

Advanced Microeconomic Theory

Subjective Probability Theory

Outline

- Objective and subjective probabilities
- States, acts, and bets
- Eliciting beliefs from choices
- Subjective expected utility
- Applications: insurance and betting
- The Ellsberg paradox

Subjective Probability Theory

- So far, probabilities were *objective*: known to the individual and to the analyst.
 - A fair coin, a roulette wheel, a lottery ticket.
- Most uncertainty in real life comes without known probabilities.
 - Will it rain tomorrow?
 - Will a new drug pass its clinical trials?
 - Will a start-up survive its first year?

Subjective Probability Theory

- Can we still use expected utility when nobody hands the individual a probability?
- *Subjective probability theory* (Savage, 1954; Anscombe and Aumann, 1963): yes, if preferences satisfy a few axioms.
- Probabilities are then *derived from choices*, not assumed.

States, Acts, and Bets

Subjective Probability Theory

- ***States of the world***: $s = 1, 2, \dots, S$.
 - Exactly one state occurs, and the individual does not control which one.
- ***Act***: a list of outcomes, one for each state,
$$f = (x_1, x_2, \dots, x_S)$$
 - Choosing an act is choosing a random variable whose probabilities are not given.

Subjective Probability Theory

- **Example**: two states, rain and sun. Should you carry an umbrella? (Payoffs in utility units.)

Act	Rain	Sun
Carry umbrella	8	8
Leave it home	0	10

Subjective Probability Theory

- If π is the probability of rain the individual has in mind, carrying the umbrella is better when
$$8 > \pi \cdot 0 + (1 - \pi) \cdot 10 \quad \text{if and only if} \quad \pi > 0.2$$
- The choice reveals whether the individual believes rain is more or less likely than 0.2.
 - Choices carry information about beliefs.

Subjective Probability Theory

- A **bet** on an event pays a prize if the event occurs, and zero otherwise.
- If the individual prefers a bet on rain to a bet on sun with the same prize, the individual acts as if rain is more likely.
- Comparing bets therefore ranks events by their likelihood.

Eliciting Beliefs from Choices

Subjective Probability Theory

- **Example:** a bet pays \$100 if it rains, \$0 otherwise.
- The individual is indifferent between the bet and \$36 for sure.
- If $u(x) = \sqrt{x}$, then
$$\pi\sqrt{100} + (1 - \pi)\sqrt{0} = \sqrt{36}$$
$$10\pi = 6, \quad \text{so } \pi = 0.6$$

Subjective Probability Theory

- A risk-neutral individual who is indifferent between the same bet and \$40 holds
$$100\pi = 40, \quad \text{so } \pi = 0.4$$
- The certainty equivalent alone does not reveal π .
 - We need the individual's utility function too: beliefs and risk attitudes are intertwined.

Subjective Probability Theory

- Once π is known, every act can be ranked.
- **Example**: with $\pi = 0.6$ and $u(x) = \sqrt{x}$, the act paying \$64 if rain and \$16 if sun has certainty equivalent

$$CE = [0.6\sqrt{64} + 0.4\sqrt{16}]^2 = 6.4^2 = 40.96$$

- Below its expected value, $0.6(64) + 0.4(16) = \$44.80$, because of risk aversion.

Subjective Expected Utility

Subjective Probability Theory

- Anscombe and Aumann (1963) allow the outcome in each state to be an objective lottery L_s .
 - Example: in state s , a ticket in a raffle with known odds.
- An act is then a list of lotteries,
 $(L_1, \dots, L_S)$

Subjective Probability Theory

- If preferences over acts satisfy continuity and the independence axiom, there are functions u_s such that

$$U(L_1, \dots, L_S) = \sum_{s=1}^S \sum_x L_s(x) u_s(x)$$

- This is *extended expected utility* (MWG, Section 6.F).
 - The utility function u_s can depend on the state.

Subjective Probability Theory

- Problem: state-dependent utility cannot be separated from probabilities.
- *Example*: these two specifications rank every act in the same way:

$$\pi = \left(\frac{1}{2}, \frac{1}{2}\right), \quad u_1(x) = 2\sqrt{x}, \quad u_2(x) = \sqrt{x}$$

$$\pi = \left(\frac{2}{3}, \frac{1}{3}\right), \quad u(x) = \sqrt{x}$$

Subjective Probability Theory

- Indeed, for every act,

$$\frac{1}{2} 2\sqrt{x_1} + \frac{1}{2} \sqrt{x_2} = \frac{3}{2} \left[\frac{2}{3} \sqrt{x_1} + \frac{1}{3} \sqrt{x_2} \right]$$

- The right side is a positive multiple of the second specification, so both rank acts identically.
 - Choices cannot tell us whether the individual believes rain is more likely or enjoys money more when it rains.

Subjective Probability Theory

- ***State uniformity***: the ranking of lotteries is the same in every state,
 $L \succsim_s L'$ for some s implies $L \succsim_{s'} L'$ for every s'
- Under state uniformity, every u_s is a positive affine transformation of a single u ,
$$u_s(x) = \pi_s u(x) + \beta_s$$
 - where $\pi_s > 0$ and β_s is any real number.

Subjective Probability Theory

- Substituting into extended expected utility,

$$\sum_s \sum_x L_s(x) u_s(x) = \sum_s \pi_s \sum_x L_s(x) u(x) + \sum_s \beta_s$$

- The last sum is a constant, so it does not affect the ranking of acts.
- Dividing every π_s by their sum does not affect it either:

$$\pi_s \text{ becomes } \frac{\pi_s}{\sum_k \pi_k}$$

Subjective Probability Theory

- *Subjective expected utility theorem*: under the axioms and state uniformity,

$$U(L_1, \dots, L_S) = \sum_{s=1}^S \pi_s \sum_x L_s(x) u(x)$$

- with $\pi_s > 0$ and $\pi_1 + \dots + \pi_S = 1$.
- π_s is the individual's *subjective probability* of state s .
 - It is unique, and u is unique up to a positive affine transformation.

Subjective Probability Theory

- With two states, an indifference curve is

$$\pi_1 u(x_1) + \pi_2 u(x_2) = \bar{u}$$

- Its slope is

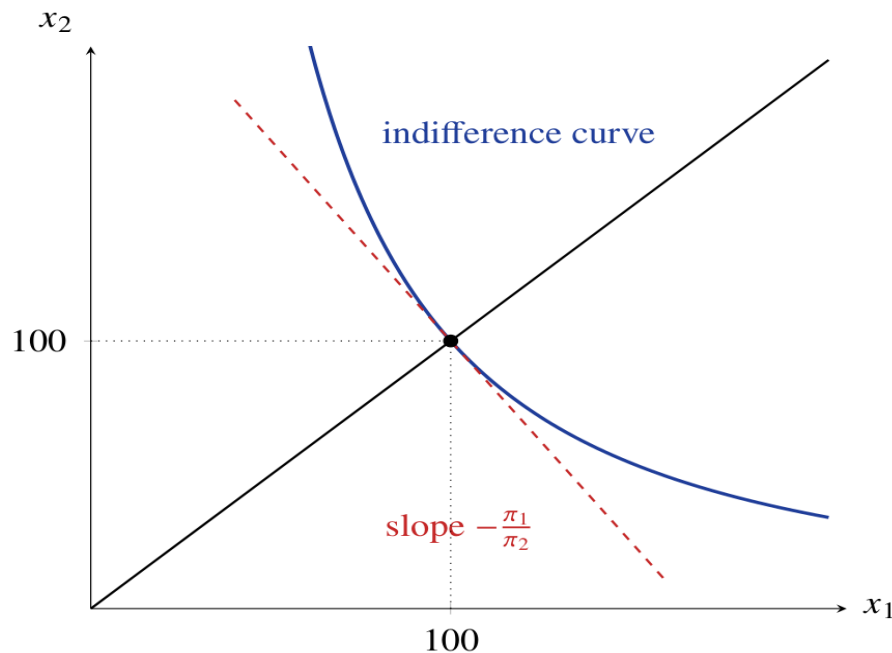
$$\text{MRS} = \frac{\pi_1 u'(x_1)}{\pi_2 u'(x_2)}$$

- Along the certainty line, $x_1 = x_2$, the slope is

$$\text{MRS} = \frac{\pi_1}{\pi_2}: \text{beliefs alone.}$$

Subjective Probability Theory

- A more optimistic belief about state 1 makes indifference curves steeper at the certainty line.



Applications: Insurance and Betting

Insurance with Subjective Beliefs

- Wealth W ; a loss L occurs with subjective probability π .
- Coverage c costs qc , where q is the price per dollar of coverage.
- The individual solves
$$\max_{c \geq 0} \pi u(W - L + (1 - q)c) + (1 - \pi)u(W - qc)$$

Insurance with Subjective Beliefs

- The first-order condition is

$$\frac{\pi(1 - q)}{W - L + (1 - q)c} = \frac{(1 - \pi)q}{W - qc}$$

- Solving for c ,

$$c^* = \frac{\pi(1 - q)W - (1 - \pi)q(W - L)}{q(1 - q)}$$

– when the right side is positive; otherwise, $c^* = 0$.

Insurance with Subjective Beliefs

- *Example:*

$$W = 100, \quad L = 50, \quad q = 0.2, \quad u(x) = \ln x$$

- Then

$$c^* = \frac{90\pi - 10}{0.16}$$

- Full insurance, $c^* = L$, arises only when $\pi = q$: the belief equals the price.

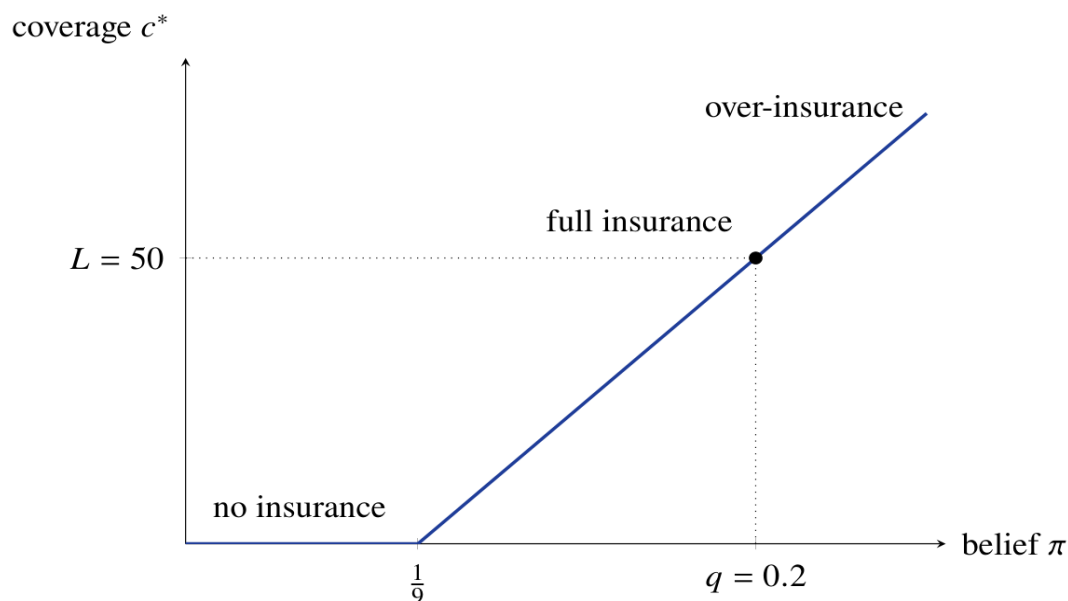
Insurance with Subjective Beliefs

- Same price, different beliefs, different coverage:

Belief π	0.10	0.12	0.15	0.18	0.20	0.22
Coverage c^*	0	5.00	21.88	38.75	50.00	61.25

Insurance with Subjective Beliefs

- No coverage if $\pi \leq \frac{1}{9}$; partial coverage between $\frac{1}{9}$ and q ; over-insurance if $\pi > q$.



Betting with Different Beliefs

- Two investors, each with \$100 in state 1 and \$100 in state 2, and $u(x) = \ln x$.
- A bet of size b gives investor 1 an extra b in state 1 and takes b in state 2.
- Investor 1 believes state 1 has probability 0.6; investor 2 believes it has probability 0.3.

Betting with Different Beliefs

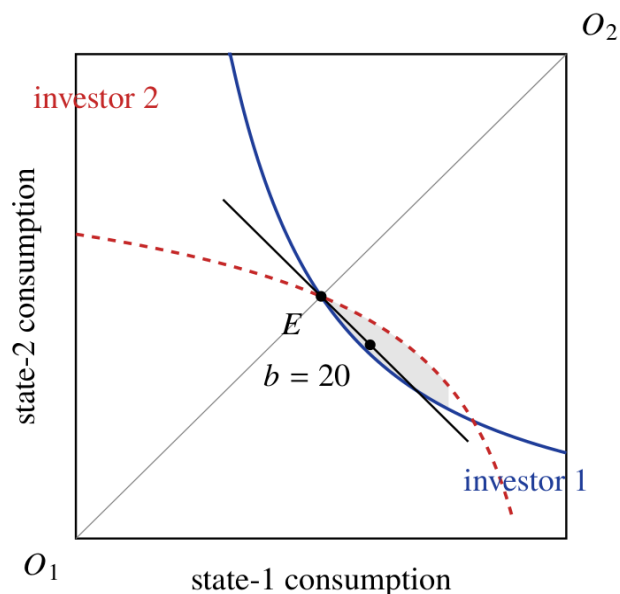
- An investor with belief π on state 1 solves
$$\max_b \pi \ln(100 + b) + (1 - \pi) \ln(100 - b)$$
$$b^* = 100(2\pi - 1)$$
- Investor 1 wants $b^* = 20$.
- Investor 2 wants $b^* = -40$: a bet of 40 on state 2.

Betting with Different Beliefs

- At $b = 20$, both investors gain relative to not betting:
$$0.6 \ln 120 + 0.4 \ln 80 \approx 4.625 > \ln 100 \approx 4.605$$
$$0.3 \ln 80 + 0.7 \ln 120 \approx 4.666 > \ln 100 \approx 4.605$$
- Trade arises from different beliefs alone: same wealth, same utility function, no risk to share.

Betting with Different Beliefs

- Slopes of the indifference curves through E:
– $-\frac{0.6}{0.4} = -1.5$ and $-\frac{0.3}{0.7} \approx -0.43$.
– The shaded lens contains the bets both investors prefer.



The Ellsberg Paradox

The Ellsberg Paradox

- An urn has 300 balls: 100 red, and 200 blue or green in an unknown proportion.
- Bets pay \$1,000 if:
 - A: red. B: blue.
 - C: blue or green. D: red or green.
- Most people choose A over B and C over D (Ellsberg, 1961).

The Ellsberg Paradox

- Payoffs, in dollars:

Bet	Red (100)	Blue	Green
A	1,000	0	0
B	0	1,000	0
C	0	1,000	1,000
D	1,000	0	1,000

The Ellsberg Paradox

- Can a single prior explain these choices? With $p_R = \frac{1}{3}$, SEU requires
 - $A \succ B$:
$$p_R > p_B$$
 - $C \succ D$:
$$p_B + p_G > p_R + p_G, \quad \text{so } p_B > p_R$$
- A contradiction: no subjective probabilities rationalize both choices.

The Ellsberg Paradox

- Bets A and B differ from C and D only in the green column, where each pair pays the same.
 - SEU (the sure-thing principle) says that common column should not matter.
- Yet A and C have known probabilities, $1/3$ and $2/3$, while B and D do not.
- People dislike unknown probabilities: *ambiguity aversion*.

Summary

- Subjective probabilities are read from choices over bets and acts.
- State uniformity separates beliefs from tastes, giving subjective expected utility.
- Different beliefs alone change insurance demand and create trade.
- The Ellsberg paradox shows choices that no single prior can explain.

References

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