

EconS 501 – Micro Theory I

Recitation #9 – Choice under Uncertainty I¹

1. **MWG 6.B.2.** Show that if the preference relation $\succsim$ on the space of lotteries $\mathcal{L}$ is represented by a utility function $U(\cdot)$ that has the *expected utility* form, then $\succsim$ satisfies the *independence axiom*.

Recall. A utility $U : \mathcal{L} \rightarrow \mathbb{R}$ has the *expected utility form* if there exist Bernoulli utilities $u_1, \dots, u_N$ over the N pure outcomes such that

$$U(L) = \sum_{n=1}^N u_n p_n \quad \text{for every lottery } L = (p_1, \dots, p_N).$$

The *independence axiom* says: for all $L, L', L'' \in \mathcal{L}$ and any $\alpha \in (0, 1)$,

$$L \succsim L' \iff \alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''.$$

Answer.

Step 1. Translate the base ranking into a numerical inequality. Because U represents $\succsim$,

$$L \succsim L' \iff U(L) \geq U(L') \iff \sum_n u_n p_n \geq \sum_n u_n p'_n.$$

Step 2. Add the same number to both sides. The utility of the “mixing lottery” L'' is $U(L'') = \sum_n u_n p''_n$. Multiplying the inequality by $\alpha > 0$ (preserving direction) and adding $(1 - \alpha)U(L'')$ to both sides,

$$\alpha \sum_n u_n p_n + (1 - \alpha) \sum_n u_n p''_n \geq \alpha \sum_n u_n p'_n + (1 - \alpha) \sum_n u_n p''_n.$$

Step 3. Recognize the linearity of U in probabilities. Because U is linear in the probability vector, the left-hand side is exactly $U(\alpha L + (1 - \alpha)L'')$ and the right-hand side is $U(\alpha L' + (1 - \alpha)L'')$. So the inequality reads

$$U(\alpha L + (1 - \alpha)L'') \geq U(\alpha L' + (1 - \alpha)L'').$$

Step 4. Translate back into preferences. U represents $\succsim$, so the utility inequality is equivalent to

$$\alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''.$$

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Every step was an equivalence, so we have shown

$$L \succsim L' \iff \alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L'',$$

which is exactly the independence axiom.

Intuitively, the expected-utility form is *linear* in probabilities. Mixing both sides of a preference with a common lottery L'' shifts every utility by the same amount $(1 - \alpha)U(L'')$, so the ordinal ranking is preserved. Any preference derived from a linear-in-probabilities utility inherits the independence axiom automatically.

2. **MWG 6.B.5.** The Allais paradox shows that many decision makers violate the independence axiom (IA). The goal of this exercise is to show that Allais-type preferences are still compatible with a *weaker* axiom, the *betweenness axiom* (BA, Dekel 1986):

$$(BA) \quad L \sim L' \implies \alpha L + (1 - \alpha)L' \sim L \quad \text{for all } \alpha \in (0, 1).$$

Suppose there are three possible outcomes.

- (a) Show that IA implies BA.
- (b) Draw the simplex representation for lotteries with three outcomes (as in MWG Fig. 6.B.1). Show that if continuity and BA hold, then indifference curves are straight lines. Do these lines need to be parallel?
- (c) Conclude that BA is strictly weaker than IA.

Answer. *Part (a): IA implies BA.*

Step 1. Apply IA with the mixing lottery equal to L . Suppose $L \sim L'$ (so in particular $L \succsim L'$). Take the IA statement with the “mixing” lottery set to L itself:

$$L \succsim L' \implies \alpha L + (1 - \alpha)L \succsim \alpha L' + (1 - \alpha)L \iff L \succsim \alpha L' + (1 - \alpha)L.$$

(The left side simplifies because $\alpha L + (1 - \alpha)L = L$.)

Step 2. Reverse direction. By the same IA applied to $L' \succsim L$ (the other half of $L \sim L'$),

$$\alpha L' + (1 - \alpha)L \succsim L.$$

Step 3. Combine. Both $L \succsim \alpha L' + (1 - \alpha)L$ and $\alpha L' + (1 - \alpha)L \succsim L$, hence $\alpha L' + (1 - \alpha)L \sim L$. Rewriting $\alpha L' + (1 - \alpha)L$ as $(1 - \alpha)L + \alpha L'$ (same convex combination, different parameter labeling), this is the BA conclusion.

Answer. *Part (b): BA implies straight indifference curves; IA additionally forces parallel.*

Simplex setup. With three outcomes, every lottery $L = (p_1, p_2, p_3)$ satisfies $p_n \geq 0$ and $p_1 + p_2 + p_3 = 1$. Parameterize by (p_1, p_3) ; then $p_2 = 1 - p_1 - p_3$ is implicit. The set of admissible pairs is the triangle with vertices at $(0,0)$ (pure outcome 2), $(1,0)$ (pure outcome 1), and $(0,1)$ (pure outcome 3). Convex combinations of two lotteries L, L' trace the straight-line segment joining them in this simplex.

BA $\Rightarrow$ straight indifference curves. Take any two indifferent lotteries $L \sim L'$ in the same indifference set. BA says every point on the segment between L and L' is also indifferent to L (equivalently to L'). So the indifference set contains the entire segment. Combined with continuity of $\succsim$ (indifference curves are closed), each indifference curve is a straight line segment inside the simplex.

Do the lines have to be parallel? No. BA restricts each indifference curve individually (each must be a straight line) but says nothing across indifference curves. Different indifference curves may cross the simplex at different angles.

IA additionally forces parallel. Suppose IA holds and take two straight indifference curves through L and L' (with $L \succ L'$). Applying IA with any third lottery L'' : mixing both curves with L'' produces a pair of parallel-shifted curves in the simplex — the parallelism is exactly the geometric content of $L \succsim L' \Leftrightarrow \alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''$. Figure 1 contrasts the two cases.

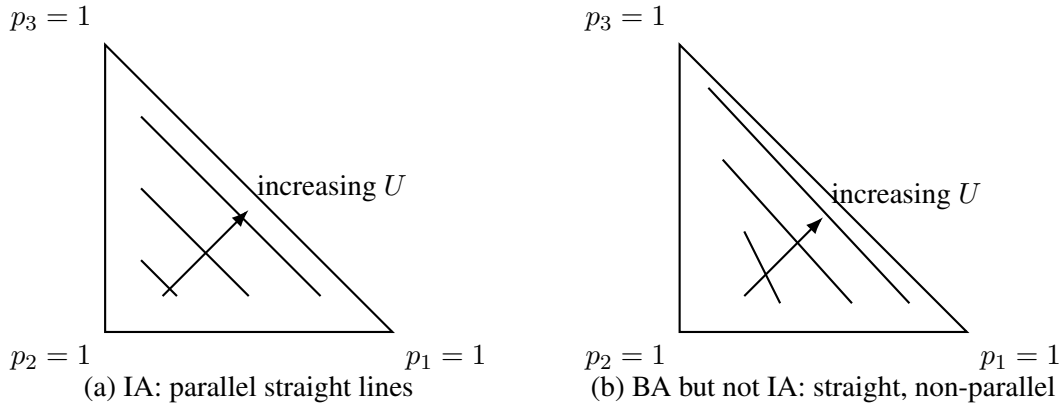


Figure 1: Simplex representation of preferences over three-outcome lotteries. Panel (a): the independence axiom forces indifference curves to be parallel straight lines. Panel (b): the betweenness axiom alone (weaker than IA) permits straight non-parallel indifference curves, which can accommodate Allais-type behavior.

Answer. *Part (c): BA is strictly weaker than IA.* By part (a), every preference satisfying IA satisfies BA. To show the reverse implication fails, exhibit a preference satisfying BA but not IA: any preference whose indifference curves in the simplex are straight lines that are *not* all parallel (panel (b) of Figure 1) qualifies. Such preferences

can rationalize Allais-paradox choices, which no expected-utility (hence IA-satisfying) preference can.

Intuitively, BA controls each indifference curve individually — “mixing an indifferent pair gives you a lottery on the same curve” — but does not tie together the shapes of different curves. IA is strictly stronger because it forces the entire family of curves to be a parallel translation of a single one. Dropping this cross-curve consistency is exactly what buys enough flexibility to accommodate Allais-style behavior while keeping the intra-curve linearity that BA provides.

3. **Evacuation planning under uncertainty.** A security agency with vNM utility function u evaluates two disaster plans for a flood-prone area. The probability of a flood is 1%. There are four possible outcomes:

- a_1 : no evacuation, no flood;
- a_2 : no evacuation, flood;
- a_3 : evacuation, no flood;
- a_4 : evacuation, flood.

The agency reports two *elicitation indifference*s: $a_3 \sim (a_1, a_2; p, 1 - p)$ and $a_4 \sim (a_1, a_2; q, 1 - q)$, for some $p, q \in (0, 1)$. Also, $u(a_1) = 1$ and $u(a_2) = 0$.

(a) Express $u(a_3)$ and $u(a_4)$ in terms of p, q .

(b) Two candidate plans have the following contingent behavior:

- *Plan 1*: evacuate in 90% of flood cases and in 10% of no-flood cases;
- *Plan 2*: evacuate in 95% of flood cases and in 15% of no-flood cases.

For each plan, derive the probability distribution over $\{a_1, a_2, a_3, a_4\}$.

(c) Compute the expected utility of each plan. When is Plan 1 strictly preferred to Plan 2?

Answer. *Part (a):* $u(a_3)$ and $u(a_4)$. Because u has the vNM (expected utility) form,

$$\begin{aligned} u(a_3) &= p u(a_1) + (1 - p) u(a_2) = p \cdot 1 + (1 - p) \cdot 0 = p, \\ u(a_4) &= q u(a_1) + (1 - q) u(a_2) = q \cdot 1 + (1 - q) \cdot 0 = q. \end{aligned}$$

So the agency’s Bernoulli utility over the four outcomes is $(1, 0, p, q)$.

Answer. *Part (b): distribution over the four outcomes under each plan.*

Let F denote flood ($\Pr F = 0.01$) and $\bar{F}$ no flood ($\Pr \bar{F} = 0.99$). Plan j specifies $\Pr(\text{evacuate} \mid F)$ and $\Pr(\text{evacuate} \mid \bar{F})$; the four contingent probabilities are the joint probabilities.

Plan 1: $\Pr(\text{evac} \mid F) = 0.90$, $\Pr(\text{evac} \mid \bar{F}) = 0.10$.

$$\begin{aligned}\Pr(a_1) &= \Pr(\bar{F}) \Pr(\text{evac} \mid \bar{F}) = 0.99 \cdot 0.90 = 0.891, \\ \Pr(a_2) &= \Pr(F) \Pr(\text{evac} \mid F) = 0.01 \cdot 0.10 = 0.001, \\ \Pr(a_3) &= \Pr(\bar{F}) \Pr(\text{no-evac} \mid \bar{F}) = 0.99 \cdot 0.10 = 0.099, \\ \Pr(a_4) &= \Pr(F) \Pr(\text{no-evac} \mid F) = 0.01 \cdot 0.90 = 0.009.\end{aligned}$$

(Check: they sum to 1.)

Plan 2: $\Pr(\text{evac} \mid F) = 0.95$, $\Pr(\text{evac} \mid \bar{F}) = 0.15$. Analogously,

$$(\Pr(a_1), \Pr(a_2), \Pr(a_3), \Pr(a_4)) = (0.8415, 0.0005, 0.1485, 0.0095).$$

Answer. *Part (c): expected utilities and when Plan 1 is preferred.*

Step 1. Compute EU of each plan.

$$\begin{aligned}EU(\text{Plan 1}) &= 0.891 \cdot 1 + 0.001 \cdot 0 + 0.099 \cdot p + 0.009 \cdot q \\ &= 0.891 + 0.099p + 0.009q. \\ EU(\text{Plan 2}) &= 0.8415 \cdot 1 + 0.0005 \cdot 0 + 0.1485 \cdot p + 0.0095 \cdot q \\ &= 0.8415 + 0.1485p + 0.0095q.\end{aligned}$$

Step 2. Compare.

$$EU(\text{Plan 1}) > EU(\text{Plan 2}) \iff 0.891 - 0.8415 > 0.1485p - 0.099p + 0.0095q - 0.009q,$$

i.e. $0.0495 > 0.0495p + 0.0005q$, or

$$q < 99(1 - p).$$

Step 3. Interpret. Since $q \in (0, 1)$, the condition $q < 99(1 - p)$ is satisfied whenever $1 < 99(1 - p)$, i.e., $p < 98/99 \approx 0.9899$. So Plan 1 is strictly preferred to Plan 2 for almost all (p, q) pairs the agency might report — only if p is extremely close to 1 (the agency is nearly indifferent between evacuation-without-flood a_3 and no-evacuation-without-flood a_1) is Plan 2 preferred.

Intuitively, Plan 2 evacuates more people in *both* states, so it does better in floods (which are rare) at the cost of more unnecessary evacuations (much more common). Plan 1 wins whenever unnecessary evacuations are costly relative to the risk of an unmanaged flood — i.e., whenever p (the utility of evacuating without a flood) is meaningfully less than 1.

4. **Complement (completeness and transitivity of three preferences over lotteries).** Exercises 1–2 above examined the *independence axiom* on lotteries, taking rationality (completeness + transitivity) for granted. This complement pulls the more basic properties into the foreground: for each of the three candidate preference rules below on independent lotteries with a finite outcome set, decide whether the rule is complete, transitive, both, or neither.

- (a) $X \succsim Y \iff E[X] \geq E[Y]$ (rank by expected value).
- (b) $X \succsim Y \iff E[X^2] \geq (E[Y])^2$ (compare second moment of X with squared mean of Y).
- (c) $X \succsim Y \iff \Pr(X \geq Y) \geq 0.5$ (“more likely than not to come out ahead”).

Answer. Part (a): *both*.

Rule (a) ranks lotteries by the real number $E[\cdot]$. Since $\geq$ on $\mathbb{R}$ is complete and transitive, the induced preference inherits both properties immediately.

Answer. Part (b): *complete but not transitive*.

Step 1. Completeness. Suppose neither $X \succsim Y$ nor $Y \succsim X$ held; then $E[X^2] < (E[Y])^2$ and $E[Y^2] < (E[X])^2$. But Jensen’s inequality applied to the convex function $t \mapsto t^2$ gives $E[Z^2] \geq (E[Z])^2$ for every Z ; chaining,

$$(E[X])^2 \leq E[X^2] < (E[Y])^2 \leq E[Y^2] < (E[X])^2,$$

a contradiction. So at least one of $X \succsim Y$ or $Y \succsim X$ holds; completeness is safe.

Step 2. Transitivity: counterexample. Let $X \equiv 2$ (constant), $Z \equiv 3$ (constant), and $Y = \pm 3$ each with probability $\frac{1}{2}$. Compute:

- $E[X^2] = 4$, $(E[Y])^2 = 0$, so $E[X^2] \geq (E[Y])^2$: $X \succsim Y$.
- $E[Y^2] = 9$, $(E[Z])^2 = 9$, so $E[Y^2] \geq (E[Z])^2$: $Y \succsim Z$.
- $E[X^2] = 4$ but $(E[Z])^2 = 9$, so $E[X^2] < (E[Z])^2$: $X \succsim Z$ *fails*.

Transitivity is violated: $X \succsim Y \succsim Z$ but not $X \succsim Z$.

Intuitively, the rule mixes two different functionals of the two lotteries (the second moment of X against the squared mean of Y). Chaining $X \succsim Y \succsim Z$ leaves you comparing $E[X^2]$ (an “ X -thing”) with $(E[Z])^2$ (a “ Z -thing”) through the intermediary Y , and the two functionals need not agree.

Answer. Part (c): *complete but not transitive*.

Step 1. Completeness. For independent X, Y ,

$$\Pr(X \geq Y) + \Pr(Y \geq X) = 1 + \Pr(X = Y) \geq 1,$$

so at least one of the two probabilities is $\geq \frac{1}{2}$. Hence at least one of $X \succsim Y$, $Y \succsim X$ holds; completeness is safe.

Step 2. Transitivity: nontransitive dice (Gardner 1970). Let A, B, C be three independent lotteries, each uniform on:

$$A \in \{1, 5, 9\}, \quad B \in \{3, 4, 8\}, \quad C \in \{2, 6, 7\}.$$

Enumerating the nine equally likely pairwise matchups,

$$\begin{aligned} \Pr(A > B) &= \Pr\{(5, 3), (5, 4), (9, 3), (9, 4), (9, 8)\} = \frac{5}{9}, \\ \Pr(B > C) &= \Pr\{(3, 2), (4, 2), (8, 2), (8, 6), (8, 7)\} = \frac{5}{9}, \\ \Pr(C > A) &= \Pr\{(2, 1), (6, 1), (6, 5), (7, 1), (7, 5)\} = \frac{5}{9}. \end{aligned}$$

So each of $A \succsim B$, $B \succsim C$, $C \succsim A$ holds. Chaining $A \succsim B \succsim C$ would suggest $A \succsim C$; but $\Pr(A \geq C) = \frac{4}{9} < \frac{1}{2}$, so $A \succsim C$ *fails*. Transitivity is violated in a cycle $A \succsim B \succsim C \succsim A$.

Intuitively, “more likely than not to come out ahead” is a *pairwise, median-like* comparison. It captures which of two lotteries the DM would pick in a head-to-head bet, but the head-to-head structure doesn’t aggregate consistently across three lotteries — the classic “nontransitive dice” phenomenon. Any preference rule that violates transitivity on lotteries cannot be induced by a real-valued utility, because $\geq$ on $\mathbb{R}$ is always transitive.