

EconS 501 - Microeconomic Theory I

Midterm Exam #1 - Answer key

1. **Rational preferences and their properties. [30 points]** A consumer's preferences over bundles $x = (x_1, x_2) \in \mathbb{R}_+^2$ are described by the relation $\succsim$ defined by $x \succsim y$ if and only if $x_1^2 + x_2^2 \geq y_1^2 + y_2^2$.
 - (a) [8 points] Show that $\succsim$ is complete, transitive, and continuous.
 - Write $g(x) = x_1^2 + x_2^2$, a real number attached to each bundle, so that $x \succsim y \Leftrightarrow g(x) \geq g(y)$.
 - *Complete.* For any x and y , either $g(x) \geq g(y)$ or $g(y) \geq g(x)$, because the order on $\mathbb{R}$ is complete, so $x \succsim y$ or $y \succsim x$.
 - *Transitive.* If $x \succsim y$ and $y \succsim z$, then $g(x) \geq g(y) \geq g(z)$, so $g(x) \geq g(z)$, that is, $x \succsim z$.
 - *Continuous.* g is a continuous function, so the upper and lower contour sets $\{x : g(x) \geq g(y)\}$ and $\{x : g(x) \leq g(y)\}$ are closed for every y . Hence $\succsim$ is continuous.
 - *Intuition.* Because the relation ranks bundles by the single number $g(x)$, it inherits completeness and transitivity directly from the order of the real line: any preference that scores bundles with a real-valued index is automatically rational. Continuity holds because a small change in the bundle moves the score only slightly, so the ranking never jumps.
 - (b) [7 points] Determine whether $\succsim$ is strongly monotone and whether it is locally nonsatiated, justifying each answer.
 - *Strongly monotone.* Suppose $x_1 \geq y_1$ and $x_2 \geq y_2$ with $x \neq y$. At least one inequality is strict, say $x_1 > y_1 \geq 0$, so $x_1^2 > y_1^2$ while $x_2^2 \geq y_2^2$. Then $g(x) > g(y)$, that is, $x \succ y$. So more of either good is strictly preferred.
 - *Locally nonsatiated.* At any x and any $\varepsilon > 0$, take $x' = (x_1 + \delta, x_2)$ with $0 < \delta < \varepsilon$. Then $g(x') = g(x) + 2x_1\delta + \delta^2 > g(x)$, so $x' \succ x$. Local nonsatiation holds (it is implied by strong monotonicity).
 - *Intuition.* The score rises whenever either good rises, so the consumer always wants more of either good and is never satiated. With no bliss point, she spends her entire budget.
 - (c) [9 points] Determine whether $\succsim$ is convex. If it is not, exhibit two bundles x and y with $x \sim y$ whose equal-weight average is strictly less preferred than both.
 - $\succsim$ is *not* convex. The upper contour set $U(y) = \{x \in \mathbb{R}_+^2 : x_1^2 + x_2^2 \geq g(y)\}$ is the set of points lying on or outside a circle centered at the origin, which is not a convex set.
 - Take $x = (2, 0)$ and $y = (0, 2)$, so that $g(x) = g(y) = 4$ and $x \sim y$. Their equal-weight average is $(1, 1)$, with $g(1, 1) = 2 < 4$, so $(1, 1) \prec x$ and $(1, 1) \prec y$. The average of two indifferent bundles is strictly worse than both, so the preferences are not convex: the consumer prefers to specialize rather than mix.

- *Intuition.* The indifference curves are quarter-circles that bow toward the origin, the reverse of the usual convex-to-the-origin shape. That is why blending two equally-liked bundles lands the consumer on a lower circle and makes a balanced mixture worse than either extreme.
- *Equation of an indifference curve.* Fix a utility level u and solve $x_1^2 + x_2^2 = u$ for x_2 . Isolating x_2^2 and taking the nonnegative root (since $x_2 \geq 0$),

$$x_2^2 = u - x_1^2 \quad \implies \quad x_2 = \sqrt{u - x_1^2}, \quad 0 \leq x_1 \leq \sqrt{u},$$

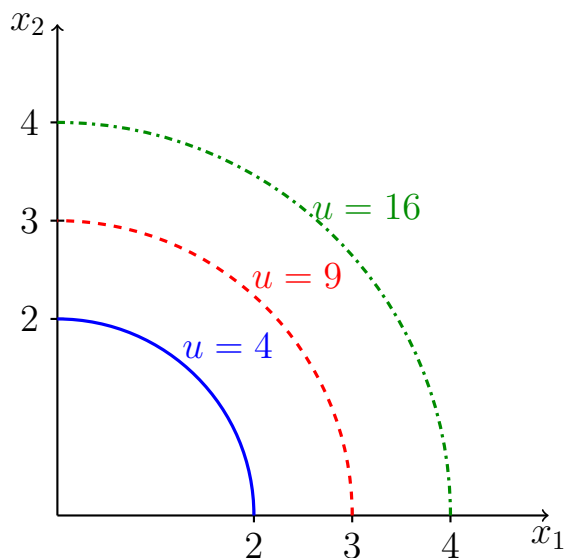
the quarter-circle of radius $\sqrt{u}$ centered at the origin.

- *Vertical intercept.* Set $x_1 = 0$ in the equation of the curve: $x_2 = \sqrt{u - 0} = \sqrt{u}$, so the curve meets the vertical axis at $(0, \sqrt{u})$.
- *Slope.* Differentiate $x_2 = (u - x_1^2)^{1/2}$ with respect to x_1 ,

$$\frac{dx_2}{dx_1} = \frac{1}{2} (u - x_1^2)^{-1/2} (-2x_1) = -\frac{x_1}{\sqrt{u - x_1^2}} = -\frac{x_1}{x_2}.$$

The slope is 0 at $x_1 = 0$ and steepens without bound as $x_1 \rightarrow \sqrt{u}$. Equivalently, the marginal rate of substitution x_1/x_2 rises as we move down the curve, the opposite of a diminishing *MRS*, which is the mark of non-convex preferences.

- *Horizontal intercept.* Set $x_2 = 0$: $0 = \sqrt{u - x_1^2}$ gives $x_1^2 = u$, so $x_1 = \sqrt{u}$ and the curve meets the horizontal axis at $(\sqrt{u}, 0)$.
- *Evaluating at $u = 4$.* The indifference curve is $x_2 = \sqrt{4 - x_1^2}$ on $0 \leq x_1 \leq 2$, with vertical intercept $(0, 2)$, horizontal intercept $(2, 0)$, and slope $-x_1/\sqrt{4 - x_1^2}$: the quarter-circle of radius 2 through the bundles $(2, 0)$ and $(0, 2)$ used above.



Indifference curves of $u = x_1^2 + x_2^2$ for $u = 4, 9, 16$.

- (d) [6 points] Exhibit a utility function that represents $\succsim$, and explain what the failure of convexity implies for the bundle the consumer chooses when she maximizes utility subject to a budget constraint $p \cdot x \leq w$.

- The function $u(x_1, x_2) = x_1^2 + x_2^2$ represents $\succsim$, since $x \succsim y \Leftrightarrow x_1^2 + x_2^2 \geq y_1^2 + y_2^2 \Leftrightarrow u(x) \geq u(y)$. It is continuous, so $\succsim$ has a continuous representation.
- Because $\succsim$ is not convex, utility maximization has a corner solution. On the budget line $p_1x_1 + p_2x_2 = w$, the objective $x_1^2 + x_2^2$ is convex, so it is maximized at an endpoint: the consumer spends all wealth on one good, choosing $(w/p_1, 0)$ when $p_1 \leq p_2$ and $(0, w/p_2)$ otherwise. She buys only the cheaper good, and her demand jumps as relative prices cross $p_1 = p_2$.
- *Intuition.* This is the mirror image of the convex case: with convex preferences the optimum is an interior tangency and demand responds smoothly to prices, whereas the outward-bowing indifference curves here send the consumer to a corner and make her demand switch abruptly from one good to the other.

2. **Recovering the Hicksian and Walrasian demands from an expenditure function.** [30 points] An econometrician estimates that a consumer's expenditure function is $e(p, u) = u(\sqrt{p_1} + \sqrt{p_2})^2$, where $p = (p_1, p_2) \gg 0$ and u is the utility level.

(a) [8 points] Verify that $e(p, u)$ has the properties of a valid expenditure function: it is homogeneous of degree one in prices, increasing in u , increasing in each price, and concave in prices.

- *Homogeneity of degree one.* $e(tp, u) = u(\sqrt{tp_1} + \sqrt{tp_2})^2 = ut(\sqrt{p_1} + \sqrt{p_2})^2 = te(p, u)$.
- *Increasing in u :* $\frac{\partial e}{\partial u} = (\sqrt{p_1} + \sqrt{p_2})^2 > 0$. *Increasing in prices:* $\frac{\partial e}{\partial p_1} = u \frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_1}} = u \left(1 + \sqrt{\frac{p_2}{p_1}}\right) > 0$, and symmetrically for p_2 .
- *Concave in prices.* The second derivatives are $\frac{\partial^2 e}{\partial p_1^2} = -\frac{u}{2}\sqrt{p_2}p_1^{-3/2} < 0$, $\frac{\partial^2 e}{\partial p_2^2} = -\frac{u}{2}\sqrt{p_1}p_2^{-3/2} < 0$, and $\frac{\partial^2 e}{\partial p_1 \partial p_2} = \frac{u}{2}(p_1p_2)^{-1/2}$. The determinant of the Hessian is

$$\left(-\frac{u}{2}\sqrt{p_2}p_1^{-3/2}\right)\left(-\frac{u}{2}\sqrt{p_1}p_2^{-3/2}\right) - \left(\frac{u}{2}(p_1p_2)^{-1/2}\right)^2 = \frac{u^2}{4}\frac{1}{p_1p_2} - \frac{u^2}{4}\frac{1}{p_1p_2} = 0.$$

With negative diagonal entries and a zero determinant, the Hessian is negative semidefinite, so e is concave in p . All the required properties hold.

(b) [7 points] Use Shephard's lemma to recover the Hicksian demands $h_1(p, u)$ and $h_2(p, u)$.

- By Shephard's lemma,

$$h_1(p, u) = \frac{\partial e}{\partial p_1} = u \frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_1}}, \quad h_2(p, u) = \frac{\partial e}{\partial p_2} = u \frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_2}}.$$

(c) [8 points] Invert e to obtain the indirect utility function $v(p, w)$, and apply Roy's identity to obtain the Walrasian demands $x_1(p, w)$ and $x_2(p, w)$.

- Setting $e(p, v) = w$ and solving for v , $v(\sqrt{p_1} + \sqrt{p_2})^2 = w$, so

$$v(p, w) = \frac{w}{(\sqrt{p_1} + \sqrt{p_2})^2}.$$

- Roy's identity, $x_i = -\frac{\partial v / \partial p_i}{\partial v / \partial w}$, with $\frac{\partial v}{\partial w} = (\sqrt{p_1} + \sqrt{p_2})^{-2}$ and $\frac{\partial v}{\partial p_1} = -\frac{w}{\sqrt{p_1}}(\sqrt{p_1} + \sqrt{p_2})^{-3}$, gives

$$x_1(p, w) = \frac{w}{\sqrt{p_1}(\sqrt{p_1} + \sqrt{p_2})} = \frac{w}{p_1 + \sqrt{p_1 p_2}}, \quad x_2(p, w) = \frac{w}{p_2 + \sqrt{p_1 p_2}}.$$

As a check, $p_1 x_1 + p_2 x_2 = \frac{w\sqrt{p_1}}{\sqrt{p_1} + \sqrt{p_2}} + \frac{w\sqrt{p_2}}{\sqrt{p_1} + \sqrt{p_2}} = w$, so Walras' law holds.

- (d) [7 points] Verify the duality identity $x_i(p, e(p, u)) = h_i(p, u)$ for $i = 1, 2$, confirming that the Walrasian and Hicksian demands recovered are mutually consistent.

- Evaluate the Walrasian demand at wealth $w = e(p, u) = u(\sqrt{p_1} + \sqrt{p_2})^2$:

$$x_1(p, e(p, u)) = \frac{u(\sqrt{p_1} + \sqrt{p_2})^2}{\sqrt{p_1}(\sqrt{p_1} + \sqrt{p_2})} = u \frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_1}} = h_1(p, u),$$

and symmetrically $x_2(p, e(p, u)) = h_2(p, u)$. The Walrasian and Hicksian demands recovered from e are mutually consistent, as duality requires (the Walrasian demand at the wealth that just attains u coincides with the Hicksian demand for u).

3. **Income and substitution effects, and welfare.** [40 points] A consumer has the quasi-linear utility function $u(x_1, x_2) = x_1 + \ln x_2$, where good 1 is the numeraire ($p_1 = 1$). She faces a price $p_2 > 0$ for good 2 and has wealth $w = 10$. The price of good 2 then triples, from p_2 to $3p_2$ (with p_1 and w unchanged), so that the answers are expressed as functions of p_2 .

- (a) [10 points] Find the Walrasian demand $x_2(p, w)$, the Hicksian demand $h_2(p, u)$, the indirect utility function $v(p, w)$, and the expenditure function $e(p, u)$.

- With good 1 the numeraire ($p_1 = 1$), the interior optimum equates the marginal utility of good 2 to its price, $\frac{1}{x_2} = p_2$. Hence the Walrasian demand for good 2 is $x_2(p, w) = \frac{1}{p_2}$, independent of wealth, and $x_1 = w - p_2 x_2 = w - 1$.
- The indirect utility function is $v(p, w) = x_1 + \ln x_2 = (w - 1) + \ln \frac{1}{p_2} = w - 1 - \ln p_2$. Inverting $v(p, e) = u$ gives the expenditure function $e(p, u) = u + 1 + \ln p_2$.
- By Shephard's lemma, $h_2(p, u) = \frac{\partial e}{\partial p_2} = \frac{1}{p_2}$, independent of u . For quasi-linear preferences the Hicksian and Walrasian demands for good 2 coincide.

- (b) [12 points] Compute the total change in the consumption of good 2 caused by the price increase, and decompose it into a substitution effect and an income effect (Hicks decomposition, holding utility at its initial level u^0). Verify that the two effects sum to the total change, and comment on the income effect obtained.

- At the initial price p_2 (with $w = 10$), $x_2^0 = \frac{1}{p_2}$. After the price triples to $3p_2$, $x_2^1 = \frac{1}{3p_2}$, so the total change is

$$\Delta x_2 = \frac{1}{3p_2} - \frac{1}{p_2} = -\frac{2}{3p_2}.$$

- The initial utility is $u^0 = v(p_2, w) = 9 - \ln p_2$. Holding utility at u^0 , the substitution effect is

$$SE = h_2(3p_2, u^0) - h_2(p_2, u^0) = \frac{1}{3p_2} - \frac{1}{p_2} = -\frac{2}{3p_2}.$$

- The income effect is the remainder,

$$IE = x_2^1 - h_2(3p_2, u^0) = \frac{1}{3p_2} - \frac{1}{3p_2} = 0.$$

The two sum to the total change, $-\frac{2}{3p_2} + 0 = -\frac{2}{3p_2} = \Delta x_2$. The income effect is zero: good 1 enters utility linearly and absorbs all wealth changes, so the demand for good 2 does not depend on wealth and the entire price response is a substitution effect.

- (c) [12 points] Compute the compensating variation (CV), the equivalent variation (EV), and the change in consumer surplus (ΔCS) of the price increase.

- The new utility level is $u^1 = v(3p_2, w) = 9 - \ln(3p_2) = 9 - \ln 3 - \ln p_2$.
- *Compensating variation.* Using $e(p_2, u) = u + 1 + \ln p_2$,

$$CV = e(3p_2, u^0) - w = [(9 - \ln p_2) + 1 + \ln(3p_2)] - 10 = \ln 3.$$

- *Equivalent variation.*

$$EV = w - e(p_2, u^1) = 10 - [(9 - \ln 3 - \ln p_2) + 1 + \ln p_2] = \ln 3.$$

- *Consumer surplus.*

$$\Delta CS = \int_{p_2}^{3p_2} x_2(s) ds = \int_{p_2}^{3p_2} \frac{1}{s} ds = \ln(3p_2) - \ln p_2 = \ln 3.$$

- (d) [6 points] Show that CV, EV, and ΔCS coincide, and explain why this happens for quasi-linear preferences. Explain why their common value depends only on the proportional size of the price increase and not on the initial level p_2 , and contrast with a consumer whose demand for good 2 has a nonzero income effect.

- All three welfare measures equal $\ln 3 \approx 1.10$. They coincide because the demand for good 2 has no income effect: its Walrasian and Hicksian demands are identical ($1/p_2$), so the area to the left of the demand curve between p_2 and $3p_2$ equals both compensated measures. Equivalently, CV and EV differ only through income effects, and here there are none.
- The common value $\ln 3$ depends only on the proportional size of the increase: $\Delta CS = \int_{p_2}^{3p_2} \frac{ds}{s} = \ln \frac{3p_2}{p_2} = \ln 3$, so the level p_2 cancels. The quantity effects, by contrast, scale with $1/p_2$ (the total change is $-2/(3p_2)$).
- For a consumer whose demand for good 2 has a nonzero income effect (for instance Cobb-Douglas preferences), the Walrasian and Hicksian demands diverge, and the three measures separate, with $CV > \Delta CS > EV$ for a normal good.