) = \alpha - \beta q_1 - \gamma q_2 \text{ for firm 1}$$
$$p_2(q_1, q_2) = \alpha - \gamma q_1 - \beta q_2 \text{ for firm 2}$$

- We assume that $\beta > 0$ and $\beta > \gamma$
  - That is, the effect of increasing q_1 on p_1 is larger than the effect of increasing q_1 on p_2
  - Intuitively, the price of a particular brand is more sensitive to changes in its own output than to changes in its rival's output
  - In other words, ***own-price effects*** dominate the ***cross-price effects***.

Product Differentiation: Cournot Model

- Firm i's PMP is (assuming no costs)

$$\max_{q_i \geq 0} \ (\alpha - \beta q_i - \gamma q_j) q_i$$

- FOC:

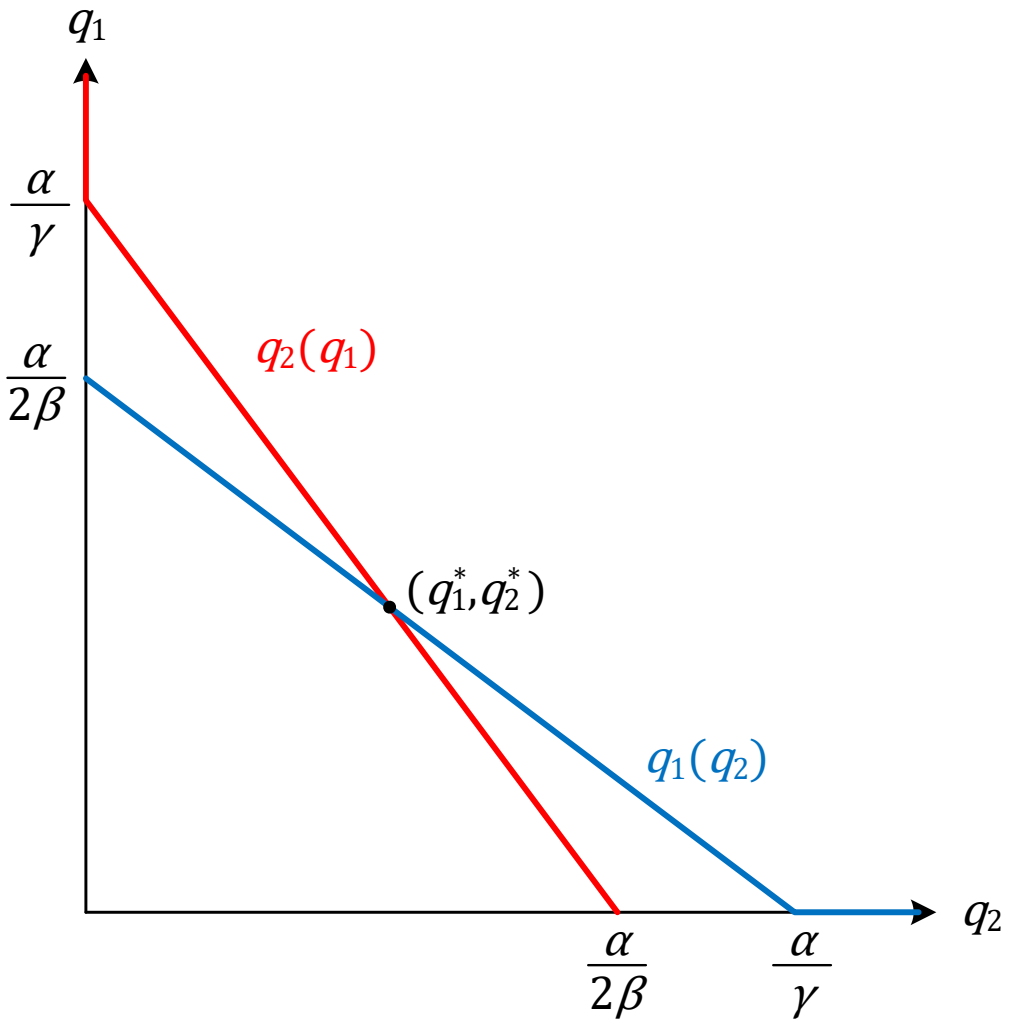
$$\alpha - 2\beta q_i - \gamma q_j = 0$$

- Solving for q_i we find firm i's BRF

$$q_i(q_j) = \frac{\alpha}{2\beta} - \frac{\gamma}{2\beta} q_j$$

- Firm j also has a symmetric BRF.

Product Differentiation: Cournot Model



Product Differentiation: Cournot Model

- Comparative statics of firm i's BRF
  - As $\beta \to \gamma$ (products become more homogeneous), BRF becomes steeper. That is, the profit-maximizing choice of q_i is more sensitive to changes in q_j (tougher competition)
  - As $\gamma \to 0$ (products become very differentiated), firm i's BRF no longer depends on q_j and becomes flat (milder competition)

Product Differentiation: Cournot Model

- Simultaneously solving the two BRF yields

$$q_i^* = \frac{\alpha}{2\beta + \gamma} \text{ for all } i = \{1,2\}$$

with a corresponding equilibrium price of

$$p_i^* = \alpha - \beta q_i^* - \gamma q_j^* = \frac{\alpha\beta}{2\beta + \gamma}$$

and equilibrium profits of

$$\pi_i^* = p_i^* q_i^* = \left(\frac{\alpha\beta}{2\beta + \gamma}\right)\left(\frac{\alpha}{2\beta + \gamma}\right) = \frac{\alpha^2\beta}{(2\beta + \gamma)^2}$$

Product Differentiation: Cournot Model

- *Note*:
  - As γ increases (products become more homogeneous), individual and aggregate output decrease, and individual profits decrease as well.
  - If $\gamma \to \beta$ (indicating undifferentiated products), then $q_i^* = \frac{\alpha}{2\beta+\beta} = \frac{\alpha}{3\beta}$ as in standard Cournot models of homogeneous products.
  - If $\gamma \to 0$ (extremely differentiated products), then $q_i^* = \frac{\alpha}{2\beta+0} = \frac{\alpha}{2\beta}$ as in monopoly.

Dynamic Competition

Dynamic Competition: Sequential Bertrand Model with Homogeneous Products

- Assume that firm 1 chooses its price p_1 first, whereas firm 2 observes that price and responds with its own price p_2.
- Since the game is a sequential-move game (rather than a simultaneous-move game), we should use ***backward induction***.

Dynamic Competition: Sequential Bertrand Model with Homogeneous Products

- Firm 2 (the follower) has a BRF given by

$$p_2(p_1) = \begin{cases} p^m \text{ if } p_1 > p^m \\ p_1 - \varepsilon \text{ if } p^m \geq p_1 > c \\ c \quad\quad \text{if } p_1 \leq c \end{cases}$$

- *Intuition*:
  - The follower charges monopoly price p^m if the leader charges prices above p^m;
  - undercuts the leader's price p_1 by a small $\varepsilon > 0$ for intermediate prices, $p^m \geq p_1 > c$; or
  - keeps its price at $p_2 = c$ if the leader sets $p_1 = c$.

Dynamic Competition: Sequential Bertrand Model with Homogeneous Products

- The leader expects that its price will be:
  - undercut by the follower when $p_1 > c$ (thus yielding no sales)
  - mimicked by the follower when $p_1 = c$ (thus entailing half of the market share)
- Hence, the leader has (weak) incentives to set a price $p_1 = c$.
- As a consequence, the equilibrium price pair remains at $(p_1^*, p_2^*) = (c, c)$, as in the simultaneous-move version of the Bertrand model.

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- Assume that firms sell differentiated products, where firm j's demand is

$$q_j = D_j(p_j, p_k)$$

  - *Example*: $q_j(p_j, p_k) = a - bp_j + cp_k$, where $a, b, c > 0$ and $b > c$

- In the second stage, firm 2 (the follower) solves following PMP

$$\max_{p_2 \geq 0} \pi_2 = p_2 q_2 - TC(q_2)$$
$$= p_2 D_2(p_2, p_1) - TC(\underbrace{D_2(p_2, p_1)}_{q_2})$$

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- FOCs wrt p_2 yield

$$D_2(p_2, p_1) + p_2 \frac{\partial D_2(p_2, p_1)}{\partial p_2} \underbrace{- \frac{\partial TC(D_2(p_2, p_1))}{\partial D_2(p_2, p_1)} \frac{\partial D_2(p_2, p_1)}{\partial p_2}}_{\text{Using the chain rule}} = 0$$

- Solving for p_2 produces the follower's BRF for every price set by the leader, p_1, i.e., $p_2(p_1)$.

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- In the first stage, firm 1 (leader) anticipates that the follower will use BRF $p_2(p_1)$ to respond to each possible price p_1, hence solves the following PMP

$$\max_{p_1 \geq 0} \ \pi_1 = p_1 q_1 - TC(q_1)$$

$$= p_1 D_1\left(p_1, \underbrace{p_2(p_1)}_{BRF_2}\right) - TC\left(\overbrace{D_1(p_1, p_2(p_1))}^{q_1}\right)$$

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- FOCs wrt p_1 yield

$$D_1(p_1,p_2)+p_1\left[\frac{\partial D_1(p_1,p_2)}{\partial p_1}+\underbrace{\frac{\partial D_1(p_1,p_2)}{\partial p_2(p_1)}\frac{\partial p_2(p_1)}{\partial p_1}}_{\text{New strategic effect}}\right]$$
$$-\frac{\partial TC(D_1(p_1,p_2))}{\partial D_1(p_1,p_2)}\left[\frac{\partial D_1(p_1,p_2)}{\partial p_1}+\underbrace{\frac{\partial D_1(p_1,p_2)}{\partial p_2(p_1)}\frac{\partial p_2(p_1)}{\partial p_1}}_{\text{New strategic effect}}\right]=0$$

- Or more compactly as

$$D_1(p_1,p_2)+\left(p_1-\frac{\partial TC(D_1)}{\partial D_1}\right)\left[\frac{\partial D_1(p_1,p_2)}{\partial p_1}+\underbrace{\frac{\partial D_1(p_1,p_2)}{\partial p_2(p_1)}\frac{\partial p_2(p_1)}{\partial p_1}}_{\text{New}}\right]=0$$

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- In contrast to the Bertrand model with simultaneous price competition, an increase in firm 1's price now produces an increase in firm 2's price in the second stage.
- Hence, the leader has more incentives to raise its price, ultimately softening price competition.
- While a softened competition benefits both the leader and the follower, the real beneficiary is the follower, as its profits increase more than the leader's.

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- ***Example***:
  - Consider a linear demand $q_i = 1 - 2p_i + p_j$, with no marginal costs, i.e., $c = 0$.
  - *Simultaneous Bertrand model*: the PMP is

$$\max_{p_j \geq 0} \ \pi_j = p_j \cdot (1 - 2p_j + p_k) \text{ for any } k \neq j$$

    where FOC wrt p_j produces firm j's BRF

$$p_j(p_k) = \frac{1}{4} + \frac{1}{4}p_k$$

  - Simultaneously solving the two BRFs yields $p_j^* = \frac{1}{3} \simeq 0.33$, entailing equilibrium profits of $\pi_j^* = \frac{2}{9} \simeq 0.222$.

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- ***Example*** (continued):
  - *Sequential Bertrand model*: in the second stage, firm 2's (the follower's) PMP is

$$\max_{p_2 \geq 0} \ \pi_2 = p_2 \cdot (1 - 2p_2 + p_1)$$

where FOC wrt p_2 produces firm 2's BRF

$$p_2(p_1) = \frac{1}{4} + \frac{1}{4}p_1$$

  - In the first stage, firm 1's (the leader's) PMP is

$$\max_{p_1 \geq 0} \ \pi_1 = p_1 \cdot \left[1 - 2p_1 + \underbrace{\left(\frac{1}{4} + \frac{1}{4}p_1\right)}_{BRF_2}\right] = p_1 \cdot \left[\frac{1}{4}(5 - 7p_1)\right]$$

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

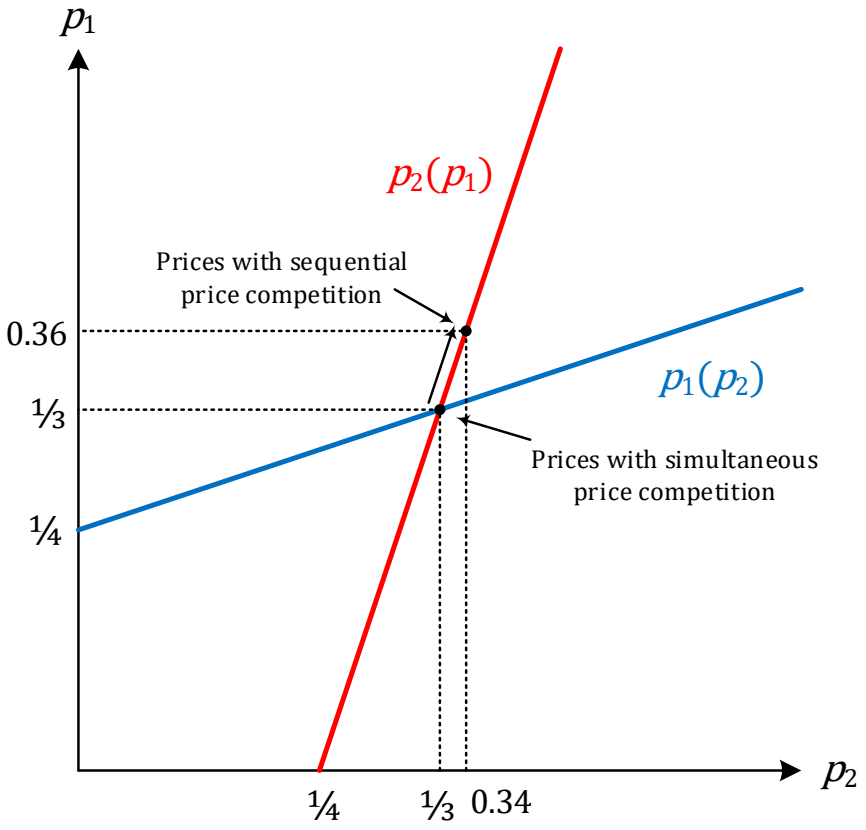
- ***Example*** (continued):
  - FOC wrt p_1, and solving for p_1, produces firm 1's equilibrium price $p_1^* = \frac{5}{14} = 0.36$.
  - Substituting p_1^* into the BRF of firm 2 yields $p_2^*(0.36) = \frac{1}{4} + \frac{1}{4}\left(\frac{5}{14}\right) = \frac{19}{56} = 0.34$.
  - Equilibrium profits are hence

$$\pi_1^* = 0.36\left[\frac{1}{4}\big(5 - 7(0.36)\big)\right] = 0.223 \text{ for firm 1}$$

$$\pi_2^* = 0.34\big(1 - 2(0.34) + (0.36)\big) = 0.23 \text{ for firm 2}$$

Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

- ***Example*** (continued):
  - Both firms' prices and profits are higher in the sequential than in the simultaneous game.
  - However, the follower earns more than the leader in the sequential game! (***second mover's advantage***)



Dynamic Competition: Sequential Cournot Model with Homogeneous Products

- ***Stackelberg model***: firm 1 (the leader) chooses output level q_1, and firm 2 (the follower), observing the output decision of the leader, responds with its own output $q_2(q_1)$.
- By backward induction, the follower's BRF is $q_2(q_1)$ for any q_1.
- Since the leader anticipates $q_2(q_1)$ from the follower, the leader's PMP is

$$\max_{q_1 \geq 0} \; p\left(q_1 + \underbrace{q_2(q_1)}_{BRF_2}\right) q_1 - TC_1(q_1)$$

Dynamic Competition: Sequential Cournot Model with Homogeneous Products

- FOCs wrt q_1 yields

$$p(q_1 + q_2(q_1)) + p'(q_1 + q_2(q_1))\left[1 + \frac{\partial q_2(q_1)}{\partial q_1}\right]q_1 - \frac{\partial TC_1(q_1)}{\partial q_1} = 0$$

or more compactly

$$p(Q) + p'(Q)q_1 + \underbrace{p'(Q)\frac{\partial q_2(q_1)}{\partial q_1}q_1}_{\text{Strategic Effect}} - \frac{\partial TC_1(q_1)}{\partial q_1} = 0$$

- This FOC coincides with that for standard Cournot model with simultaneous output decisions, except for the ***strategic effect***.

Dynamic Competition: Sequential Cournot Model with Homogeneous Products

- The strategic effect is positive since $p'(Q) < 0$ and $\frac{\partial q_2(q_1)}{\partial q_1} < 0$.
- Firm 1 (the leader) has more incentive to raise q_1 relative to the Cournot model with simultaneous output decision.
- *Intuition* (***first-mover advantage***):
  - By overproducing, the leader forces the follower to reduce its output q_2 by the amount $\frac{\partial q_2(q_1)}{\partial q_1}$.
  - This helps the leader sell its production at a higher price, as reflected by $p'(Q)$; ultimately earning a larger profit than in the standard Cournot model.

Dynamic Competition: Sequential Cournot Model with Homogeneous Products

- ***Example***:
  - Consider linear inverse demand $p = a - Q$, where $Q = q_1 + q_2$, and a constant marginal cost of c.
  - Firm 2's (the follower's) PMP is

$$\max_{q_2} \ (a - q_1 - q_2)q_2 - cq_2$$

  - FOC:

$$a - q_1 - 2q_2 - c = 0$$

  - Solving for q_2 yields the follower's BRF

$$q_2(q_1) = \frac{a - q_1 - c}{2}$$

Dynamic Competition: Sequential Cournot Model with Homogeneous Products

- ***Example*** (continued):
  - Plugging $q_2(q_1)$ into the leader's PMP, we get

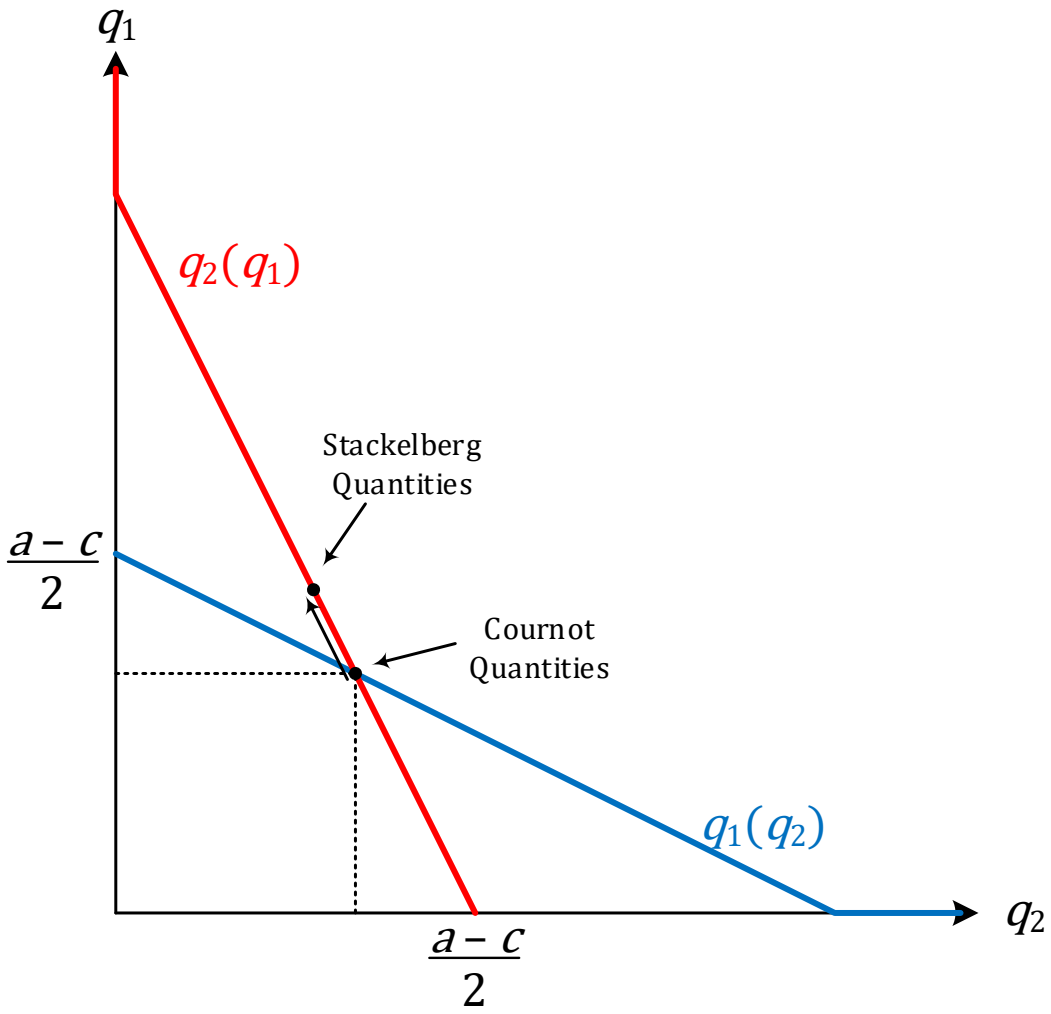
$$\max_{q_1} \left(a - q_1 - \frac{a-q_1-c}{2}\right) q_1 - cq_1 = \frac{1}{2}(a - q_1 - c)q_1$$

  - FOC:

$$\frac{1}{2}(a - 2q_1 - c) = 0$$

  - Solving for q_1, we obtain the leader's equilibrium output level $q_1^* = \frac{a-c}{2}$.
  - Substituting q_1^* into the follower's BRF yields the follower's equilibrium output $q_2^* = \frac{a-c}{4}$.

Dynamic Competition: Sequential Cournot Model with Homogeneous Products



Dynamic Competition: Sequential Cournot Model with Homogeneous Products

- ***Example*** (continued):
  - The equilibrium price is

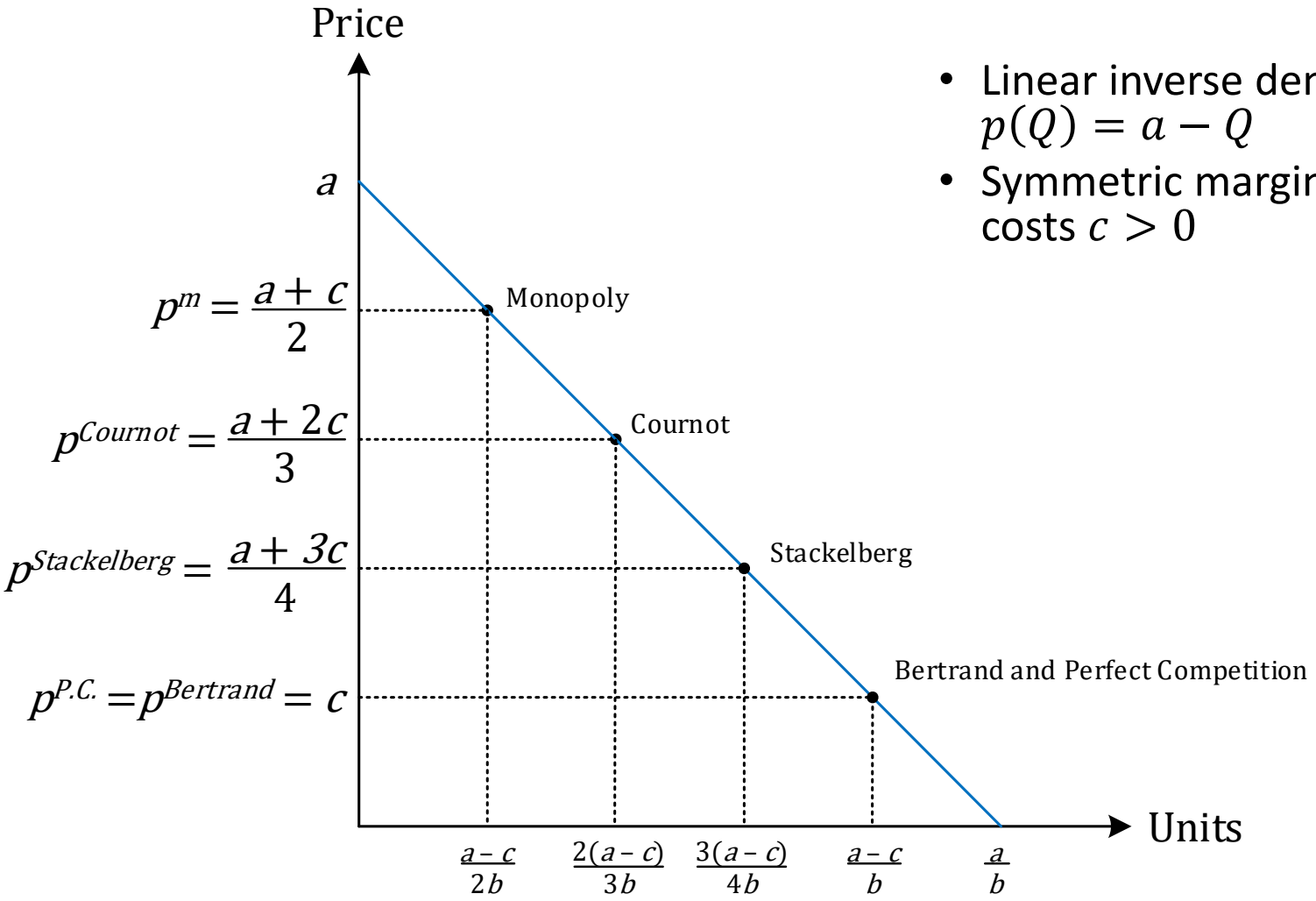
$$p = a - q_1^* - q_2^* = \frac{a + 3c}{4}$$

  - And the resulting equilibrium profits are

$$\pi_1^* = \left(\frac{a+3c}{4}\right)\left(\frac{a-c}{2}\right) - c\left(\frac{a-c}{2}\right) = \frac{(a-c)^2}{8}$$

$$\pi_2^* = \left(\frac{a+3c}{4}\right)\left(\frac{a-c}{4}\right) - c\left(\frac{a-c}{4}\right) = \frac{(a-c)^2}{16}$$

Dynamic Competition: Sequential Cournot Model with Homogeneous Products



- Linear inverse demand $p(Q) = a - Q$
- Symmetric marginal costs $c > 0$

Dynamic Competition: Sequential Cournot Model with Heterogeneous Products

- Assume that firms sell differentiated products, with inverse demand curves for firms 1 and 2

$$p_1(q_1, q_2) = \alpha - \beta q_1 - \gamma q_2 \text{ for firm 1}$$

$$p_2(q_1, q_2) = \alpha - \gamma q_1 - \beta q_2 \text{ for firm 2}$$

- Firm 2's (the follower's) PMP is

$$\max_{q_2} \ (\alpha - \gamma q_1 - \beta q_2) \cdot q_2$$

where, for simplicity, we assume no marginal costs.

- FOC:

$$\alpha - \gamma q_1 - 2\beta q_2 = 0$$

Dynamic Competition: Sequential Cournot Model with Heterogeneous Products

- Solving for q_2 yields firm 2's BRF

$$q_2(q_1) = \frac{\alpha - \gamma q_1}{2\beta}$$

- Plugging $q_2(q_1)$ into the leader's firm 1's (the leader's) PMP, we get

$$\max_{q_1} \left(\alpha - \beta q_1 - \gamma\left(\frac{\alpha - \gamma q_1}{2\beta}\right)\right) q_1 =$$

$$\max_{q_1} \left(\alpha\left(\frac{2\beta - \gamma}{2\beta}\right) - \left(\frac{2\beta^2 - \gamma^2}{2\beta}\right) q_1\right) q_1$$

- FOC:

$$\alpha\left(\frac{2\beta - \gamma}{2\beta}\right) - \left(\frac{2\beta^2 - \gamma^2}{\beta}\right) q_1 = 0$$

Dynamic Competition: Sequential Cournot Model with Heterogeneous Products

- Solving for q_1, we obtain the leader's equilibrium output level $q_1^* = \frac{\alpha(2\beta-\gamma)}{2(2\beta^2-\gamma^2)}$
- Substituting q_1^* into the follower's BRF yields the follower's equilibrium output

$$q_2^* = \frac{\alpha-\gamma q_1^*}{2\beta} = \frac{\alpha(4\beta^2-2\beta\gamma-\gamma^2)}{4\beta(2\beta^2-\gamma^2)}$$

- *Note*:
  - $q_1^* > q_2^*$
  - If $\gamma \to \beta$ (i.e., the products become more homogeneous), (q_1^*, q_2^*) convege to standard Stackelberg output $(\frac{\alpha}{2\beta}, \frac{\alpha}{4\beta})$.
  - If $\gamma \to 0$ (i.e., the products become very differentiated), (q_1^*, q_2^*) converge to the monopoly output $q^m = \frac{\alpha}{2\beta}$.

Capacity Constraints

Capacity Constraints

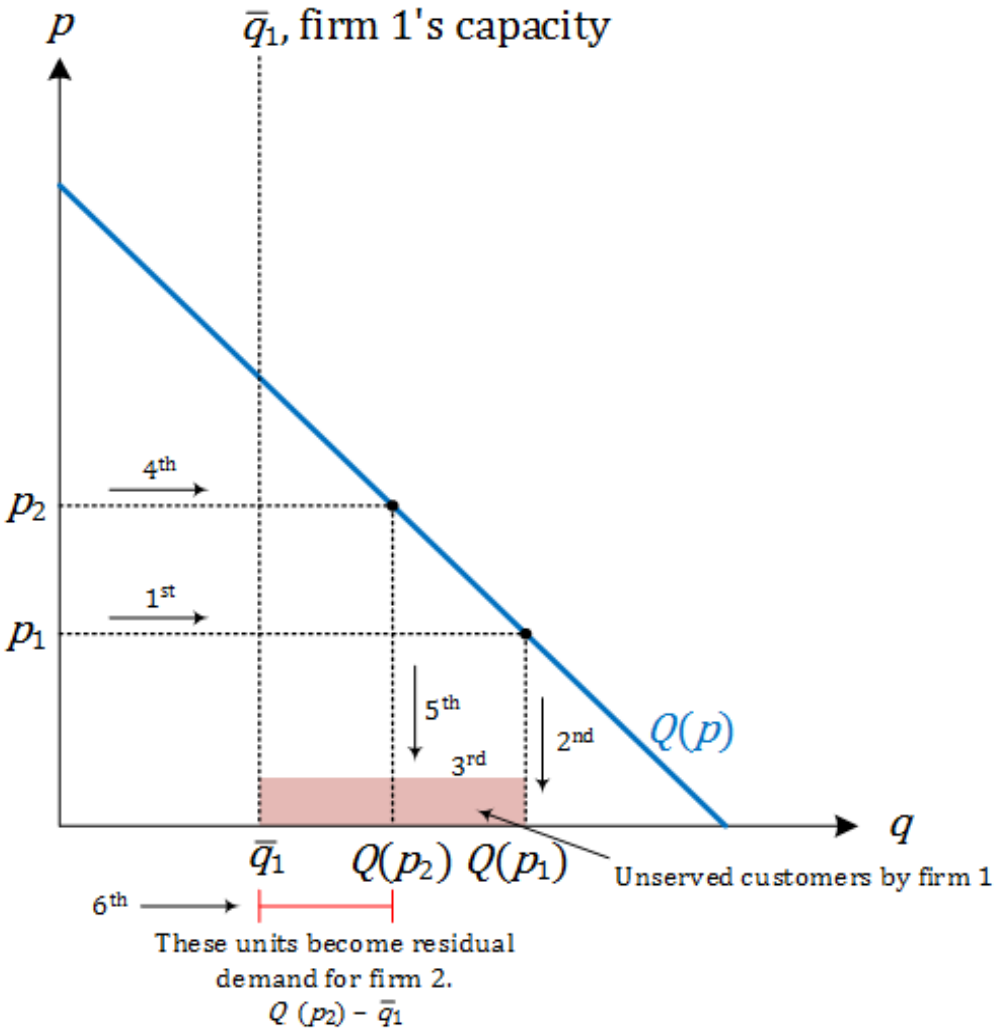
- How come are the equilibrium outcomes in standard Bertrand and Cournot models so different?
- Do firms really compete in prices without facing capacity constraints?
  - Bertrand model assumes a firm can supply infinitely large amount if its price is lower than its rivals.
- Extension of the Bertrand model:
  - **First stage**: firms set capacities, $\bar{q}_1$ and $\bar{q}_2$, with a cost of capacity $c > 0$
  - **Second stage**: firms observe each other's capacities and compete in prices, simultaneously setting p_1 and p_2

Capacity Constraints

- What is the role of capacity constraint?
  - When a firm's price is lower than its capacity, not all consumers can be served.
  - Hence, sales must be rationed through efficient rationing: the customers with the highest willingness to pay get the product first.
- Intuitively, if $p_1 < p_2$ and the quantity demanded at p_1 is so large that $Q(p_1) > \bar{q}_1$, then the first $\bar{q}_1$ units are served to the customers with the highest willingness to pay (i.e., the upper segment of the demand curve), while some customers are left in the form of residual demand to firm 2.

Capacity Constraints

- At p_1 the quantity demanded is $Q(p_1)$, but only $\bar{q}_1$ units can be served.
- Hence, the residual demand is $Q(p_1) - \bar{q}_1$.
- Since firm 2 sets a price of p_2, its demand will be $Q(p_2)$.
- Thus, a portion of the residual demand , i.e., $Q(p_2) - \bar{q}_1$, is captured.



Capacity Constraints

- Hence, firm 2's residual demand can be expressed as

$$\begin{cases} Q(p_2) - \bar{q}_1 & \text{if } Q(p_2) - \bar{q}_1 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- Should we restrict $\bar{q}_1$ and $\bar{q}_2$ somewhat?
  - Yes. A firm will never set a huge capacity if such capacity entails negative profits, independently of the decision of its competitor.

Capacity Constraints

- How to express this rather obvious statement with a simple mathematical condition?
  - The maximal revenue of a firm under monopoly is $\max_q (a-q)q$, which is maximized at $q = \frac{a}{2}$, yielding profits of $\frac{a^2}{4}$.
  - Maximal revenues are larger than costs if $\frac{a^2}{4} \geq c\bar{q}_j$, or solving for $\bar{q}_j$, $\frac{a^2}{4c} \geq \bar{q}_j$.
  - Intuitively, the capacity cannot be too high, as otherwise the firm would not obtain positive profits regardless of the opponent's decision.

Capacity Constraints: Second Stage

- By backward induction, we start with the second stage (pricing game), where firms simultaneously choose prices p_1 and p_2 as a function of the capacity choices $\bar{q}_1$ and $\bar{q}_2$.
- We want to show that in this second stage, both firms set a common price

$$p_1 = p_2 = p^* = a - \bar{q}_1 - \bar{q}_2$$

where demand equals supply, i.e., total capacity,

$$p^* = a - \bar{Q}, \text{ where } \bar{Q} \equiv \bar{q}_1 + \bar{q}_2$$

Capacity Constraints: Second Stage

- In order to prove this result, we start by assuming that firm 1 sets $p_1 = p^*$. We now need to show that firm 2 also sets $p_2 = p^*$, i.e., it does not have incentives to deviate from p^*.
- If firm 2 does not deviate, $p_1 = p_2 = p^*$, then it sells up to its capacity $\bar{q}_2$.
- If firm 2 reduces its price below p^*, demand would exceed its capacity $\bar{q}_2$. As a result, firm 2 would sell the same units as before, $\bar{q}_2$, but at a lower price.

Capacity Constraints: Second Stage

- If, instead, firm 2 charges a price above p^*, then $p_1 = p^* < p_2$ and its revenues become

$$p_2\hat{Q}(p_2) = \begin{cases} p_2(a - p_2 - \bar{q}_1) \text{ if } a - p_2 - \bar{q}_1 \geq 0 \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

- *Note*:
  - This is fundamentally different from the standard Bertrand model without capacity constraints, where an increase in price by a firm reduces its sales to zero.
  - When capacity constraints are present, the firm can still capture a residual demand, ultimately raising its revenues after increasing its price.

Capacity Constraints: Second Stage

- We now find the maximum of this revenue function. FOC wrt p_2 yields:

$$a - 2p_2 - \bar{q}_1 = 0 \iff p_2 = \frac{a - \bar{q}_1}{2}$$

- The non-deviating price $p^* = a - \bar{q}_1 - \bar{q}_2$ lies above the maximum-revenue price $p_2 = \frac{a-\bar{q}_1}{2}$ when

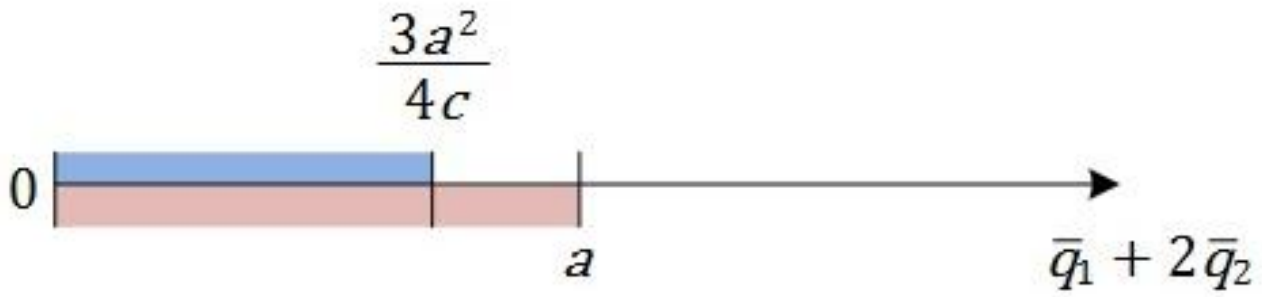
$$a - \bar{q}_1 - \bar{q}_2 > \frac{a - \bar{q}_1}{2} \iff a > \bar{q}_1 + 2\bar{q}_2$$

- Since $\frac{a^2}{4c} \geq \bar{q}_j$ (capacity constraint), we can obtain

$$\frac{a^2}{4c} + 2\frac{a^2}{4c} > \bar{q}_1 + 2\bar{q}_2 \iff \frac{3a^2}{4c} > \bar{q}_1 + 2\bar{q}_2$$

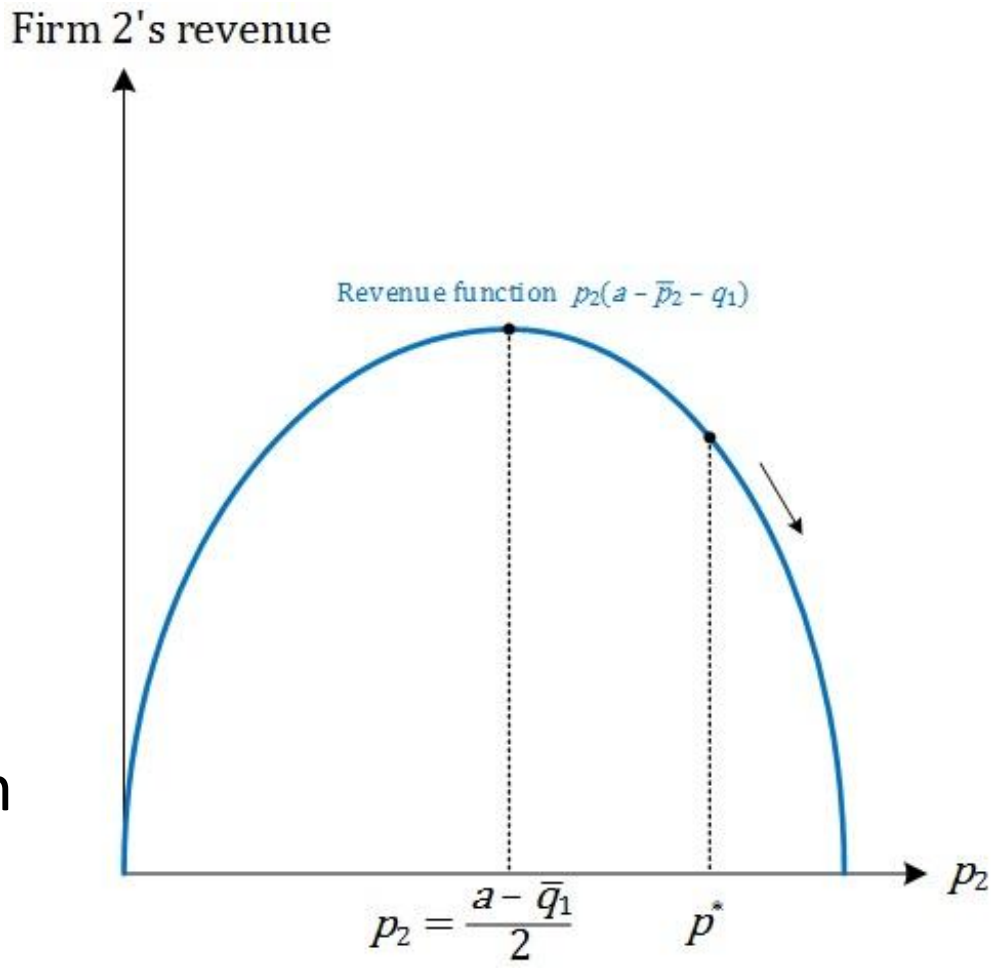
Capacity Constraints: Second Stage

- Therefore, $a > \bar{q}_1 + 2\bar{q}_2$ holds if $a > \frac{3a^2}{4c}$ which, solving for a, is equivalent to $\frac{4c}{3} > a$.



Capacity Constraints: Second Stage

- When $\frac{4c}{3} > a$ holds, capacity constraint $\frac{a^2}{4c} \geq \bar{q}_j$ transforms into $\frac{3a^2}{4c} > \bar{q}_1 + 2\bar{q}_2$, implying $p^* > p_2 = a - \frac{\bar{q}_1}{2}$.
- Thus, firm 2 does not have incentives to increase its price p_2 from p^*, since that would lower its revenues.



Capacity Constraints: Second Stage

- In short, firm 2 does not have incentives to deviate from the common price

$$p^* = a - \bar{q}_1 - \bar{q}_2$$

- A similar argument applies to firm 1 (by symmetry).
- Hence, we have found an equilibrium in the pricing stage.

Capacity Constraints: First Stage

- In the first stage (capacity setting), firms simultaneously select their capacities $\bar{q}_1$ and $\bar{q}_2$.
- Inserting stage 2 equilibrium prices, i.e.,

$$p_1 = p_2 = p^* = a - \bar{q}_1 - \bar{q}_2,$$

into firm j's profit function yields

$$\pi_j(\bar{q}_1, \bar{q}_2) = \underbrace{(a - \bar{q}_1 - \bar{q}_2)}_{p^*}\bar{q}_j - c\bar{q}_j$$

- FOC wrt capacity $\bar{q}_j$ yields firm j's BRF

$$\bar{q}_j(\bar{q}_k) = \frac{a-c}{2} - \frac{1}{2}\bar{q}_k$$

Capacity Constraints: First Stage

- Solving the two BRFs simultaneously, we obtain a symmetric solution

$$\bar{q}_j = \bar{q}_k = \frac{a - c}{3}$$

- These are the same equilibrium predictions as those in the standard Cournot model.
- Hence, capacities in this two-stage game coincide with output decisions in the standard Cournot model, while prices are set equal to total capacity.

Endogenous Entry

Endogenous Entry

- So far the number of firms was exogenous
- What if the number of firms operating in a market is endogenously determined?
- That is, how many firms would enter an industry where
  - They know that competition will be à la Cournot
  - They must incur a fixed entry cost $F > 0$.

Endogenous Entry

- Consider inverse demand function $p(Q)$, where Q denotes aggregate output
- Every firm j faces the same total cost function, $c(q_j)$, of producing q_j units
- Hence, the Cournot equilibrium must be symmetric
  - Every firm produces the same output level $q(n)$, which is a function of the number of entrants.
- Entry profits for firm j are

$$\pi_j(n) = \underbrace{p\left(\underbrace{n \cdot q(n)}_{Q}\right)}_{p(Q)} q(n) - \underbrace{c\big(q(n)\big)}_{\text{Production Costs}} - \underbrace{F}_{\text{Fixed Entry Cost}}$$

Endogenous Entry

- Three assumptions (valid under most demand and cost functions):
  - individual equilibrium output $q(n)$ is decreasing in n;
  - aggregate output $q \equiv n \cdot q(n)$ increases in n;
  - equilibrium price $p(n \cdot q(n))$ remains above marginal costs regardless of the number of entrants n.

Endogenous Entry

- ***Equilibrium number of firms***:
  - The equilibrium occurs when no more firms have incentives to enter or exit the market, i.e., $\pi_j(n^e) = 0$.
  - Note that individual profits decrease in n, i.e.,

$$\pi_j'(n) = \overbrace{\left[p\left(nq(n)\right) - c'\left(q(n)\right)\right]}^{+} \overbrace{\frac{\partial q(n)}{\partial n}}^{-} + \underbrace{q(n)p'\left(nq(n)\right)}_{-} \underbrace{\frac{\partial [nq(n)]}{\partial n}}_{+} < 0$$

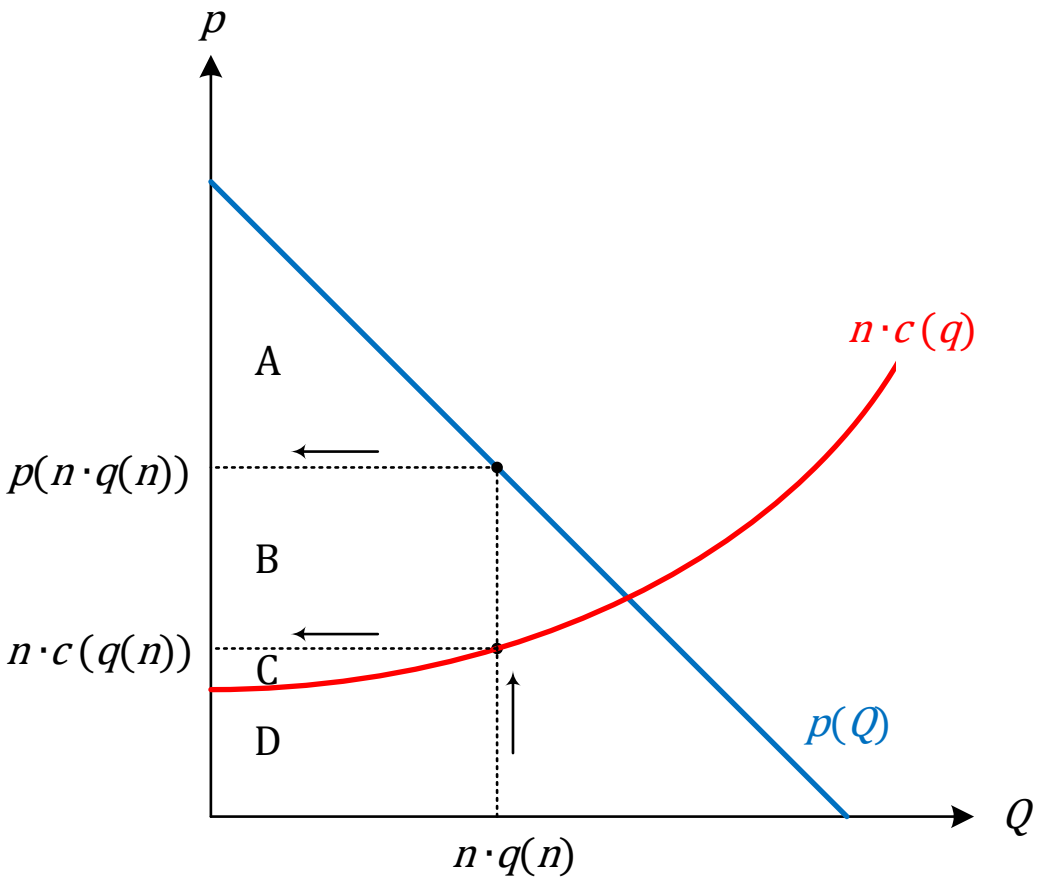
Endogenous Entry

- ***Social optimum***:
  - The social planner chooses the number of entrants n^o that maximizes social welfare

$$\max_n \; W(n) \equiv \int_0^{nq(n)} p(s)ds - n \cdot c\big(q(n)\big) - n \cdot F$$

Endogenous Entry

- $\int_0^{nq(n)} p(s)ds = A + B + C + D$
- $n \cdot c\big(q(n)\big) = C + D$
- Social welfare is thus $A + B$ minus total entry costs $n \cdot F$



Endogenous Entry

– Using Leibniz's rule, FOC wrt n yields

$$p\big(nq(n)\big)\left[n\frac{\partial q(n)}{\partial n}+q(n)\right]-c\big(q(n)\big)-nc'\big(q(n)\big)\frac{\partial q(n)}{\partial n}-F=0$$

or, rearranging,

$$\pi(n)+n\big[p\big(nq(n)\big)-c'\big(q(n)\big)\big]\frac{\partial q(n)}{\partial n}=0$$

– Hence, marginal increase in n entails two opposite effects on social welfare:

a) the profits of the new entrant increase social welfare (+, ***appropriability effect***)

b) the entrant reduces the profits of all previous incumbents in the industry as the individual sales of each firm decreases upon entry (–, ***business stealing effect***)

Endogenous Entry

- The "business stealing" effect is represented by:

$$n\left[p\left(nq(n)\right) - c'\left(q(n)\right)\right]\frac{\partial q(n)}{\partial n} < 0$$

which is negative since $\frac{\partial q(n)}{\partial n} < 0$ and $n\left[p\left(nq(n)\right) - c'\left(q(n)\right)\right] > 0$ by definition.

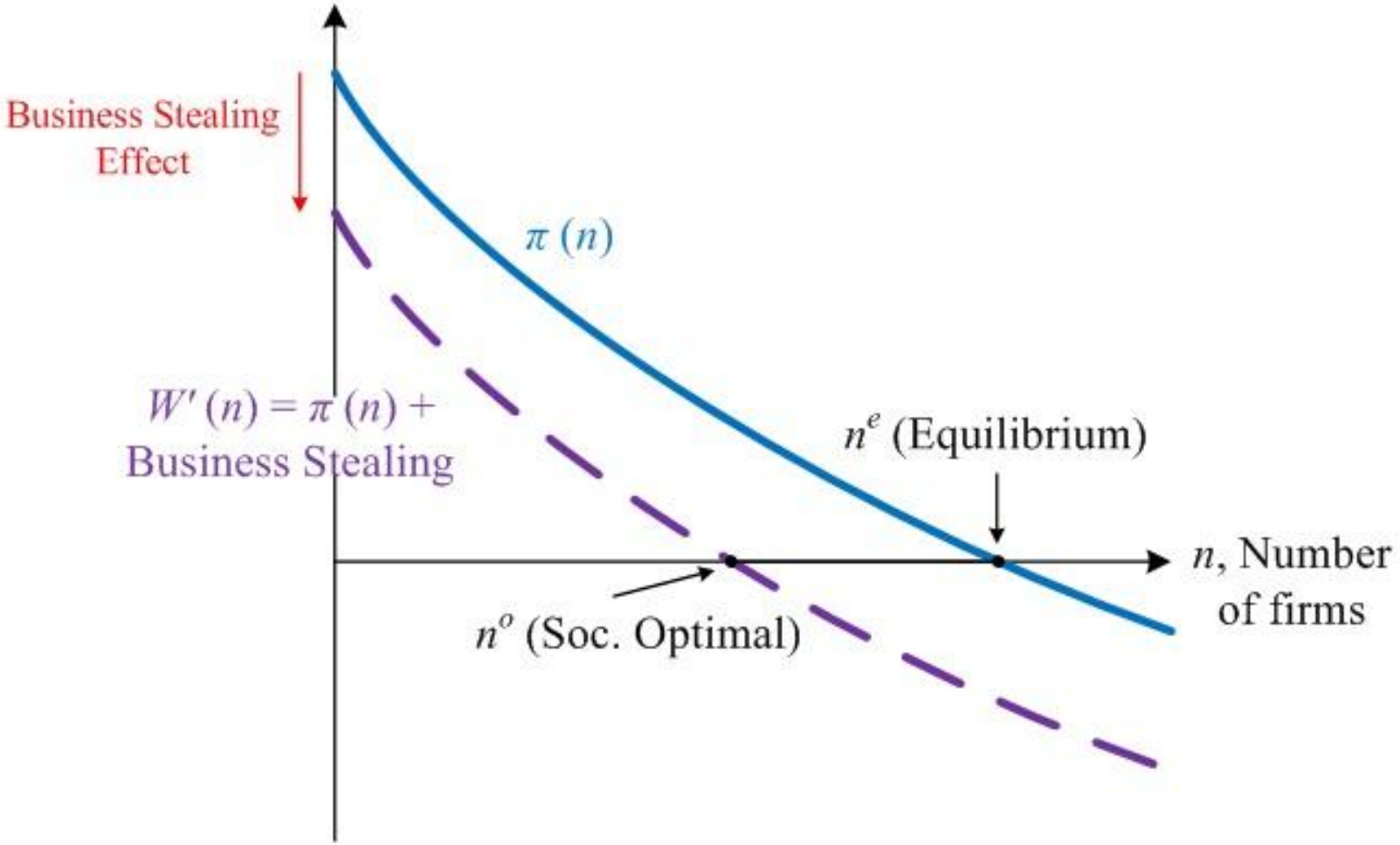
Endogenous Entry

- Given the negative sign of the business stealing effect, we can conclude that

$$W'(n) = \pi(n) + \underbrace{n\left[p\left(nq(n)\right) - c'\left(q(n)\right)\right]\frac{\partial q(n)}{\partial n}}_{-} < \pi(n)$$

  and therefore more firms enter in equilibrium than in the social optimum, i.e., $n^e > n^o$.

Endogenous Entry



Endogenous Entry

- ***Example***:
  - Consider a linear inverse demand $p(Q) = 1 - Q$ and no marginal costs.
  - The equilibrium quantity in a market with n firms that compete à la Cournot is

$$q(n) = \frac{1}{n+1}$$

  - Let's check if the three assumptions from above hold.

Endogenous Entry

- ***Example*** (continued):
  - First, individual output decreases with entry

$$\frac{\partial q(n)}{\partial n} = -\frac{1}{(n+1)^2} < 0$$

  - Second, aggregate output $nq(n)$ increases with entry

$$\frac{\partial [nq(n)]}{\partial n} = \frac{1}{(n+1)^2} > 0$$

  - Third, price lies above marginal cost for any number of firms

$$p(n) = 1 - n \cdot \frac{1}{n+1} = \frac{1}{n+1} > 0 \text{ for all } n$$

Endogenous Entry

- ***Example*** (continued):
  - Every firm earns equilibrium profits of

$$\pi(n) = \underbrace{\left(\frac{1}{n+1}\right)}_{p(n)} \underbrace{\frac{1}{n+1}}_{q(n)} - F = \frac{1}{(n+1)^2} - F$$

  - Since equilibrium profits after entry, $\frac{1}{(n+1)^2}$, is smaller than 1 even if only one firm enters the industry, $n = 1$, we assume that entry costs are in between 0 and 1, i.e., $0 \leq F \leq 1$.

Endogenous Entry

- ***Example*** (continued):
  - Social welfare is

$$W(n) = \int_0^{\frac{n}{n+1}} (1-s)ds - n \cdot F$$

$$= \left(s - \frac{s^2}{2} \right) \Bigg|_0^{\frac{n}{n+1}} - n \cdot F$$

$$= \frac{n(n+2)}{2(n+1)^2} - n \cdot F$$

Endogenous Entry

- ***Example*** (continued):
  - The number of firms entering the market in equilibrium, n^e, is that solves $\pi(n^e) = 0$,

$$\frac{1}{(n^e+1)^2} - F = 0 \iff n^e = \frac{1}{\sqrt{F}} - 1$$

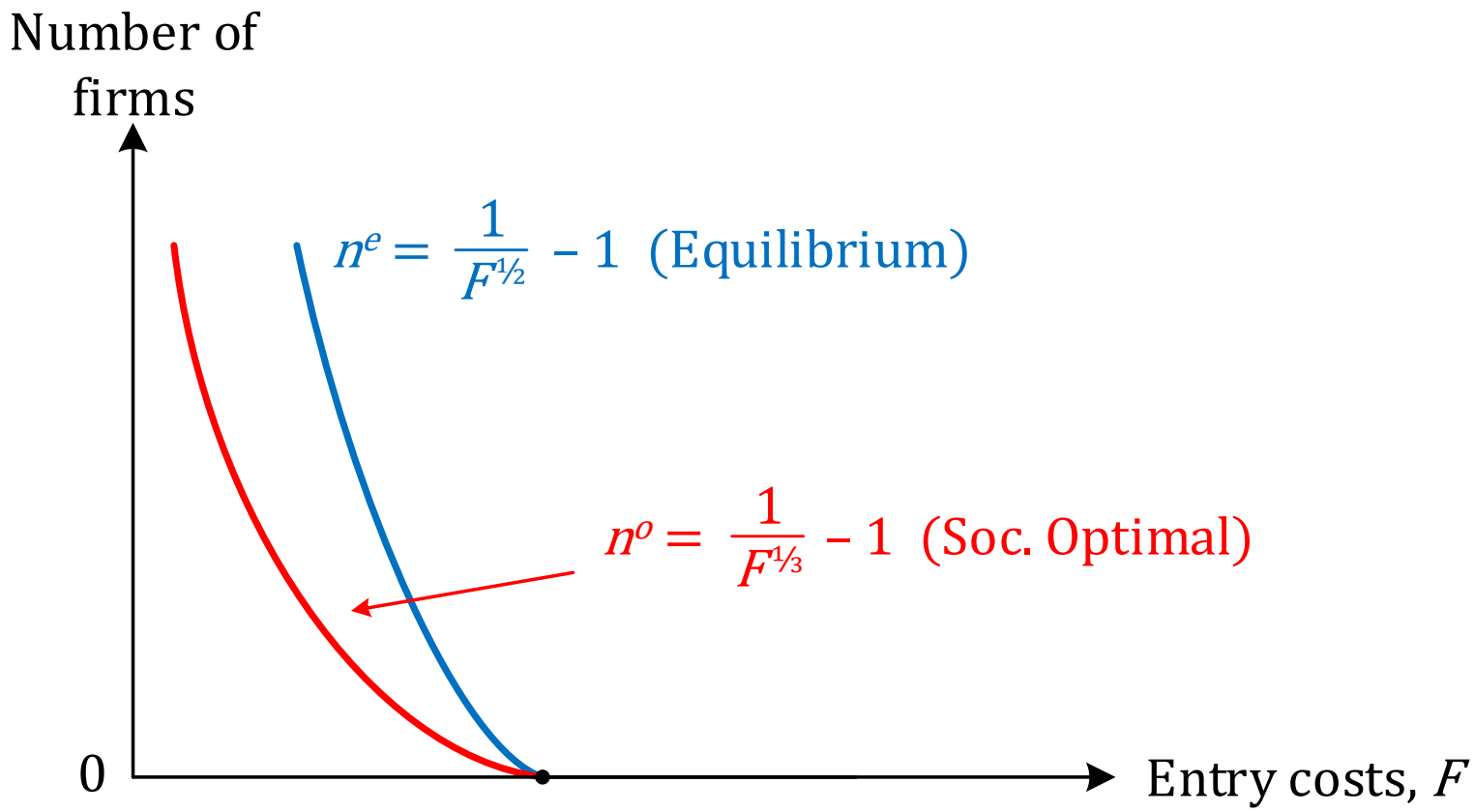
whereas the number of firms maximizing social welfare, i.e., n^o, is that solves $W'(n^o) = 0$,

$$W'(n^o) = \frac{1}{(n^o+1)^3} - F = 0 \iff n^o = \frac{1}{\sqrt[3]{F}} - 1$$

where $n^e \geq n^o$ for all admissible values of F, i.e., $F \in [0,1]$.

Endogenous Entry

- ***Example*** (continued):



Repeated Interaction

Repeated Interaction

- In all previous models, we considered firms interacting during one period (i.e., ***one-shot game***).
- However, firms compete during several periods and, in some cases, during many generations.
- In these cases, a firm's actions during one period might affect its rival's behavior in future periods.
- More importantly, we can show that under certain conditions, the strong competitive results in the Bertrand (and, to some extent, in the Cournot) model can be avoided when firms interact repeatedly along time.
- That is, collusion can be supported in the repeated game even if it could not in the one-shot game.

Repeated Interaction: Bertrand Model

- Consider two firms selling homogeneous products.
- Let $p_{j,t}$ denote firm j's pricing strategy at period t, which is a function of the history of all price choices by the two firms, $H_{t-1} = \{p_{1,t}, p_{2,t}\}_{t=1}^{t-1}$.
- Conditioning $p_{j,t}$ on the full history of play allows for a wide range of pricing strategies:
  - setting the same price regardless of previous history
  - retaliation if the rival lowers its price below a "threshold level"
  - increasing cooperation if the rival was cooperative in previous periods (until reaching the monopoly price p^m)

Repeated Interaction: Bertrand Model

- ***Finitely repeated game***:
  - Can we support cooperation if the Bertrand game is repeated for a finite number of T rounds?
    - No!
  - To see why, consider the last period of the repeated game (period T):
    - Regardless of previous pricing strategies, every firm's optimal pricing strategy in this stage is to set $p_{i,T}^{*} = c$, as in the one-shot Bertrand game.

Repeated Interaction: Bertrand Model

- Now, move to the previous to the last period ($T-1$):
  - Both firms anticipate that, regardless of what they choose at $T-1$, they will both select $p_{i,T}^{*}=c$ in period T. Hence, it is optimal for both to select $p_{i,T-1}^{*}=c$ in period $T-1$ as well.
- Now, move to period ($T-2$):
  - Both firms anticipate that, regardless of what they choose at $T-2$, they will both select $p_{i,T}^{*}=c$ in period T and $p_{i,T-1}^{*}=c$ in period $T-1$. Thus, it is optimal for both to select $p_{i,T-2}^{*}=c$ in period $T-2$ as well.
- The same argument extends to all previous periods, including the first round of play $t=1$.
- Hence, both firms behave as in the one-shot Bertrand game.

Repeated Interaction: Bertrand Model

- ***Infinitely repeated game***:
  - Can we support cooperation if the Bertrand game is repeated for an infinite periods?
    - Yes! Cooperation (i.e., selecting prices above marginal cost) can indeed be sustained using different pricing strategies.
  - For simplicity, consider the following pricing strategy

$$p_{jt}(H_{t-1}) = \begin{cases} p^m \text{ if all elements in } H_{t-1} \text{ are } (p^m, p^m) \text{ or } t = 1 \\ c \text{ otherwise} \end{cases}$$

    - In words, every firm j sets the monopoly price p^m in period 1. Then, in each subsequent period $t > 1$, firm j sets p^m if both firms charged p^m in all previous periods. Otherwise, firm j charges a price equal to marginal cost.

Repeated Interaction: Bertrand Model

- This type of strategy is usually referred to as ***Nash reversion strategy*** (NRS):
  - firms cooperate until one of them deviates, in which case firms thereafter revert to the Nash equilibrium of the unrepeated game (i.e., set prices equal to marginal cost)
- Let us show that NRS can be sustained in the equilibrium of the infinitely repeated game.
- We need to demonstrate that firms do not have incentives to deviate from it, during any period $t > 1$ and regardless of their previous history of play.

Repeated Interaction: Bertrand Model

– Consider any period $t > 1$, and a history of play for which all firms have been cooperative until $t - 1$.

– By cooperating, firm j's profits would be $(p^m - c)\frac{1}{2}\, x(p^m)$, i.e., half of monopoly profits $\frac{\pi^m}{2}$, in all subsequent periods

$$\frac{\pi^m}{2} + \delta\frac{\pi^m}{2} + \delta^2\frac{\pi^m}{2} + \cdots$$

$$= (1 + \delta + \delta^2 + \cdots)\frac{\pi^m}{2} = \frac{1}{1-\delta}\frac{\pi^m}{2}$$

where $\delta \in (0,1)$ denotes firms' discount factor

Repeated Interaction: Bertrand Model

- If, in contrast, firm j deviates in period t, the optimal deviation is $p_{j,t} = p^m - \varepsilon$, where $\varepsilon > 0$, given its rival still sets a price $p_{k,t} = p^m$.
- This allows firm j to capture all market, and obtain monopoly profits π^m during the deviating period.
- A deviation is detected in period $t + 1$, triggering a NRS from firm k (i.e., setting a price equal to marginal cost) thereafter, and entailing a zero profit for both firms.
- The discounted stream of profits for firm j is then

$$\pi^m + \delta 0 + \delta^2 0 + \cdots = \pi^m$$

Repeated Interaction: Bertrand Model

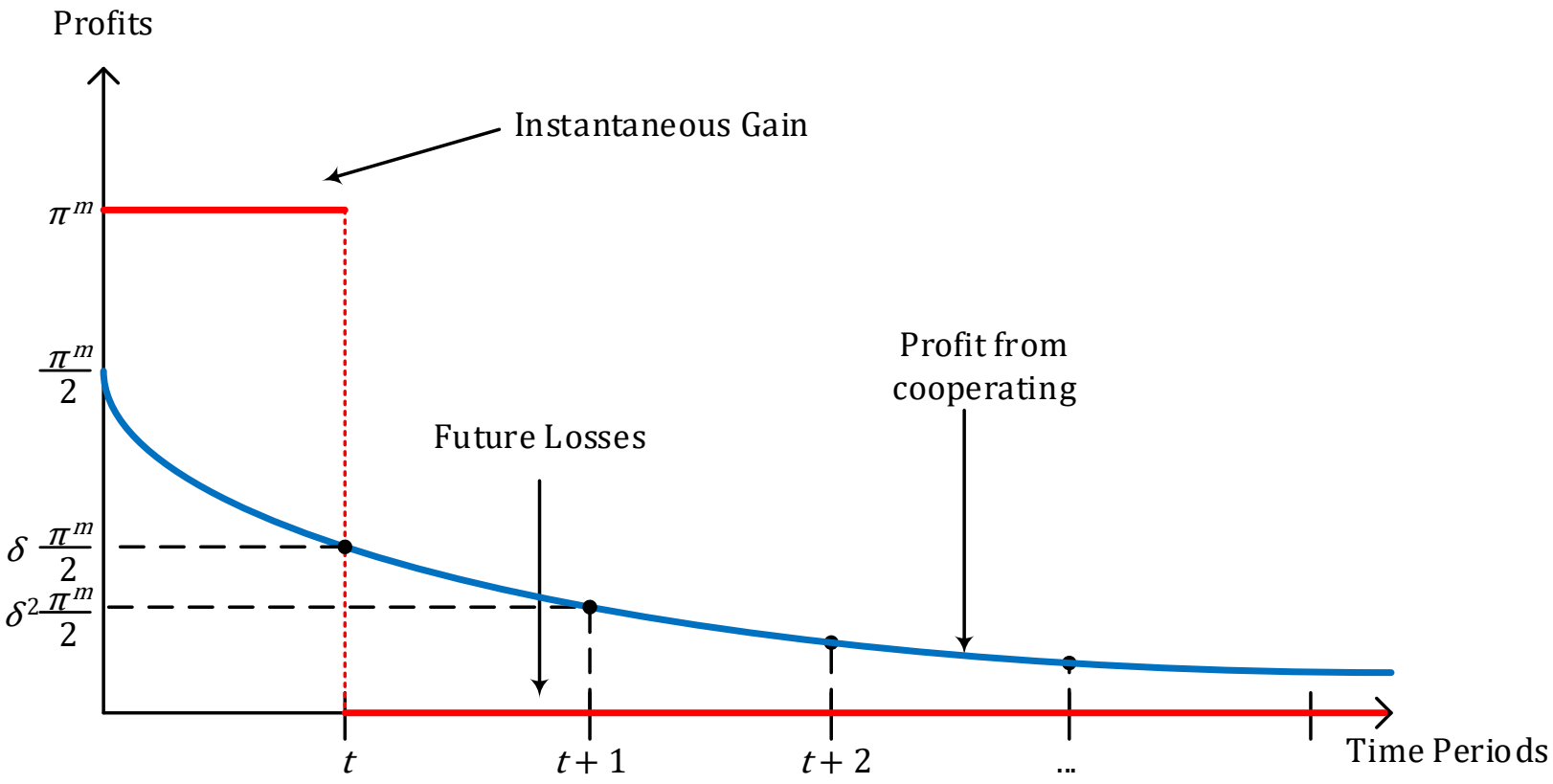
– Hence, firm j prefers to stick to the NRS at period t if

$$\frac{1}{1-\delta}\frac{\pi^m}{2} > \pi^m \quad \Leftrightarrow \quad \delta > \frac{1}{2}$$

– That is, cooperation can be sustained as long as firms assign a sufficiently high value to future profits.

Repeated Interaction: Bertrand Model

– Instantaneous gains and losses from cooperation and deviation



Repeated Interaction: Bertrand Model

- What about firm j's incentives to use NRS after a history of play in which some firms deviated?
  - NRS calls for firm j to revert to the equilibrium of the unrepeated Bertrand model.
  - That is, to implement the punishment embodied in NRS after detecting a deviation from any player.
- By sticking to the NRS, firm j's discounted stream of payoffs is

$$0 + \delta 0 + \cdots = 0$$

Repeated Interaction: Bertrand Model

- By deviating from NRS (i.e., setting a price $p_j = p^m$ while its opponent sets a punishing price $p_k = c$), profits are also zero in all periods.
- Hence, firm j has incentives to carry out the threat
  - That is, setting a punishing price of $p_j = c$, upon observing a deviation in any previous period.
- As a result, the NRS can be sustained in equilibrium, since both firms have incentives to use it, at any time period $t > 1$ and irrespective of the previous history of play.

Repeated Interaction: Bertrand Model

- ***Example***:
  - Consider an industry with only 2 firms, a linear demand $Q = 5000 - 100p$, and constant and average marginal costs of $c = \$10$.
  - If one-shot Bertrand game is played, firms would
    - charge a price of $p = c = \$10$
    - sell a total quantity of 4000 units
    - earn zero economic profits
  - If, in contrast, firms collude to fix prices at the monopoly price, can such collusion be sustained?

Repeated Interaction: Bertrand Model

- ***Example*** (continued):
  - Monopoly price is determined by solving the firms' joint PMP

$$\max_{p} \ (p-10)\cdot Q = (p-10)\,(5000-100p)$$

  - FOC:

$$5000 - 200p + 1000 = 0$$

  - Solving for p yields the monopoly price $p^m = 30$.
  - The aggregate output is $Q = 2000$ (i.e., 1000 units per firm) and the corresponding profits are $\pi^m = \$40{,}000$ ($20,000 per firm).

Repeated Interaction: Bertrand Model

- ***Example*** (continued):
  - Collusion at the monopoly price is sustainable if

$$\frac{\pi^m}{2}\frac{1}{1-\delta} \geq \pi^m + \frac{\delta}{1-\delta}\cdot 0$$

  - Since $\pi^m = \$40{,}000$, the inequality reduces to

$$20000\frac{1}{1-\delta} \geq 40000 \quad \Leftrightarrow \quad \delta \geq \frac{1}{2}$$

Repeated Interaction: Bertrand Model

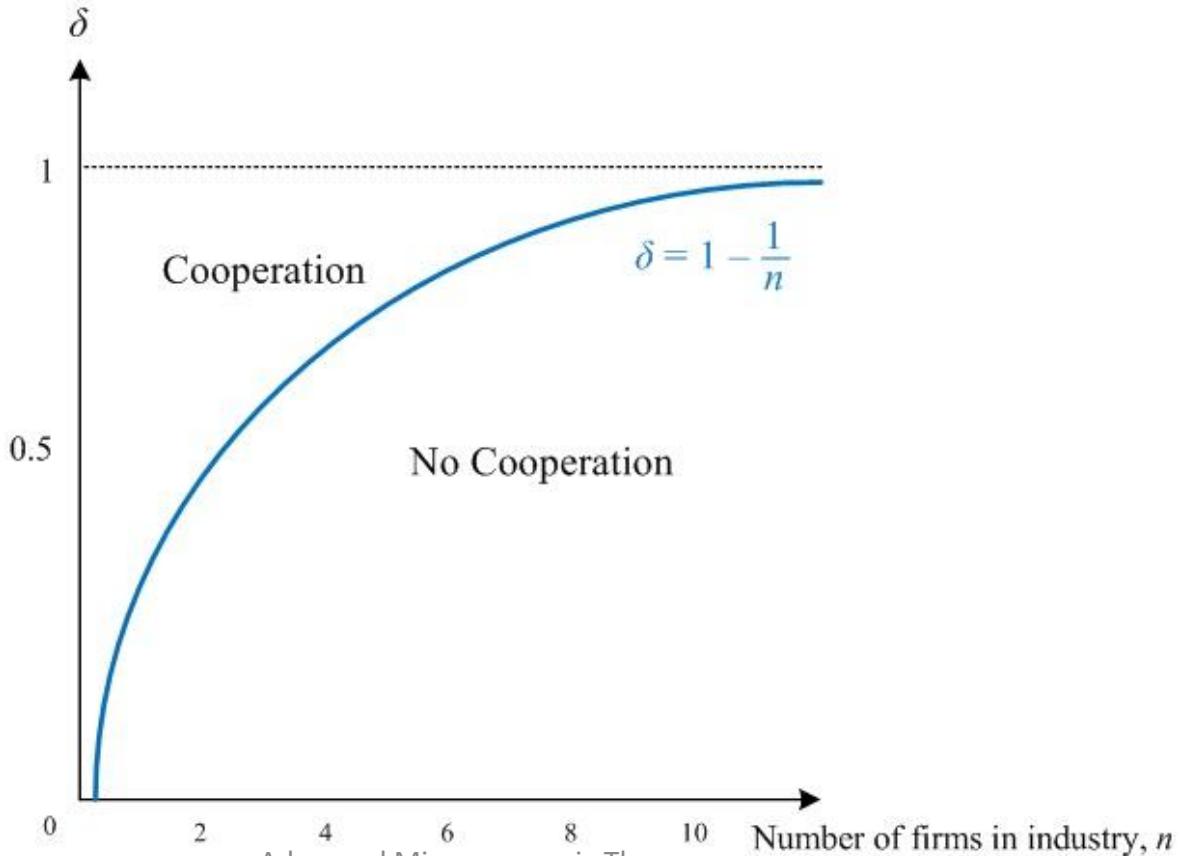
- ***Example*** (continued):
  - What would happen if there were n firms?
  - Each firm's share of the monopoly profit stream under collusion would be $\frac{\pi^m}{n} = \frac{40000}{n}$.
  - Collusion at the monopoly price is sustainable if

$$\frac{40000}{n}\frac{1}{1-\delta} \geq 40000 \Leftrightarrow \delta \geq 1 - \frac{1}{n} \equiv \bar{\delta}$$

  - Hence, as the number of firms in the industry increases, it becomes more difficult to sustain cooperation.

Repeated Interaction: Bertrand Model

- ***Example*** (continued): minimal discount factor sustaining collusion



Repeated Interaction: Cournot Model

- We can extend a similar analysis to the Cournot model of quantity competition with two firms selling homogeneous products.
- For simplicity, consider the following NRS for every firm j

$$q_{jt}(H_{t-1}) = \begin{cases} \frac{q^m}{2} \text{ if all elements in } H_{t-1} \text{ equal } \left(\frac{q^m}{2}, \frac{q^m}{2}\right) \text{ or } t = 1 \\ q_j^{Cournot} \text{ otherwise} \end{cases}$$

  - In words, firm j's strategy is to produce half of the monopoly output $\frac{q^m}{2}$ in period $t = 1$. Then, in each subsequent period $t > 1$, firm j continues producing $\frac{q^m}{2}$ if both firms produced $\frac{q^m}{2}$ in all previous periods. Otherwise, firm j reverts to the Cournot equilibrium output.

Repeated Interaction: Cournot Model

- Let us show that NRS can be sustained in the equilibrium of the infinitely repeated game.
- If firm j uses the NRS in period t, it obtains half of monopoly profits, $\frac{\pi^m}{2}$, thereafter, with a discounted stream of profits of $\frac{\pi^m}{2}\frac{1}{1-\delta}$.
- But, what if firm j deviates from this strategy? What is its optimal deviation?
  - Since firm k sticks to the NRS, and thus produces $\frac{q^m}{2}$ units, we can evaluate firm j's BRF $q_j(q_k)$ at $q_k = \frac{q^m}{2}$, or $q_j\left(\frac{q^m}{2}\right)$.

Repeated Interaction: Cournot Model

- For compactness, let $q_j^{dev} \equiv q_j\left(\frac{q^m}{2}\right)$ denote firm j's optimal deviation.
- This yields profits of

$$\pi_j^{dev} \equiv p\left(q_j^{dev}, \frac{q^m}{2}\right) \times q_j^{dev} - c_j \times q_j^{dev}$$

- By deviating firm j obtains following stream of profits

$$\pi_j^{dev} + \delta\pi_j^{Cournot} + \delta^2\pi_j^{Cournot} + \cdots$$

$$= \pi_j^{dev} + \frac{\delta}{1-\delta}\pi_j^{Cournot}$$

Repeated Interaction: Cournot Model

- Hence, firm j sticks to the NRS as long as

$$\frac{1}{1-\delta}\frac{\pi^m}{2} > \pi_j^{dev} + \frac{\delta}{1-\delta}\pi_j^{Cournot}$$

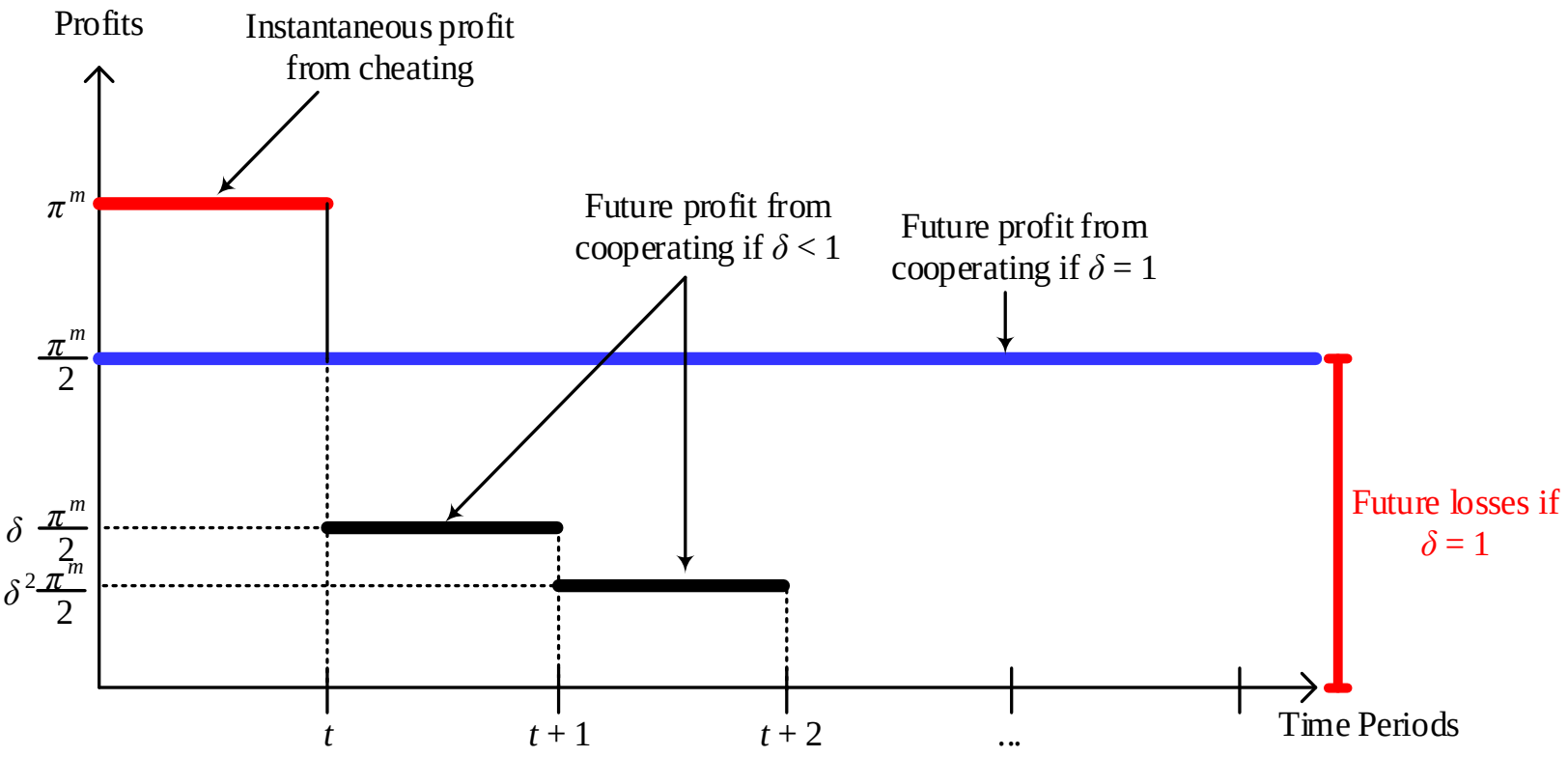
- Multiplying both sides by $(1-\delta)$ and solving for δ we obtain

$$\delta > \frac{\pi_j^{dev} - \frac{\pi^m}{2}}{\pi_j^{dev} - \pi_j^{Cournot}} \equiv \bar{\delta}$$

- Intuitively, every firm j sticks to the NRS as long as it assigns a sufficient weight to future profits.

Repeated Interaction: Cournot Model

- Instantaneous gains and losses from deviation



Repeated Interaction: Cournot Model

- What about firm j's incentives to use NRS after a history of play in which some firms deviated?
  - NRS calls for firm j to revert to the equilibrium of the unrepeated Cournot model.
  - That is, to implement the punishment embodied in NRS after detecting a deviation from any player.
- By sticking to the NRS, firm j's discounted stream of payoffs is $\frac{1}{1-\delta}\frac{\pi^{Cournot}}{2}$.
- By deviating from $q_j^{Cournot}$, while firm k produces $q_k^{Cournot}$, firm j's profits, $\hat{\pi}$, are lower than $\pi_j^{Cournot}$ since firm j's best response to its rival producing $q_k^{Cournot}$ is $q_j^{Cournot}$.

Repeated Interaction: Cournot Model

- Firm j sticks to the NRS after a history of deviations since

  $$\pi^{Cournot} + \delta\pi^{Cournot} + \cdots > \hat{\pi} + \delta\pi^{Cournot} + \cdots$$

  which holds given that $\pi^{Cournot} > \hat{\pi}$.

- Hence, no need to impose any further conditions on the minimal discount factor sustaining cooperation, $\bar{\delta}$.

Repeated Interaction: Cournot Model

- ***Example***:
  - Consider an industry with 2 firms, a linear inverse demand $p(q_1, q_2) = a - b(q_1 + q_2)$, and constant and average marginal costs of $c > 0$.
  - Firm i's PMP is

$$\max_{q_i} \big(a - b(q_i + q_j)\big)q_i - cq_i$$

  - FOCs: $a - 2bq_i - bq_j - c = 0$
  - Solving for q_i yields firm i's BRF

$$q_i(q_j) = \frac{a-c}{2b} - \frac{q_j}{2}$$

Repeated Interaction: Cournot Model

- ***Example*** (continued):
  - Solving the two BRFs simultaneously yields

$$q_i^{Cournot} = \frac{a-c}{3b}$$

with corresponding price of

$$p = a - b\left(\frac{a-c}{3b} + \frac{a-c}{3b}\right) = \frac{a+2c}{3}$$

and equilibrium profits of

$$\pi_i^{Cournot} = \left(\frac{a+2c}{3}\right)\frac{a-c}{3b} - c\left(\frac{a-c}{3b}\right) = \frac{(a-c)^2}{9b}$$

Repeated Interaction: Cournot Model

- ***Example*** (continued):
  - If, instead, each firm produced half of monopoly output, $q_i^m = \frac{q^m}{2} = \frac{a-c}{4b}$, they would face a corresponding price of $p^m = \frac{a+c}{2}$ and receive half of the monopoly profits $\pi_i^m = \frac{\pi^m}{2} = \frac{(a-c)^2}{8b}$.
  - In this setting, the optimal deviation of firm i is found by plugging q_i^m into its BRF

$$q_i^{Dev} = q_i(q_j^m) = \frac{a-c}{2b} - \frac{1}{2}\frac{a-c}{4b} = \frac{3(a-c)}{8b}$$

Repeated Interaction: Cournot Model

- ***Example*** (continued):
  - This yields price of

$$p = a - b\left(\frac{3(a-c)}{8b} + \frac{a-c}{4b}\right) = \frac{3a+5c}{8}$$

and profits of

$$\pi_i^{Dev} = \left(\frac{3a+5c}{8}\right)\frac{3(a-c)}{8b} - c\left(\frac{3(a-c)}{8b}\right) = \frac{9(a-c)^2}{64b}$$

for the deviating firm, and

$$\hat{\pi} = \left(\frac{3a+5c}{8}\right)\frac{(a-c)}{4b} - c\left(\frac{(a-c)}{4b}\right) = \frac{3(a-c)^2}{32b}$$

for the non-deviating firm.

Repeated Interaction: Cournot Model

- ***Example*** (continued):
  - Cooperation is sustainable if

$$\frac{1}{1-\delta}\frac{\pi^m}{2} > \pi_j^{dev} + \frac{\delta}{1-\delta}\pi_j^{Cournot}$$

or, in our case,

$$\frac{1}{1-\delta}\frac{(a-c)^2}{8b} > \frac{9(a-c)^2}{64b} + \frac{\delta}{1-\delta}\frac{(a-c)^2}{9b} \iff \delta > \frac{9}{17}$$

  - For the non-deviating firm, we have $\pi_i^{Cournot} > \hat{\pi}$
    - That is, if the rival firm defects, the non-defecting firm will obtain a larger profit by reverting to the Cournot output level.

Repeated Interaction: Cournot Model

- *Extensions*:
  - Temporary reversions to the equilibrium of the unrepeated game
  - (Temporary) punishments that yield even lower payoffs
  - Less “pure” forms of co-operations
  - Antitrust regulation and imperfect monitoring

Advanced Microeconomic Theory

Chapter 9: Externalities and Public Goods

Outline

- Externalities
- The Coase Theorem
- Quotas
- Pigouvian Taxes and Subsidies
- Tradable Permits
- Regulating a Polluting Monopolist or Oligopoly

Outline (continued)

- Asymmetric Information
- Tragedy of the Commons
- Pollution Abatement
- Public Goods
- Lindahl Equilibria

Externalities

Externalities

- ***Externality*** emerges when the well-being of a consumer or the production possibilities of a firm is directly affected by the actions of another agent in the economy.
  - Example: the production possibilities of a fishery are affected by the pollutants that a refinery dumps into a lake.
  - The effects from one agent to another are not captured by the price system.
- The effects transmitted through the price system are referred to as "***pecuniary externalities***."

Externalities

- Consider a polluting firm (agent 1) and an individual affected by such pollution (agent 2).
- The firm's profit function is

$$\pi(p, x)$$

where p is the price vector and x is the amount of externality generated.

Externalities

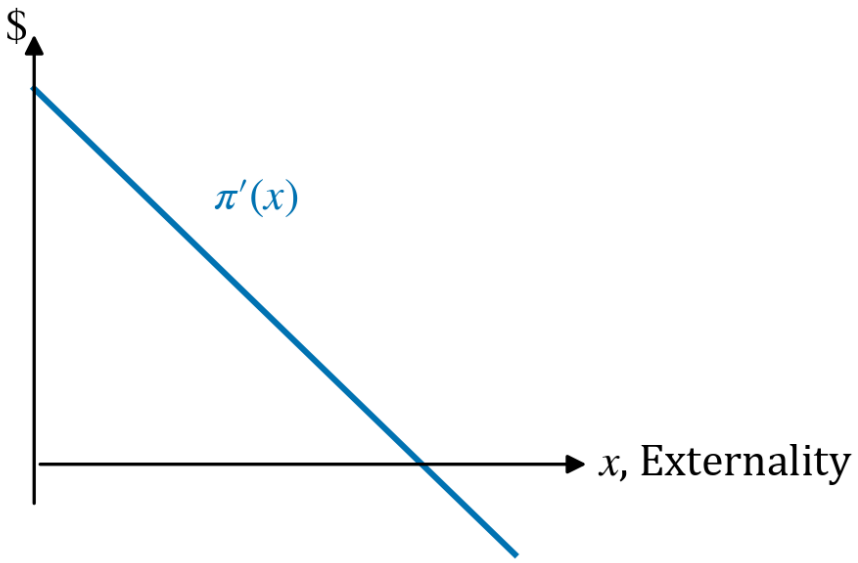
- Assume that p is given (i.e., p is a parameter). Then, the profit function becomes

$$\pi(x)$$

where $\pi'(0) > 0$ and $\pi''(x) < 0$, so marginal profit decreases in x and eventually becomes negative.

Externalities

- The firm obtains a positive and significant benefit from the first unit of the externality-generating activity.
- But the additional benefit from further units is decreasing.



Externalities

- The individual's (i.e., agent 2's) utility is given by

$$u(q, x)$$

where $q \in \mathbb{R}_+^N$ is a vector of N tradable goods and $x \in \mathbb{R}_+$ is the negative externality, with

$$\frac{\partial u}{\partial x} < 0 \quad \text{and} \quad \frac{\partial u}{\partial q_k} \geq 0 \quad \text{in every good } k$$

Externalities

- Let $q^*(p, w, x)$ denote the individual's Walrasian demand. Then,

$$v(x) = u(q^*(p, w, x), x)$$

  is the indirect utility function with $v'(x) < 0$ for all $x > 0$.
- We assume utility is quasilinear in wealth (no wealth effects), so the individual's utility is $v(x) + w$.

Externalities

- ***Example***:
  - Consider a firm whose profit function is $\pi = py - cy^2$, where $y \in \mathbb{R}_+$ is output and $p, c > 0$.
  - If every unit of output generates a unit of pollution, i.e., $x = y$, the profit function becomes $\pi(x) = px - cx^2$.
  - FOC wrt x yields $\pi'(x^*) = p - 2cx^* = 0$, producing $p = 2cx^*$ or $x^* = \frac{p}{2c}$.

Externalities

- ***Example*** (continued):
  - If every unit of output y generates α units of pollution, i.e., $x = \alpha y$ (so $y = \frac{x}{\alpha}$), where $\alpha > 0$, the profit function becomes

$$\pi(x) = p\frac{x}{\alpha} - c\left(\frac{x}{\alpha}\right)^2.$$

  - Taking FOC with respect to x yields

$$\pi'(x^*) = \frac{p}{\alpha} - 2c\frac{x^*}{\alpha}\frac{1}{\alpha} = 0,$$

with a competitive equilibrium level of pollution of

$$x^* = \alpha\frac{p}{2c}.$$

Externalities

- ***Competitive equilibrium***: All agents independently and simultaneously solve their PMP (for firms) or UMP (for consumers).
  - The firm chooses the level of the externality-generating activity, x, that solves its PMP

$$\max_{x \geq 0} \ \pi(x)$$

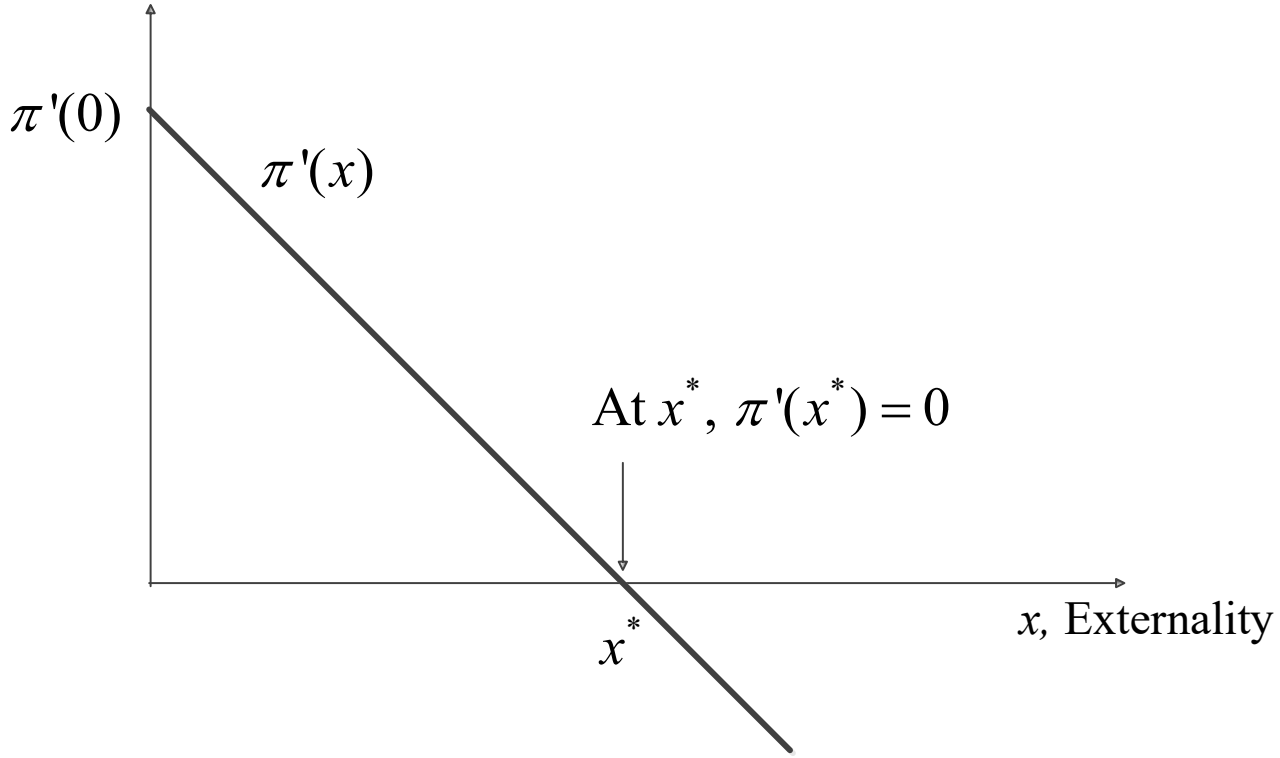
  - Taking FOC with respect to x yields

$$\pi'(x^*) \leq 0$$

with equality if $x^* > 0$ (interior solution).

Externalities

– The firm increases the externality-generating activity until the marginal benefit from an additional unit is zero, i.e., $\pi'(x^*) = 0$.



Externalities

– The UMP of the individual affected by pollution is

$$\max_{q \geq 0} \; u(q, x) \quad \text{s.t.} \quad pq \leq w$$

where $p \in \mathbb{R}_+^N$ is the given price vector.

– Notice that $q \in \mathbb{R}_+^N$ does not include pollution as one of the N tradable goods.

– Hence the individual cannot affect the level of the externality-generating activity x.

- Uninteresting case
- We relax this assumption when we assign property rights over x (Coase theorem section).

Externalities

- ***Pareto optimum***:
  - The social planner selects the level of x that maximizes social welfare (by quasilinearity, Pareto optimality is equivalent to maximizing aggregate surplus)

$$\max_{x \geq 0} \ \pi(x) + v(x)$$

  - Taking FOC with respect to x yields

$$\pi'(x^0) \leq -v'(x^0) \text{ with equality if } x^0 > 0$$

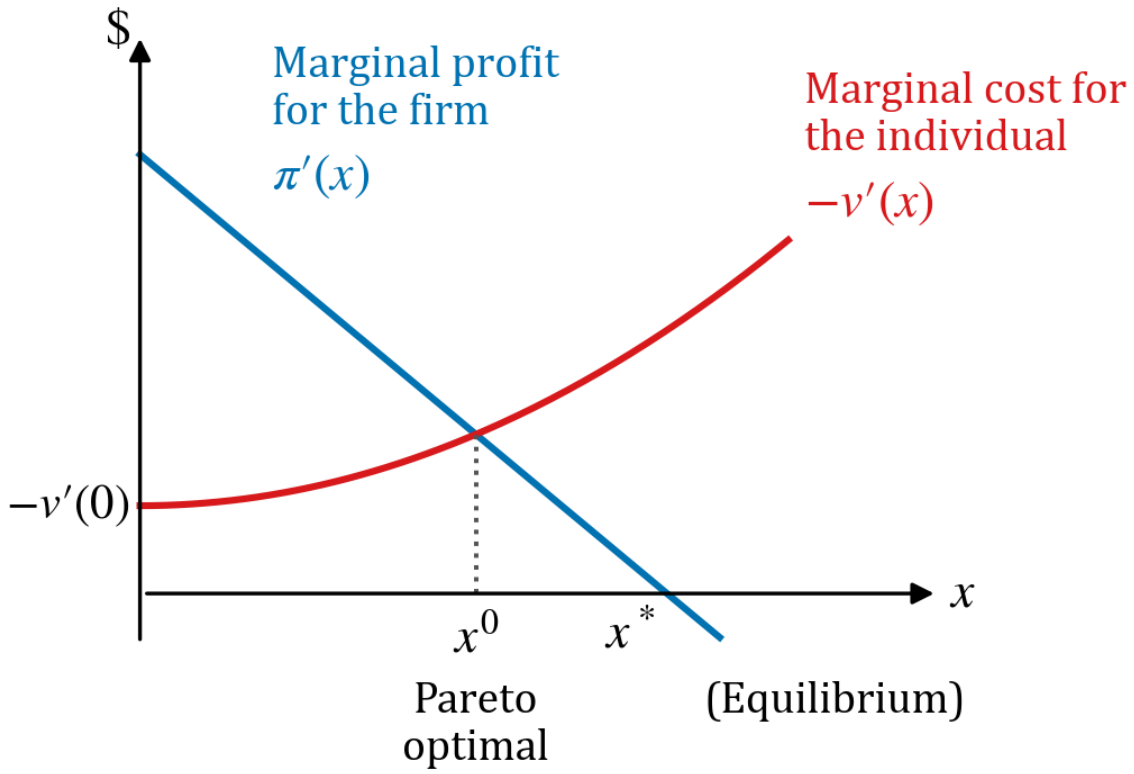
where x^0 is the Pareto optimal amount of the externality.

Externalities

- ***Pareto optimum*** (continued):
  - Intuitively, at a Pareto optimal (and interior) solution, the marginal benefit of the externality-generating activity, $\pi'(x^0)$, is equal to its marginal cost, $-v'(x^0)$.

Externalities

- Figure: Pareto optimal and equilibrium externality levels (negative externality)
- Too much externality x is produced in the competitive equilibrium relative to the Pareto optimum, i.e., $x^* > x^0$.



Externalities

- ***Example***:
  - Consider a firm with marginal profits of

$$\pi'(x) = a - bx, \text{ where } a, b > 0$$

    which is decreasing in x.

  - Assume a consumer whose marginal damage is

$$-v'(x) = c + dx, \qquad \text{where} \quad c, d > 0$$

    which is increasing in x.

Externalities

- ***Example*** (continued):
  - The competitive equilibrium amount of externality x^* solves $\pi'(x^*) = 0$, i.e., $a - bx^* = 0$. Hence,

$$x^* = \frac{a}{b}$$

  - The socially optimal level of the externality x^0 solves $\pi'(x^0) = -v'(x^0)$, i.e., $a - bx^0 = c + dx^0$. Thus,

$$x^0 = \frac{a - c}{b + d}$$

Externalities

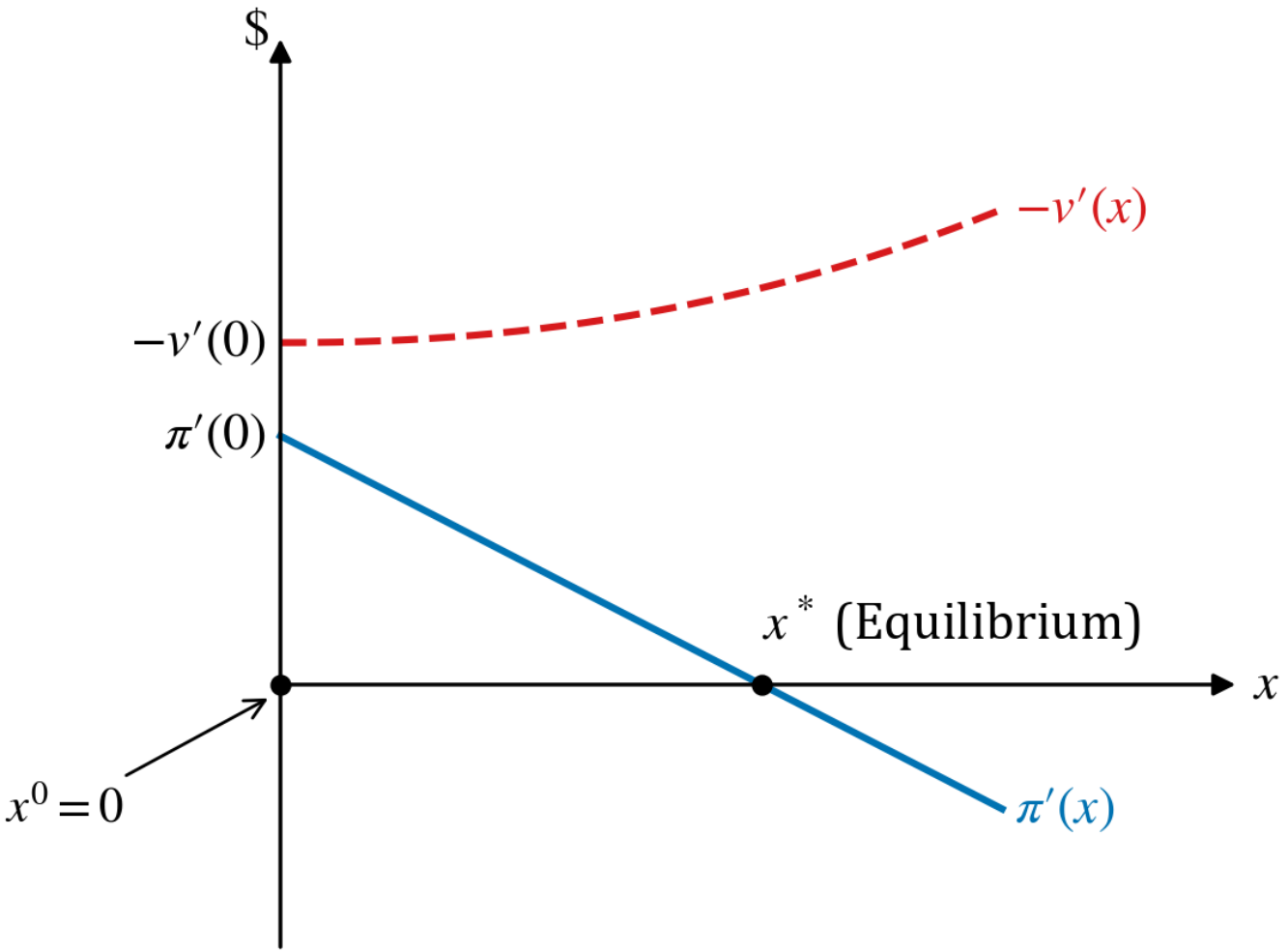
- ***Example*** (continued):
  - The level x^0 is positive if and only if $\pi'(0) > -v'(0)$, i.e., $a > c$; otherwise $x^0 = 0$.
  - Hence, for an interior optimum ($a > c$),

$$x^* - x^0 = \frac{a}{b} - \frac{a-c}{b+d} = \frac{ad+bc}{b(b+d)} > 0$$

Externalities

- Negative externalities are not necessarily eliminated at the Pareto optimal solution.
- This occurs if and only if $-v'(0) \geq \pi'(0)$.
- In this setting, the curves $\pi'(x)$ and $-v'(x)$ do not cross, and the Pareto optimal solution only occurs at the corner where $x^0 = 0$.

Externalities



Externalities

- If the firm's production activities produce a ***positive externality*** in the individual's well-being, then

$$v'(x) > 0 \text{ and } -v'(x) < 0$$

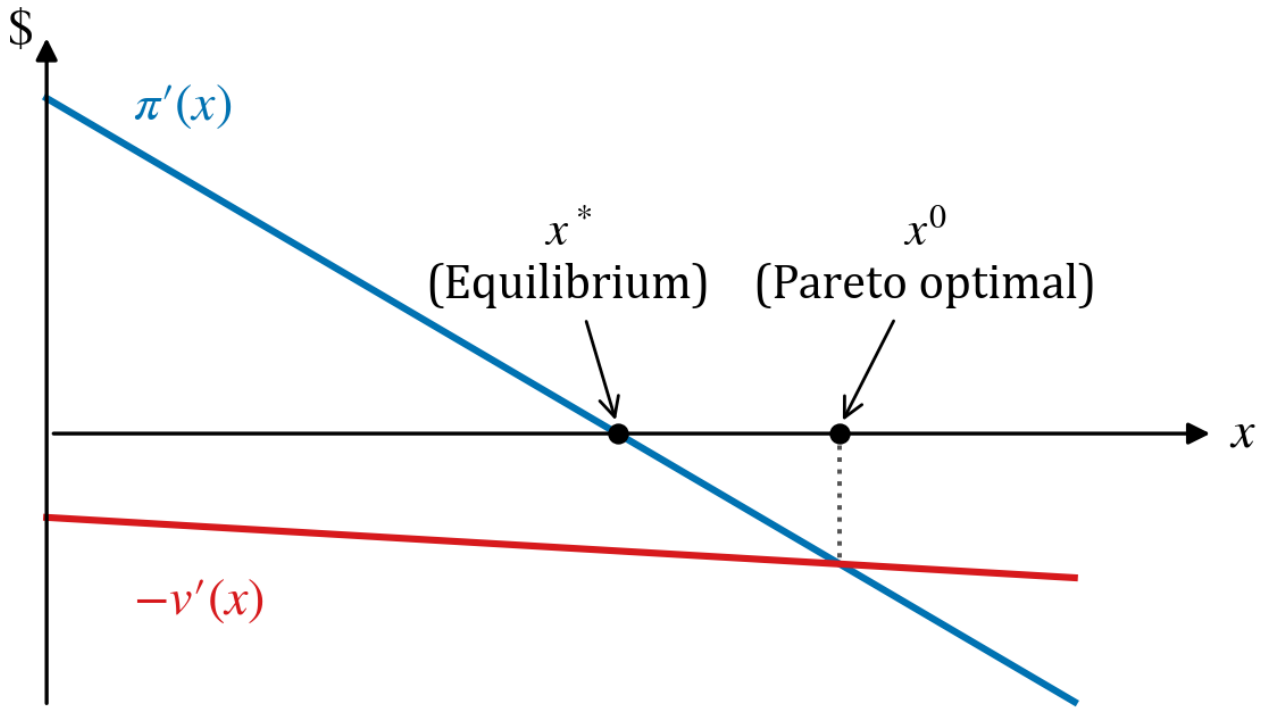
  - That is, the curve $-v'(x)$ lies below the horizontal axis.
  - $\pi'(x)$ remains unaffected.

Externalities

- In this setting, there is an underproduction of the externality-generating activity relative to the Pareto optimum, i.e., $x^* < x^0$.
- Since $\pi'(x^0) = -v'(x^0) < 0 = \pi'(x^*)$ and π' is decreasing, $x^0 > x^*$.

Externalities

- Pareto optimal and equilibrium externality level (positive externality).



Externalities

- ***Example*** (Positive externalities):
  - Consider two neighboring countries, $i \in \{1,2\}$, simultaneously choosing how many hours to spend on recycling activities, r_i.
  - The net benefit from recycling is:

$$\pi_i(r_i, r_j) = \left(a - r_i + \frac{r_j}{2}\right) r_i - b r_i$$

where $a > b > 0$, and b denotes the marginal cost of recycling.

Externalities

- ***Example*** (continued):
  - Country i's average benefit, $\left(a - r_i + \frac{r_j}{2}\right)$, is increasing in r_j because a clean environment produces positive external effects on the other country.
  - Let us first find the competitive equilibrium allocation.

Externalities

- ***Example*** (continued):
  - Taking FOC with respect to r_i yields $a - 2r_i + \frac{r_j}{2} - b = 0$, so country i's BRF is

$$r_i(r_j) = \frac{a-b}{2} + \frac{1}{4}r_j$$

  - Symmetrically, country j's BRF is

$$r_j(r_i) = \frac{a-b}{2} + \frac{1}{4}r_i$$

    - The positive slope of the BRFs indicates that countries' recycling activities are strategic complements.

Externalities

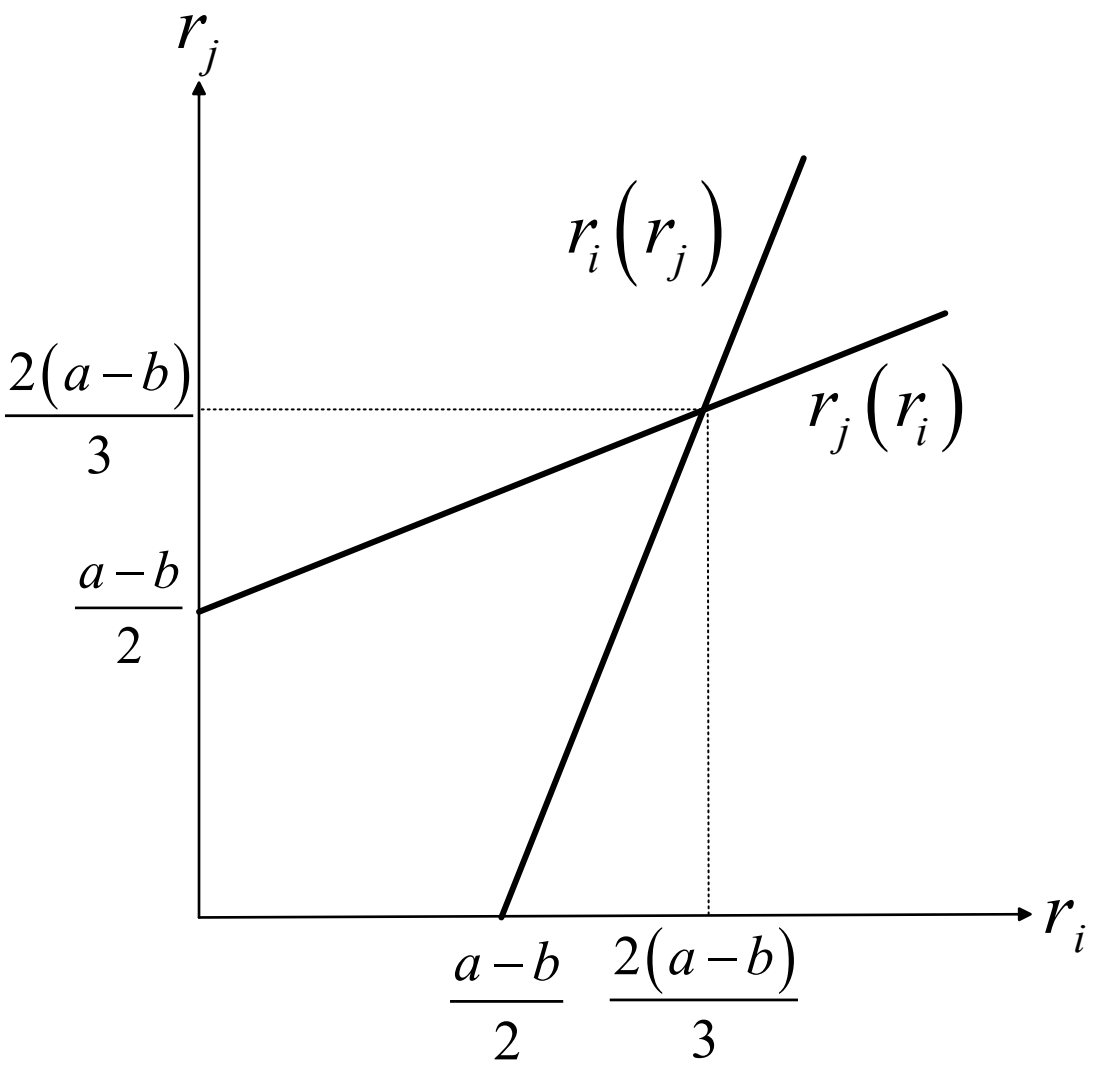
- ***Example*** (continued):
  - Simultaneously solving the two BRFs yields

$$r_i = \frac{1}{4}\left[\frac{a-b}{2} + \frac{r_i}{4}\right] + \frac{a-b}{2}$$

  - That is, $\frac{15}{16} r_i = \frac{5}{8}(a-b)$. Rearranging, we obtain an equilibrium level of recycling

$$r_i^* = \frac{2}{3}(a-b) \quad \text{for} \quad i \in \{1,2\}$$

Externalities



Externalities

- ***Example*** (continued):
  - A social planner simultaneously selects r_i and r_j in order to maximize social welfare

$$\max_{r_i, r_j}(a - r_i + \frac{r_j}{2})r_i - br_i + (a - r_j + \frac{r_i}{2})r_j - br_j$$

  - FOC with respect to r_i:

$$a - 2r_i + \frac{r_j}{2} - b + \frac{r_j}{2} = 0, \qquad \text{or} \quad a - b - 2r_i + r_j = 0$$

  and symmetrically for country j.

Externalities

- ***Example*** (continued):
  - The second $\frac{r_j}{2}$ term is the external benefit of r_i on country j, which the planner internalizes.

Externalities

- ***Example*** (continued):
  - Simultaneously solving the two FOCs yields the socially optimal levels of recycling

$$r_i^0 = a - b \quad \text{for} \quad i \in \{1,2\}$$

  - Note that, since $a > b$,

$$r_i^0 = a - b > \frac{2}{3}(a - b) = r_i^*$$

Solutions to the Externality Problem:

Property Rights

Property Rights

- This is a less intrusive approach:
  - let the parties bargain over the externality
  - no government intervention
- Key assumptions:
  - The property rights over the externality-generating activity must be:
    - Easy to identify and enforceable.
  - No bargaining costs.
  - Complete information about π and v.

Property Rights

- ***Property rights assigned to the affected individual (agent 2)***:
  - Let us assign property rights to the individual suffering the negative externality
    - "externality-free" environment: at the initial state no externality is generated, i.e., $x = 0$
  - The firm must then pay the affected individual if it wants to increase the externality above zero.

Property Rights

- ***Property rights assigned to the affected individual (agent 2)***:
  - In particular, let us assume that the affected individual makes a take-it-or-leave-it offer where the firm must pay $\$T$ in exchange for x units of pollution.

Property Rights

– The firm agrees to pay \$$T$ to the affected individual iff

$$\pi(x) + w_1 - T \geq \underbrace{\pi(0)}_{\text{status quo}} + w_1$$

where w_h is agent h's wealth, or

$$\pi(x) - T \geq \pi(0)$$

– Hence, the affected individual's UMP becomes that of choosing (x, T) that solves

$$\max_{x \geq 0, T} v(x) + w_2 + T$$
$$\text{s.t. } \pi(x) - T \geq \pi(0)$$

Property Rights

- The constraint is binding, since the affected individual will raise $\$T$ until the point where the firm is indifferent between accepting and rejecting the offer.
- Hence, $\pi(x) - T = \pi(0) \quad \text{or} \quad \pi(x) - \pi(0) = T.$
- Plugging this result into the affected individual's UMP, we obtain

$$\max_{x \geq 0} v(x) + w_2 + \underbrace{\pi(x) - \pi(0)}_{T}$$

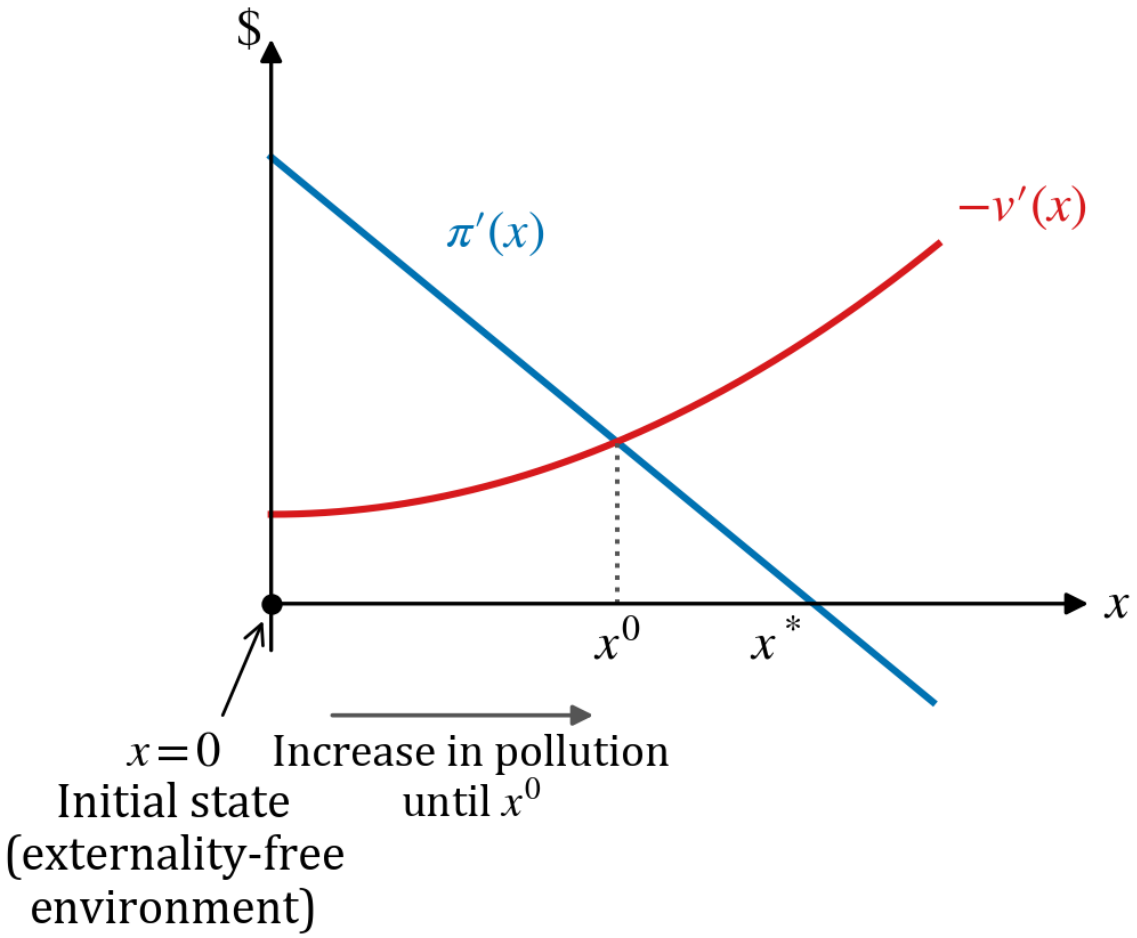
Property Rights

- The FOC with respect to x yields

$$v'(x) + \pi'(x) \leq 0 \quad \Leftrightarrow \quad \pi'(x) \leq -v'(x)$$

  with equality if $x > 0$.
- This coincides with the FOC of the social planner's problem.
- Therefore, bargaining allows for the level of the externality x to reach the optimal level, i.e., $x = x^0$.

Property Rights



Property Rights

- ***Property rights assigned to the firm***:
  - What if the property rights were assigned to the firm (i.e., the polluter)?
  - If there is no bargaining between the firm and the affected individual, the firm would pollute until $x = x^*$, where $\pi'(x^*) = 0$.
  - However, the affected individual can pay $\$T$ to the firm in exchange for a lower level of pollution, x, where $x < x^*$.

Property Rights

– The polluter is willing to accept this offer iff

$$\pi(x) + w_1 + T \geq \underbrace{\pi(x^*)}_{\text{status quo}} + w_1$$

$$\text{or} \quad \pi(x) + T \geq \pi(x^*)$$

– Thus, the affected individual's UMP becomes that of choosing (x, T) that solves

$$\max_{x \geq 0, T} v(x) + w_2 - T$$

$$\text{s.t.} \quad \pi(x) + T \geq \pi(x^*)$$

– Note that \$$T$ now enters negatively into the affected individual's utility, but positively into the firm's.

Property Rights

- The constraint is binding, since the affected individual reduces the $\$T$ until the point where the firm is indifferent between accepting and rejecting the offer T.
- Hence, $\pi(x) + T = \pi(x^*)$ or $T = \pi(x^*) - \pi(x)$.
- Inserting this result into the affected individual's UMP, we obtain

$$\max_{x \geq 0} v(x) + w_2 - \underbrace{[\pi(x^*) - \pi(x)]}_{T}$$

Property Rights

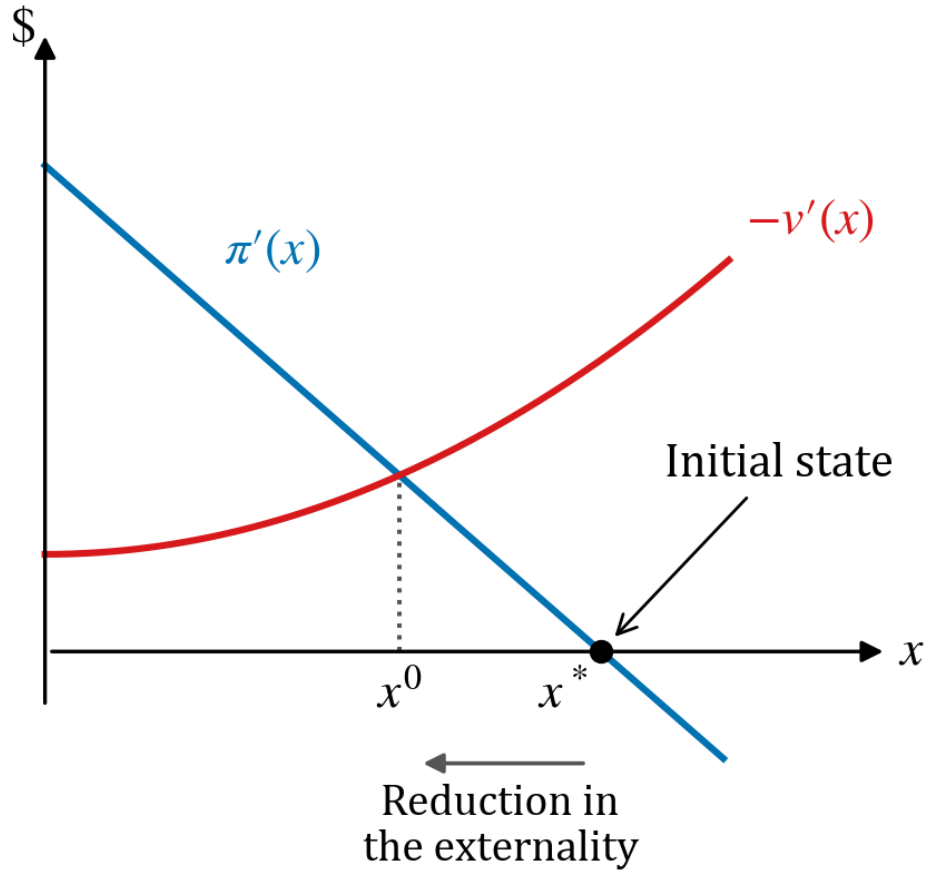
- The FOC with respect to x yields

$$v'(x) + \pi'(x) \leq 0 \quad \Leftrightarrow \quad \pi'(x) \leq -v'(x)$$

  with equality if $x > 0$.

- Again, the above coincides with the FOC at the optimal level of the externality (i.e., social planner's problem), where $x = x^0$.

Property Rights



Property Rights

- In summary, regardless of the initial assignment of property rights over the externality-generating activity…
  - agents can negotiate to increase or decrease the externality, x, until reaching the Pareto optimal level x^0.

Property Rights

- ***Coase Theorem***: If property rights are well defined and enforceable, bargaining is costless, information is complete, and preferences are quasilinear (no wealth effects), then the equilibrium level of the externality is x^0, regardless of the initial allocation of property rights.
  - That is, as long as property rights are clearly assigned, bargaining sets the level of the externality at the Pareto optimal level $x = x^0$.

Property Rights

- The allocation of property rights, however, affects the final wealth of the two agents!

Property Rights

- If property rights are assigned to the individual affected by the externality:
  - the firm must pay $T = \pi(x^0) - \pi(0)$ to the affected individual in order to increase the externality from $x = 0$ to $x = x^0$
  - the affected individual's utility is then

    $$v(x^0) + T \text{ or } v(x^0) + \pi(x^0) - \pi(0)$$

    while that of the firm is

    $$\pi(x^0) - T \text{ or } \pi(x^0) - \pi(x^0) + \pi(0) = \pi(0)$$

  - with $w_1 = w_2$, the affected individual's utility is higher than the firm's iff

    $$v(x^0) + \pi(x^0) - \pi(0) > \pi(0) \Leftrightarrow \pi(x^0) + v(x^0) > 2\pi(0)$$

Property Rights

- If property rights are assigned to the firm generating the externality:
  - the affected individual must pay $T = \pi(x^*) - \pi(x^0)$ to the firm in order to reduce the externality from $x = x^*$ to $x = x^0$
  - the firm's utility is then

    $$\pi(x^0) + T \text{ or } \pi(x^0) + \pi(x^*) - \pi(x^0) = \pi(x^*)$$

    while that of the affected individual is

    $$v(x^0) - T \text{ or } v(x^0) - \pi(x^*) + \pi(x^0)$$

  - with $w_1 = w_2$, the firm's utility is higher than the affected individual's iff

    $$\pi(x^*) > v(x^0) - \pi(x^*) + \pi(x^0) \Leftrightarrow 2\pi(x^*) > \pi(x^0) + v(x^0)$$

Property Rights

- Combining the two inequalities, we can conclude that the agent with the bargaining power (i.e., the property right over x) has a total utility higher than the agent without the bargaining power iff

$$2\pi(x^*) > \pi(x^0) + v(x^0) > 2\pi(0)$$

where $\pi(x^0) + v(x^0)$ measures the aggregate welfare at the social optimum, i.e., x^0.

Property Rights

- Each agent is better off holding the property right:
  - When the right moves from the individual to the firm, the individual's payoff falls by $\pi(x^*) - \pi(0) > 0$, and the firm's payoff rises by the same amount.

Property Rights

- ***Disadvantages of the Coase Theorem***:
  - Property rights must be perfectly defined
    - With whom should the firm bargain?
  - Bargaining must be costless.
    - This fails when many agents are involved.

Property Rights

- ***Disadvantages of the Coase Theorem*** (continued):
  - Property rights must be perfectly enforced:
    - The level of x must be perfectly observable and measurable by both parties
    - If a party does not comply with the agreement, it can be brought to court at no cost.
  - These assumptions are not satisfied in many cases, which limits the possibility of using negotiations.

Property Rights

- ***Advantages of the Coase Theorem***:
  - Only the parties involved need to know the marginal benefits and costs associated with the externality
  - The regulator only needs to assign and enforce property rights; it does not need to know $\pi(\cdot)$ or $v(\cdot)$.

Property Rights

- ***Remark***:
  - If the two parties are firms (e.g., a fishery and a refinery), a form of bargaining could be the sale of one firm to the other, i.e., a merger.
  - This is efficient as now the merged firm would internalize the effects that pollution imposes on the production process of the fishery.

Solutions to the Externality Problem:

More Intrusive Approaches

Quota

- Setting a quota (emission standard) that bans externality (emission) levels above x^0.
- The regulator must know the benefits and damages of the externality for all firms and consumers.

Pigouvian Taxation

- This policy sets a tax t_x per unit of the externality-generating activity x.
- What is the level of tax t_x that restores efficiency?
- Let us start by rewriting the firm's PMP

$$\max_{x \geq 0} \ \pi(x) - t_x \cdot x$$

Pigouvian Taxation

- FOC with respect to x:

$$\pi'(x) - t_x \leq 0, \qquad \text{or} \quad \pi'(x) \leq t_x$$

  with equality if $x > 0$.

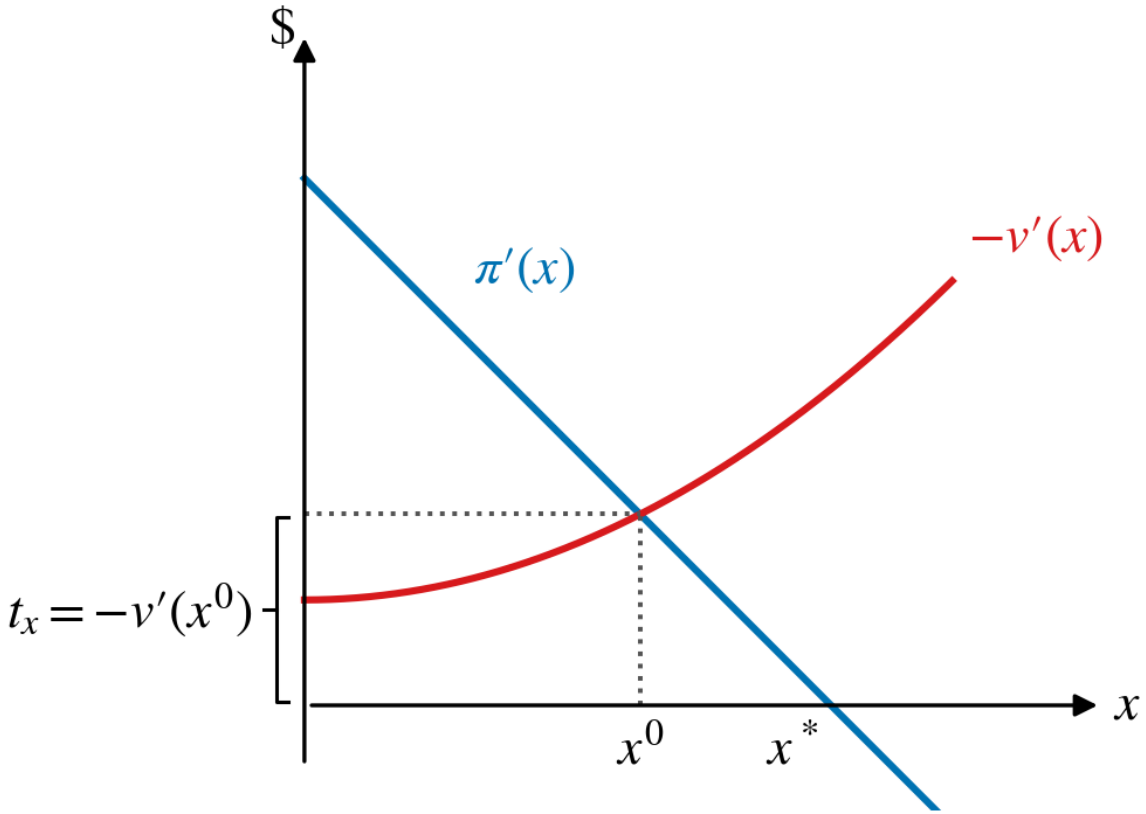
- *Intuition*: the firm increases x until the point where the marginal benefit from an additional unit of x coincides with the per-unit tax t_x.

Pigouvian Taxation

- We know that at the social optimum (i.e., x^0)
$$\pi'(x^0) = -v'(x^0)$$
- Hence, the tax t_x needs to be set at
$$t_x = -v'(x^0)$$
- This forces the firm to internalize the damage that the externality generates on the consumer's well-being at x^0.
- Since $\pi'' < 0$, $\pi'(x) = t_x$ has a unique solution, so $t_x = -v'(x^0)$ induces $x = x^0$.

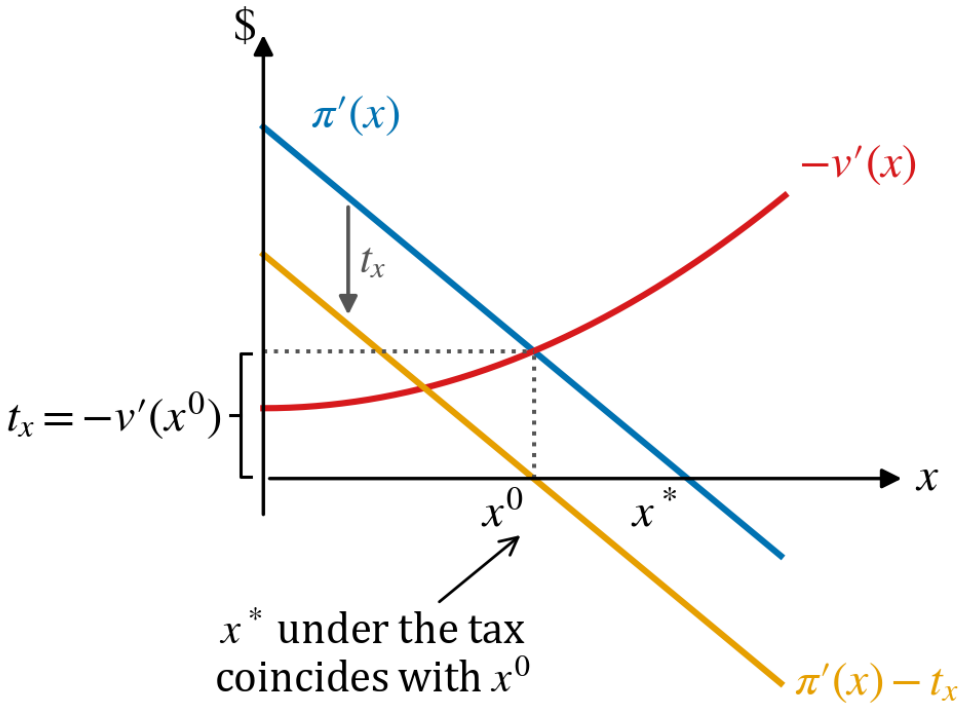
Pigouvian Taxation

- The tax t_x leads the firm to choose a level of x equal to x^0



Pigouvian Taxation

- The tax produces a downward shift in $\pi'(x)$.
- The new marginal benefit curve $\pi'(x) - t_x$ crosses the horizontal axis at x^0.

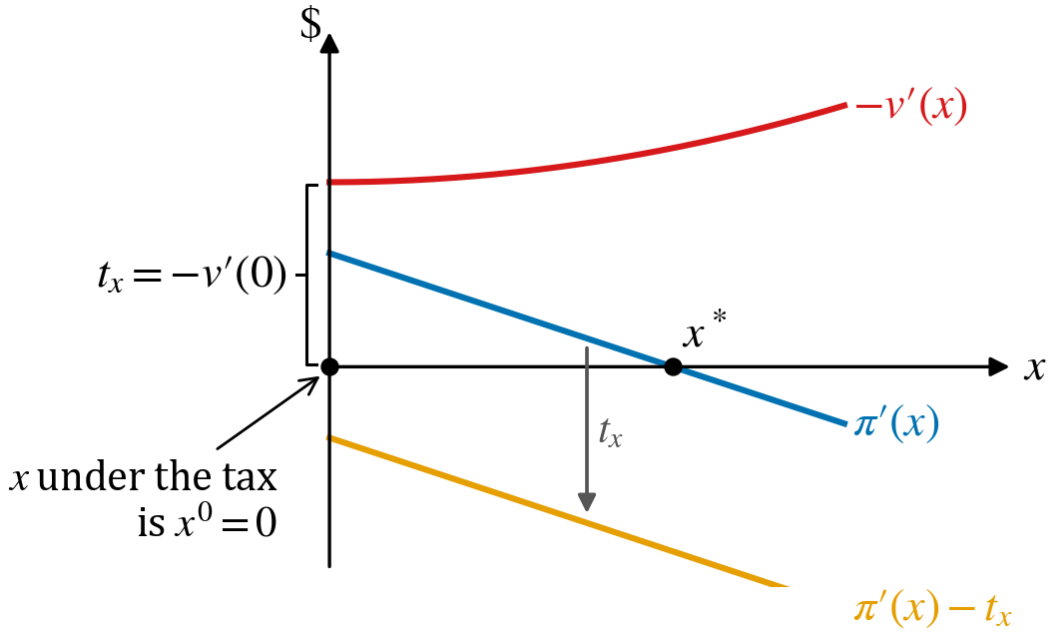


Pigouvian Taxation

- The optimality-restoring tax t_x is equal to the marginal damage at the optimal level x^0.
  - That is, it is equal to the amount of money that the affected individual would be willing to pay in order to reduce x slightly from its optimal level x^0.
- The tax t_x induces the firm to internalize the externality that it causes on the individual.

Pigouvian Taxation

- If the negative externality is large enough that $-v'(0) \geq \pi'(0)$ (see the corner case above), the social optimum is at $x^0 = 0$, and the tax $t_x = -v'(0)$ induces $x = 0$ (any $t_x \geq \pi'(0)$ does).



Pigouvian Subsidy

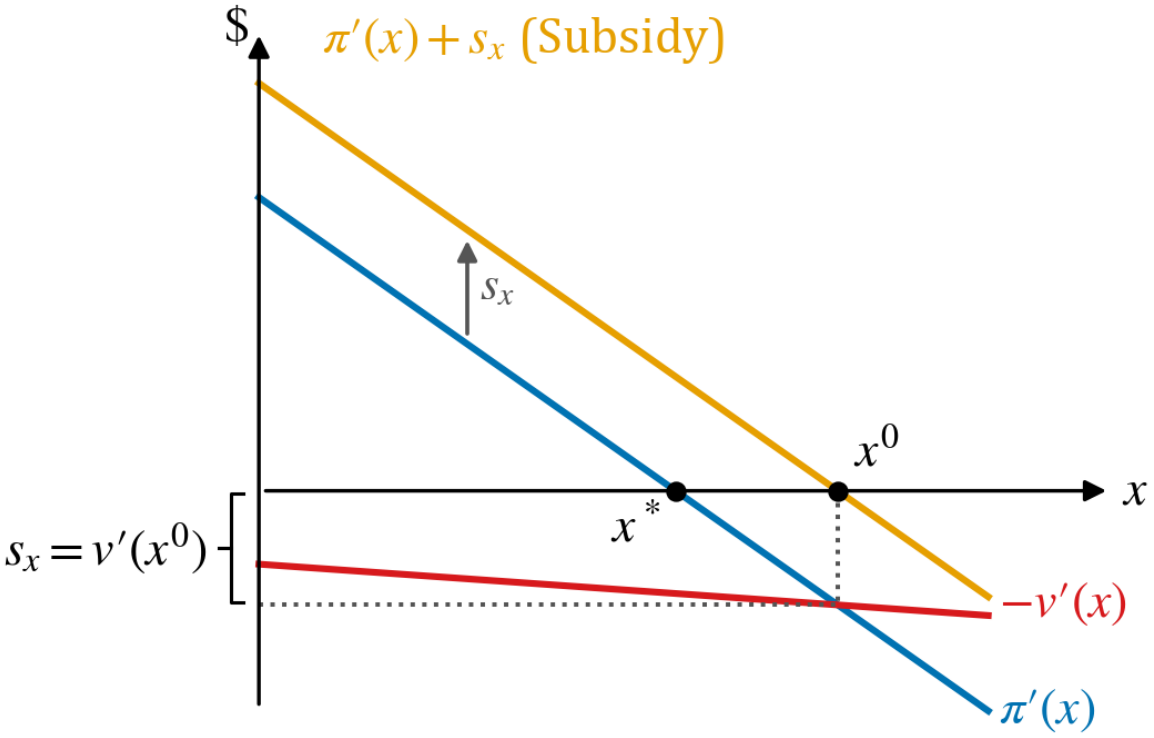
- This analysis extends to *positive externalities*.
- Since now $v'(x^0) > 0$ (i.e., x increases the individual's welfare), the optimality-restoring tax is

$$t_x = -v'(x^0) < 0$$

- We thus set "negative taxes" on the externality: a per-unit subsidy (s_x).
- The firm receives a payment of $s_x = -t_x = v'(x^0) > 0$ for each unit of the positive externality it generates.

Pigouvian Subsidy

- The per-unit subsidy produces an upward shift in the marginal benefits of the firm.
- The firm has incentives to increase x beyond the competitive equilibrium level x^* until reaching the Pareto optimal level x^0.



Pigouvian Policy: Important Points

a) A tax on the negative externality is equivalent to a subsidy inducing agents to reduce the externality.

- Consider a subsidy $s_x = -v'(x^0) > 0$ for every unit that the firm's choice of x is below x^*.
- The firm's PMP becomes:

$$\max_{x \geq 0} \pi(x) + s_x(x^* - x) = \pi(x) + \underbrace{s_x x^*}_{\text{lump-sum transfer}} - \underbrace{s_x x}_{\text{per-unit tax}}$$

Pigouvian Policy: Important Points

a) A tax on the negative externality is equivalent to a subsidy inducing agents to reduce the externality.

– FOC with respect to x yields

$$\pi'(x) - s_x \leq 0$$

with equality if $x > 0$, which holds at $x = x^0$ since $s_x = -v'(x^0)$.

Pigouvian Policy: Important Points

– This objective equals the Pigouvian-tax objective plus a lump-sum transfer $t_x x^*$ to the firm (with $s_x = t_x$), so the FOC is the same:

$$\pi(x) + s_x(x^* - x) = [\pi(x) - t_x x] + t_x x^*$$

– Hence, a subsidy for the reduction of the externality (combined with a lump-sum tax $t_x x^*$) can replicate the outcome of the Pigouvian tax.

Pigouvian Policy: Important Points

b) The Pigouvian tax levies a tax on the externality-generating activity (e.g., pollution) but not on the output that generated such pollution.

- Taxing output might lead the firm to reduce output, but it does not guarantee a reduction in pollutant emissions.
- A tax on output is equivalent to a tax on emissions only if emissions are proportional to output (as in the example with $x = \alpha y$).

Pigouvian Policy: Important Points

c) *The quota and the Pigouvian tax are equally effective under complete information.*

- They might not be equivalent when regulators face incomplete information about the benefits and costs of the externality for consumers and firms.

Solutions to the Externality Problem:

Tradable Externality Permits

Tradable Externality Permits

- Consider J firms and I consumers, where consumer i's utility is $v_i(x)$ with $x = \sum_j x_j$.
- Every permit grants the right to generate one unit of the externality.
- Suppose that $x^0 = \sum_j x_j^0$ total permits are given to the firms, with every firm receiving $\overline{x}_j$ of them.
- Let p_x^* denote the equilibrium price of these permits.

Tradable Externality Permits

- Firm j's PMP is

$$\max_{x_j \geq 0} \pi_j(x_j) + p_x^*(\overline{x}_j - x_j)$$

where firm j must pay a price p_x^* for every permit it needs to buy in excess of its initial endowment $\overline{x}_j$, and receives p_x^* for every permit it sells.

Tradable Externality Permits

- FOC wrt x_j yields

$$\pi_j'(x_j) - p_x^* \leq 0$$

with equality if $x_j > 0$ (interior solution).

- If all J firms carry out this PMP, we need the market clearing condition

$$\sum_j x_j = \sum_j \overline{x}_j = x^0$$

Tradable Externality Permits

- With the supply of permits fixed at x^0, the market-clearing price satisfies

$$p_x^* = -\sum_{i=1}^{I} v_i'(x^0)$$

Tradable Externality Permits

- Thus, firm j's FOCs become

$$\pi_j'(x_j^0) + \sum_{i=1}^{I} v_i'(x^0) \leq 0$$

  with equality if $x_j^0 > 0$ (interior solution).
- This condition coincides with the FOC that solves the social planner's problem.

Tradable Externality Permits

- The social planner solves

$$\max_{(x_1,\ldots,x_J)} \sum_j \pi_j(x_j) + \sum_i v_i\left(\sum_j x_j\right)$$

with FOC

$$\pi_j'(x_j) + \sum_i v_i'(x) \leq 0$$

with equality if $x_j > 0$.

- Therefore, every firm j is induced to voluntarily choose $x_j = x_j^0$.

Tradable Externality Permits

- The advantage of tradable externality permits:
  - Government officials need less information.
  - Relative to firm-specific quotas, the regulator only needs x^0, not each firm's π_j.
    - Specifically, data on aggregate industry profits and damages from the externality
    - But not on individual firms and consumers

Regulating a Polluting Monopolist

Regulating a Polluting Monopolist

- Consider a monopolist facing a linear inverse demand curve $p(x) = 1 - x$ and constant marginal production costs $0 \leq c < 1$.
- Each unit of output generates one unit of emissions, which cause an environmental damage $ED(x) = dx^2$, where $d > 0$.
  - Damage is convex in output.

Regulating a Polluting Monopolist

- The regulator charges an emission fee t per unit of emissions.
- The social planner seeks x_{SO} that maximizes

$$W(x) = CS(x) + PS(x) + T - ED(x)$$

where $CS(x)$ and $PS(x)$ denote consumer and producer surplus, respectively, and T represents tax revenue from the emission fee.

Regulating a Polluting Monopolist

- Let us first find the monopolist's profit-maximizing level of x for a given emission fee t.
- The PMP of the monopolist is

$$\max_{x \geq 0} (1 - x)x - (c + t)x$$

- FOC: $1 - 2x - (c + t) = 0$
- Solving for x yields an output function

$$x(t) = \frac{1 - (c + t)}{2}, \qquad x'(t) = -\frac{1}{2} < 0$$

where $x(t) > 0$ requires $t < 1 - c$.

Regulating a Polluting Monopolist

- The social planner solves

$$\max_{x \geq 0} W(x) = CS(x) + PS(x) + T - ED(x)$$

where $CS(x) \equiv \frac{1}{2}x^2$, $PS(x) \equiv (1-x)x - (c+t)x$, $T \equiv tx$, and $ED(x) \equiv dx^2$.

- T and the $-tx$ in $PS(x)$ cancel, so $W(x)$ does not depend on t directly.

Regulating a Polluting Monopolist

- FOC:

$$2\frac{1}{2}x + [1 - 2x - (c + t)] + t - 2dx = 0$$

or, rearranging, $1 - c = x(1 + 2d)$.

- The SOC holds, since $-(1 + 2d) < 0$.
- Solving for x yields

$$x_{\text{SO}} = \frac{1 - c}{1 + 2d}$$

Regulating a Polluting Monopolist

- The emission fee t that induces the monopolist to produce x_{SO} is found by solving

$$\frac{1-(c+t)}{2} = \frac{1-c}{1+2d}$$

which yields

$$t = (2d-1)\frac{1-c}{1+2d} \quad \text{or} \quad t = (2d-1)x_{SO}$$

Regulating a Polluting Monopolist

- Notice that
  - $t > 0$ (a tax) iff $d > 1/2$
  - $t < 0$ (a subsidy) iff $d < 1/2$
  - $t = 0$ iff $d = 1/2$
- The fee can be decomposed as

$$t = ED'(x_{\mathrm{SO}}) - x_{\mathrm{SO}} = 2dx_{\mathrm{SO}} - x_{\mathrm{SO}}$$

that is, the Pigouvian term minus a correction for the monopolist's market power.

Regulating a Polluting Monopolist

- *Intuition*:
  - If the market failure arising from the environmental externality is sufficiently large, i.e., $d > 1/2$, the regulator imposes a tax policy in order to reduce the production from the polluting monopolist.
  - If, in contrast, the market failure from the externality is smaller, i.e., $d < 1/2$, the regulator subsidizes the monopolist's production.

Regulating a Polluting Monopolist

- *Intuition* (continued):
  - The monopolist underproduces, $x_m = \frac{1-c}{2}$, while pollution calls for lower output.
  - Since

$$x_m - x_{\mathrm{SO}} = \frac{(2d-1)(1-c)}{2(1+2d)}$$

the monopolist overproduces relative to x_{SO}, and is taxed, if and only if $d > 1/2$.

Regulating a Polluting Oligopoly

Regulating a Polluting Oligopoly

- Consider a Cournot oligopoly. Same demand, damage $ED(X) = dX^2$ and welfare as in the monopoly case, now with an incumbent and an entrant, where $X \equiv x_{inc} + x_{ent}$ is aggregate output.
- Assume that the incumbent can
  - enjoy a cost advantage relative to the entrant, i.e., $c_{inc} < c_{ent} < 1$, or
  - face the same production costs, i.e., $c_{inc} = c_{ent} < 1$.
- We assume $d > \frac{1}{2}$ when comparing with the monopoly fee, so that both fees are positive.

Regulating a Polluting Oligopoly

- Let us first find the oligopolists' profit-maximizing output levels for a given emission fee t.
- The PMP of the incumbent is

$$\max_{x_{inc}} \ (1 - x_{inc} - x_{ent})x_{inc} - (c_{inc} + t)x_{inc}$$

  while that of the entrant's

$$\max_{x_{ent}} \ (1 - x_{ent} - x_{inc})x_{ent} - (c_{ent} + t)x_{ent}$$

- FOCs wrt x_i, where $i \in \{inc, \ ent\}$, yields

$$1 - 2x_i - x_j - (c_i + t) = 0 \ \text{ for } \ j \neq i$$

Regulating a Polluting Oligopoly

- Solving for x_i yields firm i's BRF

$$x_i(x_j) = \frac{1-(c_i+t)}{2} - \frac{1}{2}x_j$$

- Plugging firm j's BRF into firm i's, we obtain

$$x_i = \frac{1-(c_i+t)}{2} - \frac{1}{2} \cdot \left(\frac{1-(c_j+t)}{2} - \frac{1}{2}x_i \right)$$

- Solving for x_i yields an equilibrium output function

$$x_i(t) = \frac{1-2c_i+c_j-t}{3}$$

which is decreasing in c_i and t, but increasing in c_j.

Regulating a Polluting Oligopoly

- The social planner solves

$$\max_{X \geq 0} \; W(X) = CS(X) + PS(X) + T - ED(X)$$

where $X \equiv x_{inc} + x_{ent}$, $CS(X) \equiv \frac{1}{2}(X)^2$, $PS(X) \equiv (1 - X)X - (c_{inc} + t)X$, $T \equiv tX$, and $ED(X) \equiv d(X)^2$.

Regulating a Polluting Oligopoly

- FOC:

$$\frac{\partial W}{\partial X} = X + (1 - 2X) - c_{inc} - 2dX = 0$$

(the fee cancels, since $T = tX$ is a transfer), so

$$X_{SO} = \frac{1 - c_{inc}}{1 + 2d}$$

  - X_{SO} is the first-best output, produced at the lowest marginal cost c_{inc}.

Regulating a Polluting Oligopoly

- The emission fee t that induces the incumbent and entrant to produce $x_{inc,SO}$ and $x_{ent,SO}$, respectively, is found by simultaneously solving

$$x_{inc,SO} + x_{ent,SO} = \frac{1-c_{inc}}{1+2d}$$

$$x_{inc}(t) = \frac{1-2c_{inc}+c_{ent}-t}{3}$$

$$x_{ent}(t) = \frac{1-2c_{ent}+c_{inc}-t}{3}$$

Regulating a Polluting Oligopoly

- Two cases:
  - Case 1, cost symmetry ($c_{inc} = c_{ent}$): this system pins down t.
  - Case 2, cost asymmetry ($c_{inc} < c_{ent}$): a single fee cannot induce first-best production, so the regulator chooses t to maximize welfare evaluated at the equilibrium outputs $x_i(t)$, with each firm producing at its own cost (second-best).

Regulating a Polluting Oligopoly

- ***Case 1: Cost symmetry,*** $c_{inc} = c_{ent}$
  - Simultaneously solving the three equations yields

$$t = \frac{4d-1}{2}\frac{1-c_{inc}}{1+2d} \quad \text{or} \quad t = (4d-1)\frac{X_{SO}}{2}$$

where $t > 0$ iff $d > 1/4$.

- The regulator imposes emission fees on the oligopoly even in settings where a monopolist would receive a subsidy, i.e., $d \in \left(\frac{1}{4}, \frac{1}{2}\right)$.

  - Substituting t into the output function $x_i(t)$ yields

$$x_{inc,SO} = x_{ent,SO} = \frac{1}{2} \cdot \frac{1-c_{inc}}{1+2d} = \frac{X_{SO}}{2}$$

Regulating a Polluting Oligopoly

- ***Case 2: Cost asymmetry,*** $c_{inc} < c_{ent}$
  - Plugging $X(t) = \frac{2(1-t)-c_{inc}-c_{ent}}{3}$ into W (with each firm's own cost) and differentiating with respect to t yields

$$t = \frac{(2-c_{inc}-c_{ent})(4d-1)}{4A}$$

where $A \equiv 1 + 2d$ and $t > 0$ iff $d > \frac{1}{4}$.

Regulating a Polluting Oligopoly

- ***Case 2: Cost asymmetry (continued)***
  - Substituting this fee, t^*, into the output functions $x_i(t)$ yields the second-best outputs

$$x_{inc}(t^*) = \frac{2 - (3 + 4d)c_{inc} + (1 + 4d)c_{ent}}{4A}$$

$$x_{ent}(t^*) = \frac{2 - (3 + 4d)c_{ent} + (1 + 4d)c_{inc}}{4A}$$

  - The first is positive if and only if $c_{ent} > \frac{(3+4d)c_{inc}-2}{1+4d}$, and the second if and only if $c_{ent} < \frac{2+(1+4d)c_{inc}}{3+4d}$.

Regulating a Polluting Oligopoly

- Let us check whether the two conditions on c_{ent} bind.
- (1) The condition that guarantees $x_{inc}(t^*) > 0$, i.e., $c_{ent} > \frac{(3+4d)c_{inc}-2}{1+4d}$, holds for all $c_{inc} < c_{ent}$.
  - The cutoff has a negative vertical intercept, $-\frac{2}{1+4d}$, and reaches 1 at $c_{inc} = 1$, so it lies below the 45°-line for all $c_{inc} < 1$.
  - Hence, the cutoff is not binding.

Regulating a Polluting Oligopoly

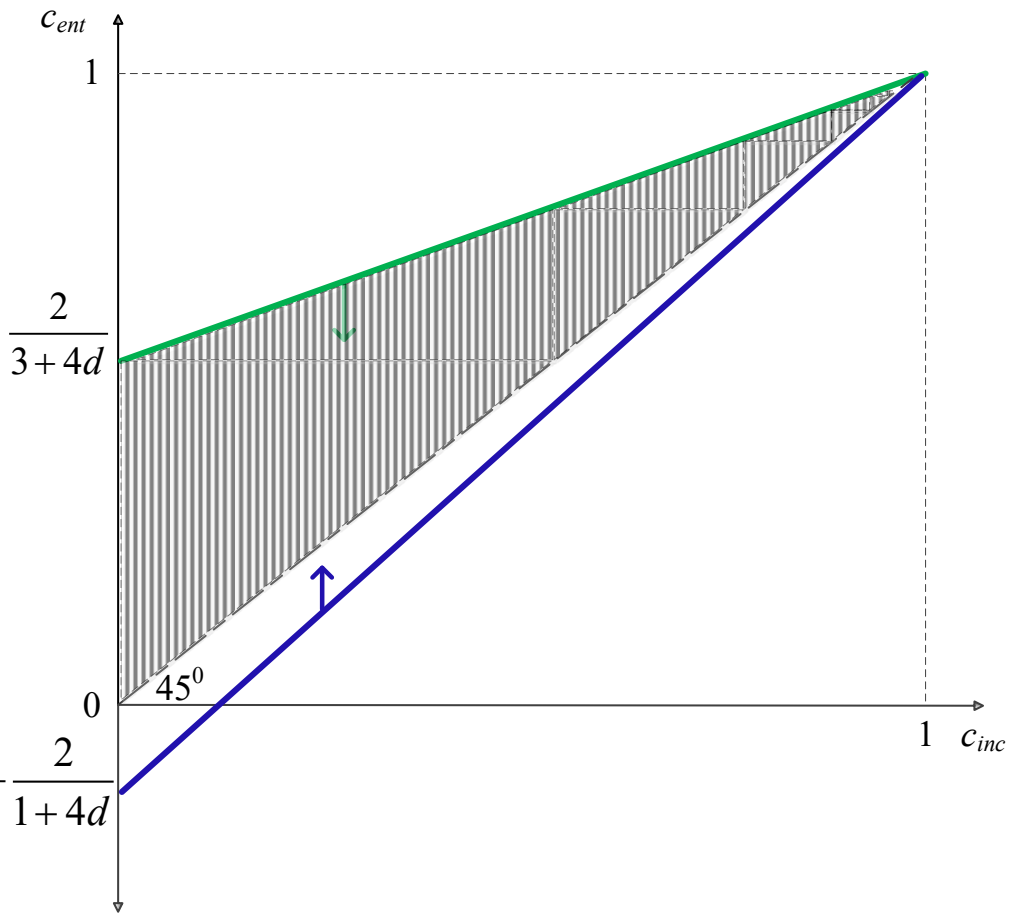
- (2) The condition that guarantees $x_{ent}(t^*) > 0$, i.e., $c_{ent} < \frac{2+(1+4d)c_{inc}}{3+4d}$, originates at $\frac{2}{3+4d}$, and reaches $c_{ent} = 1$ when $c_{inc} = 1$, thus lying above the 45°-line for all $c_{inc} < 1$.
  - Hence, the cutoff $c_{ent} < \frac{2+(1+4d)c_{inc}}{3+4d}$ is binding.

Regulating a Polluting Oligopoly

- For a positive emission fee that induces positive output from both firms, we need

$$c_{inc} < c_{ent} < \frac{2 + (1 + 4d)c_{inc}}{3 + 4d}$$

and $d > \frac{1}{4}$, where the entrant faces a higher, but not too much higher, cost than the incumbent.



Regulating a Polluting Oligopoly

- Fee comparison:
  - The regulator sets more stringent fees on the duopolists than to the monopolist
  - For the case of cost symmetry $c_{inc} = c_{ent}$,

$$(4d-1)\frac{X_{SO}}{2} > (2d-1)X_{SO}$$

$$\Leftrightarrow \; 4d-1 > 4d-2$$

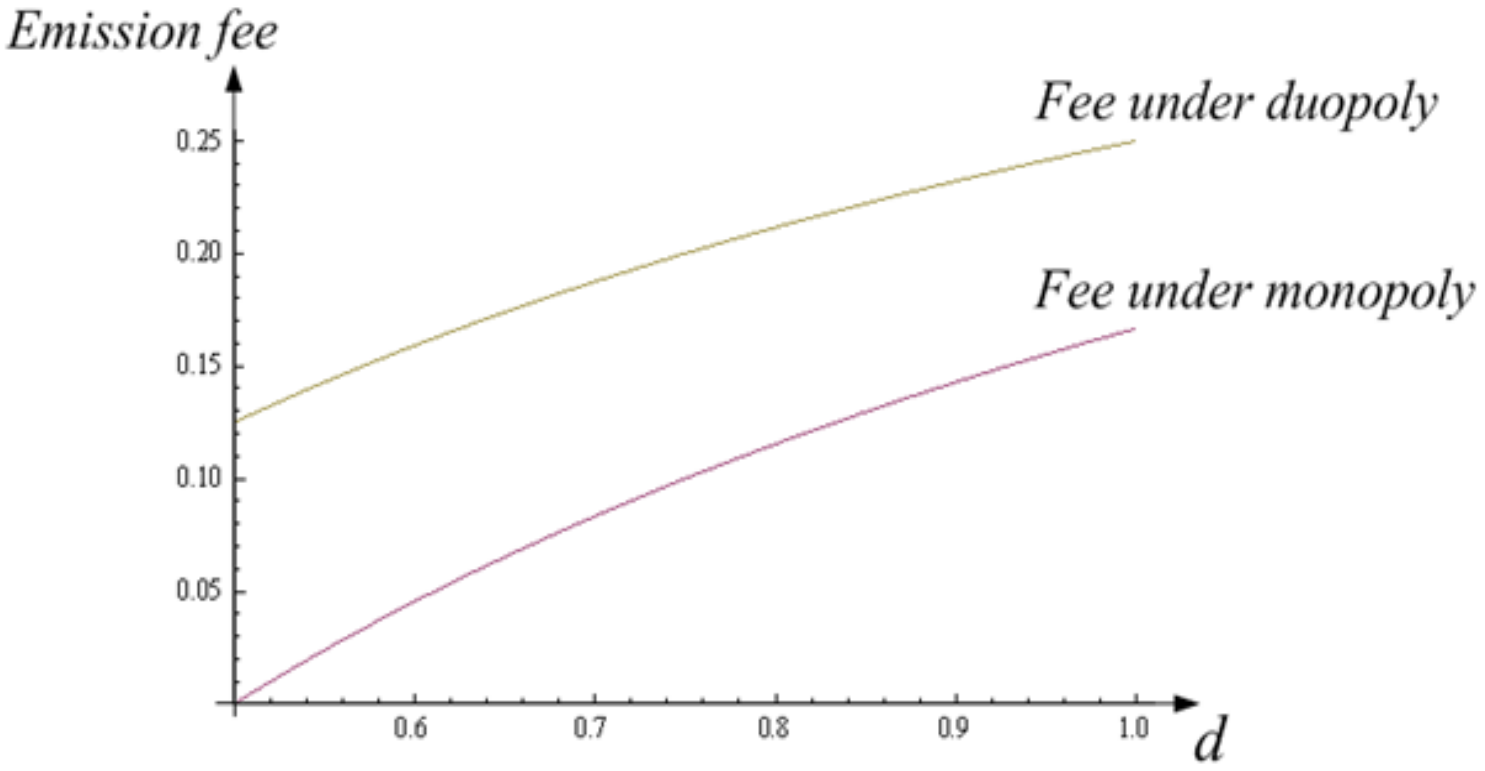
  - The same ranking holds when $c_{inc} < c_{ent}$ (compared with the monopoly fee at $c = c_{inc}$).

Regulating a Polluting Oligopoly

- Fee comparison:
  - *Intuition*:
    - The unregulated duopoly generates a larger amount of pollution than unregulated monopoly.
    - Hence, the regulator sets a more stringent fee on the former.

Regulating a Polluting Oligopoly

- Fee comparison ($c_{inc} = c_{ent} = \frac{1}{2}$):



Presence of Asymmetric Information in Externality Problems

Presence of Asymmetric Information in Externality Problems

- Consider a setting in which firms generate the externality whereas consumers are affected by that externality.
- Let $v(x, \eta)$ be the derived utility of a consumer of type $\eta \in \mathbb{R}$ from x amount of externality.
- Let $\pi(x, \theta)$ be the derived profit function of a firm of type $\theta \in \mathbb{R}$ which generates x amount of externality.
- Consider that parameters η and θ are privately observed by the consumer and firm, respectively.
- Functions $v(x, \eta)$ and $\pi(x, \theta)$ are strictly concave in the externality x for any value of η and θ.

Presence of Asymmetric Information in Externality Problems

- Agents do not observe each other's types, but know the ex-ante likelihoods of η and θ.
- For simplicity, we consider that parameters η and θ are independently distributed.

Presence of Asymmetric Information in Externality Problems

- Let us first consider the decentralized bargaining procedure.
- Bargaining in the presence of asymmetric information does not necessarily lead to an efficient level of the externality x^0.
- Suppose that the consumer has the right to an externality-free environment, and he makes a take-it-or-leave-it offer to the firm.
- Assume that there are two levels of the negative externality: $x = 0$ and $x = \bar{x}$
  - the consumer prefers $x = 0$ to $x = \bar{x}$, whereas the firm prefers $x = \bar{x}$ to $x = 0$.

Presence of Asymmetric Information in Externality Problems

- The benefits that a firm of type θ obtains from having an externality level $x = \bar{x}$ is

$$b(\theta) = \pi(\bar{x}, \theta) - \pi(0, \theta) > 0$$

- The costs that a consumer of type η bears from having an externality level $x = \bar{x}$ is

$$c(\eta) = v(0, \eta) - v(\bar{x}, \eta) > 0$$

Presence of Asymmetric Information in Externality Problems

- What matters in the negotiation between the consumer and the firm are the precise values of $b(\theta)$ and $c(\eta)$.
  - The CDFs of $b(\theta)$ and $c(\eta)$ are $G(b)$ and $F(c)$, respectively.
  - The PDFs of $b(\theta)$ and $c(\eta)$ are $g(b) > 0$ for all $b > 0$ and $f(c) > 0$ for all $c > 0$, respectively.

Presence of Asymmetric Information in Externality Problems

- In the absence of an agreement, the level of the externality remains at $x = 0$.
  - Consumer has the right to the resource

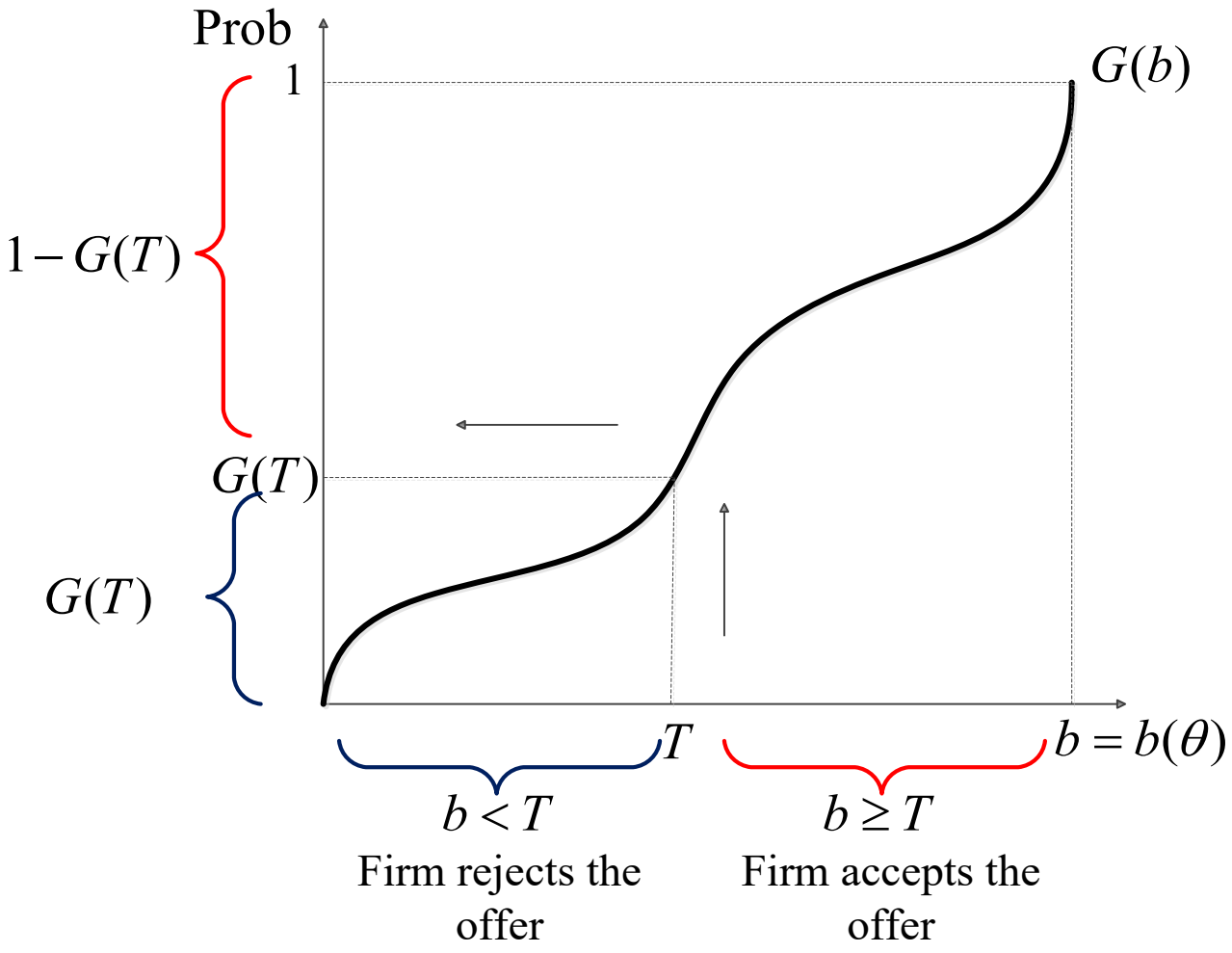
Presence of Asymmetric Information in Externality Problems

- *Pareto optimal outcome*: the firm should be allowed to set a level of the externality $x = \bar{x}$ whenever $b > c$.
  - Intuitively, the firm is willing to pay the consumer more than the damage that the consumer suffers from the externality.
  - Hence, $x = \bar{x}$ would be agreed upon by a firm and consumer if they were perfectly informed about each other's benefit b and cost c.

Presence of Asymmetric Information in Externality Problems

- Let us now start analyzing equilibrium strategies in this context.
- What amount should the consumer demand from the firm (a take-it-or-leave-it-offer) when his cost of the externality is $c(\eta) = c$?
- A θ-type firm will agree to pay T iff its benefits, $b(\theta) = b$, satisfy $b \geq T$.
- Hence, the consumer knows the probability of the firm accepting the payment of T is equal to the probability that $b \geq T$, i.e., $1 - G(T)$.

Presence of Asymmetric Information in Externality Problems



Presence of Asymmetric Information in Externality Problems

- Hence, the consumer chooses the value of the offer T that maximizes his expected net gain

$$\max_{T \geq 0} \; [1 - G(T)](T - c)$$

where

  - $1 - G(T)$ is the probability that an offer of T is accepted by the firm
  - $T - c$ is the net gain that a consumer (with cost c) obtains if the offer is accepted

Presence of Asymmetric Information in Externality Problems

- FOC wrt T yields

$$[1 - G(T_c^*)] - g(T_c^*)(T_c^* - c) \leq 0$$

and at an interior solution, $[1 - G(T_c^*)] = g(T_c^*)(T_c^* - c)$.

Presence of Asymmetric Information in Externality Problems

- Re-arranging,

$$\frac{1-G(T_c^*)}{g(T_c^*)} + c = T_c^*$$

- Since $1 - G(T_c^*) > 0$ and $g(T_c^*) > 0$, the ratio is positive, so $T_c^* > c$.

Presence of Asymmetric Information in Externality Problems

- This implies the firm rejects the consumer's offer when b satisfies $T_c^* > b > c$.
  - However, since $b > c$, Pareto optimality requires that the externality is increased until $x = \bar{x}$.
  - Hence, with probability $G(T_c^*) - G(c) > 0$, b lies in (c, T_c^*): the firm rejects the offer although agreement is efficient.

Presence of Asymmetric Information in Externality Problems

- Complete information:
  - The firm and the consumer are willing to bargain and have the externality produced when they are perfectly informed about their benefits and costs.
  - A welfare improvement for both parties.
- Asymmetric information:
  - The lack of information hinders the success of the mutually beneficial agreements.
  - Decentralized bargaining does not necessarily yield efficient outcomes.

Quotas under Incomplete Information

Quotas under Incomplete Information

- Unlike complete information settings:
  - Government intervention (quotas or taxes) does not necessarily achieve efficient outcomes when the agents are asymmetrically informed.
  - In addition, quotas or taxes are not perfect substitutes for one another.
- For given η and θ, the aggregate surplus resulting from externality level x is

$$v(x, \eta) + \pi(x, \theta)$$

Quotas under Incomplete Information

- The Pareto optimal level of the externality $x^0(\theta, \eta)$ solves

$$\max_{x \geq 0} \; v(x, \eta) + \pi(x, \theta)$$

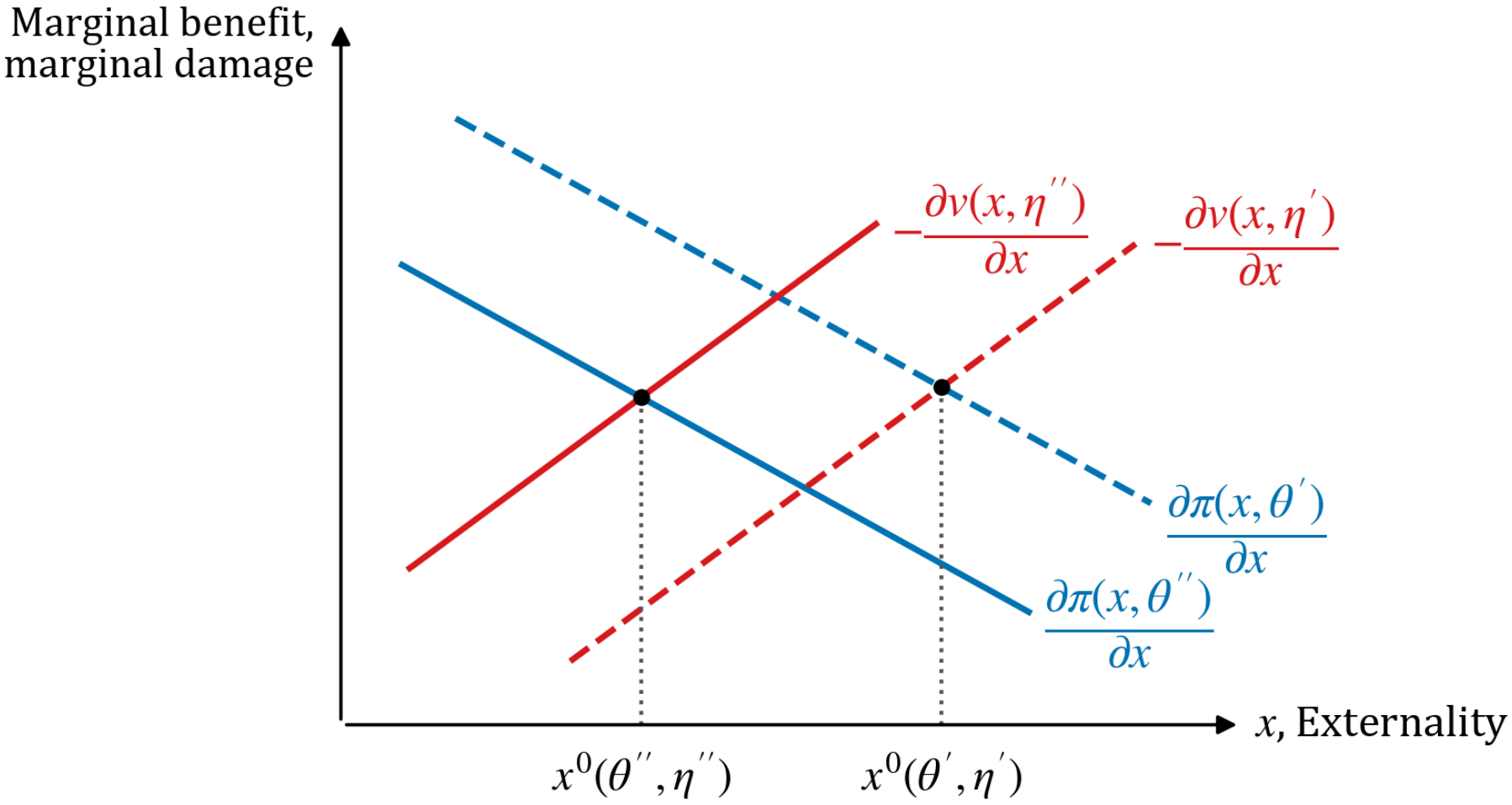
- FOC wrt x yields

$$\frac{\partial v(x,\eta)}{\partial x} + \frac{\partial \pi(x,\theta)}{\partial x} \leq 0$$

or, at an interior optimum,

$$\frac{\partial v(x,\eta)}{\partial x} + \frac{\partial \pi(x,\theta)}{\partial x} = 0$$

Quotas under Incomplete Information



Quotas under Incomplete Information

- Suppose that the regulator sets a quota $\hat{x}$ on the externality.
- The firm's PMP becomes

$$\max_{x \geq 0} \ \pi(x, \theta) \quad \text{s.t. } x \leq \hat{x}$$

- Let $x^q(\hat{x}, \theta)$ be the externality level that solves this PMP.
  - Since the PMP does not depend on η, $x^q(\hat{x}, \theta)$ is completely insensitive to η.
  - Thus, $x^q(\hat{x}, \theta)$ cannot coincide with $x^0(\theta, \eta)$ for every η.
  - The efficient level of externality is $x^0(\theta, \eta)$.

Quotas under Incomplete Information

- The regulator sets $\hat{x} = x^0(\overline{\theta}, \overline{\eta})$, where $\overline{\theta}$ and $\overline{\eta}$ are the expected values of θ and η.
- Assume the quota binds for every firm type, i.e., $\frac{\partial \pi(\hat{x},\theta)}{\partial x} > 0$ for all θ.
- Thus, the profit-maximizing level of the externality is $x^q(\hat{x}, \theta) = \hat{x}$.
- That is, the firm would like to increase the externality x beyond $\hat{x}$, but it cannot since it has already reached the quota.

Quotas under Incomplete Information

- The welfare change from quota $\hat{x}$ relative to the socially optimal level of externality $x^0(\theta,\eta)$ is

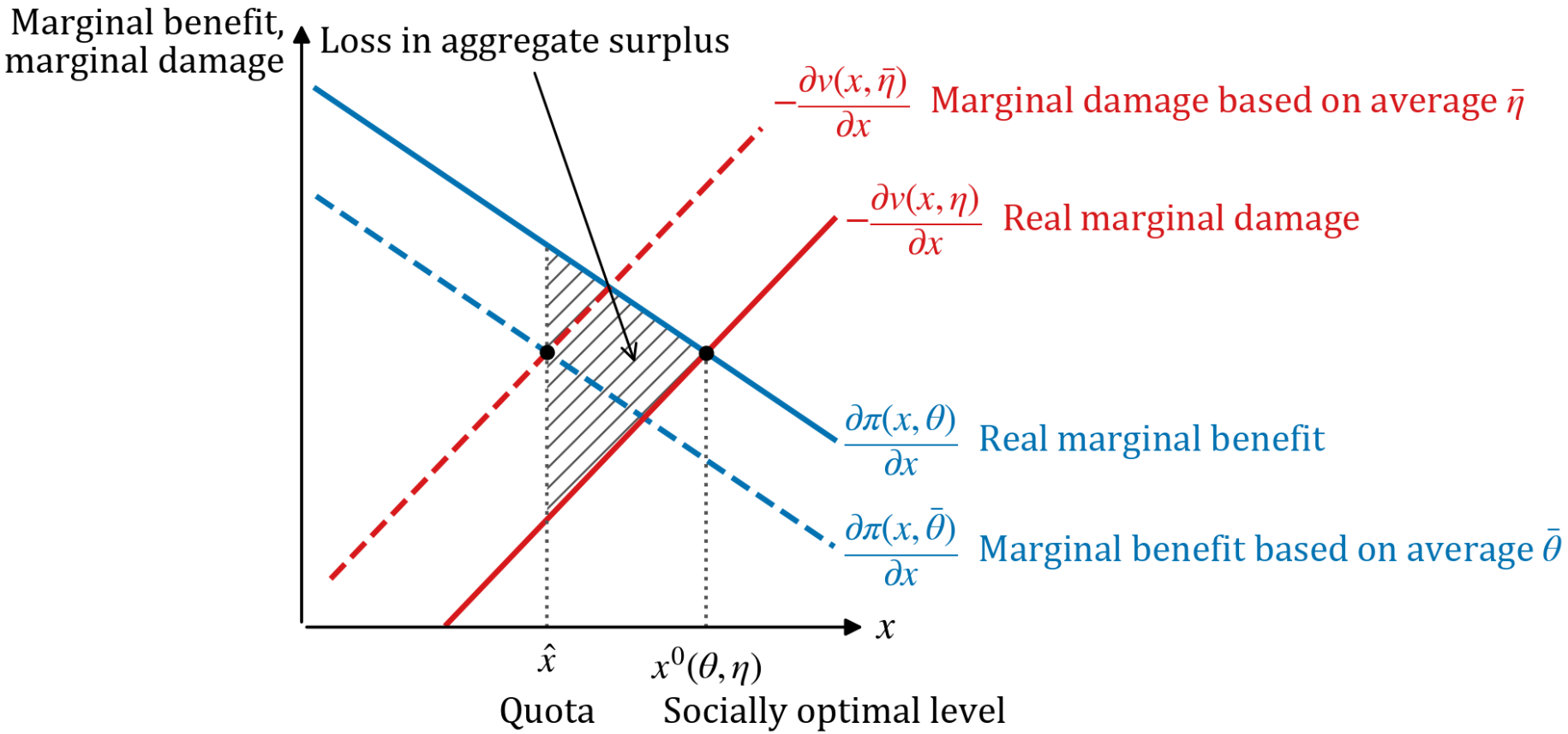
$$\overbrace{\left[v(x^q(\hat{x},\theta),\eta)+\pi(x^q(\hat{x},\theta),\theta)\right]}^{\text{Aggregate surplus with the quota } \hat{x}} - \underbrace{\left[v(x^0(\theta,\eta),\eta)+\pi(x^0(\theta,\eta),\theta)\right]}_{\text{Aggregate surplus at the PO}}$$

or, more compactly,

$$\int_{x^0(\theta,\eta)}^{x^q(\hat{x},\theta)} \left(\frac{\partial \pi(x,\theta)}{\partial x}+\frac{\partial v(x,\eta)}{\partial x}\right) dx$$

- This change is negative or zero, since $x^0(\theta,\eta)$ maximizes aggregate surplus.

Quotas under Incomplete Information



Taxes under Incomplete Information

Taxes under Incomplete Information

- Suppose that the regulator imposes a tax t per unit of the externality.
- The firm's PMP becomes

$$\max_{x \geq 0} \ \pi(x, \theta) - tx$$

- FOC yields $\frac{\partial \pi(x,\theta)}{\partial x} \leq t$ or, in an interior solution, $\frac{\partial \pi(x,\theta)}{\partial x} = t$.
- Let $x^t(t, \theta)$ denote the amount of the externality that solves the FOC (interior solution).
  - $x^t(t, \theta)$ is completely insensitive to η.
  - Thus, $x^t(t, \theta)$ cannot coincide with $x^0(\theta, \eta)$ for every η.

Taxes under Incomplete Information

- The welfare change caused by a tax t relative to the socially optimal level of externality $x^0(\theta, \eta)$ is

$$\overbrace{[v(x^t(t,\theta),\eta) + \pi(x^t(t,\theta),\theta)]}^{\text{Aggregate surplus with tax } t} - \underbrace{[v(x^0(\theta,\eta),\eta) + \pi(x^0(\theta,\eta),\theta)]}_{\text{Aggregate surplus at the PO}}$$

or, more compactly,

$$\int_{x^0(\theta,\eta)}^{x^t(t,\theta)} \left(\frac{\partial \pi(x,\theta)}{\partial x} + \frac{\partial v(x,\eta)}{\partial x} \right) dx$$

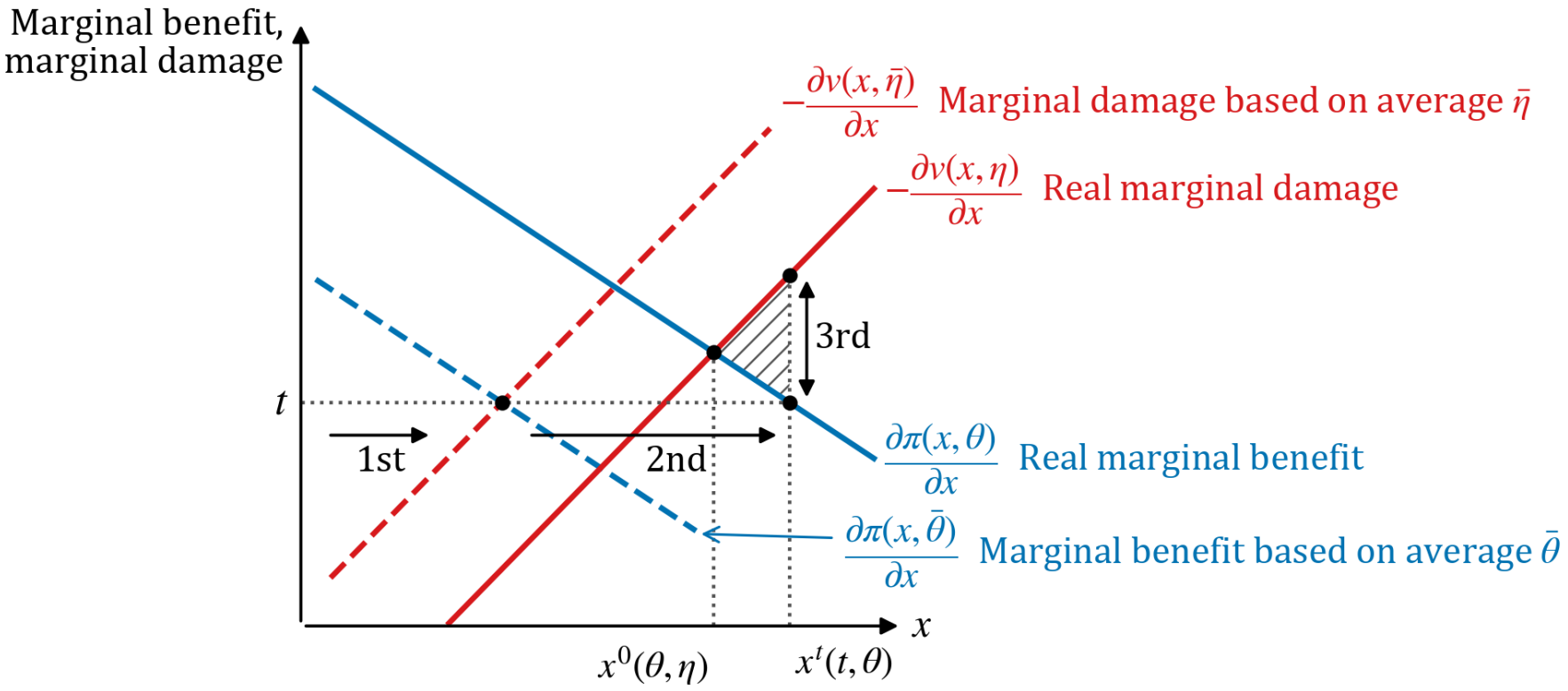
- This change is negative or zero, since $x^0(\theta, \eta)$ maximizes aggregate surplus.

Taxes under Incomplete Information

- The regulator sets the tax at the point that maximizes aggregate surplus, evaluated at the average values of θ and η, $(\overline{\theta}, \overline{\eta})$, that is

$$t = -\frac{\partial v(x^0(\overline{\theta}, \overline{\eta}), \overline{\eta})}{\partial x}$$

Taxes under Incomplete Information



1st: regulator sets t; 2nd: the firm responds along its true marginal benefit; 3rd: welfare loss

Comparing Policy Instruments under Incomplete Information

Policy Comparison

- Both quotas and emission fees create inefficiencies under incomplete information.
- Which instrument, despite being imperfect, performs better?
  - "second-best" policy
- It depends on the relative slopes of the marginal damage and marginal benefit functions.

Policy Comparison

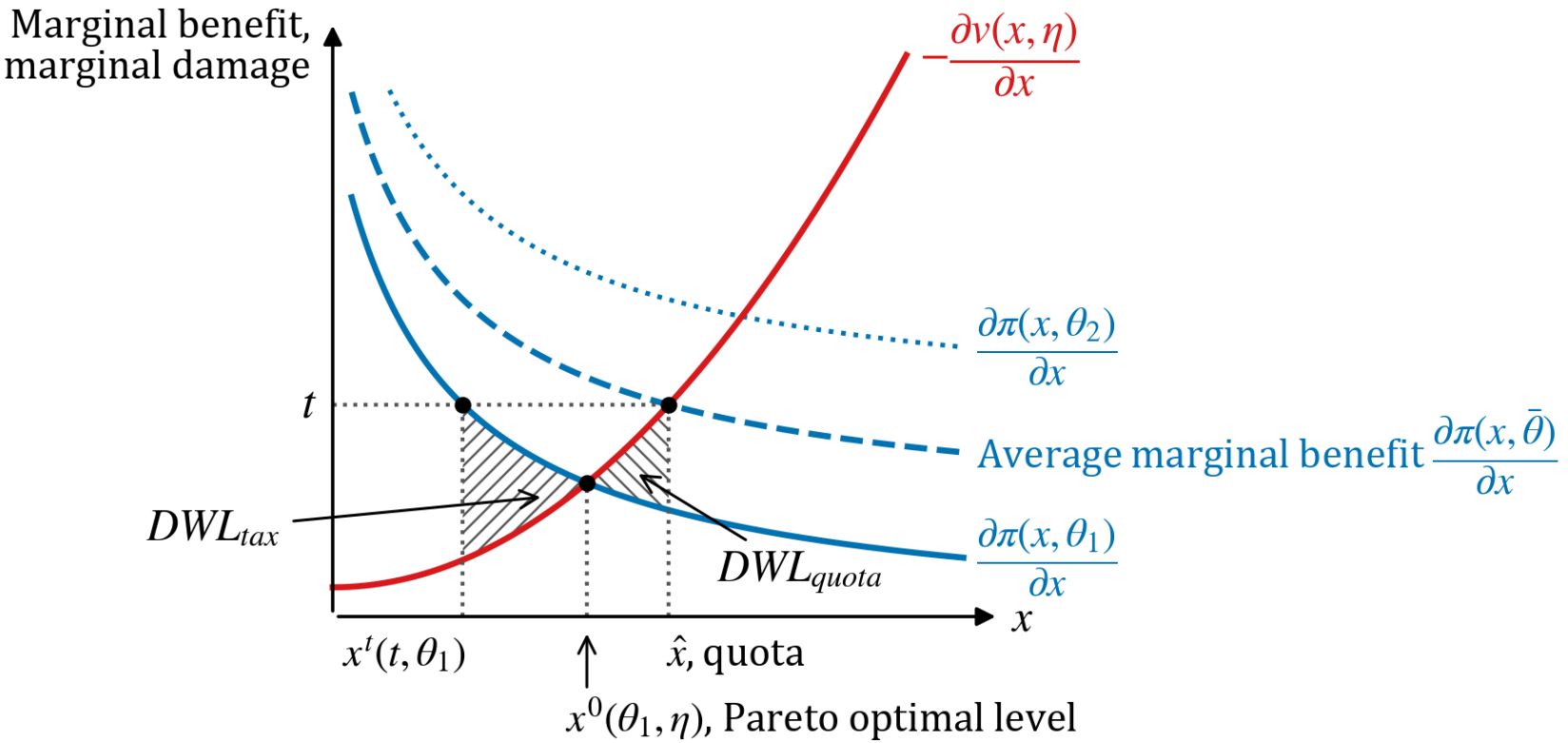
- Consider a setting where:
  - the realization of parameter θ is $\theta = \theta_1 < \overline{\theta}$ (true marginal benefit below its average)
  - the regulator has relatively precise information about the marginal damage function, but he is uncertain about the firm's marginal benefit function.

Policy Comparison

- The regulator sets:
  - a quota $\hat{x}$ at the point where the observed marginal damage function, i.e., $-\frac{\partial v(x,\eta)}{\partial x}$, crosses the average marginal benefit function, i.e., $\frac{\partial \pi(x,\overline{\theta})}{\partial x}$.
  - an emission fee t at the height at which the observed marginal damage function, i.e., $-\frac{\partial v(x,\eta)}{\partial x}$, crosses the average marginal benefit function, i.e., $\frac{\partial \pi(x,\overline{\theta})}{\partial x}$.

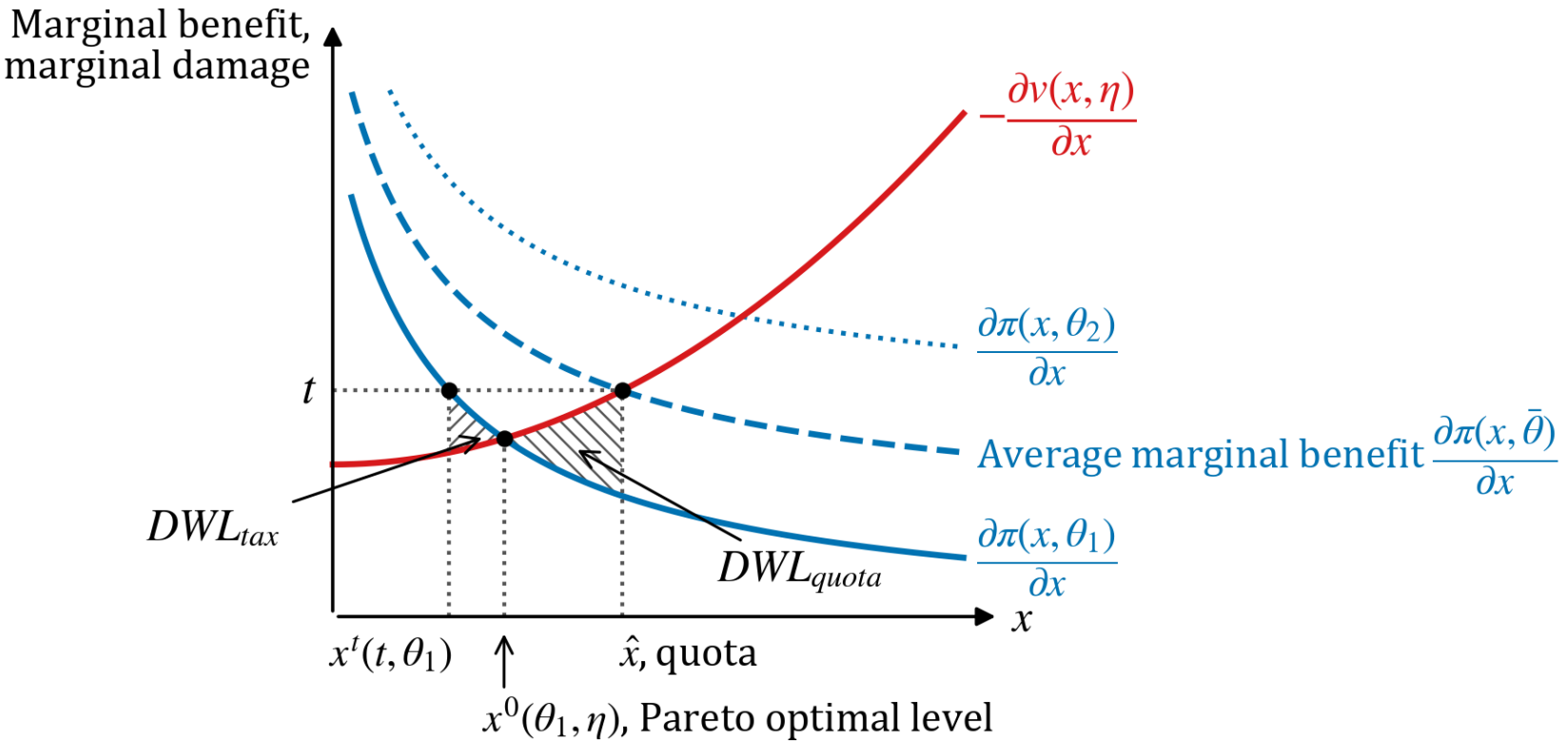
Policy Comparison

- The marginal damage function, i.e., $-\frac{\partial v(x,\eta)}{\partial x}$, is relatively sensitive to x.



Policy Comparison

- The marginal damage function, i.e., $-\frac{\partial v(x,\eta)}{\partial x}$, is not very sensitive to x.



Policy Comparison

- A quota performs better than an emission fee when the marginal damage curve is steeper (in absolute value) than the marginal benefit curve.
- The emission fee performs better when the marginal damage curve is flatter.

Tragedy of the Commons

Tragedy of the Commons

- "Tragedy of the Commons" considers a common pool resource (CPR, e.g., a fishing ground or an aquifer) exploited by $N \geq 2$ firms, and shows that firms' exploitation is socially excessive.
  - That is, the sum of the individual exploitation levels that each firm independently and simultaneously chooses exceeds the aggregate exploitation level that a social planner would select.

Tragedy of the Commons

- Consider a setting in which each firm exploiting the CPR chooses an individual appropriation level $x_i \geq 0$.
- In particular, every firm i's profit function is

$$\pi_i(x_i, X_{-i}) = p(X)x_i - c(x_i, X_{-i})$$

where $p(X)$ denotes the inverse demand function, with $p'(X) < 0$ and $p''(X) \leq 0$, and $X \equiv \sum_{i=1}^{N} x_i$ represents aggregate appropriation.

Tragedy of the Commons

- $c(x_i, X_{-i})$ represents firm i's cost of appropriating x_i units of the resource when its rivals extract $X_{-i} \equiv \sum_{j \neq i} x_j$, with

$$\frac{\partial c}{\partial x_i} > 0, \qquad \frac{\partial^2 c}{\partial {x_i}^2} \geq 0$$

- $\frac{\partial c}{\partial X_{-i}} > 0$ (scarcity: more appropriation by others raises i's cost), and
- $\frac{\partial^2 c}{\partial x_i \partial X_{-i}} > 0$, implying that the CPR becomes more scarce and more difficult to exploit by firm i.

Tragedy of the Commons

- In equilibrium, every firm i simultaneously and independently chooses its appropriation level x_i to maximize its profit $\pi_i(x_i, X_{-i})$.
- Taking FOC wrt x_i, and assuming an interior solution, we obtain

$$p(X^*) + p'(X^*)x_i^* = \frac{\partial c(x_i^*, X_{-i}^*)}{\partial x_i}$$

where $X^* = x_i^* + X_{-i}^*$.

Tragedy of the Commons

- Intuitively, the marginal revenue and marginal cost from individual appropriation x_i must coincide in equilibrium.
- Solving for x_i in the FOC above, we find firm i's BRF, $x_i^*(X_{-i}^*)$, describing firm i's appropriation as a function of its rivals' aggregate appropriation X_{-i}.

Tragedy of the Commons

- Consider a commons with:
  - linear inverse demand $p(X) = a - bX$, where $a, b > 0$; and
  - cost function $c(x_i, X_{-i}) = cx_i(1 + \alpha X_{-i})$, where $c > 0$, $a > c$, and $\alpha \geq 0$ represents how other firms' appropriation, X_{-i}, affects firm i's costs.
- The FOC for equilibrium appropriation becomes

$$a - b(x_i + X_{-i}) - bx_i = c(1 + \alpha X_{-i})$$

Tragedy of the Commons

- Solving for firm i's appropriation x_i yields a BRF of

$$x_i^*(X_{-i}) = \frac{a-c}{2b} - \frac{b+c\alpha}{2b} X_{-i}$$

which is decreasing in other firms' appropriation X_{-i}, thus reflecting that the appropriation levels of firms i and j are strategic substitutes; that is, as firm j increases its appropriation, firm i responds by decreasing its own.

Tragedy of the Commons

- In a symmetric equilibrium where $x_i^* = x_j^*$ for every pair of firms $i \neq j$, the preceding BRF becomes

$$x_i^* = \frac{a-c}{2b} - \frac{b+c\alpha}{2b}(N-1)x_i^*$$

since $X_{-i} = (N-1)x_i^*$. Solving for x_i^*,

$$x_i^* = \frac{a-c}{(N+1)b + c\alpha(N-1)}$$

which is decreasing in the number of firms N and in the external effect α.

Tragedy of the Commons

- For instance, when $N = 2$, this equilibrium appropriation becomes $x_i^* = \frac{a-c}{3b+c\alpha}$.
- When demand is perfectly elastic ($b = 0$), so firms are price takers with $p(X) = a$, and $\alpha > 0$, the FOC becomes $a = c(1 + \alpha X_{-i})$, so $X_{-i} = \frac{a-c}{c\alpha}$, and symmetry gives

$$x_i^* = \frac{a-c}{c\alpha(N-1)}$$

Tragedy of the Commons

- To more generally identify the conditions where the BRF of firm i, $x_i^*(X_{-i})$, decreases in X_{-i}, we differentiate its FOC wrt x_j,

$$\underbrace{\frac{\partial p(X^*)}{\partial x_j}}_{-} + \underbrace{\frac{\partial^2 p(X^*)}{\partial x_i \partial x_j}}_{+} \cdot x_i - \underbrace{\frac{\partial^2 c(x_i^*, X_{-i}^*)}{\partial x_i \partial x_j}}_{+}$$

  which is negative if

$$\frac{\partial^2 p(X^*)}{\partial x_i \partial x_j} \cdot x_i < -\frac{\partial p(X^*)}{\partial x_j} + \frac{\partial^2 c(x_i^*, X_{-i}^*)}{\partial x_i \partial x_j}$$

- In the special case of linear demand, $p(X) = a - bX$, $\frac{\partial p(X^*)}{\partial x_i} = -b$, implying that the cross-partial derivative is zero, $\frac{\partial^2 p(X^*)}{\partial x_i \partial x_j} = 0$, such that the inequality above unambiguously holds.

Tragedy of the Commons

- The expression above is the cross-partial $\frac{\partial^2 \pi_i}{\partial x_i \partial x_j}$.

  By the implicit function theorem,

$$\frac{dx_i}{dx_j} = -\frac{\frac{\partial^2 \pi_i}{\partial x_i \partial x_j}}{\frac{\partial^2 \pi_i}{\partial {x_i}^2}}$$

- Since the SOC holds, $\frac{\partial^2 \pi_i}{\partial {x_i}^2} < 0$, the best response decreases if and only if the cross-partial is negative.

Tragedy of the Commons

- We next show that equilibrium appropriation is excessive relative to the joint-profit benchmark.
  - This benchmark ignores consumer surplus; it captures the firms' own loss from overusing the resource.
- If all firms maximize their joint profits, their maximization problem becomes

$$\max_{x_1,\ldots,x_N} p(X)X - \sum_{i=1}^{N} c(x_i, X_{-i})$$

since $\sum_{i=1}^{N} p(X)x_i = p(X)X$.

Tragedy of the Commons

- Taking FOC wrt x_i,

$$p(X^{JP}) + p'(X^{JP})x_i^{JP} + p'(X^{JP})\sum_{j\neq i} x_j^{JP}$$

$$= \frac{\partial c(x_i^{JP}, X_{-i}^{JP})}{\partial x_i} + \sum_{j\neq i} \frac{\partial c(x_j^{JP}, X_{-j}^{JP})}{\partial x_i}$$

for every firm i and $j \neq i$.

- This FOC differs from the equilibrium condition in two ways:

Tragedy of the Commons

- It considers aggregate (rather than individual) marginal revenue on the left-hand side because firms now internalize the effect that selling more units has on the revenues of all other firms (see the third term on the left-hand side) rather than on their own revenues alone.
- It includes the increase in marginal costs that other firms experience as a result of a larger appropriation by firm i (i.e., a negative externality in costs).

Tragedy of the Commons

- Writing down the difference between firm i's FOC and the FOC from the joint maximization problem of all firms,

$$\left[p(X) + p'(X)x_i - \frac{\partial c(x_i, X_{-i})}{\partial x_i}\right]$$
$$-[p(X) + p'(X)x_i + p'(X)\sum_{j \neq i} x_j$$
$$-\frac{\partial c(x_i, X_{-i})}{\partial x_i} - \sum_{j \neq i} \frac{\partial c(x_j, X_{-j})}{\partial x_i}]$$

Tragedy of the Commons

- This difference simplifies to

$$-p'(X)\sum_{j\neq i} x_j + \sum_{j\neq i}\frac{\partial c(x_j, X_{-j})}{\partial x_i}$$

- Since $p'(X) < 0$ (by the law of demand) and $\frac{\partial c(x_j, X_{-j})}{\partial x_i} > 0$ (appropriation by firm i raises its rivals' costs), both terms in this expression are positive.

Tragedy of the Commons

- At any x, firm i's marginal profit then exceeds the joint marginal profit; with concave objectives, the Nash level exceeds the joint-profit level, $x_i^* > x_i^{JP}$ for every firm i.
- Such a result is often referred to as the "tragedy of the commons."

Tragedy of the Commons

- Consider the following two extreme cases:

1. When $\frac{\partial c(x_j, X_{-j})}{\partial x_i} = 0$, firm *i*'s appropriation does not increase its rivals' costs, but demand is negatively sloped, $p'(X) < 0$.
   - Then, the above problem coincides with that for the standard cartel, where firms collude to reduce their production and increase profits.

Tragedy of the Commons

- Consider the following two extreme cases:

2. When $p'(X) = 0$, firms take prices as given, but every firm *i*'s appropriation increases its rivals' costs, that is, $\frac{\partial c(x_j, X_{-j})}{\partial x_i} > 0$.
   - This may occur when many firms are selling the same product in the international market. In this case, the inverse demand function collapses to an exogenous price *p*, and the result $x_i^* \geq x_i^{JP}$ now indicates that the joint-profit solution internalizes the external effect that each firm's appropriation generates on its rivals' costs.

Tragedy of the Commons

- Return to the example above with $N = 2$.
- In this context, the joint profit maximization problem becomes

$$\max_{x_i, x_j} \left[a - b\left(x_i + x_j\right)\right] \cdot \left(x_i + x_j\right) - cx_i\left(1 + \alpha x_j\right) - cx_j(1 + \alpha x_i)$$

Tragedy of the Commons

- Taking FOC with respect to every x_i yields

$$a - 2b(x_i + x_j) - c(1 + 2\alpha x_j) \leq 0 \text{ for all } i \in \{1,2\} \text{ and } j \neq i$$

- Assuming an interior solution and solving for x_i, we obtain

$$x_i = \frac{a - c}{2b} - \frac{2(b + c\alpha)}{2b} x_j$$

Tragedy of the Commons

- By symmetry, we can simultaneously solve for x_i and x_j to obtain the joint-profit appropriation levels

$$x_i^{JP} = x_j^{JP} = \frac{a-c}{2(2b+c\alpha)}$$

which are lower than the equilibrium appropriation x_i^*, since

$$\frac{a-c}{2(2b+c\alpha)} < \frac{a-c}{3b+c\alpha}$$

simplifies to $-c\alpha < b$, which holds for $c, b > 0$ and $\alpha \geq 0$ by assumption.

Tragedy of the Commons

- For example, consider $a = b = 1$, $c = 1/2$, and $\alpha = 1/4$.
  - In this setting, $x_i^* = 0.16 > 0.1176 \approx x_i^{JP}$.

Tragedy of the Commons

- In both extreme cases described above, $x_i^{JP} < x_i^*$ holds:
  1. When $\alpha = 0$ but $b > 0$, since
  $$x_i^{JP} = \frac{a-c}{4b} < \frac{a-c}{3b} = x_i^*$$
  2. When $\alpha > 0$ but $b = 0$, which yields
     - $p = a$, and
     - $x_i^{JP} = \frac{a-c}{2c\alpha} < \frac{a-c}{c\alpha} = x_i^*$

Tragedy of the Commons

- Consider a setting with $n \geq 2$ symmetric firms, facing a market demand function $p(Q) = 1 - Q$, where Q is aggregate output.
- Each firm j has a convex cost function, $c(q_j) = \theta q_j^2$, where $\theta \geq 1$ is a cost parameter, unrelated to the firm type above.
- Here the externality is environmental damage (below), not a cost externality across firms.

Tragedy of the Commons

- The PMP of every firm j is

$$\max_{q_j \geq 0} (1 - q_j - q_{-j}) q_j - \theta q_j^2$$

where $q_{-j} = \sum_{k \neq j} q_k$ represents the aggregate output of all firms but j.

Tragedy of the Commons

- Taking FOC with respect to q_j :

$$1 - 2(1 + \theta)q_j - q_{-j} = 0$$

- Solving for q_j yields firm j's BRF

$$q_j(q_{-j}) = \frac{1}{2(1+\theta)} - \frac{1}{2(1+\theta)} q_{-j}$$

- Note:
  - If firm j were alone in the commons, $q_{-j} = 0$, it would produce

$$q_j(0) = \frac{1}{2(1+\theta)}$$

  - Firm j's output decreases as the aggregate output of other firms, q_{-j}, increases.

Tragedy of the Commons

- Since all firms are symmetric, in equilibrium

$$q_j = q_i = q^*$$

- Hence, the BRF of firm j can be written as

$$q^* = \frac{1}{2(1+\theta)} - \frac{n-1}{2(1+\theta)} q^*$$

which, solving for q^*, entails an equilibrium output of

$$q^* = \frac{1}{n+1+2\theta}$$

with equilibrium profits of

$$\pi^* = [1 - q^* - (n-1)q^*]q^* - \theta(q^*)^2 = \frac{1+\theta}{(n+1+2\theta)^2}$$

Tragedy of the Commons

- Aggregate output is

$$Q^* = nq^* = \frac{n}{n + 1 + 2\theta}$$

- Aggregate profits are

$$\Pi^* = n\pi^* = \frac{n(1 + \theta)}{(n + 1 + 2\theta)^2}$$

Tragedy of the Commons

- These profits reach their maximum when

$$\frac{\partial \Pi^*}{\partial n} = \frac{(1+\theta)(1+2\theta-n)}{(n+1+2\theta)^3} = 0$$

where solving for n yields $n^* = 1 + 2\theta$.

- Π^* increases in n for $n < 1 + 2\theta$ and decreases for $n > 1 + 2\theta$, so n^* maximizes industry profits.

Tragedy of the Commons

- Since the inverse demand function is linear, consumer surplus is

$$CS^* = \frac{(Q^*)^2}{2} = \frac{n^2}{2(n+1+2\theta)^2}$$

Tragedy of the Commons

- The exploitation of the commons entails a negative environmental externality (e.g., reduced biodiversity).
  - We consider a convex environmental damage function

$$ED = dQ^2,\ d > 0$$

  - Thus, the equilibrium aggregate environmental damage is

$$ED^* = d(Q^*)^2 = \frac{dn^2}{(n+1+2\theta)^2}$$

Tragedy of the Commons

- The resulting social welfare is

$$SW^* = CS^* + \Pi^* - ED^*$$

$$= \frac{n^2}{2(n+1+2\theta)^2} + \frac{n(1+\theta)}{(n+1+2\theta)^2} - \frac{dn^2}{(n+1+2\theta)^2}$$

- Consumer surplus increases in the number of firms n, i.e.,

$$\frac{\partial CS^*}{\partial n} = \frac{n(1+2\theta)}{(n+1+2\theta)^3} > 0$$

Tragedy of the Commons

- The derivative of $\Pi^* - ED^*$ with respect to n is

$$\frac{\partial[\Pi^* - ED^*]}{\partial n} = -\frac{(1+\theta)(n-1-2\theta)+2dn(1+2\theta)}{(n+1+2\theta)^3}$$

which is negative if $(1+\theta)(n-1-2\theta)+2dn(1+2\theta) > 0$, for example for all $n \geq 1+2\theta$.

Tragedy of the Commons

- *Intuition*: increasing the number of firms has two opposing effects on welfare:
  1) *Positive effect*: it increases consumer surplus (as a larger output entails lower prices);
  2) *Negative effects*: it reduces each firm's profit and, for $n > 1 + 2\theta$, industry profits AND generates more environmental damage.

Tragedy of the Commons

- The positive effect offsets the negative effects when $\frac{\partial SW}{\partial n} = 0$, which occurs when

$$\frac{n(1+2\theta)}{(n+1+2\theta)^3} = \frac{(1+\theta)(n-1-2\theta)+2dn(1+2\theta)}{(n+1+2\theta)^3}$$

Tragedy of the Commons

- Solving for n,

$$\overline{n} = \frac{(1+\theta)(1+2\theta)}{2d(1+2\theta)-\theta}$$

where $\overline{n} > 0$ requires $2d(1+2\theta) > \theta$.

- Comparing, $\overline{n} < n^*$, and solving for d, yields $d > \frac{1}{2}$.

Tragedy of the Commons

- *Intuition:*
  - If the environmental damage from exploitation is sufficiently large, i.e., $d > \frac{1}{2}$, the welfare-maximizing number of firms $\overline{n}$ is below the number that maximizes industry profits, n^*, and firm concentration would be welfare-improving.
  - The opposite applies when $d < \frac{1}{2}$.

Pollution Abatement

Pollution Abatement

- Let us analyze emission fees that induce firms to reduce their emissions in the least costly way.
  - *Examples*: using end-of-pipe technologies, redesigning their production process, or just reducing their output.
- We examine settings where environmental damages are:
  a) uniformly distributed
  b) non-uniformly distributed

Pollution Abatement: Uniform Pollutants

- Consider a regulator seeking to limit total pollution to a maximum level $\overline{x}$, so

$$\overline{x} \geq \sum_j x_j$$

- Firm j's production function is $y_j = f_j(z_j)$, where z_j is the vector of inputs, with prices w.

Pollution Abatement: Uniform Pollutants

- Firm j invests in abatement technology to limit its emissions to a given level x_j. The abatement function is

$$a_j(y_j, V_j) = x_j$$

where V_j denotes the use of abatement inputs, each with a price p_V.

Pollution Abatement: Uniform Pollutants

- Firm j can reach a target emission level x_j by
  - producing few units of output and using few abatement inputs (i.e., low y_j and V_j), or
  - producing a large amount of output but using large amounts of abatement inputs (i.e., high y_j and V_j).

- The planner below fixes each firm's output at $\overline{y}_j$.

  Fixing output rules out the output-reduction channel; we focus on abatement.

Pollution Abatement: Uniform Pollutants

- If a social planner had the ability to choose the use of production and abatement inputs across firms, i.e., $(z_1, \ldots, z_N)$ and $(V_1, \ldots, V_N)$, he would solve

$$\min_{z_j, V_j} \sum_j (w z_j + p_V V_j)$$

$$\text{s.t. } f_j(z_j) = \overline{y}_j,$$

$$\sum_j a_j(\overline{y}_j, V_j) \leq \overline{x}, \text{and}$$

$$V_j \geq 0 \text{ for every firm } j.$$

Pollution Abatement: Uniform Pollutants

- The Lagrangian of this constrained minimization problem is

$$\mathcal{L} = \sum_j \left(wz_j + p_V V_j\right) + \sum_j \lambda_j \left[\overline{y}_j - f_j(z_j)\right]$$
$$+\mu\left[\sum_j a_j(\overline{y}_j, V_j) - \overline{x}\right]$$

- FOC with respect to z_j yields

$$w = \lambda_j \frac{\partial f_j(z_j)}{\partial z_j} \text{ for every firm } j$$

Pollution Abatement: Uniform Pollutants

- FOC with respect to V_j yields

$$p_V = -\mu \frac{\partial a_j(\overline{y}_j, V_j)}{\partial V_j} \text{ for every firm } j$$

Pollution Abatement: Uniform Pollutants

- That is,
  - production input z_j should be increased until the cost of an additional unit, w, coincides with its marginal benefit in terms of additional production.
  - abatement input V_j should be increased until its cost, p_V, coincides with its marginal benefit, $-\mu \frac{\partial a_j(\overline{y}_j, V_j)}{\partial V_j}$.

Pollution Abatement: Uniform Pollutants

- For a cost-minimizing firm to voluntarily select V_j at the socially optimal level, we need to set a uniform emission fee on pollution, $t^* = \mu$, for every firm j.

Pollution Abatement: Uniform Pollutants

- Consider the cost minimization problem of firm j

$$\min_{z_j, V_j} wz_j + p_V V_j + t_j x_j$$

$$\text{s.t. } f_j(z_j) = \overline{y}_j \text{ and } a_j(\overline{y}_j, V_j) = x_j$$

which can be rewritten as

$$\min_{z_j, V_j} wz_j + p_V V_j + t_j a_j(\overline{y}_j, V_j)$$

$$\text{s.t. } f_j(z_j) = \overline{y}_j$$

Pollution Abatement: Uniform Pollutants

- Hence, the Lagrangian for firm j is

$$\mathcal{L} = wz_j + p_V V_j + t_j a_j(\overline{y}_j, V_j) + \lambda_j \left[\overline{y}_j - f_j(z_j)\right]$$

- FOC with respect to z_j yields

$$w = \lambda_j \frac{\partial f_j(z_j)}{\partial z_j}$$

which coincides with the planner's FOC when the fee equals the multiplier, $t_j = \mu$.

Pollution Abatement: Uniform Pollutants

- FOC with respect to V_j yields

$$p_V = -t_j \frac{\partial a_j(\overline{y}_j, V_j)}{\partial V_j}$$

which coincides with the FOC we found for the social planner if $t_j = \mu$.

Pollution Abatement: Uniform Pollutants

- Solving for t_j in the latter equation yields

$$t_j = -\frac{p_V}{\frac{\partial a_j(\overline{y}_j, V_j)}{\partial V_j}}$$

- For optimality, we need $t_j = \mu$. Thus, for every pair of firms j and k, $k \neq j$, we must have

$$-\frac{p_V}{\frac{\partial a_j(\overline{y}_j, V_j)}{\partial V_j}} = -\frac{p_V}{\frac{\partial a_k(\overline{y}_k, V_k)}{\partial V_k}}$$

Pollution Abatement: Uniform Pollutants

- Rearranging,

$$\frac{\partial a_k(\overline{y}_k, V_k)}{\partial V_k} = \frac{\partial a_j(\overline{y}_j, V_j)}{\partial V_j}$$

- That is, the marginal emission reduction from one more unit of abatement input is equal across firms at the social optimum.

Pollution Abatement: Non-Uniform Pollutants

- Consider now non-uniform pollution sources
  - *Example*: rivers.
- The amount of pollution measured at a particular measuring station, m_k, depends on:
  a) each firm's emissions x_j, and
  b) how pollution from a firm j transfers to the monitoring station located near firm k, as captured by d_{kj}.
- The measurement at station k, m_k, is given by

$$m_k = \sum_j d_{kj} x_j$$

Pollution Abatement: Non-Uniform Pollutants

- The regulator seeks to limit the measurement in each station k so it does not exceed a cutoff $\overline{m}_k$:

$$m_k = \sum_j d_{kj} x_j \leq \overline{m}_k$$

- The social planner problem is

$$\min_{z_j, V_j} \sum_j (w z_j + p_V V_j)$$

$$\text{s.t. } f_j(z_j) = \overline{y}_j \text{ for every firm } j,$$

$$\sum_j d_{kj} a_j(\overline{y}_j, V_j) \leq \overline{m}_k \text{ for every station } k$$

where $a_j(\overline{y}_j, V_j) = x_j$.

Pollution Abatement: Non-Uniform Pollutants

- The Lagrangian to this program is

$$\mathcal{L} = \sum_j \left(wz_j + p_V V_j\right) + \sum_j \lambda_j \left[\overline{y}_j - f_j(z_j)\right]$$
$$+ \sum_k \mu_k \left[\sum_j d_{kj} a_j(\overline{y}_j, V_j) - \overline{m}_k\right]$$

- FOC with respect to z_j yields

$$w = \lambda_j \frac{\partial f_j(z_j)}{\partial z_j} \text{ for every firm } j$$

Pollution Abatement: Non-Uniform Pollutants

- FOC with respect to V_j yields

$$p_V = -\sum_k \mu_k d_{kj} \frac{\partial a_j(\overline{y}_j, V_j)}{\partial V_j}$$

Pollution Abatement: Non-Uniform Pollutants

- In the case of two firms, the planner's FOC for firm 2 is

$$p_V = -\mu_1 d_{12} \frac{\partial a_2(\overline{y}_2, V_2)}{\partial V_2} - \mu_2 d_{22} \frac{\partial a_2(\overline{y}_2, V_2)}{\partial V_2}$$

- For firm 2's FOC to replicate the planner's, its fee must satisfy

$$-t_2 \frac{\partial a_2(\overline{y}_2, V_2)}{\partial V_2}$$
$$= -\mu_1 d_{12} \frac{\partial a_2(\overline{y}_2, V_2)}{\partial V_2} - \mu_2 d_{22} \frac{\partial a_2(\overline{y}_2, V_2)}{\partial V_2}$$

Pollution Abatement: Non-Uniform Pollutants

- Solving for t_2 yields

$$t_2 = \mu_1 d_{12} + \mu_2 d_{22}$$

- Hence, each firm faces its own fee $t_j = \sum_k \mu_k d_{kj}$, unlike the uniform fee $t^* = \mu$ under uniform pollution.

Pollution Abatement: Non-Uniform Pollutants

- The regulator can set taxes based on the pollution recorded at each monitoring point, m_k, rather than a single uniform emission fee.
- That is, firm j pays $\mu_k d_{kj}$ per unit of emissions for each station k, or $t_j = \sum_k \mu_k d_{kj}$ in total.

Public Goods

Public Goods

- Before defining public goods, let us define two properties:
  - ***Non-excludability***: If the good is provided, no consumer can be excluded from consuming it.
  - ***Non-rivalry***: Consumption of the good by one consumer does not reduce the quantity available to other consumers.

	Rivalrous	Non-rivalrous
Excludable	Private good	Club good
Non-excludable	Common property resource	Public good

Public Goods

- ***Private goods***, e.g., an apple. These goods are rival and excludable in consumption.
- ***Club goods***, e.g., a golf course. These goods are non-rival but excludable in consumption.
- ***Common property resources***, e.g., fishing grounds. These goods are rival but non-excludable in consumption.
- ***Public goods***, e.g., national defense. These goods are non-rival and non-excludable in consumption.

Public Goods

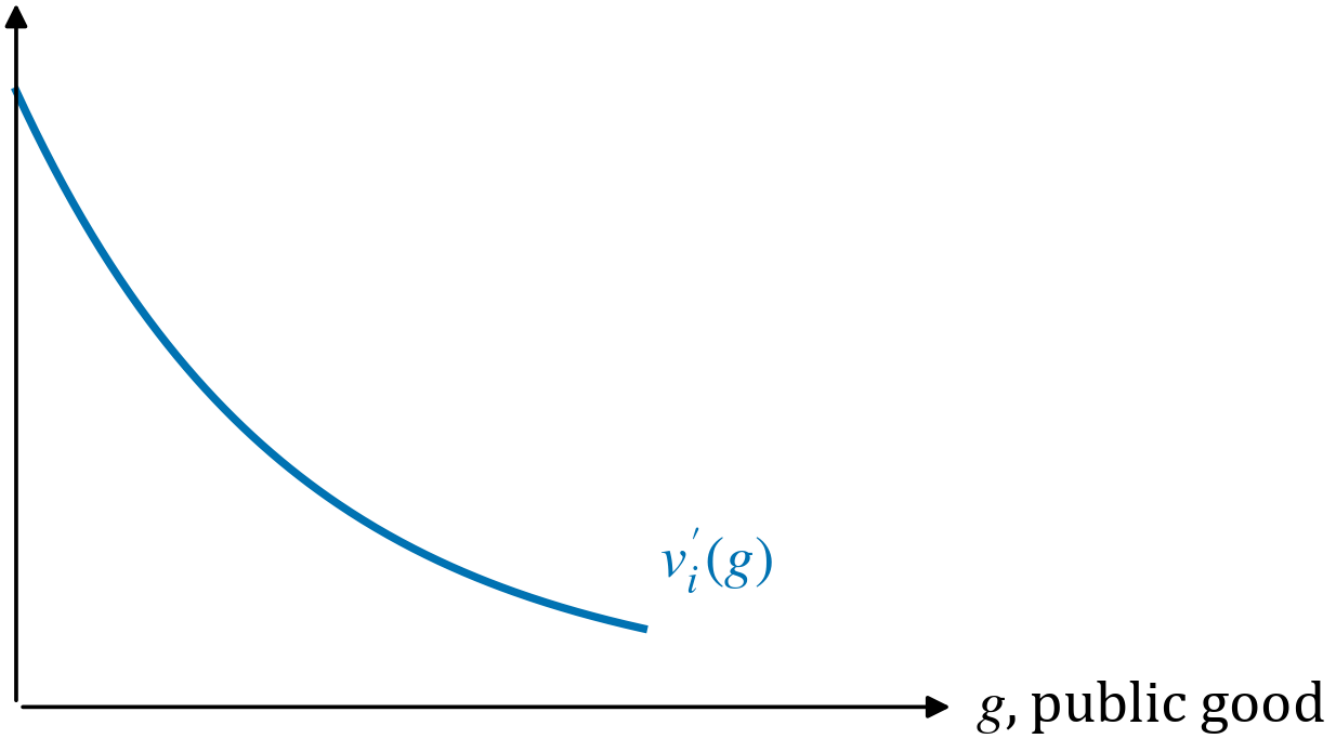
- Consider I consumers, one public good g and L traded private goods.
- Consumer i's utility from g units of the public good is $v_i(g)$, with marginal utility $v_i'(g)$.
  - Note that g does not have a subscript because of non-rivalry (every individual can enjoy g units of the public good).

Public Goods

- We consider the case of a public good, where $v_i'(g) > 0$ for every individual i.
  - A "public bad" would instead have $v_i'(g) < 0$.
- We assume that $v_i''(g) < 0$, which represents a positive but decreasing marginal utility from additional units of the public good.

Public Goods

- Marginal benefit from the public good

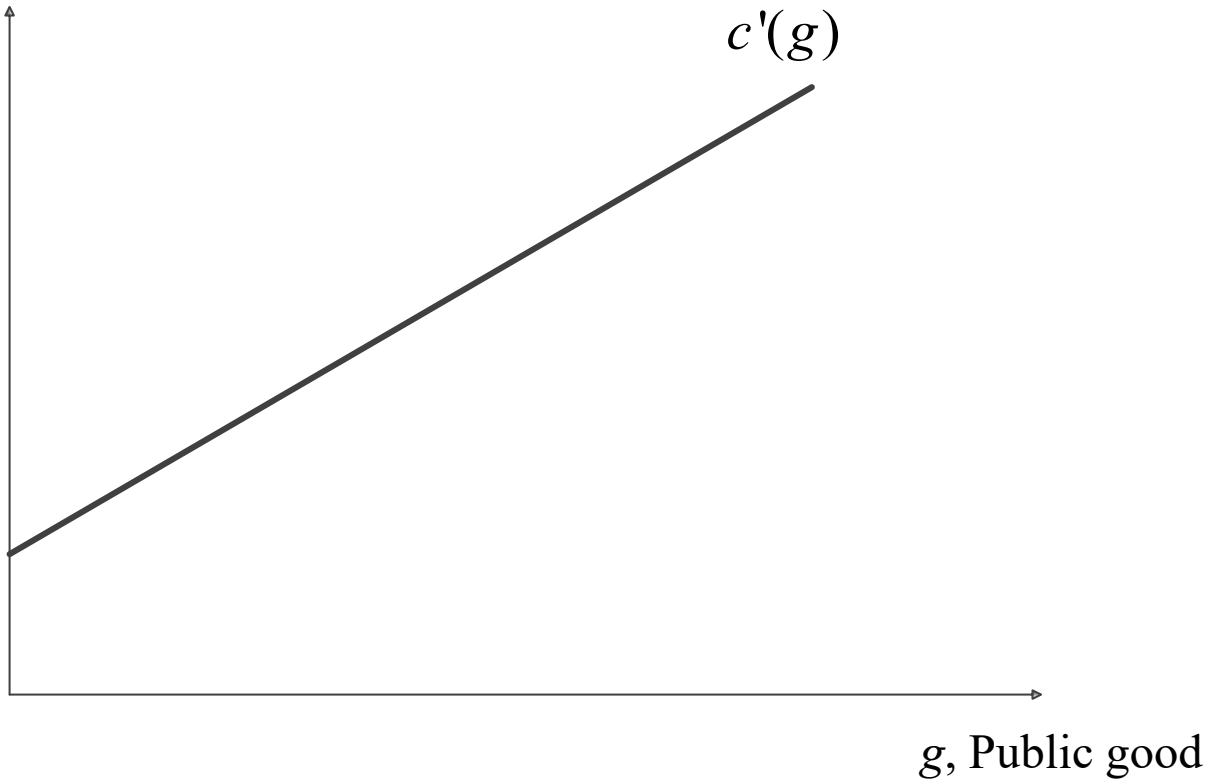


Public Goods

- We assume that the marginal utility from the public good, $v_i'(g)$, is independent of the private good (utility is quasilinear: $v_i(g) + x_i$).
- The cost of supplying g units of the public good is $c(g)$, where $c'(g) > 0$ and $c''(g) > 0$ for all g.
  - That is, the costs of providing the public good are increasing and convex in g.

Public Goods

- Marginal costs from providing the public good



Public Goods

- Let us first find the Pareto optimal allocation

$$\max_{g \geq 0} \sum_{i=1}^{I} v_i(g) - c(g)$$

- FOC with respect to g yields

$$\sum_{i=1}^{I} v_i'(g^0) - c'(g^0) \leq 0$$

with equality if $g^0 > 0$.

Public Goods

- SOCs hold since $v_i''(g) < 0$ for every i and $c''(g) > 0$, so

$$\sum_{i=1}^{I} v_i''(g) - c''(g) < 0 \quad \text{for all} \quad g$$

Public Goods

- In case of an interior solution, the optimal level of public good is achieved for the level of g^0 that solves

$$\sum_{i=1}^{I} v_i'(g^0) = c'(g^0)$$

- That is, the sum of the consumers' marginal benefits from an additional unit of the public good is equal to its marginal cost (***Samuelson rule***).

Public Goods

- The Pareto optimal level of public goods does not coincide with that of private goods.
- If g were a private good, each consumer would buy until

$$v_i'(g_i^*) = c'(g_i^*)$$

so only his own marginal benefit would count.

Public Goods

- ***Example*** (Discrete public good):
  - Consider a public good with $g \in \{0,1\}$, i.e., it is either produced or not.
  - Individual i's valuation of the good is $v_i(g) = \alpha_i g$, where $\alpha_i \geq 0$.
  - Total cost of producing the public good is cg, where $c > 0$.
  - Producing the good is Pareto optimal if and only if

$$\sum_{i=1}^{I} \alpha_i \geq c$$

Public Goods

- ***Example*** (continued):
  - In the discrete setting, the public good is produced if

$$\sum_{i=1}^{I} \alpha_i \geq c$$

  - That is, if the sum of individual valuations weakly exceeds the cost of the good.

Inefficiency of the Private Provision of Public Goods

Inefficiency of the Private Provision of Public Goods

- Let us consider the case in which a market exists for the public good and that each consumer i chooses how much of the public good to buy, denoted as $g_i \geq 0$, taking as given a market price of p.
- The total amount of the public good purchased by all I individuals is $g = \sum_{i=1}^{I} g_i$.
- Consider a single producer of the public good with a cost function $c(g)$.

Inefficiency of the Private Provision of Public Goods

- Formally, at a competitive equilibrium price p^*, each consumer i's purchase of the public good, g_i^*, must solve

$$\max_{g_i \geq 0} \; v_i\left(g_i + \sum\nolimits_{k \neq i} g_k^*\right) + w_i - p^* g_i$$

where w_i denotes i's wealth.

Inefficiency of the Private Provision of Public Goods

- The first term reflects that individual i benefits from both the g_i units of the public good he purchases and the $\sum_{k \neq i} g_k^*$ units that all other individuals acquire.
- Individual i takes others' purchases $\sum_{k \neq i} g_k^*$ as given and pays $p^* g_i$.

Inefficiency of the Private Provision of Public Goods

- FOC with respect to g_i yields
$$v_i'(g_i^* + \sum_{k \neq i} g_k^*) - p^* \leq 0$$
with equality if $g_i^* > 0$ (interior solution).
- For compactness, let g^* denote the total purchases of the public good, that is,
$$g^* = g_i^* + \sum_{k \neq i} g_k^*.$$
- Hence, the above FOC can be expressed as
$$v_i'(g^*) - p^* \leq 0$$
with equality if $g_i^* > 0$ (interior solution)

Inefficiency of the Private Provision of Public Goods

- On the other hand, the firm's PMP is

$$\max_{g \geq 0} \; p^* g - c(g)$$

- FOC with respect to g yields

$$p^* - c'(g^*) \leq 0$$

with equality if $g^* > 0$ (interior solution).

Inefficiency of the Private Provision of Public Goods

- Finally, the market clearing condition implies that the total amount of the public good produced coincides with the amount consumed by all individuals,

$$g^* = \sum_{i=1}^{I} g_i^*$$

Inefficiency of the Private Provision of Public Goods

- Combining the FOCs for consumers and the firm, we obtain

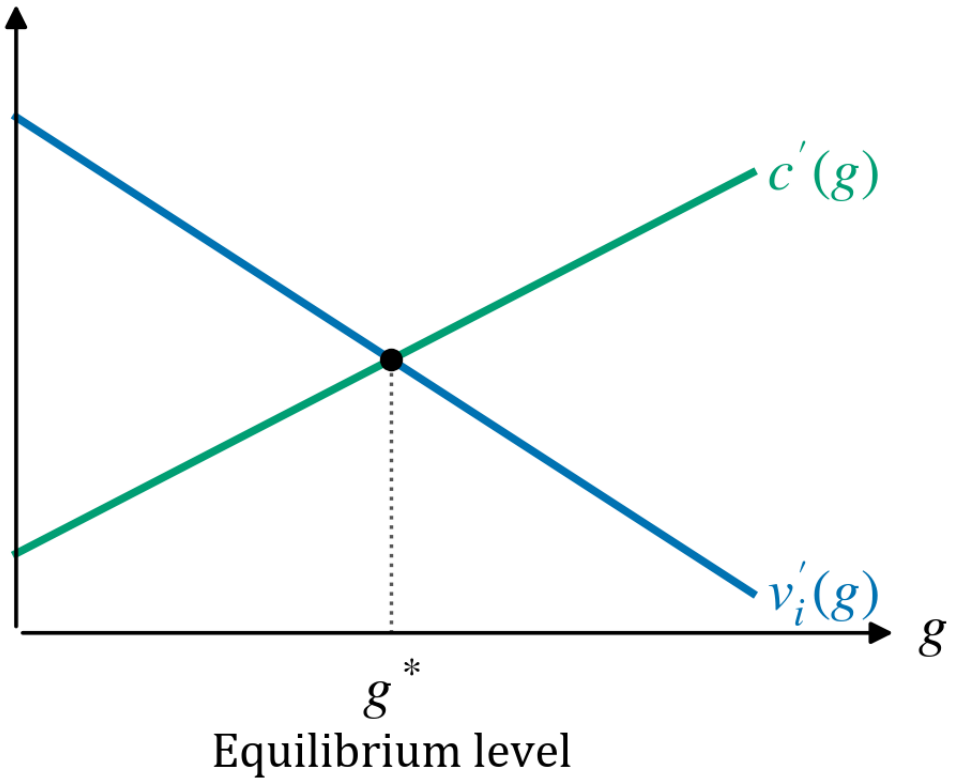
$$v_i'(g^*) \leq c'(g^*)$$

for every i, with equality if $g_i^* > 0$.

- Intuitively, individual i increases his consumption of the public good until his marginal benefit from the public good equals its marginal cost.

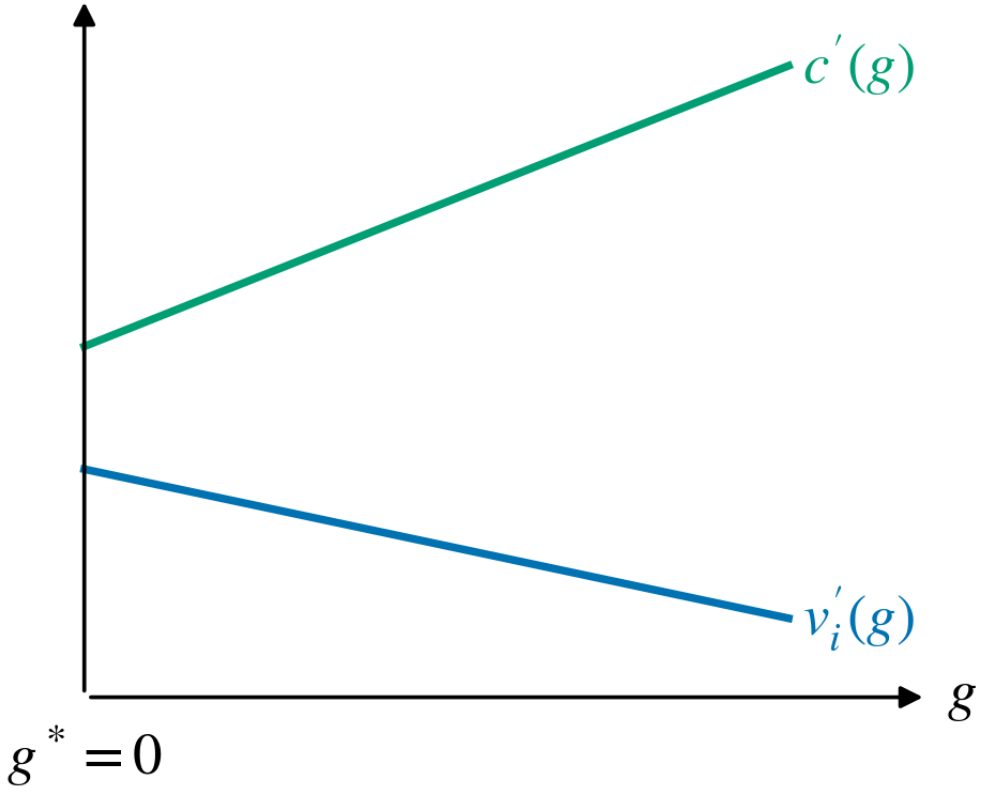
Inefficiency of the Private Provision of Public Goods

- Equilibrium level of public good (interior solution).



Inefficiency of the Private Provision of Public Goods

- Equilibrium level of public good (corner solution).



Inefficiency of the Private Provision of Public Goods

- However, at the Pareto optimum, we must have

$$\sum_{i=1}^{I} v_i'(g^0) = c'(g^0)$$

- That is, the sum of the marginal benefits that all individuals obtain from the public good must equal the marginal cost.

Inefficiency of the Private Provision of Public Goods

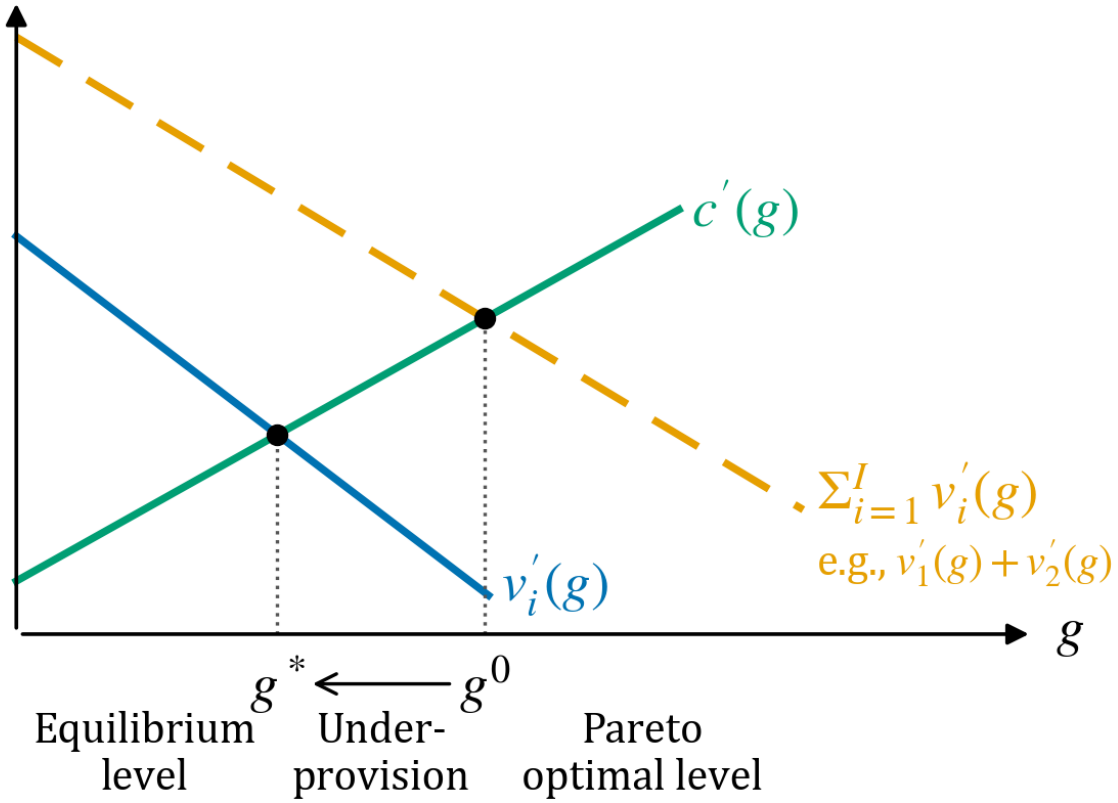
- At g^*, a buyer i has $v_i'(g^*) = c'(g^*)$, and $v_j'(g^*) > 0$ for all $j \neq i$, so

$$\sum_{j=1}^{I} v_j'(g^*) > c'(g^*)$$

- Since $\sum_j v_j' - c'$ is decreasing, $g^0 > g^*$: there is an *underprovision* of the public good.
  - *Exception*: when the marginal cost becomes vertical at a capacity $\overline{g}$ ($c'(g) = +\infty$ for $g > \overline{g}$). In this case, $g^* = g^0$.

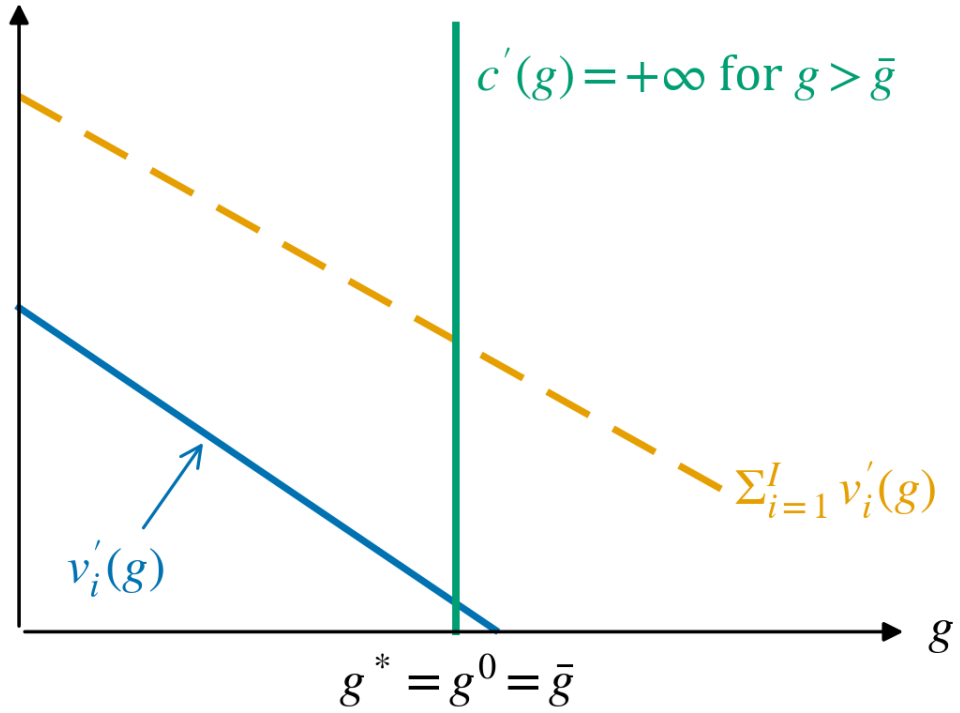
Inefficiency of the Private Provision of Public Goods

- Pareto optimal and equilibrium level of public good



Inefficiency of the Private Provision of Public Goods

- Pareto optimal level coincides with the equilibrium level of public good



Inefficiency of the Private Provision of Public Goods

- *Intuition*:
  - Each individual's purchase of the public good benefits not only him, but also all other individuals in the economy.
  - No individual internalizes the positive externality that his purchase generates on others.
  - Hence, each individual does not have enough incentives to purchase sufficient amounts of the public good.
  - This leads to the *free-rider problem*, whereby the public good is underprovided.

Inefficiency of the Private Provision of Public Goods

- ***Example*** (Private contributions to a public good):
  - Consider an economy with two individuals $i \in \{1,2\}$, with quasilinear utility function
$$u_i(G, y_i) = y_i + m_i \log(G)$$
  - where $m_i > 0$ denotes the value that individual i assigns to total contributions to the public good, $G = g_i + g_j$, and y_i is a composite private good.

Inefficiency of the Private Provision of Public Goods

- ***Example*** (continued):
  - Assume that $m_1 > m_2$.
  - For simplicity, the price of both private and public good is 1, so each individual i faces the budget constraint $g_i + y_i = w$, where $w > m_1$.
  - Individual i's UMP is then

$$\max_{g_i \geq 0} \; y_i + m_i \log\left(g_i + g_j\right)$$
$$\text{s.t.} \;\; g_i + y_i = w$$

Inefficiency of the Private Provision of Public Goods

- ***Example*** (continued):
  - Using the budget constraint, $y_i = w - g_i$, and the fact that $G = g_i + g_j$, we can rewrite this UMP as the unconstrained program

$$\max_{g_i \geq 0} \; w - g_i + m_i \log\left(g_i + g_j\right)$$

  - Taking FOC with respect to g_i yields

$$-1 + \frac{m_i}{g_i + g_j} \leq 0$$

with equality if $g_i > 0$.

Inefficiency of the Private Provision of Public Goods

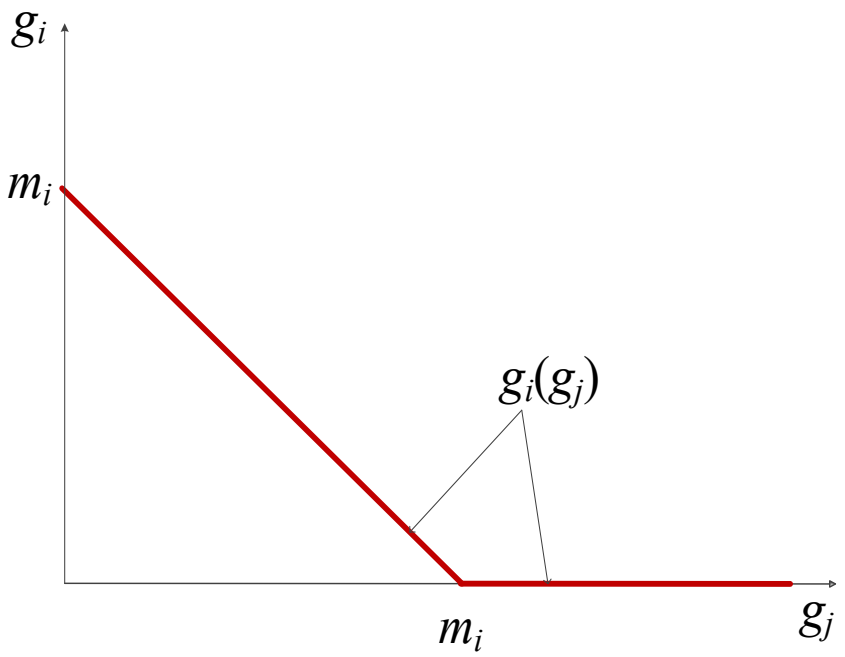
- ***Example*** (continued):
  - Solving for g_i produces BRF $g_i(g_j)$

$$g_i(g_j) = \begin{cases} m_i - g_j & \text{if } g_j < m_i \\ 0 & \text{otherwise} \end{cases}$$

  - If $g_j \geq m_i$, the left-hand side of the FOC is negative at every $g_i > 0$, so $g_i = 0$.

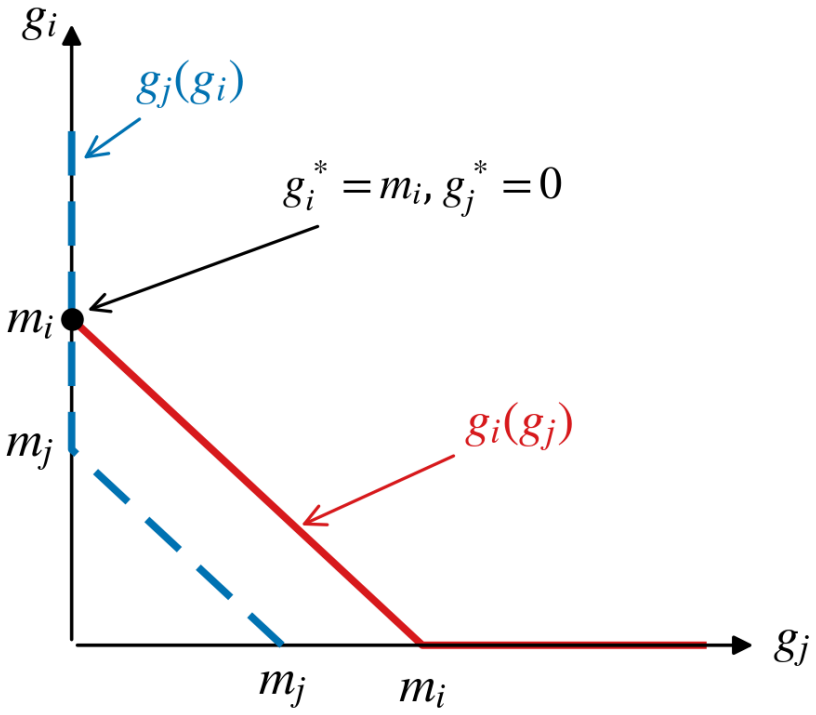
Inefficiency of the Private Provision of Public Goods

- ***Example*** (continued):
  - Individual i's BRF $g_i(g_j)$.
  - Individual j's BRF $g_j(g_i)$ is analogous.



Inefficiency of the Private Provision of Public Goods

- ***Example*** (continued):
  - The equilibrium level of (g_i^*, g_j^*) is obtained by simultaneously solving the two BRFs, $g_i(g_j)$ and $g_j(g_i)$.
  - Hence, $g_1^* = m_1 > 0$ and $g_2^* = 0$, since $m_1 > m_2$.



Inefficiency of the Private Provision of Public Goods

- ***Example*** (continued):
  - Both individuals cannot contribute: that would require $g_1 + g_2 = m_1$ and $g_1 + g_2 = m_2$, impossible since $m_1 > m_2$.
  - With $g_2^* = 0$, individual 1 contributes $g_1^* = g_1(0) = m_1$; and individual 2's best response is $g_2(m_1) = \max\{m_2 - m_1, 0\} = 0$.

Inefficiency of the Private Provision of Public Goods

- ***Example*** (continued):
  - In contrast, a social planner would maximize total welfare by solving

$$\max_{g_i, g_j} \; w - g_i + m_i \log(g_i + g_j) + w - g_j + m_j \log(g_j + g_i)$$

  - FOC:

$$-1 + \frac{m_i + m_j}{g_i + g_j} = 0$$

Inefficiency of the Private Provision of Public Goods

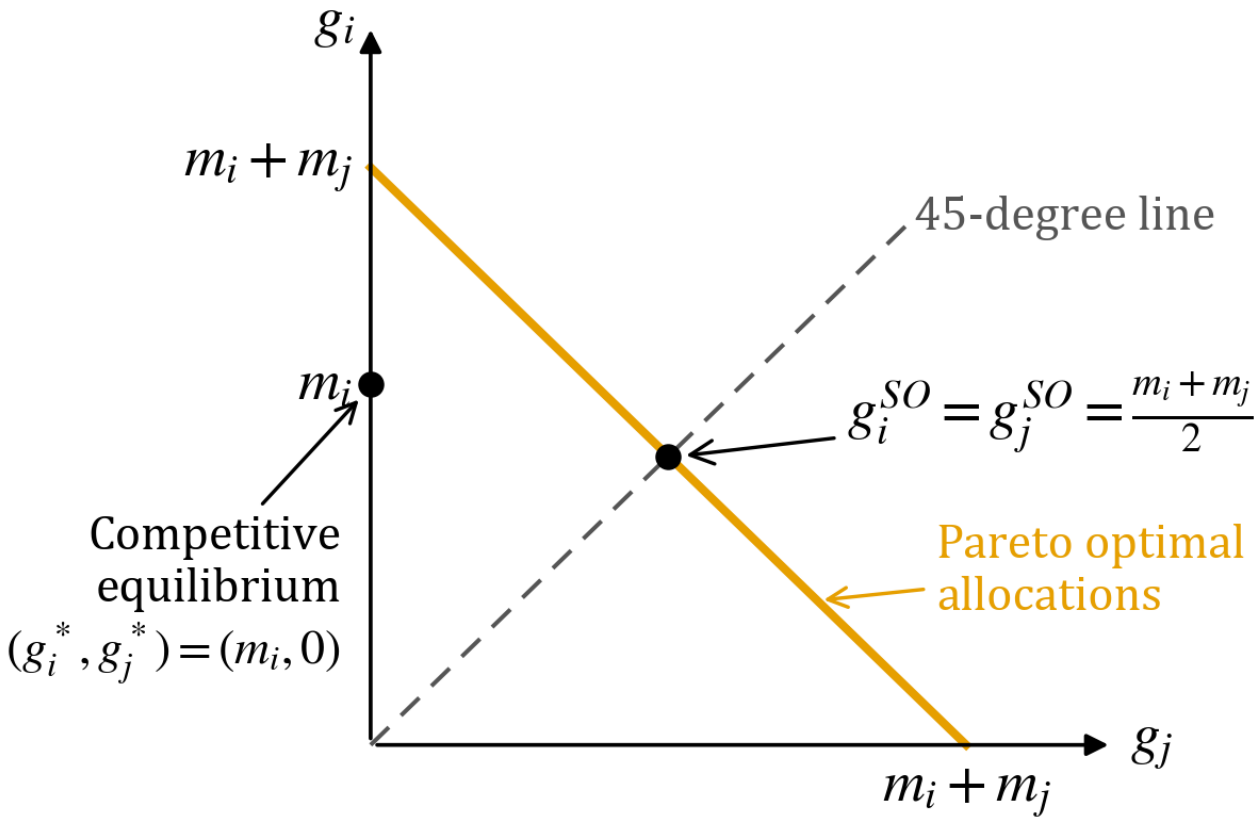
- ***Example*** (continued):
  - Solving for g_i, we obtain a continuum of Pareto optimal allocations

$$g_i^{SO} = m_i + m_j - g_j^{SO}$$

  - Hence $G^{SO} = m_1 + m_2 > m_1 = G^*$: the public good is underprovided.

Inefficiency of the Private Provision of Public Goods

- ***Example*** (continued):



Neutrality and Crowding out

Neutrality and Crowding out

- ***Revenue neutral policies***: an income tax on one individual whose revenue is transferred in full to another individual.
- Consider a Cobb-Douglas utility function

$$u_i(x_i, G) = x_i^{\alpha} G^{1-\alpha}$$

  where
  - x_i denotes the private good
  - G represents total contributions to the public good.
- Assume for simplicity that the price of the private and public good is 1, and that individual i's income is w_i.

Neutrality and Crowding out

- Every individual $i \in \{1,2\}$ solves the utility maximization problem

$$\max_{x_i, g_i} \; x_i^{\alpha}\left(g_i + g_j\right)^{1-\alpha}$$
$$\text{s.t.} \;\; x_i + g_i = w_i$$

- Since $x_i = w_i - g_i$, the above maximization problem can be reduced to an unconstrained program

$$\max_{g_i \geq 0} \; (w_i - g_i)^{\alpha}\left(g_i + g_j\right)^{1-\alpha}$$

Neutrality and Crowding out

- FOC wrt g_i yields

$$-\alpha(w_i - g_i)^{\alpha-1}\left(g_i + g_j\right)^{1-\alpha}$$
$$+(1-\alpha)(w_i - g_i)^{\alpha}\left(g_i + g_j\right)^{-\alpha} \leq 0$$

  with equality if $g_i > 0$.

- In the case of interior solutions, solving for g_i produces individual i's BRF

$$g_i\left(g_j\right) = (1-\alpha)w_i - \alpha g_j$$

Neutrality and Crowding out

- Inserting j's BRF into i's BRF yields

$$g_i = (1-\alpha)w_i - \alpha[(1-\alpha)w_j - \alpha g_i]$$

so that

$$g_i(1-\alpha^2) = (1-\alpha)(w_i - \alpha w_j)$$

- Hence, the equilibrium contribution is

$$g_i^* = \frac{w_i - \alpha w_j}{1+\alpha}$$

where $g_i^* > 0$ if and only if $w_i > \alpha w_j$.

Neutrality and Crowding out

- Equilibrium aggregate donation is

$$G^* = g_i^* + g_j^* = \frac{(1-\alpha)(w_1 + w_2)}{1+\alpha}$$

which increases in w_1 and w_2 but decreases in α because

$$\frac{\partial G^*}{\partial \alpha} = -\frac{2(w_1 + w_2)}{(1+\alpha)^2} < 0$$

Neutrality and Crowding out

- A benevolent planner maximizes $\sum_{i=1}^{2} {x_i}^{\alpha} G^{1-\alpha}$ subject to $x_1 + x_2 + G = w_1 + w_2$.
- For a given G, strict concavity of ${x_i}^{\alpha}$ gives $x_1 = x_2 = \frac{w_1+w_2-G}{2}$, and the FOC with respect to G becomes

$$\frac{\alpha}{w_1 + w_2 - G} = \frac{1-\alpha}{G}$$

so the socially optimal donation is

$$G^{SO} = (1-\alpha)(w_1 + w_2) > G^*$$

Neutrality and Crowding out

- Now, consider transfer $dw_1 > 0$ to individual 1 and a tax $dw_2 < 0$ so that

$$dw_1 + dw_2 = 0 \text{ or } dw_1 = -dw_2$$

- Individual i's equilibrium contribution g_i^* is affected as follows:

$$dg_i^* = \frac{dw_i - \alpha dw_j}{1+\alpha}$$

- Since $dw_i = -dw_j$, the above expression can be re-written as

$$dg_i^* = \frac{dw_i + \alpha dw_i}{1+\alpha} = \frac{(1+\alpha)dw_i}{1+\alpha} = dw_i$$

Neutrality and Crowding out

- Hence, individual i's contribution increases by the transfer he receives (if $dw_i > 0$) or falls by the tax he pays (if $dw_i < 0$).
- In addition, his contribution change dg_i^* is unaffected by the initial income distribution (i.e., w_i and w_j).

Neutrality and Crowding out

- As a consequence, aggregate donations are unaffected by income redistributions, since
$$dG^* = dg_1^* + dg_2^* = dw_1 + dw_2$$
which is zero by definition (i.e., $dw_1 = -dw_2$).
- This holds for redistributions that keep both individuals contributing, i.e., $w_i + dw_i > \alpha(w_j + dw_j)$ for $i, j \in \{1,2\}, i \neq j$ (Warr, 1983; Bergstrom, Blume and Varian, 1986).

Neutrality and Crowding out

- This condition shows the crowding out effect of levying taxes to fund the production of the public good.
- That is, a \$1 tax, i.e., $dw_i = -1 < 0$, reduces individual i's private contribution to the public good by \$1 (as long as i keeps contributing), since $dg_i^* = dw_i = -1$.

Neutrality and Crowding out

- ***Example*** (Increasing the number of contributors):
  - Let us extend the previous setting of two individuals with Cobb-Douglas preferences to a context with N individuals.
  - Assume all donors have the same income

$$w_1 = w_2 = \cdots = w_N = w$$

  - Every individual i chooses g_i to solve

$$\max_{g_i \geq 0} \alpha \log(w - g_i) + (1 - \alpha) \log\left(g_i + \sum_{j \neq i} g_j\right)$$

Neutrality and Crowding out

- ***Example*** (continued):
  - Differentiating with respect to g_i, we obtain

$$-\frac{\alpha}{w-g_i}+\frac{1-\alpha}{g_i+\sum_{j\neq i}g_j}\leq 0$$

with equality if $g_i > 0$. Rearranging the equality yields

$$\alpha g_i + \alpha G_{-i} = (1-\alpha)w - (1-\alpha)g_i$$

where $G_{-i} \equiv \sum_{j\neq i} g_j$ stands for the aggregate donations of all individuals other than i.

Neutrality and Crowding out

- ***Example*** (continued):
  - Solving for g_i, every individual i's best response function becomes

$$g_i(G_{-i}) = \begin{cases} (1-\alpha)w - \alpha G_{-i} & \text{if } G_{-i} < \dfrac{1-\alpha}{\alpha} w \\ 0 & \text{otherwise} \end{cases}$$

Neutrality and Crowding out

- ***Example*** (continued):
  - Invoking symmetry in equilibrium, i.e., $g_1 = g_2 = \cdots = g_N = g$, we have $G_{-i} = (N-1)g$, yielding
$$g = (1-\alpha)w - \alpha(N-1)g$$
  - Solving for g, we obtain an equilibrium donation
$$g^* = \frac{(1-\alpha)w}{1+\alpha(N-1)}$$
  - The aggregate contribution is
$$G^* = Ng^* = \frac{N(1-\alpha)w}{1+\alpha(N-1)}$$

Neutrality and Crowding out

- ***Example*** (continued):
  - The individual contribution is decreasing in α because

$$\frac{\partial g^*}{\partial \alpha} = -\frac{Nw}{[1+\alpha(N-1)]^2} < 0$$

    so as the individual's utility is more sensitive to the private good, his contribution to the public good falls.
  - The individual contribution is increasing in w because

$$\frac{\partial g^*}{\partial w} = \frac{1-\alpha}{1+\alpha(N-1)} > 0$$

    so as the individual becomes more wealthy, he can afford to make more contributions to the public good.

Neutrality and Crowding out

- ***Example*** (continued):
  - The effect of increasing the number of contributors N on the equilibrium results:

$$\frac{\partial g^*}{\partial N} = -\frac{\alpha(1-\alpha)w}{[1+\alpha(N-1)]^2} < 0$$

$$\frac{\partial G^*}{\partial N} = \frac{(1-\alpha)^2 w}{[1+\alpha(N-1)]^2} > 0$$

  - That is, while individual contributions decrease as a result of more donors potentially contributing to the public good, the overall effect of adding more donors is still positive.

Neutrality and Crowding out

- ***Example*** (continued):
  - In the limit of infinitely many contributors,

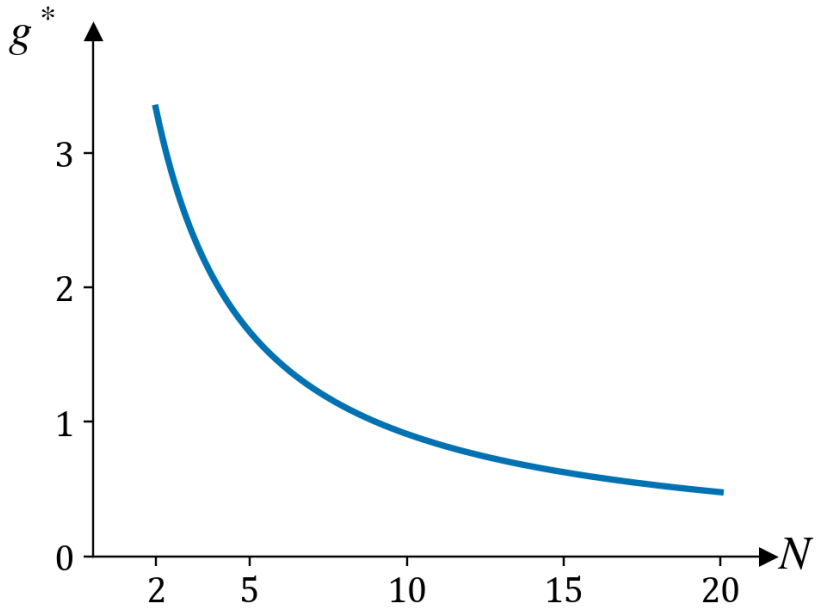
$$\lim_{N\to\infty} g^* = 0$$

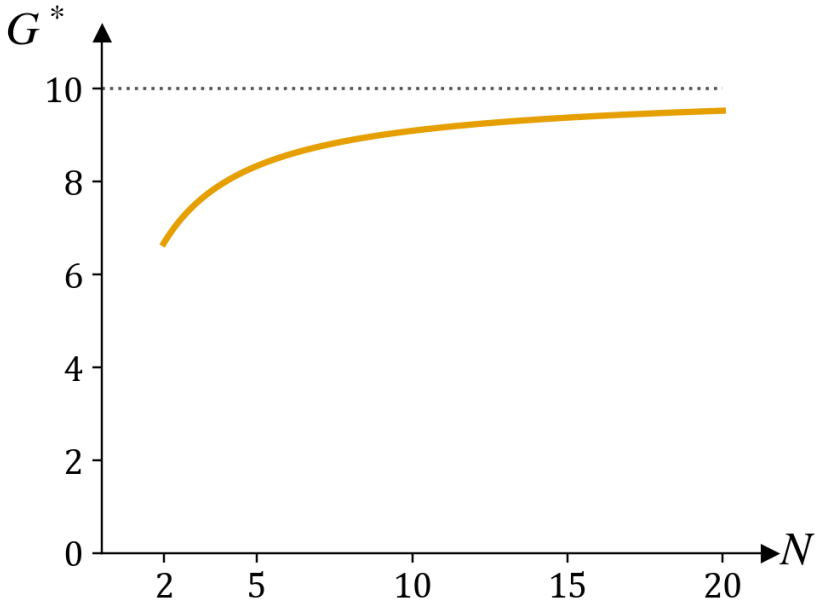
so each individual's contribution converges to zero, while

$$\lim_{N\to\infty} G^* = \frac{(1-\alpha)w}{\alpha}$$

Neutrality and Crowding out

- ***Example*** (continued):
  - We assume $\alpha = \frac{1}{2}$ and $w = 10$. Hence, the above expressions become $g^* = \frac{10}{1+N}$ and $G^* = N\frac{10}{1+N}$.





Neutrality and Crowding out

- In contrast, the social planner chooses $(g_1, \dots, g_N)$ to solve

$$\max_{g_1,\dots,g_N \geq 0} \alpha \log(w - g_i) + \alpha \sum_{j \neq i} \log(w - g_j)$$
$$+ N(1-\alpha) \log\left(g_i + \sum_{j \neq i} g_j\right)$$

- Differentiating with respect to g_i, an interior solution yields

$$-\frac{\alpha}{w - g_i} + \frac{N(1-\alpha)}{g_i + \sum_{j \neq i} g_j} = 0$$

Neutrality and Crowding out

- Rearranging and solving for g_i, we obtain

$$[\alpha + N(1-\alpha)]g_i = N(1-\alpha)w - \alpha \sum_{j \neq i} g_j$$

- Invoking symmetry, $g_i = g_j$ for all $j \neq i$, the socially optimal contribution is

$$g^{so} = \frac{N(1-\alpha)w}{\alpha + N(1-\alpha) + \alpha(N-1)}$$

Since $\alpha + N(1-\alpha) + \alpha(N-1) = N$, we obtain

$$g^{so} = (1-\alpha)w$$

Neutrality and Crowding out

- Every individual's contribution falls short of the socially optimal one,

$$g^{SO} = (1-\alpha)w > \frac{(1-\alpha)w}{1+\alpha(N-1)} = g^{*}$$

since this inequality reduces to $1+\alpha(N-1) > 1$, which holds because $N \geq 2$ and $0 < \alpha < 1$.

Neutrality and Crowding out

- The difference in contribution becomes

$$g^{SO} - g^* = (1-\alpha)w - \frac{(1-\alpha)w}{1+\alpha(N-1)}$$
$$= \frac{\alpha(1-\alpha)(N-1)w}{1+\alpha(N-1)}$$

which is increasing in w and N since

$$\frac{\partial(g^{SO} - g^*)}{\partial w} = \frac{\alpha(1-\alpha)(N-1)}{1+\alpha(N-1)} > 0$$
$$\frac{\partial(g^{SO} - g^*)}{\partial N} = \frac{\alpha(1-\alpha)w}{[1+\alpha(N-1)]^2} > 0$$

Neutrality and Crowding out

- Furthermore, this difference in contribution varies in α as

$$\frac{\partial(g^{SO} - g^{*})}{\partial\alpha} = \frac{(N-1)[1 - 2\alpha - \alpha^2(N-1)]w}{[1 + \alpha(N-1)]^2}$$

which is negative if

$$\alpha > \hat{\alpha} \equiv \frac{\sqrt{N} - 1}{N - 1} = \frac{1}{1 + \sqrt{N}}$$

Neutrality and Crowding out

- This is a condition that becomes less restrictive as N increases, since $\hat{\alpha}$ decreases in N:

$$\frac{\partial \hat{\alpha}}{\partial N} = -\frac{1}{2\sqrt{N}(1+\sqrt{N})^2} < 0$$

Neutrality and Crowding out

- When utility is more sensitive to the consumption of the private good (α increases), the difference between the equilibrium and socially optimal contributions narrows when $\alpha > \hat{\alpha}$.
- In addition,

$$\lim_{N\to\infty} \hat{\alpha} = \lim_{N\to\infty} \frac{1}{1+\sqrt{N}} = 0$$

so, in the limit of infinitely many contributors, $\frac{\partial(g^{SO}-g^{*})}{\partial\alpha} < 0$ holds for all values of α.

Neutrality and Crowding out

- Both the socially optimal contribution g^{SO} and the equilibrium contribution g^* decrease in α, but the former decreases faster when $\alpha > \hat{\alpha}$:

$$\frac{\partial g^{SO}}{\partial \alpha} = -w < \frac{\partial g^*}{\partial \alpha} = -\frac{Nw}{[1+\alpha(N-1)]^2}$$

so their difference shrinks.

Remedies to the Under-Provision of Public Goods

Remedies to the Under-Provision

- ***Quantity-based intervention***: a direct governmental provision of the public good
- ***Price-based intervention***: taxes or subsidies
  - We now denote the public good by x.
  - Assume two consumers with benefit functions $v_1(x_1 + x_2)$ and $v_2(x_1 + x_2)$, respectively, where x_i denotes the amount of the public good purchased by consumer i.

Remedies to the Under-Provision

- As in our analysis of externalities, we can design a subsidy s_i per unit of the public good purchased by every consumer i that induces him to take into account the positive external effect of his purchases of the public good on the other individual's welfare.
- We next show that the subsidy must be $s_i = v'_{-i}(x^0)$.

Remedies to the Under-Provision

- Here, $v'_{-i}(x) \equiv \sum_{j \neq i} v'_j(x)$ is the marginal benefit that all other individuals obtain from x units of the public good.
- Equivalently, the regulator can impose a per-unit tax $t_i = -v'_{-i}(x^0) < 0$ (a negative tax).

Remedies to the Under-Provision

- Every consumer i's UMP becomes that of selecting $\tilde{x}_i$ for a given level of $\tilde{x}_j$

$$\max_{x_i \geq 0} \underbrace{v_i\left(x_i + \tilde{x}_j\right) + \underbrace{s_i x_i}_{\text{subsidy}}}_{\text{benefit from } x_i} - \underbrace{\tilde{p} x_i}_{\text{cost}}$$

- Taking FOC with respect to x_i yields

$$v_i'\left(\tilde{x}_i + \tilde{x}_j\right) + s_i - \tilde{p} \leq 0$$

with equality if $\tilde{x}_i > 0$ (interior solution).

Remedies to the Under-Provision

- Using the market clearing condition $\tilde{x} = \tilde{x}_i + \tilde{x}_j$, and the fact that in a competitive equilibrium the firm's PMP implies $\tilde{p} = c'(\tilde{x})$ (if $\tilde{x} > 0$), the above FOC becomes

$$v_i'(\tilde{x}) + s_i \leq c'(\tilde{x})$$

Remedies to the Under-Provision

- For the equilibrium to reach $\tilde{x} = x^0$, the FOC must coincide with the Samuelson condition

$$v_i'(x^0) + v_j'(x^0) = c'(x^0)$$

which requires

$$s_i = v_j'(x^0)$$

Remedies to the Under-Provision

- Hence, we need a subsidy $s_i = v'_{-i}(x^0)$ which, for the case of only two consumers i and j, implies $s_i = v'_j(x^0)$.
- In the case of N individuals, the subsidy to consumer i would be

$$s_i = v'_j(x^0) + v'_k(x^0) + \cdots = \sum_{j \neq i} v'_j(x^0)$$

- The subsidies are financed with lump-sum taxes to avoid new distortions.

Remedies to the Under-Provision

- The introduction of a subsidy might seem an effective and easy solution to the underprovision problem in public goods.
- However, the regulator might not have access to information about the marginal benefits of the public good for every consumer.

Lindahl Equilibria

Lindahl Equilibria

- Private provision of a public good results in inefficiencies, i.e., $x^* < x^0$.
  - This inefficiency can be corrected by the use of quantity-based or price-based regulation.
- There is, however, a market solution that *in principle* can achieve optimality.
- Consider a market where every individual's consumption of the public good is a different commodity with its own market.
- Denote the price of this personalized good by p_i, which can differ across consumers.

Lindahl Equilibria

- If consumer i faces a price p_i^{**}, his UMP is

$$\max_{x_i \geq 0} \; v_i(x_i) + w_i - p_i^{**} x_i$$

where x_i denotes the amount of the public good consumer i demands at his personalized price p_i^{**}.

Lindahl Equilibria

- FOC wrt x_i yields

$$v_i'(x_i^{**}) - p_i^{**} \leq 0$$

with equality if $x_i^{**} > 0$.

- Hence, at the aggregate level,

$$\sum_{i=1}^{I} v_i'(x_i^{**}) \leq \sum_{i=1}^{I} p_i^{**}$$

Lindahl Equilibria

- On the other hand, the firm produces a bundle of I goods (one for each consumer), with PMP

$$\max_{x \geq 0} \underbrace{\sum_{i=1}^{I} p_i^{**} x}_{\text{Total revenue}} - c(x)$$

- Each unit of x jointly supplies one unit of every personalized good, so revenue is $(\sum_i p_i^{**})x$.

Lindahl Equilibria

- FOC wrt x yields

$$\sum_{i=1}^{I} p_i^{**} - c'(x^{**}) \leq 0, \qquad \text{or}$$

$$\sum_{i=1}^{I} p_i^{**} \leq c'(x^{**})$$

with equality if $x^{**} > 0$ (interior solution).

Lindahl Equilibria

- Market clearing in every personalized market gives $x_i^{**} = x^{**}$ for all i.
- For $x^{**} > 0$, the firm's FOC and each consumer's FOC hold with equality, so

$$\sum_{i=1}^{I} v_i'(x^{**}) = \sum_{i=1}^{I} p_i^{**} = c'(x^{**})$$

  which is the Samuelson condition; hence, $x^{**} = x^0$.
- This equilibrium is known as the ***Lindahl equilibrium***.

Lindahl Equilibria

- *Why do we obtain efficiency?*
  - First, we define personalized markets for the public good.
  - Second, each consumer, taking the price of his personalized good as given, fully determines his own level of consumption of the public good.
  - Each consumer pays for every unit he enjoys, so the externality is internalized and no one free-rides.

Lindahl Equilibria

- *Are these personalized markets for the public good realistic?*
  - We need excludability between the different personalized public goods, which might only be applicable to very specific public goods
    - e.g., some forms of health care, college education, etc.
  - Even if excludability were possible, personalized markets would be monopsonistic (there is only one buyer on the demand side)
    - Thus, the price-taking assumption is difficult to support.

Lindahl Equilibria

- ***Example*** (Calculating a Lindahl equilibrium):
  - Consider three first-year economics graduate students, Eric (E), Chris (C), and Matt (M) deciding to purchase a microwave (a public good) for their office.
  - The utility function of each student is

$$u_i(x_i, y) = \ln x_i + \alpha_i \ln y$$

Lindahl Equilibria

- ***Example*** (continued):
  - x_i denotes student i's consumption of the private good, $i \in \{E, C, M\}$;
  - y denotes the amount of the public good (spending on the microwave);
  - α_i measures how much student i values the microwave.

Lindahl Equilibria

- ***Example*** (continued):
  - For simplicity, assume that the price of the private good, x_i, and the wealth of each student are 1.
  - The microwave's unit cost is 1, so Lindahl prices must sum to marginal cost: $p_E + p_C + p_M = 1$.

Lindahl Equilibria

- ***Example*** (continued):
  - The UMP for student i is

$$\max_{x_i, y} \ln x_i + \alpha_i \ln y$$
$$\text{s.t.} \quad x_i + p_i y \leq 1$$

where p_i represents the Lindahl price each student pays.

Lindahl Equilibria

- ***Example*** (continued):
  - Since the budget constraint holds with equality, the UMP becomes

$$\max_{y} \ \ln(1 - p_i y) + \alpha_i \ln y$$

  - FOC wrt y yields

$$\frac{p_i}{1 - p_i y} = \frac{\alpha_i}{y}$$

  - Rearranging, we obtain

$$p_i y = \frac{\alpha_i}{1 + \alpha_i}$$

Lindahl Equilibria

- ***Example*** (continued):
  - Summing across all three students yields

$$(p_E + p_C + p_M)y = y = \frac{\alpha_E}{1+\alpha_E} + \frac{\alpha_C}{1+\alpha_C} + \frac{\alpha_M}{1+\alpha_M}$$

  - Substituting this value for y in equation $p_i y = \frac{\alpha_i}{1+\alpha_i}$ and solving for the Lindahl prices produces

$$p_i^{**} = \frac{\frac{\alpha_i}{1+\alpha_i}}{\frac{\alpha_E}{1+\alpha_E} + \frac{\alpha_C}{1+\alpha_C} + \frac{\alpha_M}{1+\alpha_M}}$$

Lindahl Equilibria

- ***Example*** (continued):
  - For instance, if $\alpha_E = 1$ and $\alpha_C = \alpha_M = 0.6$, then $y = 1.25$, with Lindahl prices $p_E^{**} = 0.4$ and $p_C^{**} = p_M^{**} = 0.3$.

Public Goods that Experience Congestion

Public Goods that Experience Congestion

- Consider that consumption by other individuals *reduces* the benefit each user i enjoys from the good.
- Hence, the utility function becomes

$$v_i(x_i, x_{-i}) + w_i$$

where x_i is individual i's use of the good, and $x_{-i} = (x_j)_{j \neq i}$ is the use by all others.

Public Goods that Experience Congestion

- The use by others, x_{-i}, enters $v_i(\cdot)$ negatively, thus capturing the ***congestion effect*** (as in CPR goods, e.g., the tragedy of the commons).
- Utility increases in x_i, but decreases in the use of the good by other individuals, x_{-i}.

Public Goods that Experience Congestion

- Let $x \equiv \sum_{i=1}^{N} x_i$ denote total use. The social planner's maximization problem is

$$\max_{x_1,\ldots,x_N} \sum_{i=1}^{N} [v_i(x_i, x_{-i}) + w_i] - c(x)$$

- Since $\frac{\partial c(x)}{\partial x_i} = c'(x)$, the FOC wrt x_i yields

$$\frac{\partial v_i(x_i, x_{-i})}{\partial x_i} + \sum_{j \neq i} \frac{\partial v_j(x_j, x_{-j})}{\partial x_i} - c'(x) \leq 0$$

for all i, with equality in the case of interior solutions.

Public Goods that Experience Congestion

- Each FOC equates individual i's marginal benefit, minus the marginal congestion cost that his use imposes on all others, to the marginal cost, where this congestion term is

$$\sum_{j \neq i} \frac{\partial v_j(x_j, x_{-j})}{\partial x_i} < 0$$

Public Goods that Experience Congestion

- In equilibrium, each individual ignores this term,

$$\frac{\partial v_i(x_i, x_{-i})}{\partial x_i} = c'(x)$$

so the socially optimal use is lower than the equilibrium use.

Behavioral Motives in Public Good Games

Warm-Glow Benefits

- Individuals obtain a "warm glow" benefit from their donations to the public good (see Andreoni, 1990)
- Individual i's utility function $u_i(x_i, G, g_i)$ is increasing in:
  - his private consumption, x_i;
  - total contributions to the public good, G; and
  - his own donation, g_i (the source of warm glow).

Warm-Glow Benefits

- Warm glow makes crowding-out incomplete: a $1 tax-financed transfer no longer reduces private donations by $1 (compare with the neutrality result above).

Warm-Glow Benefits

- Donor i's UMP is

$$\max_{x_i, g_i, G} \; u_i(x_i, G, g_i)$$
$$\text{s.t.} \quad x_i + g_i = w_i$$
$$g_i + G_{-i} = G$$

  where $G_{-i} = \sum_{j \neq i} g_j$ is total donations from all other individuals.
- Since $g_i = G - G_{-i}$ and $x_i = w_i - g_i = w_i - G + G_{-i}$, the UMP becomes

$$\max_{G_{-i} \leq G \leq w_i + G_{-i}} \; u_i(w_i - G + G_{-i}, G, G - G_{-i})$$

Warm-Glow Benefits

- FOC wrt G yields

$$-\frac{\partial u_i}{\partial x_i}+\frac{\partial u_i}{\partial G}+\frac{\partial u_i}{\partial g_i}=0$$

evaluated at $(w_i - G + G_{-i}, G, G - G_{-i})$, which defines a total donations function

$$G = f_i(w_i + G_{-i}, G_{-i})$$

where $w_i + G_{-i}$ is i's social income, and G_{-i} enters separately only because of warm glow ($g_i = G - G_{-i}$).

Warm-Glow Benefits

- The individual donation function of player i, $g_i = G - G_{-i}$, is

$$g_i = f_i(w_i + G_{-i}, G_{-i}) - G_{-i}$$

  - The first argument of $f_i(\cdot)$ is common to public good games in which donors do not benefit from warm glow.
  - In contrast, the second argument of $f_i(\cdot)$ arises because of the warm glow benefits that donors obtain.

Warm-Glow Benefits

- *Intuition*:
  - the first argument captures *altruistic* motivations in public goods; whereas
  - the second argument measures *egoistic* motivations (because the warm glow benefit is private).

Warm-Glow Benefits

- Let f_{ia} (f_{ie}) denote the partial derivative of $f_i(\cdot)$ wrt the first argument (altruism) and the second argument (egoism), where

$$f_{ia} \in (0,1),\ f_{ie} \geq 0 \text{ and } 0 < f_{ia} + f_{ie} \leq 1$$

Warm-Glow Benefits

- This notation helps us obtain the following "altruism coefficient"

$$\alpha_i = \frac{\frac{\partial f_i}{\partial w_i}}{\frac{\partial f_i}{\partial G_{-i}}} = \frac{f_{ia}}{f_{ia}+f_{ie}}$$

  - $\alpha_i = 1$ for pure altruists, $u_i(x_i, G)$, where $f_{ie} = 0$
  - $\alpha_i = f_{ia}$ for pure egoists, $u_i(x_i, g_i)$, since $f_{ia} + f_{ie} = 1$

Warm-Glow Benefits

- Let us now consider a transfer from individual 2 to individual 1, i.e., $dw_1 > 0$ and $dw_2 < 0$ where $dw_1 = -dw_2$.
- Let us examine how total donations G and individual contributions are affected by the transfer.
- Totally differentiating g_i yields

$$dg_i = f_{ia}(dw_i + dG_{-i}) + f_{ie}dG_{-i} - dG_{-i}$$

- Factoring out dG_{-i} yields

$$dg_i = f_{ia}\, dw_i + (f_{ia} + f_{ie} - 1)dG_{-i}$$

Warm-Glow Benefits

- Since $G_{-i} = G - g_i$ (by definition), substituting $dG_{-i} = dG - dg_i$ in the above expression yields

$$dg_i = f_{ia}dw_i + (f_{ia} + f_{ie} - 1)dG - (f_{ia} + f_{ie} - 1)dg_i$$

- Rearranging,

$$(f_{ia} + f_{ie})dg_i = (f_{ia} + f_{ie} - 1)dG + f_{ia}dw_i$$

- Dividing both sides by $(f_{ia} + f_{ie})$ and using the definition of α_i yields

$$dg_i = \frac{f_{ia}+f_{ie}-1}{f_{ia}+f_{ie}}dG + \alpha_i dw_i$$

Warm-Glow Benefits

- This captures how the donation of individual i is affected by a change in his wealth dw_i.
- Aggregating across donors

$$\sum_{i=1}^{I} dg_i = \sum_{i=1}^{I} \frac{f_{ia}+f_{ie}-1}{f_{ia}+f_{ie}} dG + \sum_{i=1}^{I} \alpha_i \, dw_i$$

- Since $\sum_{i=1}^{I} dg_i = dG$, then the change in aggregate contributions, dG, is

$$dG = \sum_{i=1}^{I} \frac{f_{ia}+f_{ie}-1}{f_{ia}+f_{ie}} dG + \sum_{i=1}^{I} \alpha_i \, dw_i$$

Warm-Glow Benefits

- We can now rearrange and solve for dG to obtain

$$dG = \frac{1}{1-\sum_{i=1}^{I} \frac{f_{ia}+f_{ie}-1}{f_{ia}+f_{ie}}} \sum_{i=1}^{I} \alpha_i \, dw_i$$

- For compactness, let

$$\kappa \equiv \frac{1}{1-\sum_{i=1}^{I} \frac{f_{ia}+f_{ie}-1}{f_{ia}+f_{ie}}}$$

Warm-Glow Benefits

- Since $f_{ia} + f_{ie} \leq 1$, every term $\frac{f_{ia}+f_{ie}-1}{f_{ia}+f_{ie}} \leq 0$, so $\kappa \in (0, 1]$.
- Thus, the expression for dG can be expressed as

$$dG = \kappa \sum_{i=1}^{I} \alpha_i \, dw_i$$

Warm-Glow Benefits

- Finally, since $dw_1 = -dw_2$ and $dw_j = 0$ for all other individuals $j \notin \{1, 2\}$, the above expression becomes

$$dG = \kappa\left[\alpha_1 dw_1 + \alpha_2(-dw_1)\right] = \kappa(\alpha_1 - \alpha_2)dw_1$$

- Since $\kappa > 0$, the transfer from individual 2 to individual 1 increases total donations if and only if individual 1 is more altruistic ($\alpha_1 > \alpha_2$), leaves them unchanged if $\alpha_1 = \alpha_2$, and reduces them otherwise.

Warm-Glow Benefits

- ***Example*** (Warm glow in public goods):
  - Consider a public good game with two individuals $i \in \{1, 2\}$, each with Cobb-Douglas utility function

$$u_i(x_i, G, g_i) = a \log x_i + b \log G + c_i \log g_i$$

where $c_i > 0$ represents the warm-glow benefit that individual i obtains from his contribution to the public good, with $c_1 > c_2$.

Warm-Glow Benefits

- ***Example*** (continued):
  - Since $x_i = w_i - g_i$ and $G = g_i + g_j$, we can rewrite this utility function as

$$u_i(w_i - g_i, g_i + g_j, g_i)$$
$$= a\log(w_i - g_i) + b\log(g_i + g_j) + c_i \log g_i$$

  - FOC wrt g_i yields

$$-\frac{a}{w_i - g_i} + \frac{b}{g_i + g_j} + \frac{c_i}{g_i} = 0$$

Warm-Glow Benefits

- ***Example*** (continued):
  - For simplicity, consider $w_i = 10$ for both individuals, $a = 1, b = \frac{1}{3}$, and warm-glow parameters $c_1 = \frac{1}{4}$ and $c_2 = \frac{1}{5}$, so player 1 obtains a larger warm-glow benefit than player 2.
  - Solving for g_1 (the positive root of a quadratic equation in g_1) yields player 1's best response function, $g_1(g_2)$. Operating similarly for player 2, we find his best response function $g_2(g_1)$.

Warm-Glow Benefits

- ***Example*** (continued):
  - Simultaneously solving for g_1 and g_2 yields equilibrium contribution levels of

$$g_1^* = 2.996 \quad \text{and} \quad g_2^* = 2.623$$

with aggregate donations $G^* = 5.619$.

Warm-Glow Benefits

- ***Example*** (continued):
  - At the equilibrium above, player 1 is the more altruistic donor in the sense of the previous result ($\alpha_1 \approx 0.42 > \alpha_2 \approx 0.39$): a larger warm-glow parameter makes his donation respond more to his own wealth.
  - Hence, a transfer from player 2 to player 1 should increase total donations.

Warm-Glow Benefits

- ***Example*** (continued):
  - A \$2 transfer from player 2 to player 1 gives

$$w_1 = \$12 \quad \text{and} \quad w_2 = \$8$$

which changes individual donations to

$$g_1^* = 3.857 \quad \text{and} \quad g_2^* = 1.892$$

with aggregate donations $G^* = 5.749 > 5.619$.

Social Preferences

- Consider the public good game where two players simultaneously and independently choose between contributing (C) and not contributing (NC) to the public good.

		Player 2	
		C	NC
Player 1	C	a, a	c, b
	NC	b, c	d, d

- Both players' payoffs satisfy $b > a > d > c$, thus indicating that both players have incentives to free ride.

Social Preferences

- Every player's best response is NC, both when his opponent contributes (since $b > a$), and when he does not (since $d > c$).
- In fact, both players find C to be strictly dominated by NC, ultimately implying that the strategy profile (NC,NC) is the unique Nash equilibrium of the game.

Social Preferences

- Consider that players exhibit Fehr and Schmidt (1999)-type social preferences:

$$U_i(x_i, x_j) = x_i - \alpha_i \max\{x_j - x_i, 0\} - \beta_i \max\{x_i - x_j, 0\}$$

where

  - x_i and x_j are player i's and j's monetary payoffs, respectively
  - α_i is player i's disutility from *envy*, which occurs when $x_i < x_j$ implying $U_i(x_i, x_j) = x_i - \alpha_i(x_j - x_i)$
  - β_i is player i's disutility from *guilt* which occurs when $x_i > x_j$ implying $U_i(x_i, x_j) = x_i - \beta_i(x_i - x_j)$
- Fehr and Schmidt (1999) assume that players' envy is at least as strong as their guilt, i.e., $\alpha_i \geq \beta_i$ and $0 \leq \beta_i < 1$.

Social Preferences

- Hence, the above payoff matrix can be reformulated as follows

		Player 2: C	Player 2: NC
Player 1	C	a, a	$c - \alpha_1(b-c), b - \beta_2(b-c)$
	NC	$b - \beta_1(b-c), c - \alpha_2(b-c)$	d, d

- Every player's utility level decreases when he is either:
  - the player with the higher payoff (due to guilt), e.g., player 1 under outcome (NC,C), or
  - the player with the lower payoff (due to envy), e.g., player 1 under outcome (C,NC).

Social Preferences

- Let us first analyze player i's BRF:
  - When player j chooses C, player i prefers NC if $a \leq b - \beta_i(b-c)$, or, more compactly, $\beta_i \leq \frac{b-a}{b-c}$, where $\frac{b-a}{b-c} \in (0,1)$ since $a > c$.
  - When player j chooses NC, player i prefers NC if $c - \alpha_i(b-c) \leq d$, equivalently, $\alpha_i \geq \frac{c-d}{b-c}$, which holds since $\frac{c-d}{b-c} < 0 \leq \alpha_i$.

Social Preferences

- Therefore, if $\beta_i < \frac{b-a}{b-c}$, NC is a strictly dominant strategy for player i.
- If, instead, $\beta_i > \frac{b-a}{b-c}$, then player i's best response to C is C, but his best response to NC is still NC.

Social Preferences

- Hence, Nash equilibria of the game in pure strategies are:
  - (NC,NC) is the unique equilibrium if

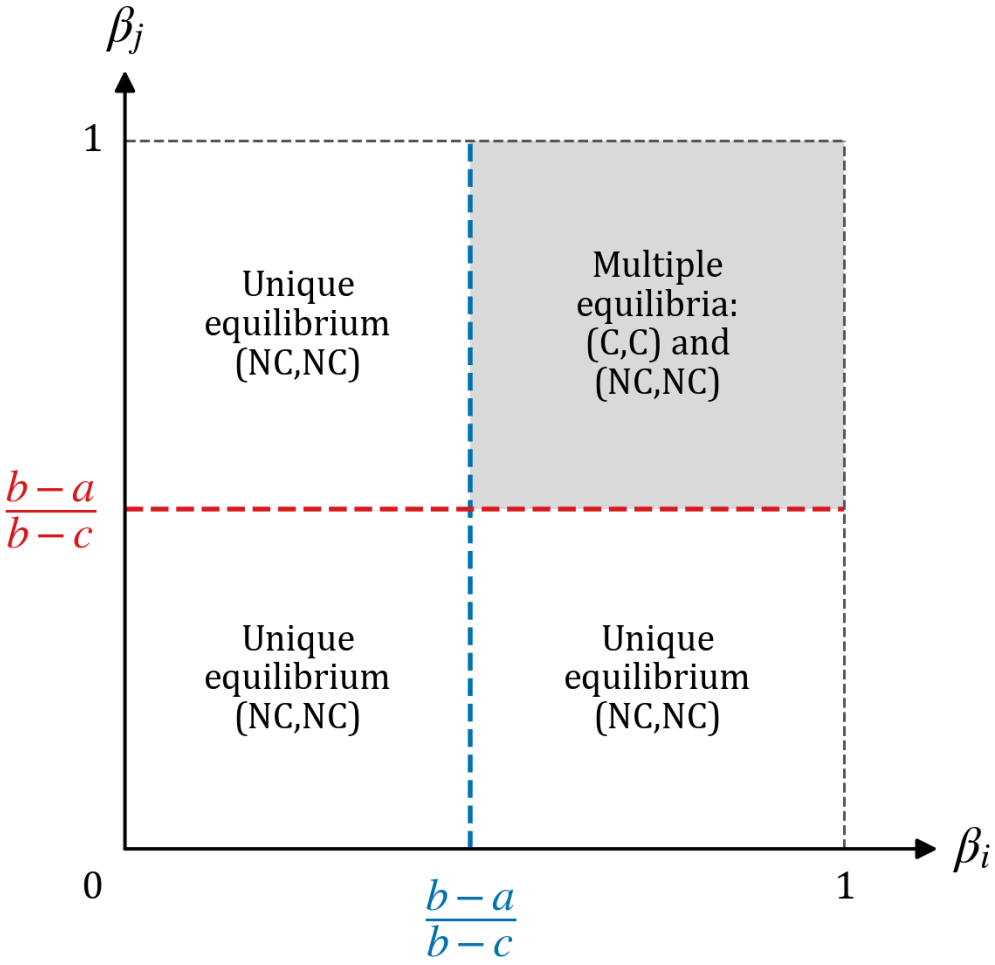
$$\beta_i < \frac{b-a}{b-c} \quad \text{or} \quad \beta_j < \frac{b-a}{b-c}$$

  - (C,C) and (NC,NC) if $\beta_i, \beta_j \geq \frac{b-a}{b-c}$

Social Preferences

- Intuitively,
  - if at least one player has a guilt parameter below $\frac{b-a}{b-c}$, the unique Nash equilibrium of the game coincides with that where players have no concerns about the fairness of the payoff distribution (standard preferences).
  - if both individuals are sufficiently concerned about fairness, we can identify two different Nash equilibria. In this region, each player's best response is to choose the same action as his opponent.

Social Preferences



Competition for Status Acquisition

Competition for Status Acquisition

- Consider a public good game with two agents privately contributing to its provision.
- Let g_i denote individual i's voluntary contributions to the public good, and $x_i \geq 0$ represent his consumption of private goods.
- Every player obtains the return $m > 0$ from total contributions to the public good, i.e., $G = g_i + g_j$.

Competition for Status Acquisition

- In addition, each player benefits from *status acquisition* if his donation is larger than his rival's.
- Assume that every player is endowed with w monetary units that can be distributed between private and public good consumption, with w large enough that $g_i \leq w$ never binds.
- The marginal utility individual i derives from his consumption of the private good is one.

Competition for Status Acquisition

- Contributor i's utility function is

$$u_i(x_i, g_i, g_j) = x_i + \ln[mG + \alpha_i(g_i - g_j)]$$

- In this setting, the *status* subject i acquires by contributing g_i is given by $g_i - g_j$.
  - That is, subject i enhances his relative status if his contribution is greater than individual j's; otherwise, subject i perceives himself as an individual with lower status than subject j.

Competition for Status Acquisition

- The difference is scaled by $\alpha_i \in [0, +\infty)$, indicating the importance of relative status for subject i.
- Players' parameters (m, α_i, α_j) are common knowledge.

Competition for Status Acquisition

- The UMP of every player i is

$$\max_{x_i, g_i} \; x_i + \ln\left[mG + \alpha_i\left(g_i - g_j\right)\right]$$

$$\text{s. t.} \;\; x_i + g_i = w,$$

$$g_i + g_j = G, \text{ and}$$

$$g_i, g_j \geq 0$$

- Using $x_i = w - g_i \geq 0$ and $g_i + g_j = G$, we obtain the following problem in g_i alone

$$\max_{g_i \geq 0} \; w - g_i + \ln\left[m\left(g_i + g_j\right) + \alpha_i\left(g_i - g_j\right)\right]$$

Competition for Status Acquisition

- Note that an increase in player j's contribution, g_j, imposes two types of externalities on player i's utility level:
  - ***Positive externality***: arising from the public good nature of player j's contributions.
  - ***Negative externality***: player j's donations reduce the status perception of player i.

Competition for Status Acquisition

- FOC wrt g_i yields player i's BRF

$$g_i(g_j) = \begin{cases} 1 + \frac{\alpha_i - m}{\alpha_i + m} g_j & \text{if } g_j \in \left[0, \frac{m+\alpha_i}{m-\alpha_i}\right] \\ 0 & \text{if } g_j > \frac{m+\alpha_i}{m-\alpha_i} \end{cases}$$

- This expression applies when $\alpha_i < m$.

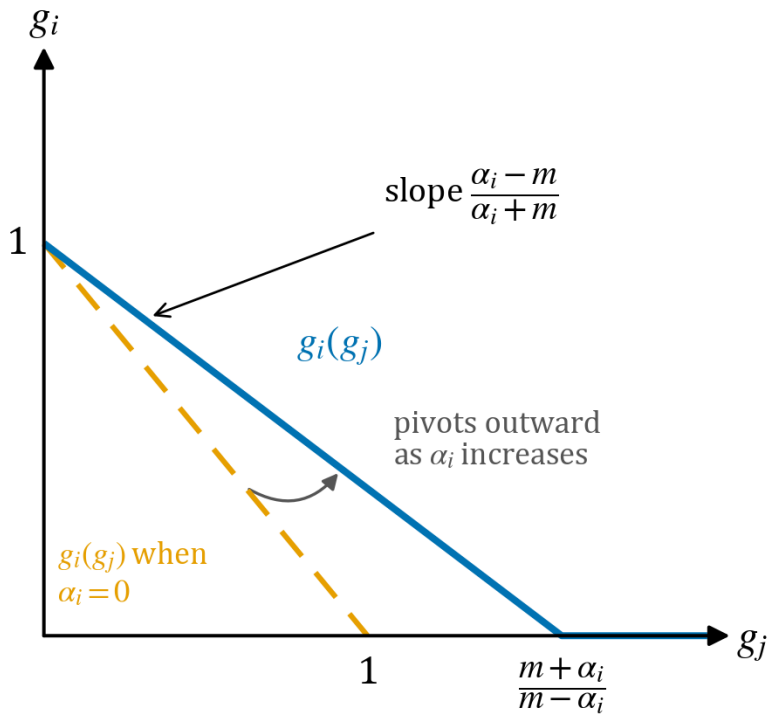
Competition for Status Acquisition

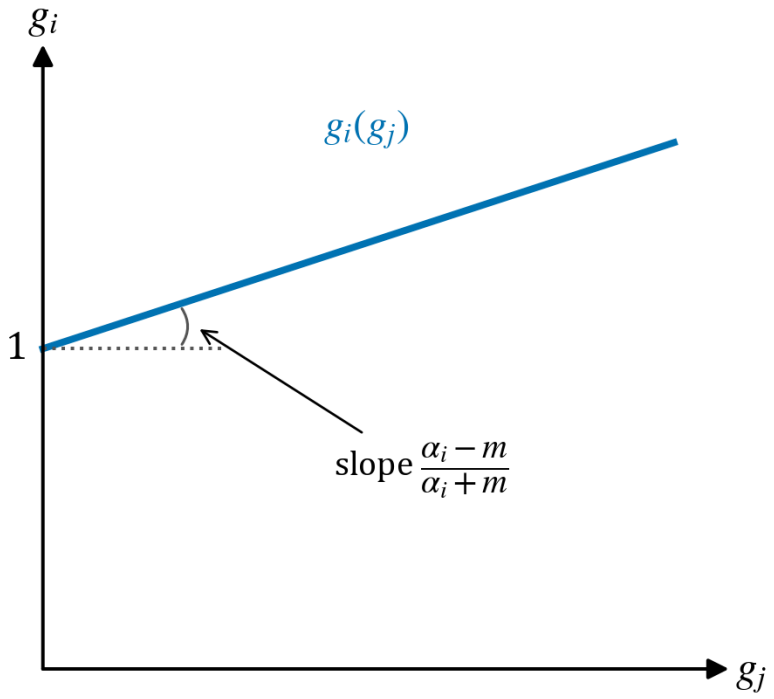
- When $\alpha_i \geq m$, the BRF is

$$g_i(g_j) = 1 + \frac{\alpha_i - m}{\alpha_i + m} g_j \quad \text{for all} \quad g_j \geq 0$$

- When $\alpha_i = m$, $g_i = 1$ regardless of g_j.
- Note that player i's BRF is
  - weakly decreasing in g_j if $\alpha_i < m$
  - increasing in g_j if $\alpha_i > m$
  - weakly increasing in α_i (strictly if $g_j > 0$)

Competition for Status Acquisition





$g_i(g_j)$ when $\alpha_i < m$

$g_i(g_j)$ when $\alpha_i > m$

Competition for Status Acquisition

- *Remarks*:
  - When $\alpha_i < m$, the positive externality that player j's contributions impose on player i's utility dominates the negative externality.
    - Player i views player j's contributions as strategic *substitutes* for his own.
  - When $\alpha_i > m$, the negative externality that player j's contributions impose on player i's utility dominates the positive externality.
    - Player i views player j's donations as strategic *complements* for his own.

Competition for Status Acquisition

- Solving the two linear BRFs $g_i = 1 + s_i g_j$, where $s_i = \frac{\alpha_i - m}{\alpha_i + m}$, gives, whenever $\alpha_i + \alpha_j > 0$,

$$g_i = \frac{1 + s_i}{1 - s_i s_j} = \frac{\alpha_i(\alpha_j + m)}{m(\alpha_i + \alpha_j)}$$

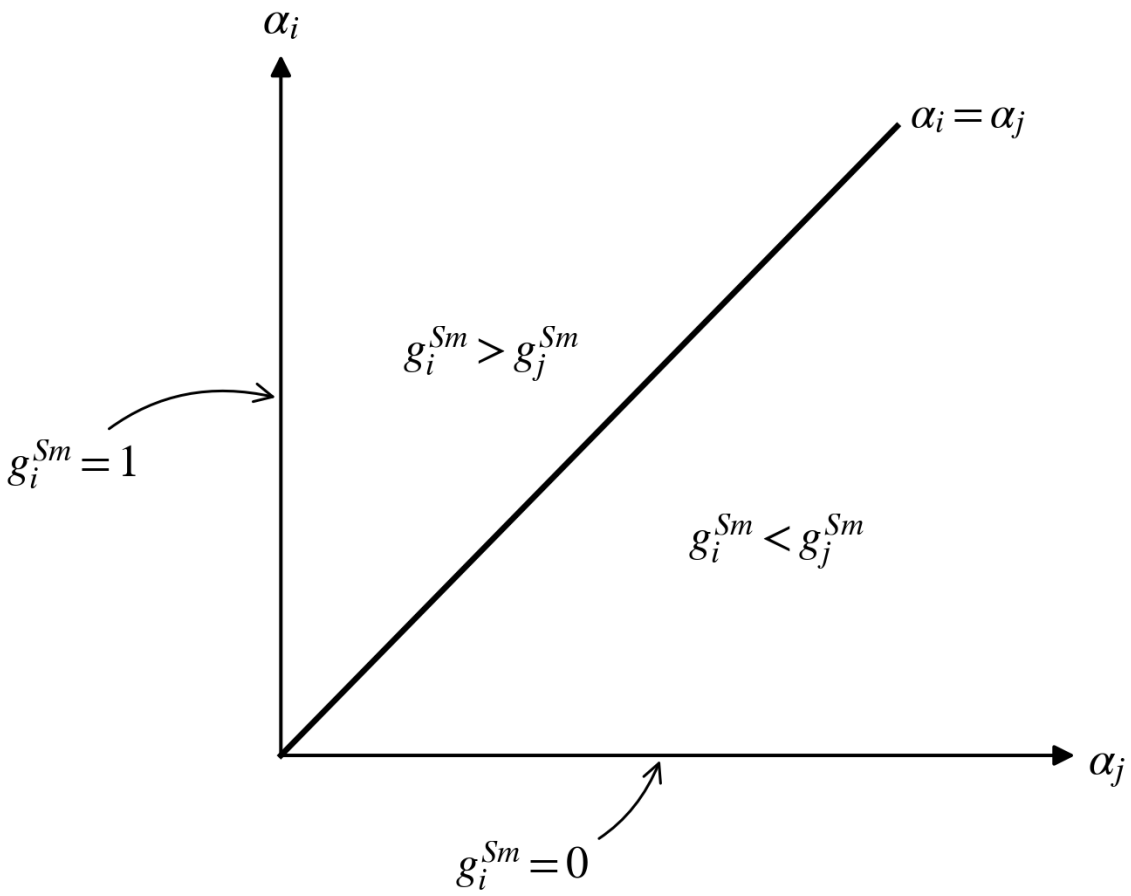
Competition for Status Acquisition

- Nash equilibrium contribution of player i (superscript Sm: status motive):

$$g_i^{Sm} = \begin{cases} 1 \quad \text{if} \quad \alpha_i > 0 \text{ and } \alpha_j = 0 \\ \dfrac{\alpha_i(\alpha_j + m)}{(\alpha_i + \alpha_j)m} \quad \text{if} \quad \alpha_i > 0 \text{ and } \alpha_j > 0 \\ 0 \quad \text{if} \quad \alpha_i = 0 \text{ and } \alpha_j > 0 \end{cases}$$

where $g_i^{Sm} + g_j^{Sm} = 1$ if $\alpha_i = \alpha_j = 0$.

Competition for Status Acquisition



Competition for Status Acquisition

- Total contributions to the public good are

$$G^{Sm} = \begin{cases} 1 \quad \text{if} \quad \alpha_i > 0 \text{ and } \alpha_j = 0 \\ 1 + \dfrac{2\alpha_i\alpha_j}{(\alpha_i+\alpha_j)m} \quad \text{if} \quad \alpha_i > 0 \text{ and } \alpha_j > 0 \\ 1 \quad \text{if} \quad \alpha_i = 0 \text{ and } \alpha_j > 0 \end{cases}$$

where G^{Sm} is weakly increasing in α_i and α_j and, for a given total status concern $\alpha_i + \alpha_j$, is highest when $\alpha_i = \alpha_j$.

Competition for Status Acquisition

- *Intuition*:
  - Total contributions when *only one* player values status coincide with total contributions when *neither* player does, $G^{Sm} = 1$.
  - An increase in the status concerns of only one individual (α_i or α_j) raises total contributions, provided the other individual also values status.
  - Finally, for a given total status concern, G^{Sm} is highest when $\alpha_i = \alpha_j$.

Appendix 1:
More General Policy Mechanisms

More General Policy Mechanisms

- In the presence of incomplete information, standard policy tools generally entail welfare losses (see quotas and taxes under incomplete information above).
- Let us examine more general policy mechanisms that try to maximize social surplus in the context of incomplete information.
- We consider mechanisms in which we ask agents to self-report their types.

More General Policy Mechanisms

- Recall the setting with incomplete information above: benefit $b(\theta)$, cost $c(\eta)$, and externality level $\overline{x}$.
- We ask the firm:
  - What is your benefit from increasing the externality level from $x = 0$ to $x = \bar{x}$, i.e., $b = b(\theta)$, given your private observation of θ?
- We ask the consumer:
  - What is your damage from the externality, i.e., $c = c(\eta)$, given your private observation of η?

More General Policy Mechanisms

- The mechanism we are interested in focuses on providing incentives to all parties to guarantee that a truth-telling equilibrium emerges.
- ***Groves-Clarke-Vickrey (GCV) mechanism***:
  - The regulator declares that it will set the level of the externality at $x = \bar{x}$ if $\hat{b} > \hat{c}$.
    - If this is the case, the regulator pays $\hat{b}$ to the consumer and charges $\hat{c}$ to the firm.
    - Not a typo!
  - Otherwise, the regulator keeps the level of the externality at $x = 0$.

More General Policy Mechanisms

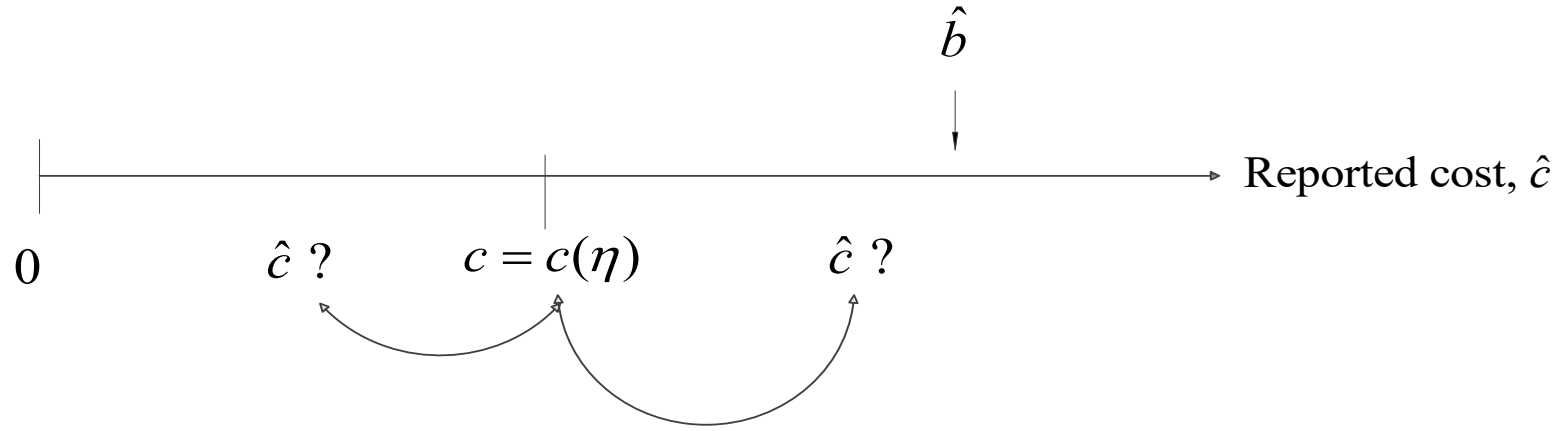
- ***Payoffs*** under the GCV mechanism:
  - Consumer: $\hat{b} - c$ if $\hat{c} < \hat{b}$, and 0 otherwise.
  - Firm: $b - \hat{c}$ if $\hat{c} < \hat{b}$, and 0 otherwise.

More General Policy Mechanisms

- Wouldn't this type of mechanism induce firms to *underreport* their benefits?
  - That is, stating a benefit $\hat{b} < b$, in order to reduce the compensation that they have to provide to those consumers affected by the externality.
- Also, wouldn't this type of mechanism induce consumers to *overreport* their damages?
  - That is, stating a cost $\hat{c} > c$, in order to guarantee that the externality is not allowed or, if allowed, they are substantially compensated for the cost they suffer.

More General Policy Mechanisms

- ***Consumer***:
  - Consider a consumer with a real cost $c = c(\eta)$.
  - Let us examine the consumer's optimal announcement, $\hat{c}$, given a firm's report of a benefit $\hat{b} > c$.



More General Policy Mechanisms

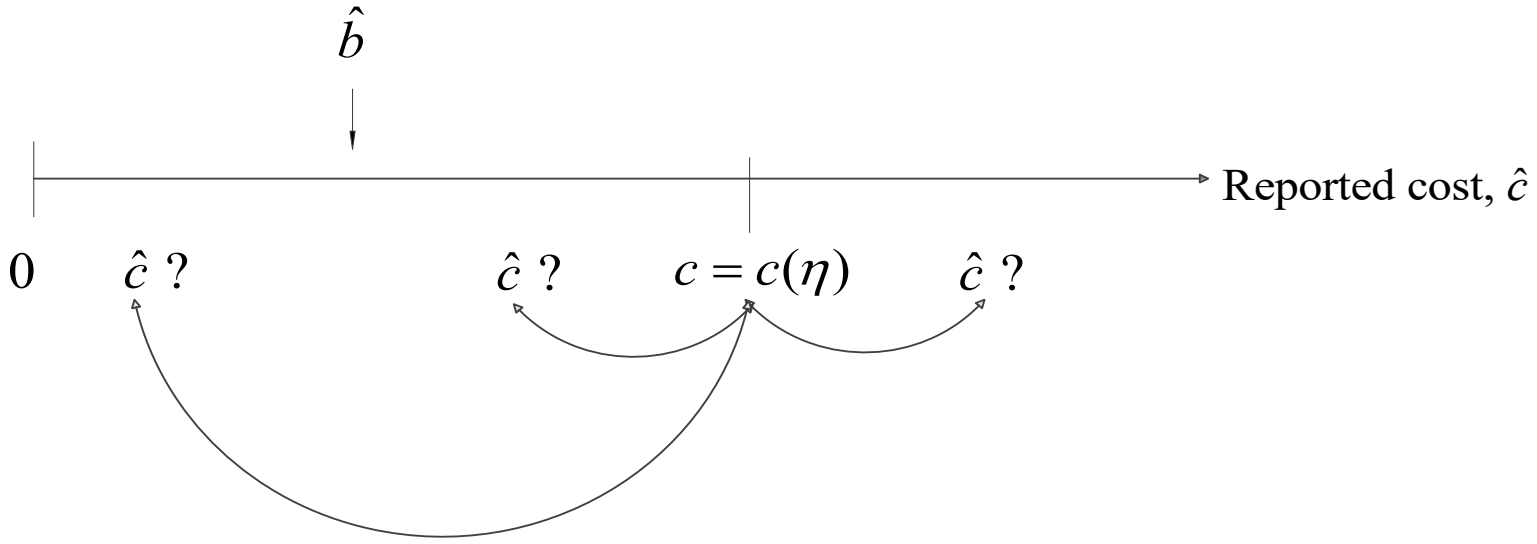
- The consumer does not have incentives to overreport her cost while keeping $\hat{c} < \hat{b}$, i.e., $c < \hat{c} < \hat{b}$, or underreport it, i.e., $\hat{c} < c$, since in both cases the payment she receives is $\hat{b}$.
  - The payment that the consumer receives is unaffected by her report, so truthful reporting is optimal; any other report $\hat{c} < \hat{b}$ does equally well.

More General Policy Mechanisms

- If the consumer overreports her costs, i.e., $\hat{c} > \hat{b}$, the regulator would decide to not allow the externality, i.e., $x = 0$.
  - Such an outcome yields a lower payoff for the consumer than the above outcomes, whereby a report $\hat{c} = c$ yields a payment of $\hat{b}$ from the regulator.

More General Policy Mechanisms

– Let us now examine the consumer's optimal announcement, $\hat{c}$, given a firm's announcement of a benefit $\hat{b} < c$.



More General Policy Mechanisms

- If the consumer overreports her costs, i.e., $\hat{c} > \hat{b}$, the regulator would decide to not allow the externality, i.e., $x = 0$.
- If the consumer slightly underreports her cost $\hat{c}$, i.e., $\hat{b} < \hat{c} < c$, the externality is still not allowed by the regulator, given that reports satisfy $\hat{c} > \hat{b}$.

More General Policy Mechanisms

– Finally, an extreme underreport of her costs, i.e., $\hat{c} < \hat{b}$, is not sensible either:

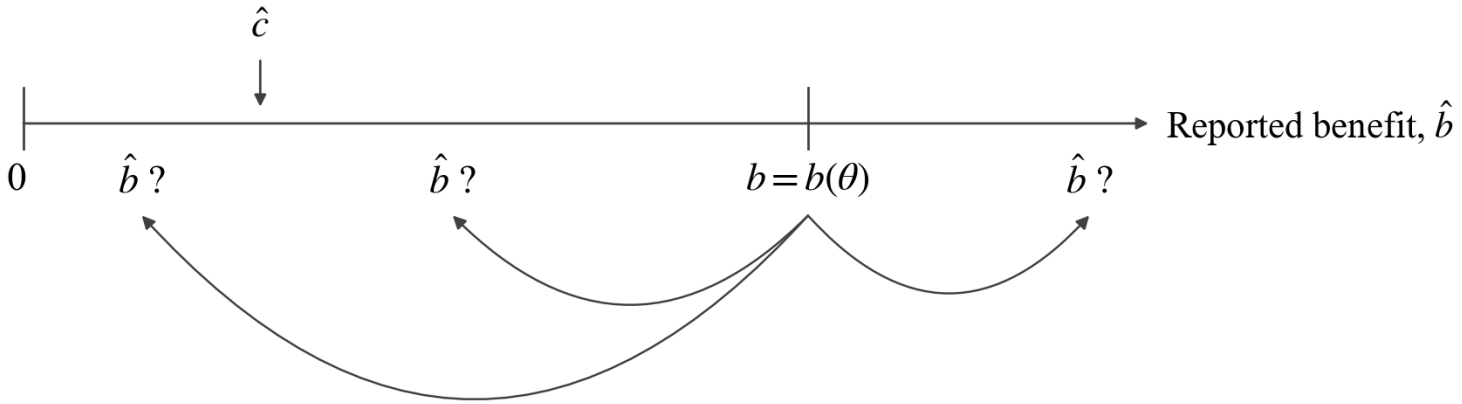
  - While the externality is now allowed (since $\hat{b} > \hat{c}$), the consumer receives a payment $\hat{b}$ below her true cost c, i.e., $c > \hat{b}$.

More General Policy Mechanisms

- Hence, the consumer has incentives to truthfully reveal the damage she suffers from the externality, $\hat{c} = c(\eta)$, regardless of the precise report $\hat{b}$ that the firm makes.
  - That is, truthfully reporting her cost is a weakly dominant strategy for the consumer.

More General Policy Mechanisms

- ***Firm***:
  - Consider a firm with a real benefit $b = b(\theta)$.
  - Let us first examine the firm's optimal announcement, $\hat{b}$, given a consumer's report of a cost $\hat{c} < b$.



More General Policy Mechanisms

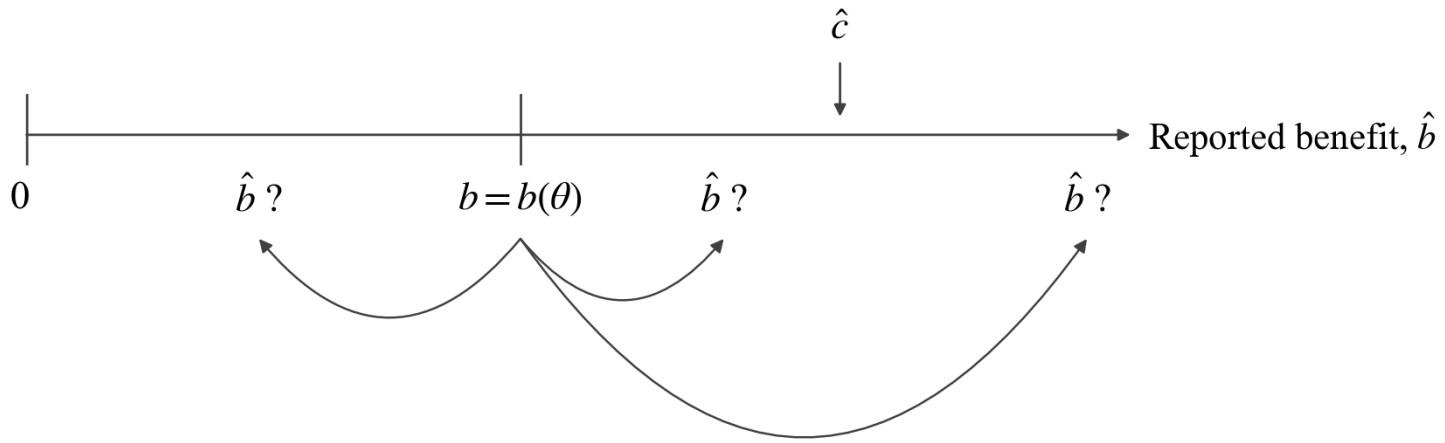
- The firm has no incentives to overreport its true benefit b, i.e., $\hat{b} > b$.
  - The firm would have to pay the same charge, $\hat{c}$, to the regulator, and the externality would still be allowed since reports satisfy $\hat{b} > \hat{c}$.
- The firm has no incentives to slightly underreport its true benefit, i.e., $\hat{c} < \hat{b} < b$.
  - The charge that the firm has to pay is still $\hat{c}$ and the externality is allowed, since reports still satisfy $\hat{b} > \hat{c}$.

More General Policy Mechanisms

- Finally, the firm has no incentives to extremely underreport its true benefit, i.e., $\hat{b} < \hat{c}$.
  - In this case, the externality would not be allowed by the regulator given that reports satisfy $\hat{b} < \hat{c}$, so the firm forgoes $b - \hat{c} > 0$.

More General Policy Mechanisms

– Let us now consider the case where consumer's report $\hat{c}$ lies above the firm's true benefit b, i.e., $\hat{c} > b$.



More General Policy Mechanisms

- If the firm overreports its benefit, i.e., $b < \hat{c} < \hat{b}$, the externality would be allowed (since $\hat{b} > \hat{c}$).
  - However, the firm has to pay a charge $\hat{c}$ to the regulator, which is higher than the real benefit that the firm obtains from the externality, i.e., $b < \hat{c}$.
- If the firm slightly overreports its benefits, i.e., $b < \hat{b} < \hat{c}$, or underreports it, i.e., $\hat{b} < b < \hat{c}$, the externality will not be allowed given that reports would now satisfy $\hat{b} < \hat{c}$.

More General Policy Mechanisms

- Hence, the firm prefers no externality whatsoever, i.e., $x = 0$.
  - The true benefit that the firm obtains from the externality b lies below the cost $\hat{c}$ that the consumer reported.
- Therefore, truthfully reporting its benefit from the externality is a weakly dominant strategy for the firm.

More General Policy Mechanisms

- Truth-telling by both parties gives $x = \overline{x}$ if and only if $b > c$, the efficient outcome.
- When $x = \overline{x}$, the regulator pays $\hat{b}$ to the consumer and collects $\hat{c}$ from the firm, a net outlay of $b - c > 0$: the mechanism is not budget balanced.

Advanced Microeconomic Theory

Chapter 10: Contract Theory

Outline

- Moral Hazard
- Moral Hazard with a Continuum of Effort Levels—The First-Order Approach
- Moral Hazard with Multiple Signals
- Adverse Selection—The “Lemons” Problem
- Adverse Selection—The Principal–Agent Problem
- Application of Adverse Selection—Regulation

Moral Hazard

Moral Hazard

- **Moral hazard**: settings in which an agent does not observe the actions of the other individual(s).
  - Also referred to as "hidden action"
- Example:
  - A manager in a firm cannot observe the effort of employees in the firm even if the manager is perfectly informed about the worker's ability or productivity.
  - The worker might have incentives to *slack* from exerting a costly effort, thus giving rise to moral hazard problems.

Moral Hazard

- The manager can offer contracts that provide incentives to the worker to work hard
  - Paying a higher salary (bonus) if the worker's output is high but a low salary otherwise.
- Providing incentives to work hard is costly for the manager
- The manager only induces a high effort if the firm's expected profits are higher than those of inducing a low effort

Moral Hazard

- Consider a principal with benefit function

$$B(\pi - w)$$

where π is the profit that arises from the agent's effort and w is the salary that the principal pays to the agent.

- The benefit function satisfies $B' \geq 0$ and $B'' \leq 0$.
- The agent's (quasi-linear) utility function is

$$U(w, e) = u(w) - g(e)$$

where $u(w)$ is utility from the agent's salary, for $u' > 0$ and $u'' \leq 0$, and $g(e)$ is the agent's disutility from effort (e), for $g' > 0$ and $g'' \geq 0$.

Moral Hazard

- The agent's effort level e affects the probability that a certain profit occurs.
- For a given effort e, the conditional probability that a profit $\pi = \pi_i$ is

$$f(\pi_i|e) = \text{Prob}\{\pi = \pi_i|e\} \geq 0$$

where $i = \{1,2, \dots, N\}$ is the profits that can emerge for a given effort e.

- Hence a high profit could arise even if the worker slacks
  - That is, a given profit level $\pi = \pi_i$ can arise from every effort level

Symmetric Information

- The principal **can observe** the agent's effort level e.
- The principal's maximization problem is

$$\max_{\{e,w(\pi_i)\}_{i=1}^{N}} \sum_{i=1}^{N} f(\pi_i|e) \cdot B(\pi_i - w(\pi_i))$$

$$\text{s.t.} \quad \sum_{i=1}^{N} f(\pi_i|e)\, u(w(\pi_i)) - g(e) \geq \bar{u}$$

- The principal seeks to maximize expected profits, subject to the agent participating in the contract.
  - The constraint guarantees the agent's voluntary participation in the contract.
  - Hence it is referred to as the participation constraint (PC) or the "individual rationality" condition.
- The constraint must be binding (holding with equality).

Symmetric Information

- The Lagrangian that solves the maximization problem is

$$\mathcal{L} = \sum_{i=1}^{N} f(\pi_i|e) \cdot B\big(\pi_i - w(\pi_i)\big) + \lambda\left[\sum_{i=1}^{N} f(\pi_i|e)\, u\big(w(\pi_i)\big) - g(e) - \bar{u}\right]$$

- Take FOC with respect to w to obtain

$$f(\pi_i|e) \cdot B'\big(\pi_i - w(\pi_i)\big) \cdot (-1) + \lambda f(\pi_i|e) u'\big(w(\pi_i)\big) = 0$$

where B' and u' are the derivative of $B(\cdot)$ and $u(\cdot)$ with respect to w, respectively.

Symmetric Information

- Rearranging

$$\lambda u'\big(w(\pi_i)\big) = B'\big(\pi_i - w(\pi_i)\big)$$

- Solving for λ

$$\lambda = \frac{B'\big(\pi_i - w(\pi_i)\big)}{u'\big(w(\pi_i)\big)} \qquad \textbf{(1)}$$

  which is positive since $B'(\cdot) > 0$ and $u'(\cdot) > 0$.

- $\lambda > 0$ entails that the agent's participation constraint must bind (i.e., hold with equality)

$$\sum_{i=1}^{N} f(\pi_i|e)u\big(w(\pi_i)\big) - g(e) = \bar{u}$$

Symmetric Information

- Example 1:
  - Consider a risk-neutral principal hiring a risk-averse agent with utility function $u(w) = \sqrt{w}$, disutility of effort $g(e) = e$, and reservation utility $\bar{u} = 9$.
  - There are two effort levels $e_H = 5$ and $e_L = 0$.
  - When $e_H = 5$, the principal's sales are \$0 with probability 0.1, \$100 with probability 0.3, and \$400 with probability 0.6.
  - When $e_L = 0$, the principal's sales are \$0 with probability 0.6, \$100 with probability 0.3, and \$400 with probability 0.1.
  - In the case of $e_H = 5$, the expected profit is \$270, while in the case of $e_L = 0$, the expected profit is \$70.

Symmetric Information

- Example 1: (con't)
  - When effort is observable, the principal can induce an effort $e_H = 5$ by paying a wage w_e^* that solves

$$u(w_e^*) = \bar{u} + g(e)$$

$$\sqrt{w_e^*} = 9 + 5$$

$$w_e^* = 14^2 = 196$$

  - Similarly, the principal can induce a low effort $e_L = 0$ by offering a wage

$$\sqrt{w_e^*} = 9 + 0$$

$$w_e^* = 9^2 = 81$$

Symmetric Information

- Example 1: (con't)
  - Given these salaries, the profits that the principal obtains are

$$\$270 - \$196 = \$74 \text{ from } e_H = 5$$

$$\$70 - \$81 = -\$11 \text{ from } e_L = 0$$

  - Thus the principal prefers to induce e_H when effort is observable.

Risk Attitudes

- Continuing with the moral hazard setting under symmetric information, let us now consider the role of **risk aversion**.
- Three cases:
  1. The principal is risk neutral but the agent is risk averse
  2. The principal is risk averse but the agent is risk neutral
  3. Both the principal and the agent are risk averse

Risk Attitudes: Case 1

- The principal is risk neutral but the agent is risk averse.
- The principal's benefit function is

$$B\big(\pi_i - w(\pi_i)\big) = \pi_i - w(\pi_i)$$

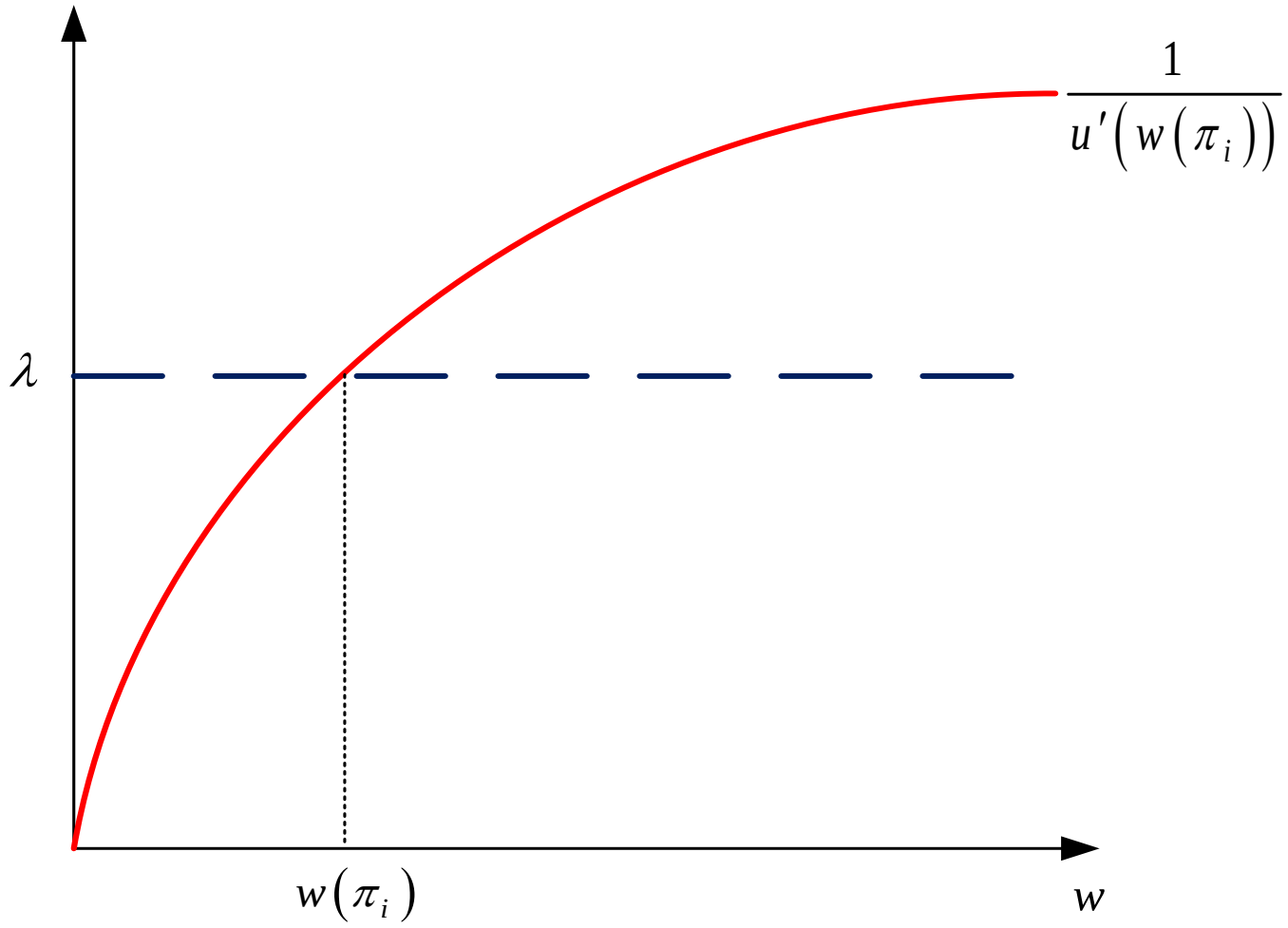
- Hence,

$$B'\big(\pi_i - w(\pi_i)\big) = 1$$

- In this context, FOC in expression **(1)** becomes

$$\lambda = \frac{1}{u'\big(w(\pi_i)\big)} \text{ for all } \pi_i \qquad \textbf{(2)}$$

Risk Attitudes: Case 1



Risk Attitudes: Case 1

- FOC in **(2)** entails that the principal pays a **fixed wage** level for all profit realizations.
- For any $\pi_i \neq \pi_j$,

$$\lambda = \frac{1}{u'\big(w(\pi_i)\big)} = \frac{1}{u'\left(w\big(\pi_j\big)\right)}$$

$$u'\big(w(\pi_i)\big) = u'\left(w\big(\pi_j\big)\right)$$

$$w(\pi_i) = w\big(\pi_j\big) \text{ given } u' > 0$$

- This is a **standard risk-sharing result**
  - The risk-neutral principal offers a contract to the risk-averse agent that guarantees the latter a fixed salary of w_e^* regardless of the specific profit realization that emerges.
  - The risk-neutral principal bears all the risk.

Risk Attitudes: Case 1

- Since the agent's PC binds, we can express it

$$u(w_e^*) - g(e) = \bar{u}$$

- Rearranging the PC expression

$$u(w_e^*) = \bar{u} + g(e)$$

- Applying the inverse

$$w_e^* = u^{-1}(\bar{u} + g(e))$$

- This expression helps to identify the salary that the principal needs to offer in order to induce a specific effort level e from the agent.

Risk Attitudes: Case 1

- For two effort levels e_L and e_H, the disutility of effort function satisfies

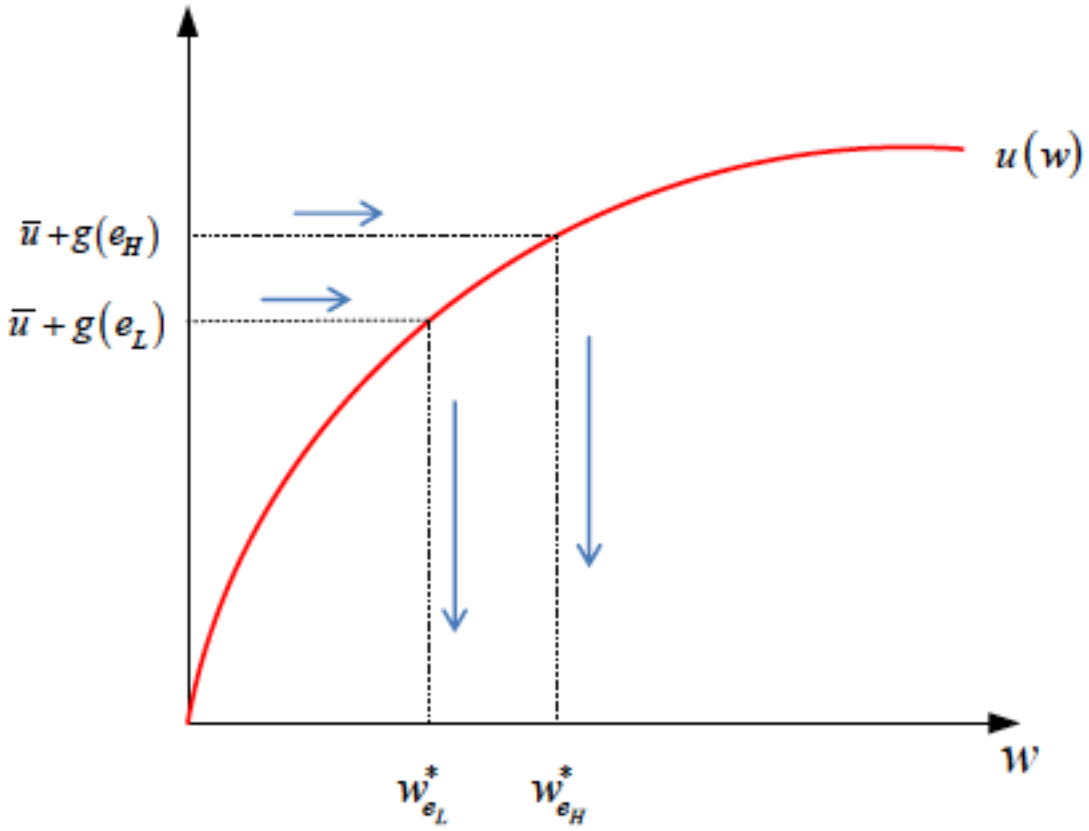
$$g(e_L) < g(e_H)$$

- This entails

$$w_{e_L}^* = u^{-1}\big(\bar{u} + g(e_L)\big) < u^{-1}\big(\bar{u} + g(e_H)\big) = w_{e_H}^*$$

- In order to induce e_H, we need to evaluate the utility function at a height of $\bar{u} + g(e_H)$.
- Inducing a higher effort implies offering a higher salary.

Risk Attitudes: Case 1



Risk Attitudes: Case 1

- We can plug in salary $w_e^* = u^{-1}\big(\bar{u} + g(e)\big)$ into the principal's objective function in order to find the effort level that maximizes the principal's expected profits

$$\max_e \sum_{i=1}^{N} f(\pi_i|e)\big(\pi_i - w(\pi_i)\big)$$

$$= \max_e \sum_{i=1}^{N} f(\pi_i|e)\cdot\pi_i - \underbrace{u^{-1}\big(\bar{u} + g(e)\big)}_{w_e^*}$$

  where w_e^* does not depend on π_i.
- This helps to reduce the number of choice variables to only the effort level e.

Risk Attitudes: Case 1

- Taking FOC with respect to e yields

$$\sum_{i=1}^{N} f'(\pi_i|e) \cdot \pi_i - \frac{\partial u^{-1}\left(\bar{u} + g(e)\right)}{\partial e} g'(e) = 0$$

where $\frac{\partial u^{-1}\left(\bar{u}+g(e)\right)}{\partial e}$ can be expressed as $(u^{-1})'\left(\bar{u} + g(e)\right)$.

- By the implicit function theorem,

$$(u^{-1})'\left(\bar{u} + g(e)\right) = (u')^{-1}\left(\bar{u} + g(e)\right)$$

- Hence the above FOC can be rewritten as

$$\sum_{i=1}^{N} f'(\pi_i|e) \cdot \pi_i = \frac{g'(e)}{u'\left(\bar{u} + g(e)\right)}$$

Risk Attitudes: Case 1

- Intuition:
  - Effort e is increased until the point at which its marginal expected profit (left-hand side) coincides with its certain costs (in the right-hand side), which stems from a larger marginal disutility of effort for the agent (numerator) that needs to be compensated with a more generous salary (denominator).
- See textbook for the second-order condition that guarantees concavity.

Risk Attitudes: Case 2

- The principal is risk averse but the agent is risk neutral.
- The principal's benefit function is $B\big(\pi_i - w(\pi_i)\big)$, with $B' > 0$ and $B'' < 0$.
- The agent's utility function is

$$u(w_i) - g(e) = w_i - g(e)$$

- In this context, FOC in expression **(1)** becomes

$$\lambda = B'\big(\pi_i - w(\pi_i)\big)$$

where $u'\big(w(\pi_i)\big) = 1$.

Risk Attitudes: Case 2

- FOC entails that it is now the principal who obtains a **fixed payoff** for all profit realizations.
- For any $\pi_i \neq \pi_j$,

$$\lambda = B'\big(\pi_i - w(\pi_i)\big) = B'\left(\pi_j - w\big(\pi_j\big)\right)$$

$$\pi_i - w(\pi_i) = \pi_j - w\big(\pi_j\big) = K \text{ given } B' > 0$$

- That is, the risk-averse principal receives the same payoff regardless of the profit realization π, whereas the risk-neutral agent now bears all the risk.

Risk Attitudes: Case 2

- The agent's salary is

$$w(\pi_i) = \pi_i - K$$

where K is found by making the agent indifferent between accepting and rejecting the franchise contract

- Fee K solves

$$\sum_{i=1}^{N} f(\pi_i|e)[\pi_i - K] - g(e) = \bar{u}$$

$$K = \sum_{i=1}^{N} f(\pi_i|e)\pi_i - \bar{u} - g(e)$$

Risk Attitudes: Case 2

- The principal's expected profit is

$$\sum_{i=1}^{N} f(\pi_i|e)B\big(\pi_i - w(\pi_i)\big)$$
$$= \sum_{i=1}^{N} f(\pi_i|e)B\big(\pi_i - (\pi_i - K)\big)$$
$$= \sum_{i=1}^{N} f(\pi_i|e)B(K) = B(K)$$

- The principal's problem can then be written as

$$\max_{e} \; B(K) = B\left(\sum_{i=1}^{N} f(\pi_i|e)\pi_i - \bar{u} - g(e)\right)$$

Risk Attitudes: Case 2

- Taking FOC with respect to e yields

$$B'\left(\sum_{i=1}^{N} f(\pi_i|e)\pi_i - \bar{u} - g(e)\right)\left(\sum_{i=1}^{N} f'(\pi_i|e)\pi_i - g'(e)\right) = 0$$

which simplifies to

$$\sum_{i=1}^{N} f'(\pi_i|e)\pi_i = g'(e)$$

- Intuition:
  - Effort e is increased until the point where marginal expected profit from having the agent exert more effort (left-hand side) coincides with his marginal disutility (right-hand side).

Risk Attitudes: Case 2

- The second-order condition is

$$\sum_{i=1}^{N} f''(\pi_i|e)\pi_i - g''(e) \leq 0$$

where $g''(e) \geq 0$.

Risk Attitudes: Case 3

- Both the principal and the agent are risk averse.
- Recall the FOCs with respect to w

$$B'\big(\pi_i - w(\pi_i)\big) \cdot (-1) + \lambda u'\big(w(\pi_i)\big) = 0 \qquad \textbf{(3)}$$

$$\lambda = \frac{B'\big(\pi_i - w(\pi_i)\big)}{u'\big(w(\pi_i)\big)} \qquad \textbf{(4)}$$

- To better understand how the profit-maximizing salary is affected by the profit realization π_i, differentiate **(3)** with respect to π_i

$$-B''\big(\pi_i - w(\pi_i)\big) + B''\big(\pi_i - w(\pi_i)\big)w'(\pi_i) + \lambda u''\big(w(\pi_i)\big)w'(\pi_i) = 0$$

Risk Attitudes: Case 3

- Plugging λ from **(4)** yields

$$-B''\big(\pi_i - w(\pi_i)\big) + B''\big(\pi_i - w(\pi_i)\big)w'(\pi_i) + \frac{B'\big(\pi_i - w(\pi_i)\big)}{u'\big(w(\pi_i)\big)}u''\big(w(\pi_i)\big)w'(\pi_i) = 0$$

- Factoring out $w'(\pi_i)$ yields

$$B''\big(\pi_i - w(\pi_i)\big) = \left[B''\big(\pi_i - w(\pi_i)\big) + B'\big(\pi_i - w(\pi_i)\big)\frac{u''\big(w(\pi_i)\big)}{u'\big(w(\pi_i)\big)}\right]w'(\pi_i)$$

Risk Attitudes: Case 3

- Solving for $w'(\pi_i)$ yields

$$w'(\pi_i) = \frac{B''\big(\pi_i - w(\pi_i)\big)}{B''(\cdot) + \dfrac{u''\big(w(\pi_i)\big)}{u'\big(w(\pi_i)\big)} B'(\cdot)}$$

- Dividing numerator and denominator by $B'(\cdot)$ yields

$$w'(\pi_i) = \frac{B''(\cdot)/B'(\cdot)}{B''(\cdot)/B'(\cdot) + \dfrac{u''(\cdot)}{u'(\cdot)}} = \frac{r_P}{r_P + r_A} \qquad \textbf{(5)}$$

where r_P and r_A denote the **Arrow–Pratt coefficient of absolute risk aversion** of the principal and the agent, respectively.

Risk Attitudes: Case 3

- Differentiating $\frac{r_P}{r_P+r_A}$ with respect to r_A, we find

$$\frac{\partial}{\partial r_A}\left[\frac{r_P}{r_P+r_A}\right] = -\frac{r_P}{(r_P+r_A)^2} < 0$$

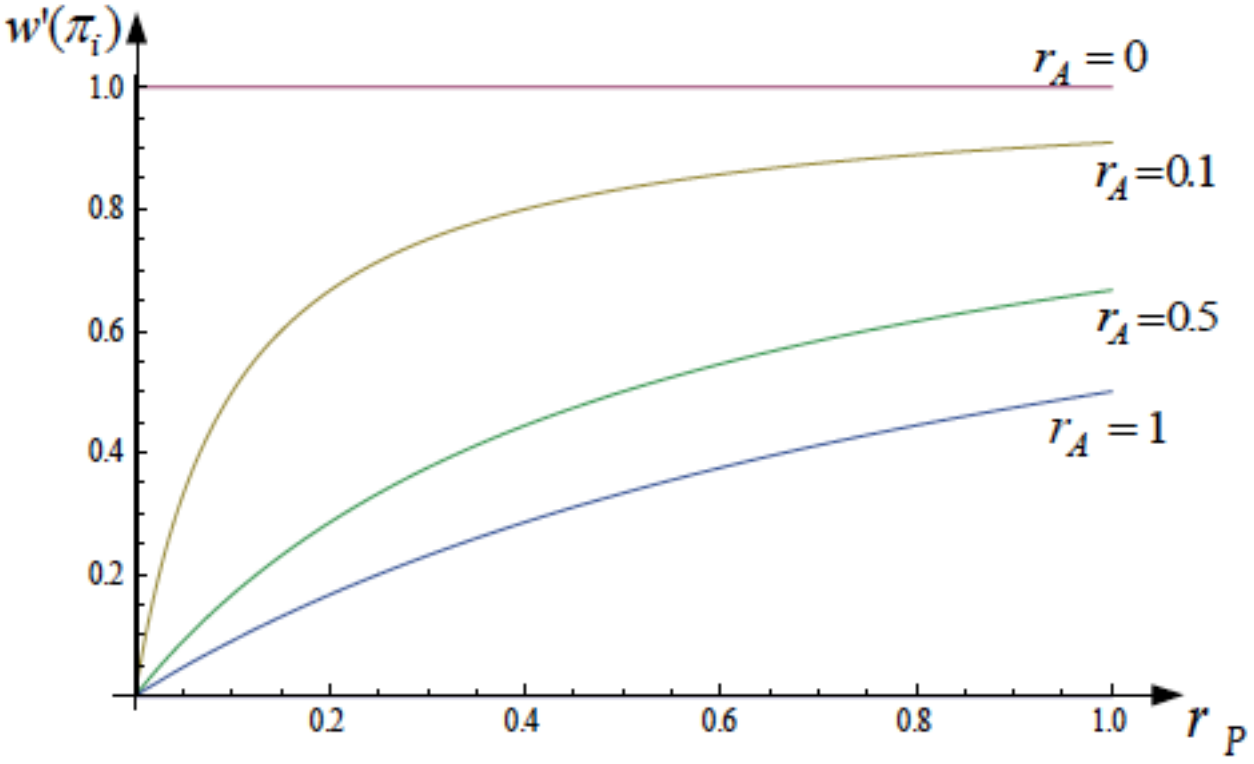
  - The agent bears less risk when he becomes more risk averse, as seen from the flattening of the marginal wage curves.

- Differentiating $\frac{r_P}{r_P+r_A}$ with respect to r_{P}, we have

$$\frac{\partial}{\partial r_P}\left[\frac{r_P}{r_P+r_A}\right] = \frac{r_A}{(r_P+r_A)^2} > 0$$

  - The agent bears more risk when the principal becomes more risk averse, as seen from the positive slope of the curves.

Risk Attitudes: Case 3



Risk Attitudes: Case 3

- Let us next evaluate the ratio in expression **(5)** at different values of r_P and r_A.
- **Risk neutral principal**: $r_P = 0$
  - The expression in **(5)** becomes $w'(\pi_i) = 0$.
  - This result holds regardless of the agent's coefficient of risk aversion $r_A > 0$, as seen from the origin of the curves.
- This setting coincides with that in Case 1, where the agent receives a fixed wage to insure him against the profit realization π_i, whereas the risk-neutral principal bears all the risk.

Risk Attitudes: Case 3

- **Risk neutral agent**: $r_A = 0$
  - The expression in **(5)** becomes $w'(\pi_i) = 1$.
  - This holds regardless of principal's coefficient of risk aversion $r_P > 0$, as seen from the flat curve on the top.
  - This setting coincides with that in Case 2, where the risk-neutral agent bears all the risk while the principal receives a fixed payment K that insures him against different profit realization π_i.

Risk Attitudes: Case 3

- **Agent is more risk averse than principal**: $r_A > r_P > 0$
  - The expression in **(5)** becomes $w'(\pi_i) < 1/2$.
  - It is optimal for the agent's salary $w'(\pi_i)$ to exhibit small variations in the profit realization π_i.
  - The more risk-averse agent bears less payoff volatility.

Risk Attitudes: Case 3

- **Principal is more risk averse than agent**: $r_P > r_A > 0$
  - The expression in **(5)** becomes $w'(\pi_i) > 1/2$.
  - The less risk-averse agent bears more payoff volatility.

Risk Attitudes: Case 3

- **Same degree of risk aversion**: $r_A = r_P = r > 0$
  - The expression in **(5)** becomes $w'(\pi_i) = 1/2$.
  - Both the agent and the principal bear the same risk in the contract.

Risk Attitudes: Case 3

- **Extremely risk averse agent**: $r_A \to \infty$
  - The expression in **(5)** becomes

$$\lim_{r_A \to \infty} \frac{r_P}{r_P + r_A} = 0$$

  - The agent cannot bear any risk so he receives a fixed wage.
- **Extremely risk averse principal**: $r_P \to \infty$
  - The expression in **(5)** becomes

$$\lim_{r_P \to \infty} \frac{1}{1 + r_A / r_P} = 1$$

  - The agent bears all the risk so he pays a fixed franchise fee.

Asymmetric Information

- The principal **cannot observe** the agent's effort level e.
- The principal needs to offer to the agent enough incentives to exert the profit-maximizing effort level.
- How can the principal achieve this objective?
  - Make the salary an increasing function of the realized profit.
  - This is optimal even if the agent is risk averse.

Asymmetric Information

- The principal's problem is

$$\max_{\{e,w(\pi_i)\}_{i=1}^{N}} \sum_{i=1}^{N} f(\pi_i|e) \cdot B(\pi_i - w(\pi_i))$$

s.t. $\sum_{i=1}^{N} f(\pi_i|e)\,[u(w(\pi_i)) - g(e)] \geq \bar{u}$

$$e^* \in \arg\max_{e} \sum_{i=1}^{N} f(\pi_i|e)\,[u(w(\pi_i)) - g(e)]$$

- The principal seeks to maximize its expected profits subject to:
  1. The voluntary participation of the agent (PC condition);
  2. The effort that he anticipates the agent will optimally choose in order to maximize his expected utility after receiving the contract from the principal (incentive compatibility, IC, condition).

Asymmetric Information

- Assume there are only two different effort levels available to the agent (e_L and e_H, where $e_H > e_L$).
- The agent can choose to work a positive number of hours or completely slack from the job ($e_H = e > 0$ and $e_L = 0$).
- Consider that the principal seeks to induce the high effort level e_H and that the principal is **risk neutral** while the agent is **risk averse**.

Asymmetric Information

- The principal's problem reduces to

$$\max_{\{e,w(\pi_i)\}_{i=1}^{N}} \sum_{i=1}^{N} f(\pi_i|e_H) \cdot [\pi_i - w(\pi_i)]$$

s.t. $\sum_{i=1}^{N} f(\pi_i|e_H)\,[u(w(\pi_i)) - g(e_H)] \geq \bar{u}$ (PC)

$$\sum_{i=1}^{N} f(\pi_i|e_H)\left[u\left(w(\pi_i)\right) - g(e_H)\right] \geq \sum_{i=1}^{N} f(\pi_i|e_L)\,[u(w(\pi_i)) - g(e_L)] \quad \text{(IC)}$$

where the IC condition induces the agent to choose effort level e_H as such effort yields a higher expected utility than e_L for the agent.

Asymmetric Information

- The Lagrangian becomes

$$\mathcal{L} = \sum_{i=1}^{N} f(\pi_i|e_H) \cdot [\pi_i - w(\pi_i)]$$
$$+ \lambda \left[\sum_{i=1}^{N} f(\pi_i|e_H) \left[u\big(w(\pi_i)\big) - g(e_H) \right] - \bar{u} \right]$$
$$+ \mu \left\{ \sum_{i=1}^{N} f(\pi_i|e_H) \left[u\big(w(\pi_i)\big) - g(e_H) \right] \right.$$
$$\left. - \left[\sum_{i=1}^{N} f(\pi_i|e_L) \left[u\big(w(\pi_i)\big) - g(e_L) \right] \right] \right\}$$

Asymmetric Information

- Taking FOC with respect to w yields

$$-f(\pi_i|e_H) + \lambda f(\pi_i|e_H)u'\big(w(\pi_i)\big) + \mu\big[f(\pi_i|e_H)u'\big(w(\pi_i)\big) - f(\pi_i|e_L)u'\big(w(\pi_i)\big)\big] = 0$$

- Rearranging

$$\lambda + \underbrace{\mu\left[1 - \frac{f(\pi_i|e_L)}{f(\pi_i|e_H)}\right]}_{\text{New}} = \frac{1}{u'\big(w(\pi_i)\big)} \qquad \textbf{(6)}$$

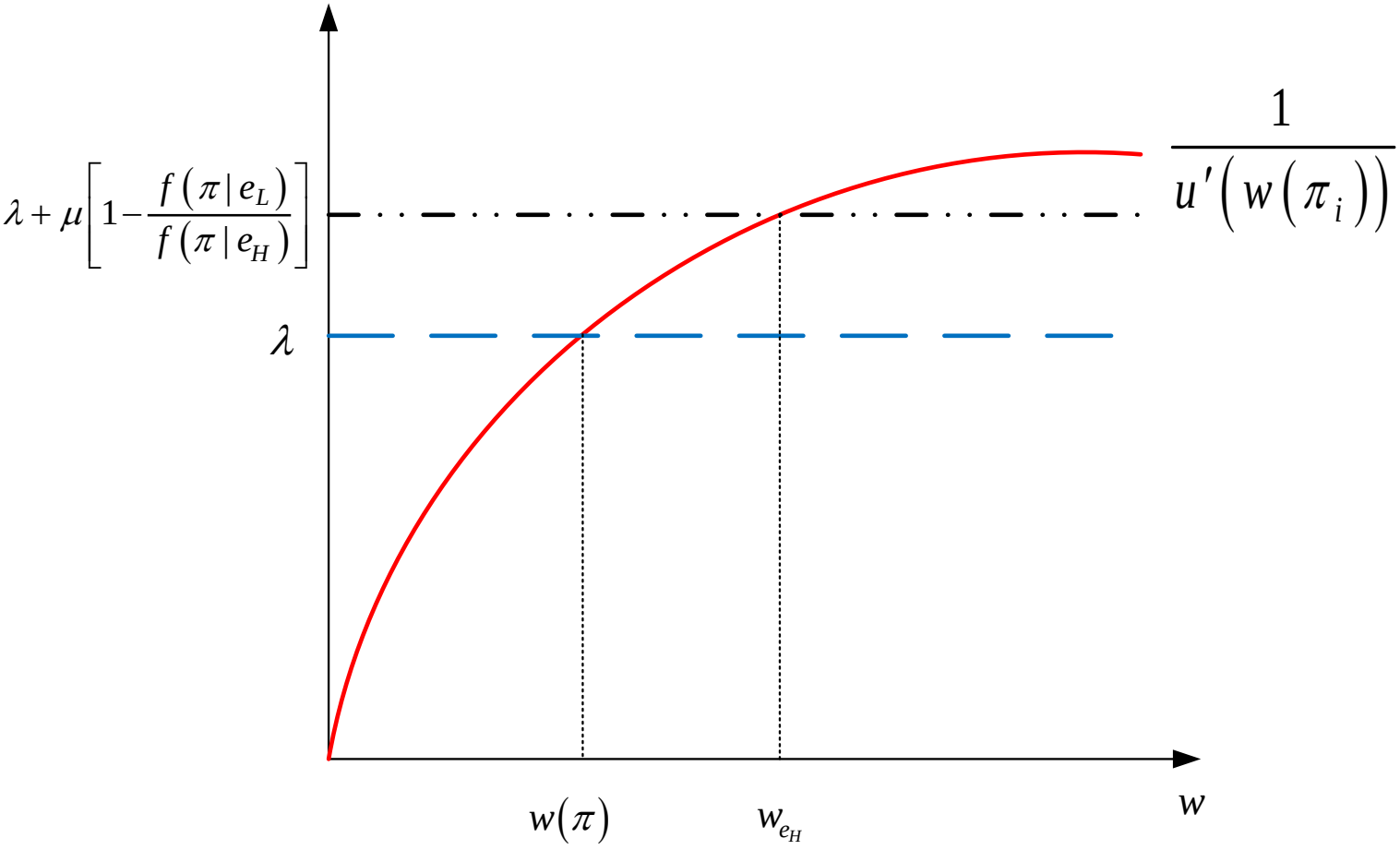
- Compare **(6)** with expression **(2)** in case 1, where the principal was risk neutral but the agent was risk averse.

Asymmetric Information

- Because $\lambda > 0$, $\mu > 0$, and $\frac{f(\pi_i|e_L)}{f(\pi_i|e_H)} < 1$, then

$$\underbrace{\lambda + \mu\left[1 - \frac{f(\pi_i|e_L)}{f(\pi_i|e_H)}\right]}_{\text{asymmetric info.}} > \underbrace{\lambda}_{\text{symmetric info.}}$$

Asymmetric Information



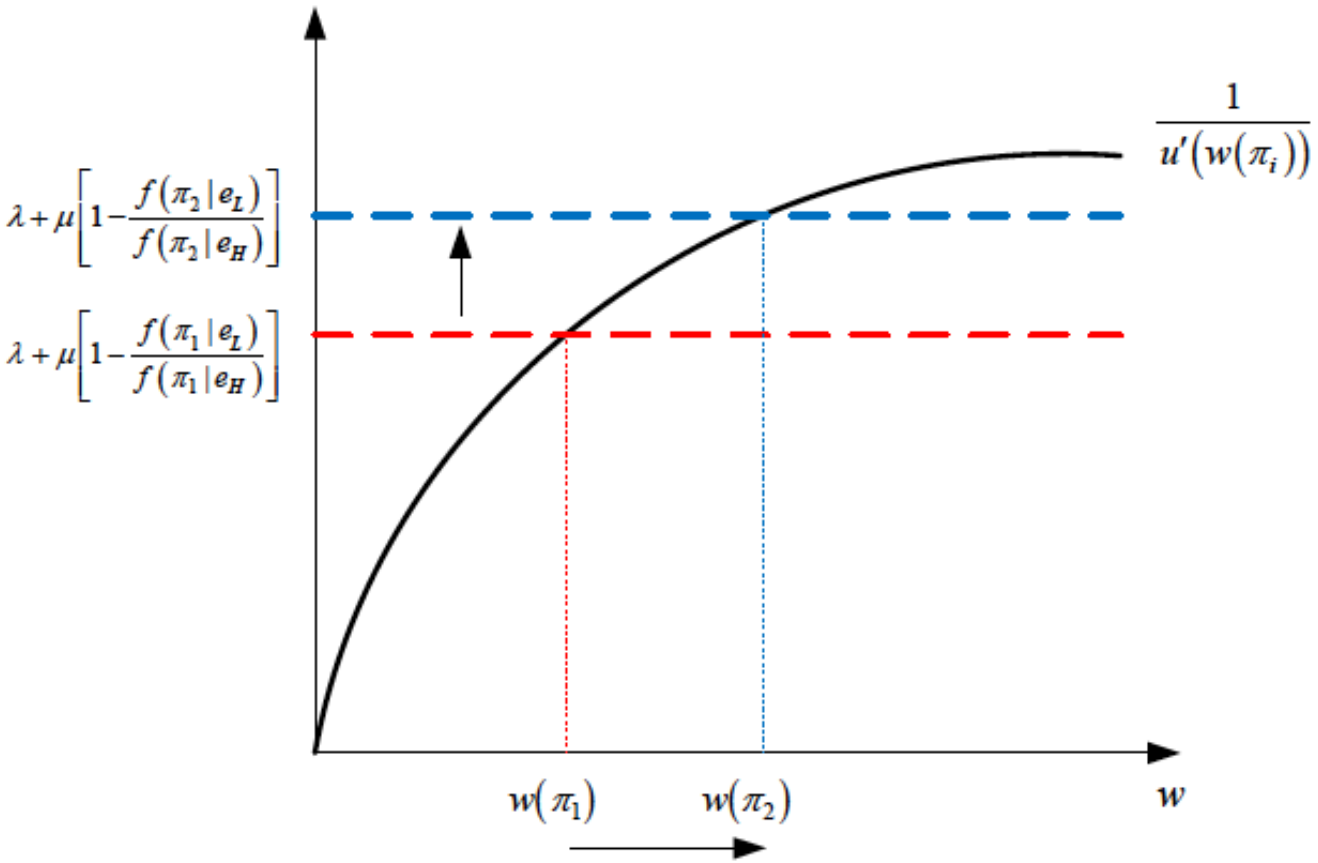
Asymmetric Information

- When deciding which effort to implement, the principal compares the effects of inducing a high effort level e_H.
  - Effort e_H yields a **positive effect** on profits since it increases the likelihood of higher profits.
  - This positive effect emerges under both symmetric and asymmetric information.
  - Effort e_H also entails a **negative effect** on profits since the salary that induces such effort is higher under asymmetric than under symmetric information $w_{e_H} > w(\pi)$.
  - Hence the principal is less willing to induce e_H when the agent's effort is unobservable than when it is observable.

Comparative Statics

- How does the salary above change as a function of the profit realization?
  - For that to happen, the left-hand side of **(6)** needs to increase in π.
  - This occurs if the likelihood ratio $\frac{f(\pi_i|e_L)}{f(\pi_i|e_H)}$ decreases in π.
  - Intuitively, as profits increase, the likelihood of obtaining a profit level of π from effort e_H increases faster than the probability of obtaining such a profit level from e_L.
  - This property is commonly known as the **monotone likelihood ratio property**, MLRP.

Comparative Statics



Comparative Statics

- Example 2:
  - Consider Example 1, but assuming that the principal cannot observe the agent's effort.
  - In this incomplete information setting, the principal must offer a salary that increases in profit if he seeks to induce $e_H = 5$.
  - The principal's maximization problem becomes

$$\max_{\{w(\pi_i)\}_{i=1}^{3}} \; 270 - [0.1w(\pi_1) + 0.3w(\pi_2) + 0.6w(\pi_3)]$$

$$\text{s.t.} \quad 0.1\sqrt{w(\pi_1)} + 0.3\sqrt{w(\pi_2)} + 0.6\sqrt{w(\pi_3)} - 5 \geq 9 \quad \text{(PC)}$$

$$0.1\sqrt{w(\pi_1)} + 0.3\sqrt{w(\pi_2)} + 0.6\sqrt{w(\pi_3)} - 5 \geq$$
$$0.6\sqrt{w(\pi_1)} + 0.3\sqrt{w(\pi_2)} + 0.1\sqrt{w(\pi_3)} \quad \text{(IC)}$$

Comparative Statics

- Example 2: (con't)
  - Since the principal's revenue is a constant ($270), he can alternatively minimize his expected costs

$$\min_{\{w(\pi_i)\}_{i=1}^{3}} \quad 0.1w(\pi_1) + 0.3w(\pi_2) + 0.6w(\pi_3)$$

$$\text{s.t. } 0.1\sqrt{w(\pi_1)} + 0.3\sqrt{w(\pi_2)} + 0.6\sqrt{w(\pi_3)} - 14 \geq 0 \quad \text{(PC)}$$

$$-0.5\sqrt{w(\pi_1)} + 0.5\sqrt{w(\pi_3)} - 5 \geq 0 \quad \text{(IC)}$$

where the IC constraint has been simplified.

Comparative Statics

- Example 2: (con't)
  - The associated Lagrangian is

$$\mathcal{L} = \quad 0.1w(\pi_1) + 0.3w(\pi_2) + 0.6w(\pi_3)$$
$$- \lambda\left[0.1\sqrt{w(\pi_1)} + 0.3\sqrt{w(\pi_2)} + 0.6\sqrt{w(\pi_3)} - 14\right]$$
$$- \mu\left[-0.5\sqrt{w(\pi_1)} + 0.5\sqrt{w(\pi_3)} - 5\right]$$

  - Taking FOC with respect to $w(\pi_1)$, $w(\pi_2)$, and $w(\pi_3)$ yields

$$\frac{\partial \mathcal{L}}{\partial w(\pi_1)} = 0.1 - \frac{0.1\lambda}{2\sqrt{w(\pi_1)}} + \frac{0.5\mu}{2\sqrt{w(\pi_1)}} = 0 \quad \textbf{(7)}$$

$$\frac{\partial \mathcal{L}}{\partial w(\pi_2)} = 0.3 - \frac{0.3\lambda}{2\sqrt{w(\pi_2)}} = 0 \quad \textbf{(8)}$$

$$\frac{\partial \mathcal{L}}{\partial w(\pi_3)} = 0.6 - \frac{0.6\lambda}{2\sqrt{w(\pi_3)}} - \frac{0.5\mu}{2\sqrt{w(\pi_3)}} = 0 \quad \textbf{(9)}$$

Comparative Statics

- Example 2: (con't)
  - Rearranging **(7)** and **(8)**

$$\lambda = 2\sqrt{w(\pi_2)}$$

$$\mu = 0.4\sqrt{w(\pi_2)} - 0.4\sqrt{w(\pi_1)}$$

  - Plugging these values into **(9)** and rearranging

$$0.1\sqrt{w(\pi_1)} - 0.7\sqrt{w(\pi_2)} + 0.6\sqrt{w(\pi_3)} = 0 \quad \textbf{(10)}$$

  - Combining equation **(10)** with the (PC) and (IC) equations, we have three equations and three unknowns $w(\pi_1)$, $w(\pi_2)$, and $w(\pi_3)$.

Comparative Statics

- Example 2: (con't)
  - The (IC) equation yields
$$\sqrt{w(\pi_3)} = 10 + \sqrt{w(\pi_1)}$$
  - Substituting this into the (PC) equation
$$3\sqrt{w(\pi_2)} = 80 - 7\sqrt{w(\pi_1)}$$
  - Last, substituting the values of $w(\pi_2)$ and $w(\pi_3)$ in equation **(10)**
$$w(\pi_1) = \$29.47,\ w(\pi_2) = \$196,\ w(\pi_3) = \$238.04$$
  - The principal's expected profit is then
$$270 - [0.1 \cdot 29.47 + 0.3 \cdot 196 + 0.6 \cdot 238.04] = \$65.43$$
which is lower than its profit when effort is observable ($74).

Moral Hazard with a Continuum of Effort Levels—The First-Order Approach

Continuum of Effort Levels

- So far we assumed that a worker could only have a discrete number of effort levels.
- Let us now consider a continuum of effort levels.
- The principal seeks to maximize its expected profits by anticipating the effort level that the agent selects in the second stage of the game:

$$\max_{\{e,w(\pi_i)\}_{i=1}^{N}} \sum_{i=1}^{N} f(\pi_i|e) \cdot [\pi_i - w(\pi_i)]$$

s.t. $\sum_{i=1}^{N} f(\pi_i|e)\, u(w(\pi_i)) - g(e) \geq \bar{u}$ (PC)

$e^* \in \arg\max_{e} \sum_{i=1}^{N} f(\pi_i|e) \left[u\left(w(\pi_i)\right) - g(e)\right]$ (IC)

Continuum of Effort Levels

- Difference/similarities between discrete and continuum effort levels
  - The objective function of the principal and the PC condition for the agent coincide.
  - The agent's IC condition, however, differs as it now allows him to choose among a continuum of effort levels.
  - Intuitively, the IC condition represents the agent's UMP where, for a given salary $w(\pi_i)$, the agent selects an effort level e that maximizes his expected utility.

Continuum of Effort Levels

- Differentiating the agent's expected utility with respect to e yields

$$\sum_{i=1}^{N} f'(\pi_i|e)\, u\big(w(\pi_i)\big) - g'(e) = 0$$

- The agent's FOC above can be used as the IC condition in the principal's problem.
- This approach is known as the **first-order approach**.

Continuum of Effort Levels

- The principal's problem, using a "first-order approach," is then

$$\max_{\{e,w(\pi_i)\}_{i=1}^{N}} \sum_{i=1}^{N} f(\pi_i|e) \cdot [\pi_i - w(\pi_i)]$$

s.t. $\sum_{i=1}^{N} f(\pi_i|e)\, u(w(\pi_i)) - g(e) \geq \bar{u}$ (PC)

$\sum_{i=1}^{N} f'(\pi_i|e)\, u\big(w(\pi_i)\big) - g'(e) = 0$ (IC)

Continuum of Effort Levels

- The Lagrangian becomes

$$\mathcal{L} = \sum_{i=1}^{N} f(\pi_i|e) \cdot [\pi_i - w(\pi_i)] + \lambda \left[\sum_{i=1}^{N} f(\pi_i|e)\, u\big(w(\pi_i)\big) - g(e) - \bar{u} \right] + \mu \left[\sum_{i=1}^{N} f'(\pi_i|e)\, u\big(w(\pi_i)\big) - g'(e) \right]$$

- Taking FOC with respect to w yields

$$\frac{\partial \mathcal{L}}{\partial w} = -f(\pi_i|e) + \lambda f(\pi_i|e) u'\big(w(\pi_i)\big) + \mu f'(\pi_i|e) u'\big(w(\pi_i)\big) = 0$$

Continuum of Effort Levels

- Dividing both sides by $f(\pi_i|e)$

$$-1 + \lambda u'\big(w(\pi_i)\big) + \mu \frac{f'(\pi_i|e)}{f(\pi_i|e)} u'\big(w(\pi_i)\big) = 0$$

- Factoring out $u'\big(w(\pi_i)\big)$ on the left-hand side and rearranging

$$\lambda + \mu \frac{f'(\pi_i|e)}{f(\pi_i|e)} = \frac{1}{u'\big(w(\pi_i)\big)}$$

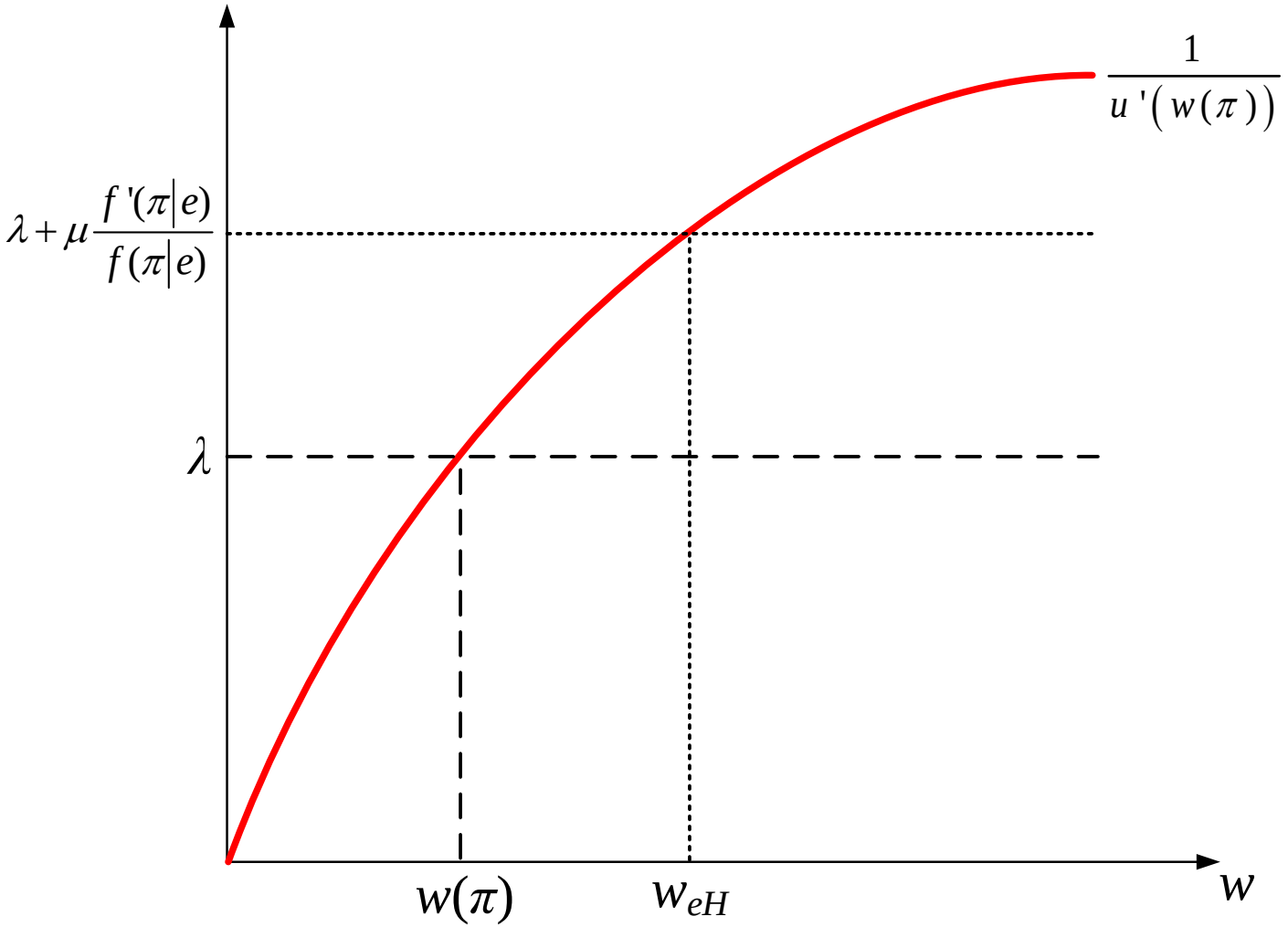
- This result is similar to that in previous sections.
- Because $\lambda > 0$ and $\mu > 0$ (since PC and IC bind), the left-hand side satisfies

$$\lambda + \mu \frac{f'(\pi_i|e)}{f(\pi_i|e)} > \lambda$$

Continuum of Effort Levels

- Since u' is decreasing in w (by concavity), its inverse, $1/u'$, is increasing in w.
- Hence the principal offers a larger salary under asymmetric information than symmetric information.
- $\frac{f'(\pi_i|e)}{f(\pi_i|e)}$ is the likelihood ratio, which measures how a marginally higher effort entails a larger probability of obtaining a given profit level π_i relative to an initial effort level.

Continuum of Effort Levels



Continuum of Effort Levels

- Taking FOC with respect to e yields

$$\frac{\partial \mathcal{L}}{\partial e} = \sum_{i=1}^{N} f'(\pi_i|e) \cdot [\pi_i - w(\pi_i)] + \mu \left[\sum_{i=1}^{N} f''(\pi_i|e)\, u\big(w(\pi_i)\big) - g''(e) \right] + \lambda \left[\sum_{i=1}^{N} f'(\pi_i|e)\, u\big(w(\pi_i)\big) - g'(e) \right] = 0$$

- Rearranging

$$\sum_{i=1}^{N} f'(\pi_i|e)\, \pi_i = \sum_{i=1}^{N} f'(\pi_i|e)\, w(\pi_i) - \mu \left[\sum_{i=1}^{N} f''(\pi_i|e)\, u\big(w(\pi_i)\big) - g''(e) \right] - \lambda \left[\sum_{i=1}^{N} f'(\pi_i|e)\, u\big(w(\pi_i)\big) - g'(e) \right] \quad \textbf{(11)}$$

- Intuitively, effort is increased until the point where its expected profits (left-hand side) coincide with its associated costs (right-hand side).

Continuum of Effort Levels

- The cost of inducing a higher effort originates from two sources:
  1. A higher effort increases the probability of obtaining a higher profit, and thus the salary that the principal pays the agent once the profit is realized (first term on the right-hand side).
  2. The principal must provide more incentives (higher salary) in order for the agent to exert the effort level that the principal intended (second term on the right-hand side).

Continuum of Effort Levels

- Example 3:
  - Moral hazard with continuous effort but only two possible outcomes.
  - Consider a setting in which the conditional probability satisfies

$$f(\pi_i|e) = ef_H(\pi_i) + (1-e)f_L(\pi_i),\ e \in [0,1]$$

  - When effort is relatively high $e \to 1$, the probability of obtaining a profit level π_i is $f_H(\pi_i)$, where $f_H(\pi_i) > f_L(\pi_i)$.
  - When $e \to 0$, the probability of obtaining a profit level π_i is $f_L(\pi_i)$.

Continuum of Effort Levels

- Example 3: (con't)
  - The agent's expected utility is

$$EU(e) = \sum_{i=1}^{N} [e f_H(\pi_i) + (1-e) f_L(\pi_i)] u(w(\pi_i)) - g(e)$$

  - Since

$$e f_H(\pi_i) + (1-e) f_L(\pi_i) = e[f_H(\pi_i) - f_L(\pi_i)] + f_L(\pi_i)$$

  - Then

$$EU(e) = \sum_{i=1}^{N} e[f_H(\pi_i) - f_L(\pi_i)] u(w(\pi_i)) + \sum_{i=1}^{N} f_L(\pi_i) u(w(\pi_i)) - g(e)$$

  - Differentiating $EU(e)$ twice with respect to effort e, yields $-g''(e)$, which is negative by definition.
  - So we can use the first-order approach.

Continuum of Effort Levels

- Example 3: (con't)
  - The agent's FOC with respect to e is

$$\sum_{i=1}^{N}[f_H(\pi_i) - f_L(\pi_i)]u(w(\pi_i)) = g'(e)$$

  - Plugging this FOC into the principal's problem

$$\max_{\{e,w(\pi_i)\}_{i=1}^{N}} \sum_{i=1}^{N}[ef_H(\pi_i) + (1-e)f_L(\pi_i)] \cdot [\pi_i - w(\pi_i)]$$

s.t. $\sum_{i=1}^{N}[ef_H(\pi_i) + (1-e)f_L(\pi_i)]\, u\big(w(\pi_i)\big) - g(e) \geq \bar{u}$ (PC)

$\sum_{i=1}^{N}[f_H(\pi_i) - f_L(\pi_i)]u(w(\pi_i)) = g'(e)$ (IC)

Continuum of Effort Levels

- Example 3: (con't)
  - The Lagrangian of this program is

$$\mathcal{L} = \sum_{i=1}^{N} [e f_H(\pi_i) + (1-e) f_L(\pi_i)] \cdot [\pi_i - w(\pi_i)]$$
$$+ \lambda \left[\sum_{i=1}^{N} [e f_H(\pi_i) + (1-e) f_L(\pi_i)]\, u\big(w(\pi_i)\big) - g(e) - \bar{u} \right]$$
$$+ \mu \left[\sum_{i=1}^{N} [f_H(\pi_i) - f_L(\pi_i)] u\big(w(\pi_i)\big) - g'(e) \right]$$

  - Taking FOC with respect to w and rearranging

$$\lambda + \mu \frac{f_H(\pi_i) - f_L(\pi_i)}{e f_H(\pi_i) + (1-e) f_L(\pi_i)} = \frac{1}{u'\big(w(\pi_i)\big)}$$

Continuum of Effort Levels

- Example 3: (con't)
  - Taking FOC with respect to e and rearranging

$$\sum_{i=1}^{N}\left[f_H(\pi_i)-f_L(\pi_i)\right]\pi_i$$

$$=\sum_{i=1}^{N}\left[f_H(\pi_i)-f_L(\pi_i)\right]w(\pi_i)+\mu g''(e)$$

$$-\lambda\left[\sum_{i=1}^{N}\left[f_H(\pi_i)-f_L(\pi_i)\right]u\big(w(\pi_i)\big)-g'(e)\right]$$

Continuum of Effort Levels

- Example 3: (con't)
  - From the binding (IC), we can further simplify and obtain

$$\sum_{i=1}^{N} [f_H(\pi_i) - f_L(\pi_i)]\,\pi_i$$

$$= \sum_{i=1}^{N} [f_H(\pi_i) - f_L(\pi_i)]\,w(\pi_i) + \mu g''(e)$$

  - The expected profit to the principal (left-hand side) is exactly balanced by the expected cost of inducing effort e from the agent (right-hand side).

Continuum of Effort Levels

- Example 4:
  - Moral hazard using the first-order approach
  - Assume the expected utility function of the agent is

$$u(w,e) = E(w) - \frac{1}{2}\rho Var(w) - c(e)$$

    where:
    - ρ is the Arrow–Pratt coefficient of absolute risk aversion for utility function $u(w) = -\mathrm{e}^{-\rho w}$,
    - $e \in [0,1]$ is the agent's effort, and
    - $c(e) = 0.5e^2$ is the cost of effort.
  - The outcome of the project, x, is stochastic and given by

$$x = f(e,\varepsilon) = e + \varepsilon, \text{ where } \varepsilon \sim N(0,\sigma^2)$$

Continuum of Effort Levels

- Example 4: (con't)
  - The agent's reservation utility is $\bar{u} = \frac{1}{2}$.
  - The principal offers a linear contract to the agent
$$w(x) = a + bx$$
  - where $a \geq 0$ is a fixed payment, and $b \in [0,1]$ is the share of profits that the agent receives (bonus).
  - The principal's expected profits are
$$E(\pi) = E(x - w) = E(x) - E(w)$$
$$= E(x) - [a + bE(x)] = (1 - b)e - a$$

Continuum of Effort Levels

- Example 4: (con't)
  - Since $E(x) = e$, the expected utility of the agent when he exerts effort level e is

$$E[u(w,e)] = E(w) - \frac{1}{2}\rho Var(w) - c(e)$$

$$= a + be - \frac{1}{2}\rho b^2\sigma^2 - \frac{1}{2}e^2$$

where $E(w) = a + be$, $Var(w) = b^2\sigma^2$, and $c(e) = \frac{1}{2}e^2$.

Continuum of Effort Levels

- Example 4: (con't)
  - Taking FOC with respect to e, we can find the effort that the agent chooses

$$\frac{\partial E[u(w,e)]}{\partial e} = b - e = 0$$

$$e = b$$

  - The principal's problem is to choose the fixed payment, a, and the bonus, b, to solve

$$\max_{e,a,b} \ (1-b)e - a$$

$$\text{s.t.} \quad a + be - \frac{1}{2}\rho b^2\sigma^2 - \frac{1}{2}e^2 \geq \frac{1}{2} \qquad \text{(PC)}$$

$$e = b \qquad \text{(IC)}$$

Continuum of Effort Levels

- Example 4: (con't)
  - Plugging $e = b$ into the program and simplifying

$$\max_{a,b} (1-b)b - a$$

$$\text{s.t.} \qquad a + \frac{1}{2}b^2(1-\rho\sigma^2) \geq \frac{1}{2} \qquad \text{(PC)}$$

  - The Lagrangian is

$$\mathcal{L} = (1-b)b - a + \lambda\left[a + \frac{1}{2}b^2(1-\rho\sigma^2) - \frac{1}{2}\right]$$

Continuum of Effort Levels

- Example 4: (con't)
  - The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial a} = -1 + \lambda = 0 \rightarrow \lambda = 1 \quad \textbf{(12)}$$

$$\frac{\partial \mathcal{L}}{\partial b} = 1 - 2b + \lambda b(1 - \rho\sigma^2) = 0 \quad \textbf{(13)}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = a + \frac{1}{2}b^2(1 - \rho\sigma^2) - \frac{1}{2} = 0 \quad \textbf{(14)}$$

  - Plugging **(12)** into **(13)** yields

$$1 - 2b + b(1 - \rho\sigma^2) = 0$$

$$b = \frac{1}{1 + \rho\sigma^2}$$

Continuum of Effort Levels

- Example 4: (con't)
  - Plugging $b = \frac{1}{1+\rho\sigma^2}$ into the binding (PC) constraint yields

$$a + \frac{1 - \rho\sigma^2}{2(1 + \rho\sigma^2)^2} = \frac{1}{2}$$

  - Solving for the fixed payment a

$$a = \frac{1}{2}\left[1 - \frac{1 - \rho\sigma^2}{(1 + \rho\sigma^2)^2}\right]$$

Continuum of Effort Levels

- Example 4: (con't)
  - If $\sigma^2 = 0$, effort e is deterministic (a perfect predictor of profits)

$$x = f(e) = e$$

  - Then,

$$b = \frac{1}{1 + \rho \cdot 0} = 1$$

$$a = \frac{1}{2}\left[1 - \frac{1 - \rho \cdot 0}{(1 + \rho \cdot 0)^2}\right] = 0$$

  - Intuitively, the principal does not offer a fixed payment, and the agent is benefited from high-powered incentives.

Continuum of Effort Levels

- Example 4: (con't)
  - If $\sigma^2 = 1$, effort e is an imprecise predictor of outcomes.
  - Then,

$$b = \frac{1}{1+\rho}$$

$$a = \frac{\rho(\rho+3)}{2(1+\rho)^2}$$

Continuum of Effort Levels

- Example 4: (con't)
  - When the agent becomes more risk averse (ρ increases), the agent is offered a higher fixed payment but a lower bonus, since

$$\frac{\partial b}{\partial \rho} = -\frac{1}{(1+\rho)^2} < 0$$

$$\frac{\partial a}{\partial \rho} = \frac{3-\rho}{2(1+\rho)^3} > 0$$

Continuum of Effort Levels

- Example 4: (con't)
  - In general, for $0 \leq \sigma^2 \leq 1$, we show that a increases but b decreases in σ^2 since

$$\frac{\partial b}{\partial \sigma^2} = -\frac{\rho}{(1+\rho\sigma^2)^2} < 0$$

$$\frac{\partial a}{\partial \sigma^2} = -\frac{-\rho(1+\rho\sigma^2)^2 - 2\rho(1+\rho\sigma^2)(1-\rho\sigma^2)}{2(1+\rho\sigma^2)^4}$$

$$= \frac{\rho(3-\rho\sigma^2)}{2(1+\rho\sigma^2)^3} > 0$$

Continuum of Effort Levels

- Example 4: (con't)
- When σ^2 is low (i.e., all effort levels yield a similar outcome x), the fixed payment a is low while the bonus b is high, which we call **high-powered incentives**.
- When σ^2 is high (i.e., an effort level may yield many different outcomes x), the fixed payment a is high while the bonus b is low, which we call **low-powered incentives**.

Continuum of Effort Levels

- Example 4: (con't)
- When $\rho = 0$, the agent is risk neutral, yielding

$$b = \frac{1}{1 + 0 \cdot \sigma^2} = 1$$

$$a = \frac{1}{2}\left[1 - \frac{1 - 0 \cdot \sigma^2}{(1 + 0 \cdot \sigma^2)^2}\right] = 0$$

- The agent is totally commission-driven and does not receive a fixed payment, which applies to jobs like sales representatives and insurance agents.

Continuum of Effort Levels

- Example 4: (con't)
- When $\rho = 1/\sigma^2$, the agent is risk averse, yielding

$$b = \frac{1}{1+1} = \frac{1}{2}$$

$$a = \frac{1}{2}\left[1 - \frac{1-1}{(1+1)^2}\right] = \frac{1}{2}$$

- The agent receives a fixed wage to compensate for reservation utility $\bar{u}$, and splits the payoff with the principal, such as car dealers, bankers, and realtors.

Moral Hazard with Multiple Signals

Multiple Signals

- Consider a setting in which the principal, still not observing effort e, observes:
  - the profits π of the firm;
  - a signal s, based on a middle management report about the agent's performance.
- Signal s provides no intrinsic economic value but it provides information about effort e.
- Hence the probability density function has two observables, π and s.
- Then, similar to equation **(6)**, we have

$$\frac{1}{u'(w)} = \gamma + \mu\left[1 - \frac{f(\pi, s|e_L)}{f(\pi, s|e_H)}\right]$$

Multiple Signals

- Hence variations in s affect wages only if
$$f(\pi, s|e) \neq f(\pi|e)$$
- That is, if π is not a sufficient statistic of e.
- Intuitively, the pair (π, s) contains more information about the agent's exerted effort e than π alone.
- Signal s is uninformative (provides no more information than π alone), if
$$f(\pi, s|e) = f(\pi|e)$$
- We can examine under which conditions w increases in signal s.

Multiple Signals

- For two signals s_1 and s_2, where $s_2 > s_1$, if salary increases in the signal, $w(\pi, s_2) > w(\pi, s_1)$, then $u'(w)$ decreases and its inverse, $1/u'(w)$, increases.
- Therefore,

$$\gamma + \mu\left[1 - \frac{f(\pi, s_2|e_L)}{f(\pi, s_2|e_H)}\right] > \gamma + \mu\left[1 - \frac{f(\pi, s_1|e_L)}{f(\pi, s_1|e_H)}\right]$$

- Simplifying this inequality to express it in terms of the likelihood ratio, $\frac{f(\pi,s|e_L)}{f(\pi,s|e_H)}$, we obtain

$$\frac{f(\pi, s_2|e_L)}{f(\pi, s_2|e_H)} < \frac{f(\pi, s_1|e_L)}{f(\pi, s_1|e_H)}$$

- In words, this condition says that, for the salary to increase in the intermediate signal s that the principal receives, we need such a signal to have a *decreasing* likelihood ratio.

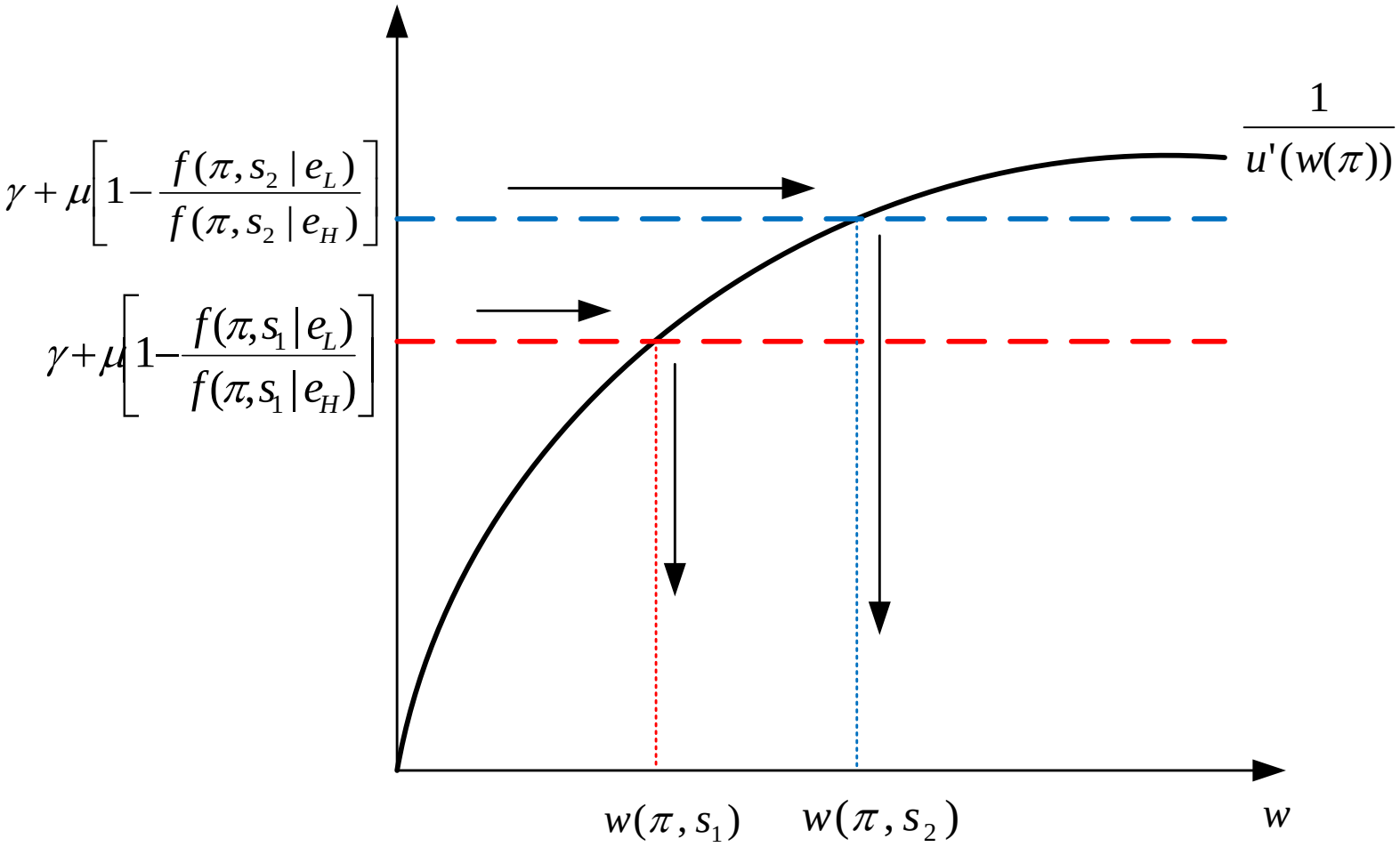
Multiple Signals

- Alternatively, we can rearrange expression $\frac{f(\pi,s_2|e_L)}{f(\pi,s_2|e_H)} < \frac{f(\pi,s_1|e_L)}{f(\pi,s_1|e_H)}$ as follows

$$\frac{f(\pi,s_1|e_H)}{f(\pi,s_1|e_L)} < \frac{f(\pi,s_2|e_H)}{f(\pi,s_2|e_L)}$$

- Intuitively, signal s_2 is more likely to originate from the high than the low effort, relative to signal s_1.

Multiple Signals



Adverse Selection
The “Lemons” Problem

Adverse Selection

- **Adverse selection**: settings in which an agent does not observe the payoff of the other individual.
  - Also referred to as "hidden information"
- Example:
  - A manager in a firm might not observe the worker's ability
  - The manager could err in its selection of candidates for a job if he does not observe their ability, thus giving rise to adverse selection
- Under symmetric information markets often work well.
- Under asymmetric information, however, markets do not necessarily work well.

Adverse Selection

- Akerlof's (1970) model:
  - Consider a market of used cars, whose quality is denoted by q, where $q \in U[0, Q]$ and $Q \in (1,2)$.
  - A car of quality q is valued as such by the buyer, and as q/Q by the seller.
  - Since $\frac{q}{Q} < q$, the buyer assigns a higher value to the car than the seller.
  - This allows both parties to exchange the car at a price p between q/Q and q and make a profit (for the seller) and a surplus (for the buyer).

Adverse Selection

- Akerlof's (1970) model:
  - If a car of quality q is exchanged at price p the buyer obtains a utility

    $$u(p, q) = q - p$$

    while the seller makes a profit of

    $$\pi(p, q, Q) = p - \frac{q}{Q}$$

  - Assume that there are a sufficient number of buyers so that all gains from trade are appropriated by the seller.

Symmetric Information

- When the buyer can **perfectly** observe the car quality q, he buys at a price p if and only if

$$q - p \geq 0$$

- That is, his utility from such a trade is positive.
- A seller with a car of quality q anticipates such an acceptance rule by the buyer and sets a price p that solves

$$\max_{p \geq 0} \; p - \frac{q}{Q}$$

$$\text{s.t. } p \leq q$$

where $p \leq q$ is the buyer's participation constraint (PC).

Symmetric Information

- Since condition (PC) must bind, $p = q$, the seller's objective function can be represented as unconstrained problem:

$$\max_{p \geq 0} \; p - \frac{p}{Q}$$

- Taking the FOC with respect to p yields

$$1 - \frac{1}{Q} > 0 \;\text{ or }\; \frac{Q-1}{Q} > 0$$

- Since $Q > 1$ by definition, a corner solution exists whereby the seller raises the price p as much as possible

$$p^{SI} = q$$

Asymmetric Information

- When the buyer is **unable** to observe the car's true quality q, he forms an expectation $E(q)$.
- The buyer accepts a trade if the car's asking price p satisfies

$$p = E(q)$$

- The seller anticipates such an acceptance rule by the buyer and sets a price p that solves

$$\max_{p \geq 0} \; p - \frac{q}{Q}$$

$$\text{s.t. } \; p \leq E(q)$$

where $p \leq E(q)$ is the buyer's PC constraint.

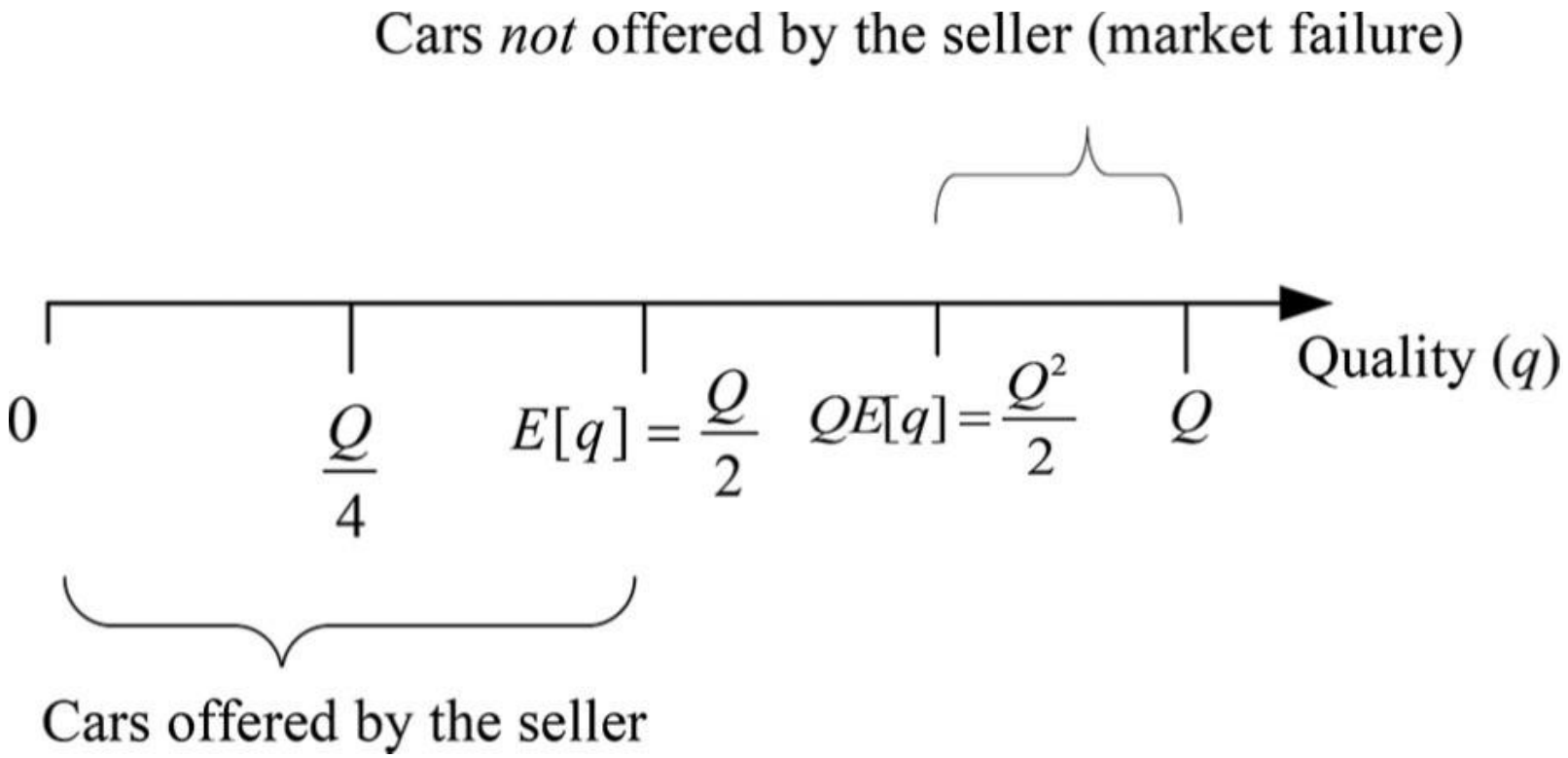
Asymmetric Information

- Since condition (PC) must bind, $p = E(q)$, the price that the seller sets

$$p - \frac{q}{Q} = E(q) - \frac{q}{Q} \geq 0$$

$$q \leq Q \cdot E(q)$$

Asymmetric Information



Asymmetric Information

- When q is uniformly distributed, that is, $q \sim U[0, Q]$, its expected value becomes

$$E(q) = \frac{Q - 0}{2} = \frac{Q}{2}$$

- Then, $Q \cdot E(q) = Q^2/2$.
- Hence all cars with relatively **low quality**, $q \leq Q^2/2$, are offered by the seller at a price

$$p = E(q) = \frac{Q}{2}$$

yielding profit of $\frac{Q}{2} - \frac{q}{Q}$ for the seller and a zero (expected) utility for the buyer since $p = E(q)$.

Asymmetric Information

- Cars with relatively **high quality**, $q \geq Q^2/2$, are not offered by the seller since the highest price he can charge to the uninformed buyer, $p = E(q)$, does not compensate the seller's costs.
- This is problematic.
- The buyer's inability to observe q leads to the non-existence of the market for good cars ("peaches"), whereas only bad cars ("lemons") exist in the market.

Asymmetric Information

- A fully rational buyer would anticipate such a pricing decision by the seller
  - That the seller finds it worthwhile to only offer low quality cars, $p \leq Q^2/2$.
- In that case, the buyer anticipates that only cars of quality $q \in (0, Q^2/2)$ are offered.
- Then, if $q \sim U[0, Q]$, buyers can compute the expected quality of those offered cars

$$E\left[q \middle| q \leq \frac{Q^2}{2}\right] = \frac{\frac{Q^2}{2} - 0}{2} = \frac{Q^2}{4}$$

Asymmetric Information

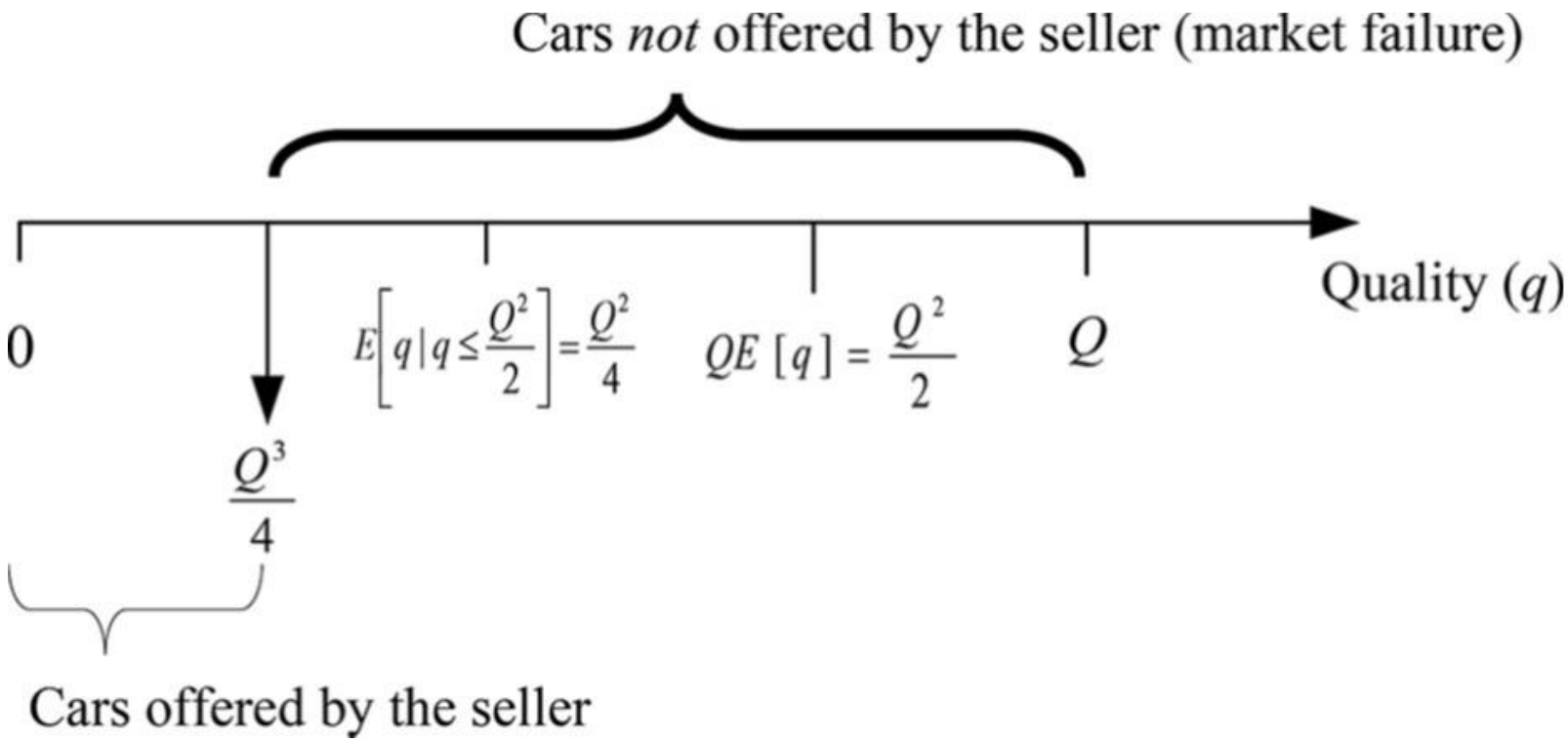
- Hence the buyer would only buy cars whose price satisfies $p = Q^2/4$.
- The seller would then set the price at $p = Q^2/4$, yielding a profit of

$$p - \frac{q}{Q} = \frac{Q^2}{4} - \frac{q}{Q}$$

which is positive only if quality q satisfies

$$q \leq \frac{Q^3}{4}$$

Asymmetric Information



Asymmetric Information

- A rational buyer would now update its expected car quality to those satisfying $Q^3/4$
- This yields an expected quality of only

$$E\left[q\middle|q \leq \frac{Q^3}{4}\right] = \frac{\frac{Q^3}{4} - 0}{2} = \frac{Q^3}{8}$$

- The seller offers cars that yield a positive profit, that is, those with quality q satisfying

$$p - \frac{q}{Q} = \frac{Q^3}{8} - \frac{q}{Q} \geq 0 \quad \text{or} \quad q \leq \frac{Q^4}{8}$$

which lies closer to zero than cutoff $\frac{Q^3}{4}$.

Asymmetric Information

- Intuition:
  - The seller would shift the set of offered cars even more to the left of the quality line toward worse cars (closer to zero).
  - Repeating the same argument enough times, we find that the market "unravels."
  - It only offers cars of the worst possible quality, $q = 0$.
  - The buyer is only willing to pay a price of $p = 0$, leaving all other types of cars unsold.

Asymmetric Information

- Example 5:
  - Consider a market of used cars with maximum available quality $Q = 1.9$, and that $q \sim U[0, Q]$.
  - Recall that $Q \in (1,2)$, i.e., the availability of several cars of relatively good quality.
  - The buyer's expected value is $\frac{1.9}{2} = 0.95$.
  - The cutoff $Q \cdot E(q)$ of cars offered by the seller is $1.9 \cdot 0.95 = 1.805$.
  - Unoffered cars $(1.805, 1.9)$.
  - Under complete information, these cars would have been bought by the buyer who values them at q, and sold by the seller who values them at only $\frac{q}{1.9} = 0.52q$.

Asymmetric Information

- Example 5: (con't)
  - A rational buyer will anticipate that cars in the interval $(1.805, 1.9)$ are unoffered by the seller.
  - Thus buyer updates expected value of offered cars to

$$E[q|q \leq 1.805] = \frac{1.805 - 0}{2} = 0.9$$

  - This leads the seller to only offer those cars with quality

$$q \leq \frac{Q^3}{4} = 1.71$$

  - The set of offered cars is thus restricted from $(0, 1.805)$ to $(0, 1.71)$.
  - A similar argument applies to further iterations in the buyer's expected car quality.
  - The presence of asymmetric information between buyer and seller prevents mutually beneficial trades from occurring.

Asymmetric Information

- Application to Labor Markets
  - Consider a competitive labor market with many firms seeking to hire a worker for a specific position.
  - The worker (seller of labor services) privately observes his own productivity θ, but firms (the buyer of labor) cannot observe it.
  - Firms offer a wage according to the worker's expected productivity

$$E(\theta) = 1/2,\ \theta \sim U[0,1]$$

  - For this salary, only workers with a productivity $\theta \leq 1/2$ would be interested in accepting the position, while those with $\theta > 1/2$ will be left unemployed.

Asymmetric Information

- Application to Labor Markets
  - A fully rational manager will only offer a salary of

$$w = E\left(\theta | \theta \leq \frac{1}{2}\right) = \frac{1}{4}$$

  - Then only those workers with productivity $\theta \leq \frac{1}{4}$ accept the job.
  - Extending the argument infinite times, workers with lowest productivity level $\theta = 0$ are employed, while the labor market for all other worker types $\theta > 0$ unravels.

Solutions to Adverse Selection

- The market failure described above can be overcome by a number of tools.
  - Sellers can offer warranties for their cars in order to signal their quality.
  - **Screening**: The principal (buyer) offers a menu of contracts to the agent (seller) that induce each type of agent to voluntarily select only one contract, whereby the contracts induce self-selection.

Adverse Selection
The Principal–Agent Problem

The Principal–Agent Problem

- Consider a setting where a firm (the principal) seeks to hire a worker (an agent).
- The firm cannot observe the worker's cost of effort
  - This affects the amount of effort that the worker exerts and thus the firm's profits.
- The firm's manager would like to know the worker's cost of effort in order to design his salary.
- The firm's profit function is

$$\pi(e, w) = x(e) - w$$

where $x(e)$ is the benefit that the firm obtains when the worker supplies e units of effort, $x'(e) \geq 0$, $x''(e) \leq 0$.

The Principal–Agent Problem

- The worker's utility function is
$$v(w, e|\theta) = u(w) - c(e, \theta)$$
where $u(w)$ is the value from the salary w, $u'(w) > 0$, $u''(w) \leq 0$; $c(e, \theta)$ is the worker's cost of exerting e units of effort when his type is θ.
- Assume the worker can only be of two types, θ_L and θ_H, where $\theta_L < \theta_H$, with probabilities p and $1 - p$.
- A high-type worker faces a higher total and marginal cost of effort
$$c(e, \theta_L) < c(e, \theta_H)$$
$$c'(e, \theta_L) < c'(e, \theta_H)$$
for every e.

Symmetric Information

- When the principal (firm) **knows** that the agent is type $i = \{L, H\}$, it solves

$$\max_{w_i, e_i} \; x(e_i) - w_i$$

$$\text{s.t.} \quad u(w_i) - c(e_i, \theta_i) \geq 0 \qquad \text{(PC)}$$

- (PC) constraint guarantees that the worker willingly accepts the contract.
- Since the firm can reduce w_i until (PC) holds with equality, (PC) must bind

$$u(w_i) = c(e_i, \theta_i)$$

$$w_i = u^{-1}[c(e_i, \theta_i)]$$

Symmetric Information

- The principal's unconstrained maximization problem can then be written as

$$\max_{e_i} \; x(e_i) - u^{-1}[c(e_i, \theta_i)]$$

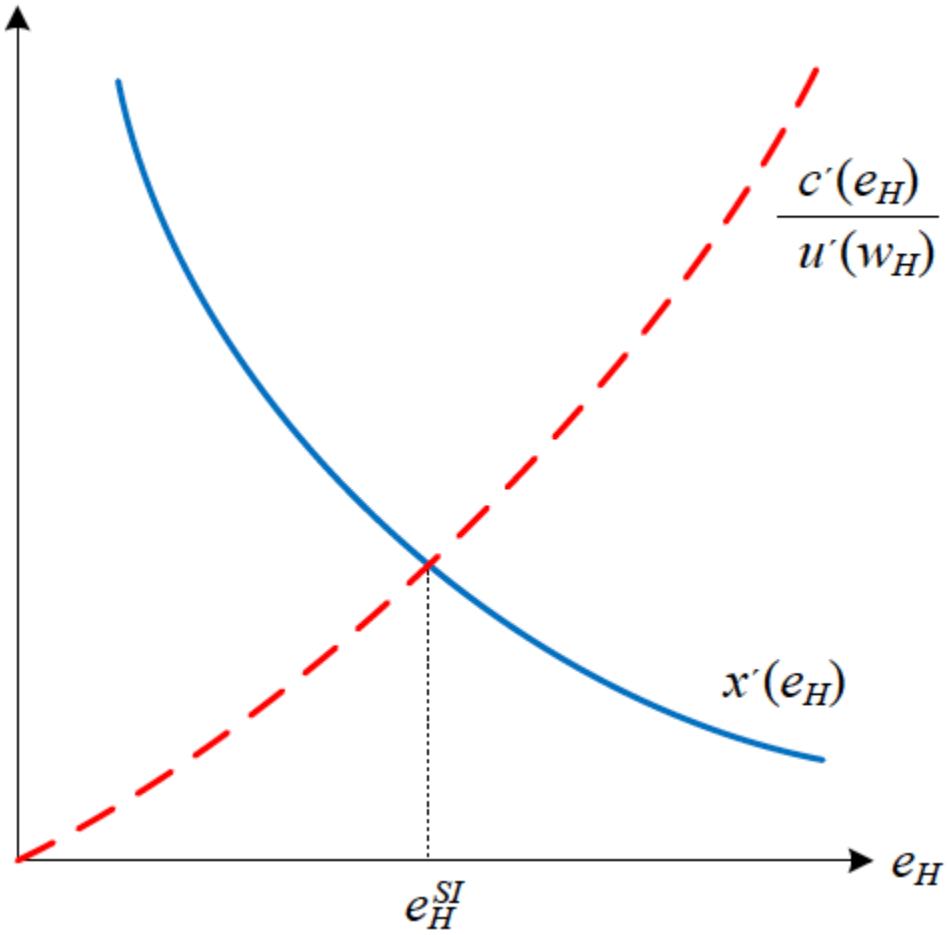
- Taking FOC with respect to e_i yields

$$x'(e_i) = \frac{1}{u'\{u^{-1}[c(e_i, \theta_i)]\}} c'(e_i, \theta_i)$$

$$x'(e_i) = \frac{c'(e_i, \theta_i)}{u'\{w_i\}}$$

- Hence effort is increased until the point at which the marginal rate of substitution of effort and wage for the firm (left-hand side) coincides with that of the worker (right-hand side).

Symmetric Information



Symmetric Information

- Example 6:
  - Consider a principal and an agent of type $\theta_L = 1$, $\theta_H = 2$.
  - The probability of facing a low type is $p = 1/2$.
  - Productivity of effort is $x(e) = \log(e)$, and $u(w) = w$.
  - The cost of effort is $c(e, \theta) = \theta_i e^2$, with the marginal cost of effort of $2\theta_i e$, which is positive and increasing in e.
  - The principal's profit function is
$$\pi(e, w) = \log(e) - w$$
  - The agent's utility is
$$v(w, e|\theta_i) = w - \theta_i e^2$$

Symmetric Information

- Taking FOC

$$x'(e_i) = \frac{c'(e_i, \theta_i)}{u'\{w_i\}} \Rightarrow \frac{1}{e_i} = \frac{2\theta_i e_i}{1}$$

- Solving for e_i

$$e_i^2 = \frac{1}{2\theta_i} \rightarrow e_i^{SI} = \left(\frac{1}{2\theta_i}\right)^{1/2}$$

- Use the (PC) constraint, $u(w_i) = c(e_i, \theta_i)$, to find optimal salary

$$w_i = \theta_i\left(e_i^{SI}\right)^2 = \theta_i\left(\left(\frac{1}{2\theta_i}\right)^{1/2}\right)^2 = \frac{1}{2}$$

Symmetric Information

- Plugging in $\theta_L = 1$ and $\theta_H = 2$, we find optimal contracts

$$(w_H^{SI}, e_H^{SI}) = \left(\frac{1}{2}, \frac{1}{2}\right) = (0.5, 0.5)$$

$$(w_L^{SI}, e_L^{SI}) = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right) = (0.5, 0.707)$$

- The firm will pay both types of workers the same wage under symmetric information, but expect a higher effort level from the low-cost worker, $e_L^{SI} > e_H^{SI}$.

Asymmetric Information

- When the firm **cannot** observe the worker's type, it seeks to maximize the expected profits by designing a pair of contracts, (w_H, e_H) and (w_L, e_L), that satisfy four constraints:
  1. voluntary participation of the high-type worker;
  2. voluntary participation of the low-type worker;
  3. the high-type worker prefers the contract (w_H, e_H) rather than that for the low-type, (w_L, e_L);
  4. the low-type worker prefers the contract (w_L, e_L) rather than that for the high-type, (w_H, e_H).
- Since every type of worker has an incentive to select the contract meant for him, these contracts induce "**self-selection**."

Asymmetric Information

- The firm solves the following profit maximization problem

$$\max_{w_L, e_L, w_H, e_H} \; p[x(e_L) - w_L] + (1-p)[x(e_H) - w_H]$$

$$\text{s.t.} \quad u(w_H) - c(e_H, \theta_H) \geq 0 \qquad (PC_H)$$

$$u(w_L) - c(e_L, \theta_L) \geq 0 \qquad (PC_L)$$

$$u(w_H) - c(e_H, \theta_H) \geq u(w_L) - c(e_L, \theta_H) \qquad (IC_H)$$

$$u(w_L) - c(e_L, \theta_L) \geq u(w_H) - c(e_H, \theta_L) \qquad (IC_L)$$

Asymmetric Information

- Note that (PC_L) is implied by (IC_L) and (PC_H)

$$u(w_L) - c(e_L, \theta_L) \geq u(w_H) - c(e_H, \theta_L)$$
$$> u(w_H) - c(e_H, \theta_H) \geq 0$$

  - The first (weak) inequality stems from (IC_L).
  - The second (strict) inequality stems from the assumption $c(e_H, \theta_L) < c(e_H, \theta_H)$.
  - The third (weak) inequality stems from (PC_H).
- Hence we obtain (PC_L)

$$u(w_L) - c(e_L, \theta_L) > 0$$

Asymmetric Information

- The Lagrangian is

$$\begin{aligned}\mathcal{L} \quad &= p[x(e_L) - w_L] + (1-p)[x(e_H) - w_H] \\ &+ \lambda_1[u(w_H) - c(e_H, \theta_H)] \\ &+ \lambda_2[u(w_H) - c(e_H, \theta_H) - u(w_L) + c(e_L, \theta_H)] \\ &+ \lambda_3[u(w_L) - c(e_L, \theta_L) - u(w_H) + c(e_H, \theta_L)]\end{aligned}$$

Asymmetric Information

- Taking FOCs

$$\frac{\partial \mathcal{L}}{\partial w_L} = -p - \lambda_2 u'(w_L) + \lambda_3 u'(w_L) = 0$$

$$\frac{\partial \mathcal{L}}{\partial w_H} = -(1-p) + \lambda_1 u'(w_H) + \lambda_2 u'(w_H) - \lambda_3 u'(w_H) = 0$$

$$\frac{\partial \mathcal{L}}{\partial e_L} = p x'(e_L) + \lambda_2 c'(e_L, \theta_H) - \lambda_3 c'(e_L, \theta_L) = 0$$

$$\frac{\partial \mathcal{L}}{\partial e_H} = (1-p) x'(e_H) - \lambda_1 c'(e_H, \theta_H) - \lambda_2 c'(e_H, \theta_H) + \lambda_3 c'(e_H, \theta_L) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = u(w_H) - c(e_H, \theta_H) \geq 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = u(w_H) - c(e_H, \theta_H) - u(w_L) + c(e_L, \theta_H) \geq 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_3} = u(w_L) - c(e_L, \theta_L) - u(w_H) + c(e_H, \theta_L) \geq 0$$

Asymmetric Information

- For simplicity, consider that the cost of effort takes the following form

$$c(e, \theta_i) = \theta_i c(e) \quad \text{for all} \quad i = \{H, L\}$$

where $c(e)$ is increasing and convex in effort, $c'(e) \geq 0$ and $c''(e) \geq 0$.

- Rearranging the first two FOCs yields

$$-\lambda_2 + \lambda_3 = \frac{p}{u'(w_L)}$$

$$\lambda_1 + \lambda_2 - \lambda_3 = \frac{1-p}{u'(w_H)}$$

- Then adding them together

$$\lambda_1 = \frac{p}{u'(w_L)} + \frac{1-p}{u'(w_H)}$$

Asymmetric Information

- Hence $\lambda_1 > 0$, implying that the constraint associated with Lagrange multiplier λ_1, (PC_H), binds:

$$u(w_H) - c(e_H, \theta_H) = 0$$

Asymmetric Information

- The third FOC can be written as

$$px'(e_L) = \lambda_3\theta_L c'(e_L) - \lambda_2\theta_H c'(e_L)$$

- Rearranging

$$\frac{px'(e_L)}{c'(e_L)} = \lambda_3\theta_L - \lambda_2\theta_H$$

- The fourth FOC can be written as

$$(1-p)x'(e_H) = \lambda_1\theta_H c'(e_H) - \lambda_3\theta_L c'(e_H) + \lambda_2\theta_H c'(e_H)$$

- Rearranging

$$\frac{(1-p)x'(e_H)}{c'(e_H)} = \lambda_1\theta_H - (\lambda_3\theta_L - \lambda_2\theta_H)$$

Asymmetric Information

- Combining the two (rearranged) FOCs yields

$$\frac{(1-p)x'(e_H)}{c'(e_H)} = \lambda_1\theta_H - \frac{px'(e_L)}{c'(e_L)}$$

- Solving for $\lambda_1\theta_H$ and using $\lambda_1 = \frac{p}{u'(w_L)} + \frac{1-p}{u'(w_H)}$ from our results above, we obtain

$$\left[\frac{p}{u'(w_L)} + \frac{1-p}{u'(w_H)}\right]\theta_H = \frac{px'(e_L)}{c'(e_L)} + \frac{(1-p)x'(e_H)}{c'(e_H)}$$

Asymmetric Information

- Moreover, $\lambda_3 > \lambda_2$, since otherwise the first FOC, $(\lambda_3 - \lambda_2)u'(w_L) = p$, could not hold.
- Therefore, $\lambda_3 > 0$, which means (IC_L) binds:

$$u(w_L) - \theta_L c(e_L) = u(w_H) - \theta_L c(e_H)$$

- Rearranging the right-hand side

$$u(w_L) - \theta_L c(e_L) = u(w_H) - \theta_H c(e_H) + (\theta_H - \theta_L)c(e_H)$$

- Since (PC_H), binds, $u(w_H) - c(e_H, \theta_H) = 0$, hence

$$u(w_L) - \theta_L c(e_L) = (\theta_H - \theta_L)c(e_H)$$

Asymmetric Information

- Intuition:
  - The most efficient agent, θ_L, obtains in equilibrium a positive utility level, $(\theta_H - \theta_L)c(e_H)$, that increases in his difference with respect to the least efficient worker, $\theta_H - \theta_L$.

Asymmetric Information

- The incentive compatibility condition of the least efficient worker, (IC_H) , does not bind, implying that its associated Lagrange multiplier $\lambda_2 = 0$.
- Using this result in the first and third FOCs yields

$$\lambda_3 = \frac{p}{u'(w_L)} \quad \text{and} \quad \frac{px'(e_L)}{c'(e_L)} = \lambda_3 \theta_L$$

- Solving for λ_3 and combining the two FOCs

$$\frac{p}{u'(w_L)} = \frac{px'(e_L)}{\theta_L c'(e_L)}$$

- Solving for $x'(e_L)$

$$x'(e_L) = \frac{\theta_L c'(e_L)}{u'(w_L)}$$

Asymmetric Information

- Intuition:
  - For the most efficient worker, the equilibrium outcome under asymmetric information **coincides** with the socially optimal result under symmetric information.

Asymmetric Information

- Using $\lambda_1 = \frac{p}{u'(w_L)} + \frac{1-p}{u'(w_H)}$, $\lambda_2 = 0$, $\lambda_3 = \frac{p}{u'(w_L)}$ in the fourth FOC, we obtain

$$(1-p)x'(e_H) - \left[\frac{p}{u'(w_L)} + \frac{1-p}{u'(w_H)}\right]\theta_H c'(e_H) + \left[\frac{p}{u'(w_L)}\right]\theta_L c'(e_H) = 0$$

- Rearranging

$$\frac{(\theta_H - \theta_L)p}{1-p}\frac{c'(e_H)}{u'(w_L)} + \frac{\theta_H c'(e_H)}{u'(w_H)} = x'(e_H)$$

- The effort level that solves this equation is the **optimal** effort under asymmetric information, e_H^{AI}.

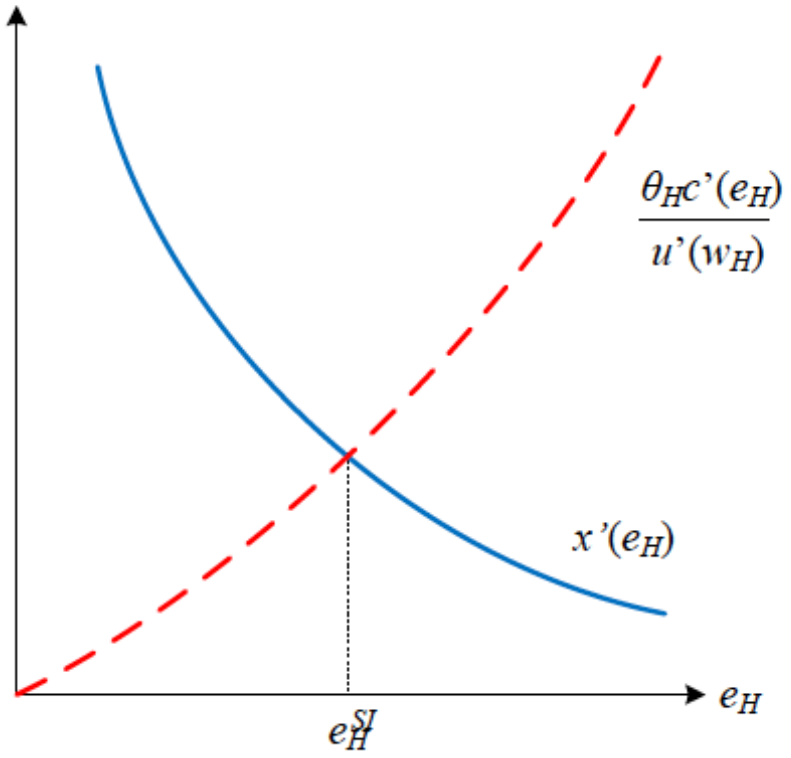
Asymmetric Information

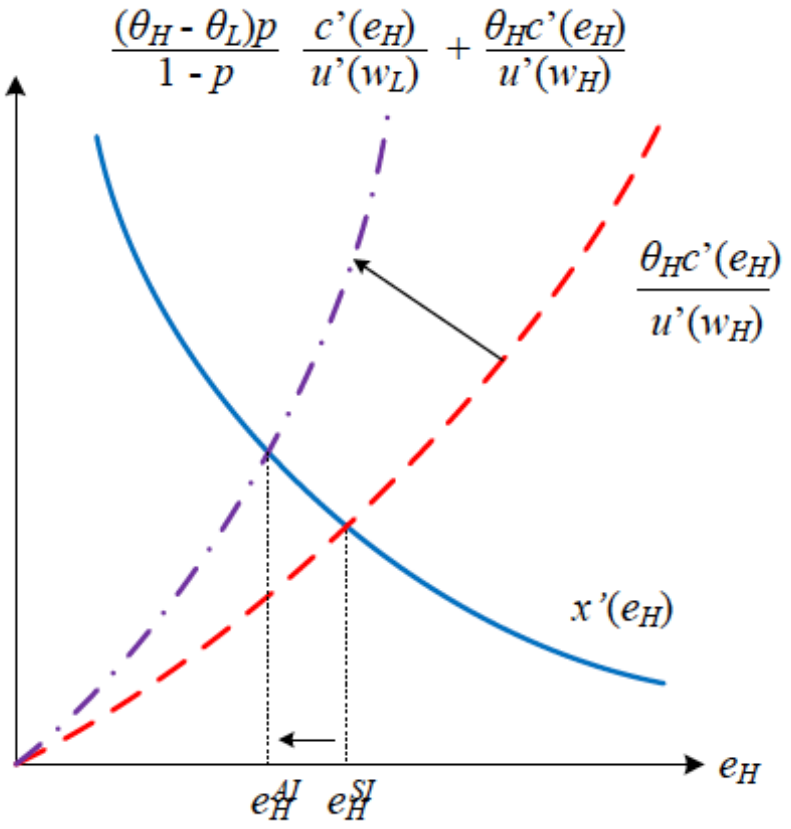
- Compare e_H^{AI} against the effort arising under symmetric information e_H^{SI}, $\frac{\theta_H c'(e_H)}{u'(w_H)} = x'(e_H)$.
- Given $\theta_H - \theta_L > 0$, $p > 0$, $c'(e_H) > 0$ and $u'(w_L) > 0$,

$$\frac{(\theta_H - \theta_L)p}{1-p}\frac{c'(e_H)}{u'(w_L)} + \frac{\theta_H c'(e_H)}{u'(w_H)} > \frac{\theta_H c'(e_H)}{u'(w_H)}$$

- Hence the effort level under asymmetric information is **lower** than that under symmetric information, $e_H^{AI} < e_H^{SI}$.

Asymmetric Information





Asymmetric Information

- In summary, the pair of contracts (w_H, e_H) and (w_L, e_L) must satisfy the following equations

$$u(w_L) - \theta_L c(e_L) = (\theta_H - \theta_L)c(e_H)$$

$$u(w_H) - c(e_H, \theta_H) = 0$$

$$\frac{\theta_L c'(e_L)}{u'(w_L)} = x'(e_L)$$

$$\frac{(\theta_H - \theta_L)p}{1-p}\frac{c'(e_H)}{u'(w_L)} + \frac{\theta_H c'(e_H)}{u'(w_H)} = x'(e_H)$$

Monotonicity in Effort

- Consider that effort levels satisfy $e_L \geq e_H$.
  - That is, the worker with the lowest cost of effort exerts a larger effort level than the worker with a high cost of effort.
- Combining (IC_L) and (IC_H) to obtain

$$u(w_L) - c(e_L, \theta_L) \geq u(w_H) - c(e_H, \theta_L) > u(w_H) - c(e_H, \theta_H) \geq u(w_L) - c(e_L, \theta_H)$$

  - The first inequality stems from (IC_L).
  - The second inequality is due to $c(e_L, \theta_L) < c(e_H, \theta_H)$.
  - The third inequality is due to (IC_H).
- Hence, the above inequality can be rearranged as

$$c(e_H, \theta_L) - c(e_L, \theta_L) \geq u(w_H) - u(w_L) > c(e_H, \theta_H) - c(e_L, \theta_H)$$

Monotonicity in Effort

- Multiplying this expression by -1, and using the first and last terms

$$c(e_L, \theta_L) - c(e_H, \theta_L) < c(e_L, \theta_H) - c(e_H, \theta_H)$$

- This condition indicates that the marginal cost of increasing effort from e_H to e_L is higher for the high-type than for the low-type worker.
- Evaluating this condition in the cost of effort function $c(e, \theta) = \theta c(e)$

$$\theta_L[c(e_L) - c(e_H)] < \theta_H[c(e_L) - c(e_H)]$$

- Since $\theta_L < \theta_H$, we must have $c(e_L) > c(e_H)$.
- Hence effort is larger for the worker with the low cost of effort, $e_L > e_H$

Monotonicity in Effort

- Example 7:
  - Let us use Example 6 to calculate the optimal contracts under asymmetric information.
  - Taking FOCs from above

$$u(w_L) - \theta_L c(e_L) = (\theta_H - \theta_L)c(e_H) \Rightarrow w_L - e_L^2 = e_H^2$$

$$u(w_H) - \theta_H c(e_H) = 0 \Rightarrow w_H = 2e_H^2$$

$$x'(e_L) = \frac{\theta_L c'(e_L)}{u'(w_L)} \Rightarrow \frac{1}{e_L} = \frac{2e_L}{1} \rightarrow 2e_L^2 = 1 \rightarrow e_L = \frac{1}{\sqrt{2}}$$

$$\frac{(\theta_H - \theta_L)p}{1-p}\frac{c'(e_H)}{u'(w_L)} + \frac{\theta_H c'(e_H)}{u'(w_H)} = x'(e_H) \Rightarrow \frac{2e_H}{1} + \frac{4e_H}{1}$$

$$= \frac{1}{e_H} \rightarrow e_H = \frac{1}{\sqrt{6}}$$

Monotonicity in Effort

- Example 7: (con't)
  - From the last two FOCs, we obtain the equilibrium effort levels $e_L = \frac{1}{\sqrt{2}}$ and $e_H = \frac{1}{\sqrt{6}}$.
  - From the first equation

$$w_L - \frac{1}{2} = \frac{1}{6} \rightarrow w_L = \frac{2}{3}$$

  - From the second equation

$$w_H = 2 \cdot \frac{1}{6} \rightarrow w_H = \frac{1}{3}$$

  - Therefore, the optimal pair of contracts is

$$\left(w_H^{AI}, e_H^{AI}\right) = \left(\frac{1}{3}, \frac{1}{\sqrt{6}}\right) = (0.333, 0.408)$$

$$\left(w_L^{AI}, e_L^{AI}\right) = \left(\frac{2}{3}, \frac{1}{\sqrt{2}}\right) = (0.667, 0.707)$$

Monotonicity in Effort

- Example 7: (con't)
  - The introduction of asymmetric information entails:
    - *No changes* in effort for the low-cost worker relative to symmetric information

$$e_L^{AI} = e_L^{SI} = 0.707$$

    - *Lower* effort for the high-cost worker than under symmetric information

$$e_H^{AI} = 0.408 < 0.5 = e_H^{SI}$$

    - *Higher* salaries for the low-cost worker than under symmetric information

$$w_L^{AI} = 0.667 > 0.5 = w_L^{SI}$$

    - *Lower* salaries for the high-cost worker

$$w_H^{AI} = 0.333 < 0.5 = w_H^{SI}$$

Monotonicity in Effort

- Example 7: (con't)
  - The net utility that each type of worker obtains under asymmetric information is

$$u_H^{AI} = w_H - 2e_H^2 = 0$$
$$u_L^{AI} = w_L - e_L^2 = 0.167$$

  - Hence the worker with a low cost of effort captures an information rent

$$u_L^{AI} - u_L^{SI} = 0.167 - 0 = 0.167$$

  - The worker with a high cost of effort does not

$$u_H^{AI} = u_H^{SI} = 0$$

  - Intuitively, the firm must compensate the low-cost worker above symmetric information terms in order for him to reveal his type.

Application of Adverse Selection—Regulation

Regulation

- Regulatory agencies often cannot observe some characteristics of the regulated firm or of an individual.
- Examples:
  - A firm's production costs
  - A firm's costs from pollution abatement
  - A consumer's willingness to pay for certain products
- In these scenarios the privately informed party (e.g., firm) has incentives to overstate its costs.
- Hence the regulator cannot directly ask firms about their production costs since responses would be unreliable.
- Adverse selection models offer an alternative contracting tool to extract information from privately informed firms (or consumers).

Regulation

- Consider that a government regulating a monopoly with cost function

$$c(q) = C + cq$$

  where C is fixed costs and $c > 0$ is marginal costs.
- The consumer pays F for the bulk of q units consumed, and the monopolist may receive a lump-sum subsidy from the government of S.
- Assume that the shadow cost of raising public funds is $g \in (0,1)$, thus implying that the total cost of providing a subsidy S to the monopolist is $(1 + g)S$.
- Analyze settings where government has symmetric and asymmetric information about the monopolist's costs.

Regulation- Symmetric Information

- Consider that the government can **perfectly** observe the monopolist's marginal cost of production c.
- The government solves the following problem subject to PCs of both the monopolist and the consumer:

$$\max_{F,S,q} \; [u(q) - F] + [F + S - C - cq] - (1 + g)S$$

$$\text{s.t.} \quad F + S - C - cq \geq 0 \qquad (PC_{Monop})$$

$$u(q) - F \geq 0 \qquad (PC_{Consum})$$

where $u(q) - F$ is the consumer's utility after paying F for q units; and $F + S - C - cq$ is the monopolist's profits.

- The Lagrangian is

$$\mathcal{L} = [u(q) - F] + [F + S - C - cq] - (1 + g)S + \lambda_1[F + S - C - cq] + \lambda_2[u(q) - F]$$

Regulation- Symmetric Information

- Taking FOCs yields

$$\frac{\partial \mathcal{L}}{\partial F} = \lambda_1 - \lambda_2 = 0 \rightarrow \lambda_1 = \lambda_2$$

$$\frac{\partial \mathcal{L}}{\partial S} = 1 - (1 + g) + \lambda_1 = 0 \rightarrow \lambda_1 = g$$

$$\frac{\partial \mathcal{L}}{\partial q} = u'(q) - c - \lambda_1 c + \lambda_2 u'(q) = 0$$

$$\lambda_1[F + S - C - cq] = 0$$

$$\lambda_2[u(q) - F] = 0$$

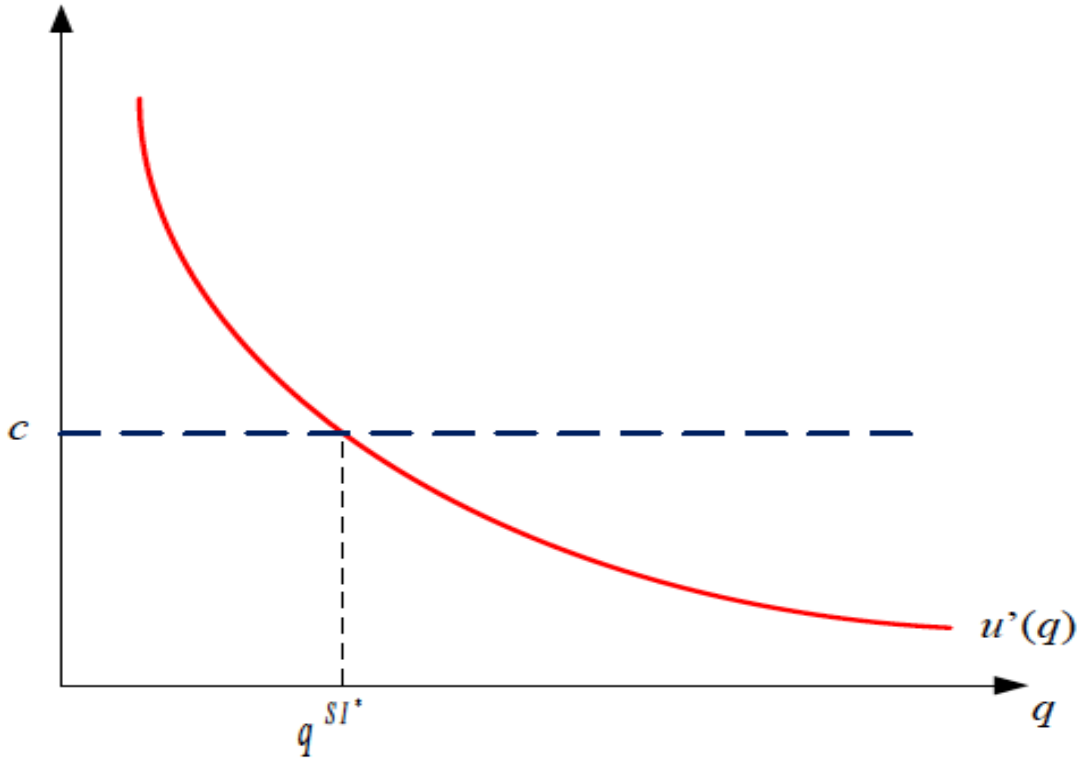
- Combining the first and second FOC

$$\lambda_1 = \lambda_2 = g$$

Regulation- Symmetric Information

- Plugging this result into the third FOC yields
$$u'(q) - c - gc + gu'(q) = 0$$
- Rearranging
$$(1+g)u'(q) = (1+g)c \Leftrightarrow u'(q) = c$$
- That is, q is increased until the point where marginal utility from further units coincides with its marginal cost.
- Hence, under symmetric information, the monopolist's production is **efficient**.

Regulation- Symmetric Information



Regulation- Asymmetric Information

- Consider now that the government **cannot** observe the monopolist's marginal cost of production c.
- Marginal cost can be low or high $c = \{c_L, c_H\}$, where $c_L < c_H$, with associated probabilities p and $1 - p$, respectively.
- The government offers two menus (F_L, S_L, q_L) and (F_H, S_H, q_H) to maximize the expected social welfare subject to PCs of both the monopolist and the consumer.

Regulation- Asymmetric Information

- The government's maximization problem is

$$\max_{(F_L,S_L,q_L),(F_H,S_H,q_H)} p[u(q_L) - F_L + F_L + S_L - C - c_L q_L - (1+g)S_L]$$
$$+ (1-p)[u(q_H) - F_H + F_H + S_H - C - c_H q_H - (1+g)S_H]$$

s.t.

$$F_L + S_L - C - c_L q_L \geq 0 \qquad (PC_{Monop,L})$$
$$F_H + S_H - C - c_H q_H \geq 0 \qquad (PC_{Monop,H})$$
$$F_L + S_L - C - c_L q_L \geq F_H + S_H - C - c_L q_H \qquad (IC_{Monop,L})$$
$$F_H + S_H - C - c_H q_H \geq F_L + S_L - C - c_H q_L \qquad (IC_{Monop,H})$$
$$u(q_L) - F_L \geq 0 \qquad (PC_{Consum,L})$$
$$u(q_H) - F_H \geq 0 \qquad (PC_{Consum,H})$$

Regulation- Asymmetric Information

- Timeframe:
  - The government offers contracts
  - The monopolist chooses one of the contracts, and then the K-type monopolist offers q_K units to the consumer at a lump-sum price of F_K where $K = \{L, H\}$.
  - The consumer can accept or reject the offer.
- *Practice:* Solve the problem on your own.
  - Output of the low type coincides with that under symmetric information, whereas, that of the high type is smaller.
  - However, the subsidy that the high-cost firm receives is lower than under symmetric information, while that of the low-cost firm is the same.

Regulation- Asymmetric Information

- Example 8:
  - Consider consumers with utility function $u(q) = \sqrt{q}$, a monopoly with cost function $c(q) = \frac{1}{4} + cq$, where marginal costs can be high $c_H = \frac{1}{8}$ or low $c_L = \frac{1}{16}$, with probability $p = \frac{1}{2}$.
  - The shadow cost of raising public funds is $g = \frac{1}{24}$.
  - Symmetric information entails an output level that solves

$$\frac{1}{2\sqrt{q}} = c_K$$

    which yields $q_H^{SI} = 16$ and $q_L^{SI} = 64$.
  - Asymmetric information entails output levels

$$q_L^{AI} = q_L^{SI} = 64$$
$$q_H^{AI} \cong 15.38 < q_H^{SI} = 16$$