

Label Confusion and the Groucho Effect

A step-by-step exercise in the spirit of Harbaugh et al. (2011)

Companion note to Harbaugh, R., J. W. Maxwell, and B. Roussillon (2011), “Label Confusion: The Groucho Effect of Uncertain Standards,” *Management Science* 57(9): 1512–1527. Full reference at the end.

Overview. An eco-label certifies that a product meets a pass-fail standard, but consumers rarely know how demanding that standard is. A label then carries news about two unknowns at once: the product is at least as good as the standard, and the standard is no harder than the product. Harbaugh et al. (2011) call the second inference the *Groucho effect*, after Groucho Marx’s quip that he would not join any club willing to accept him as a member. When a product with a poor reputation displays a label, consumers conclude that the label must be easy to get. This note builds the mechanism in the simplest setting that keeps it intact: the standard is drawn uniformly on $[0, 1]$, which is the paper’s own leading example. Under this normalization a firm of quality q passes with probability q , so the label reweights the prior distribution of quality by q , and the equilibrium cutoffs have closed forms. Step (a) sets up the benchmark with a known standard; Step (b) derives the joint updating behind the Groucho effect; Step (c) measures how much a label reveals; Step (d) finds the labeling and non-labeling equilibria; Step (e) adds firm reputations; Step (f) studies label proliferation and focal labels; Step (g) turns to several firms and the strategic choice among labels; and Step (h) lists what the toy model keeps and drops.

1. The Toy Model

Quality. A firm sells a product of environmental quality $Q \in [0, 1]$, drawn from a cumulative distribution F with full support, mean $\mu \equiv E[Q]$, and variance $\sigma^2 \equiv \text{Var}(Q)$. The firm observes the realization q ; consumers know only F . Quality is fixed, so the firm makes no investment in it.

The label. A single eco-label has a standard $S \in [0, 1]$, drawn uniformly and independently of Q . The firm observes the realization s . If $q \geq s$ the firm may certify at a fixed cost $c > 0$ (certification fees, testing, documentation); if $q < s$ it cannot certify. The standard is *certain* if consumers also observe s , and *uncertain* if they know only that S is uniform.

Payoffs. The firm earns consumers’ expected quality given what they observe, minus c if it certifies. (The paper notes that its results hold whenever the firm’s payoff is a strictly increasing function of expected quality.)

The key normalization. Because S is uniform, a firm of quality q meets the standard with probability

$$\Pr(S \leq q) = q. \quad (1)$$

Figure 1 depicts (1). Panel (a) plots the density of the standard, $g(s) = 1$ on $[0, 1]$, and shades its part to the left of a quality level q : the shaded area is the probability that the standard is no harder than q . Panel (b) plots this probability against q , which gives the 45-degree line: a firm of higher quality is more likely to meet the standard, and a firm of quality 1 meets it for sure. A label therefore reweights each quality level q by its pass probability q .

The numerical examples below also take F uniform, so that $\mu = \frac{1}{2}$, $E[Q^2] = \frac{1}{3}$, and $\sigma^2 = \frac{1}{12}$.

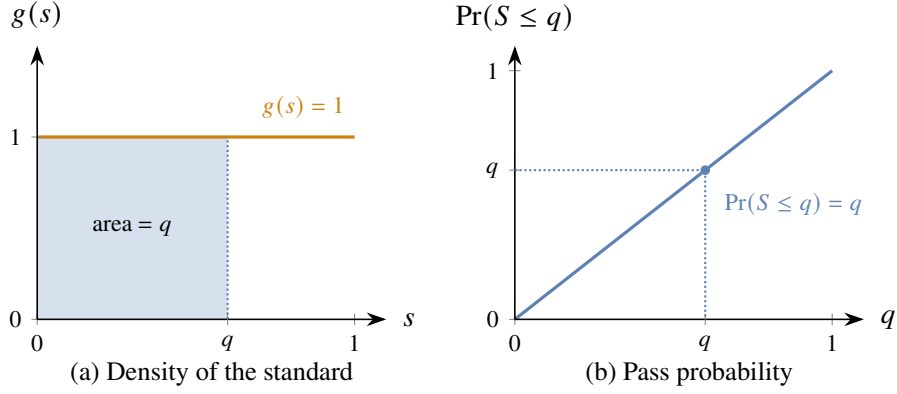


Figure 1: Probability that a uniform standard is no harder than quality q (shown for $q = 0.6$).

2. Step (a): The Benchmark with a Known Standard

Consider first a certain standard s , and assume that every firm able to certify does so. Consumers who see a label learn $Q \geq s$, and consumers who see no label learn $Q < s$. Write $\bar{q}(s) \equiv E[Q \mid Q \geq s]$ and $\underline{q}(s) \equiv E[Q \mid Q < s]$. Figure 2 depicts the two events on the quality interval $[0, 1]$ for a given s . With uniform F , quality is spread evenly over each event, so the expected quality in an event is the midpoint of its interval: $\underline{q}(s)$ is the midpoint of $[0, s]$ and $\bar{q}(s)$ is the midpoint of $[s, 1]$.

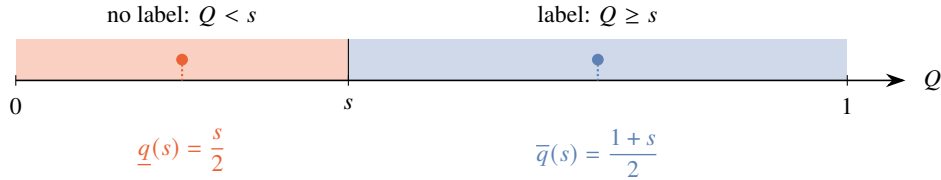


Figure 2: Expected quality given $Q < s$ and given $Q \geq s$, with uniform quality (shown for $s = 0.4$).

In formulas,

$$\bar{q}(s) = \frac{1+s}{2}, \quad \underline{q}(s) = \frac{s}{2}. \quad (2)$$

Averaging over the possible standards,

$$E[\bar{q}(S)] = \int_0^1 \frac{1+s}{2} ds = \frac{3}{4}, \quad E[\underline{q}(S)] = \int_0^1 \frac{s}{2} ds = \frac{1}{4}. \quad (3)$$

Figure 3 depicts the two integrals in (3): the shaded area under $\bar{q}(s)$ in panel (a) equals $\frac{3}{4}$, and the shaded area under $\underline{q}(s)$ in panel (b) equals $\frac{1}{4}$. Because the standard has density 1 on $[0, 1]$, each area is the average of the line over the standards.

These are the posterior means consumers would form if the conditional distribution of the standard did not change with the firm's labeling decision. Step (b) shows that it does change.

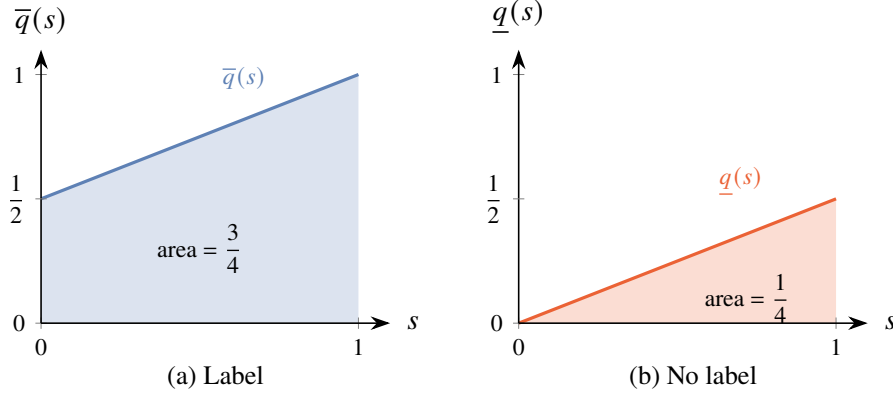


Figure 3: Expected quality given a known standard s , with uniform quality; shaded areas are $E[\bar{q}(S)]$ and $E[\underline{q}(S)]$.

3. Step (b): Joint Updating and the Groucho Effect

When the standard is uncertain, a label tells consumers only that $Q \geq S$ for some unobserved S . By (1), the expected quality of a labeled product is

$$E[Q \mid Q \geq S] = \frac{E[Q \Pr(S \leq Q \mid Q)]}{\Pr(S \leq Q)} = \frac{E[Q \cdot Q]}{E[Q]} = \frac{E[Q^2]}{\mu}, \quad (4)$$

and, by the same steps, the expected quality of an unlabeled product is

$$E[Q \mid Q < S] = \frac{E[Q(1 - Q)]}{1 - \mu} = \frac{\mu - E[Q^2]}{1 - \mu}. \quad (5)$$

With uniform F , $E[Q^2] = \frac{1}{3}$ and $\mu = \frac{1}{2}$, so these equal $\frac{2}{3}$ and $\frac{1}{3}$. Figure 4 depicts the reweighting behind the two means. A label multiplies the prior density $f(q)$ by the pass probability q and divides by μ , the probability of a label, which gives the density $2q$; no label multiplies it by $1 - q$ and divides by $1 - \mu$, which gives $2(1 - q)$. The first density puts more weight on high qualities, so its mean lies above the prior mean $\frac{1}{2}$, and the second puts more weight on low qualities, so its mean lies below it.

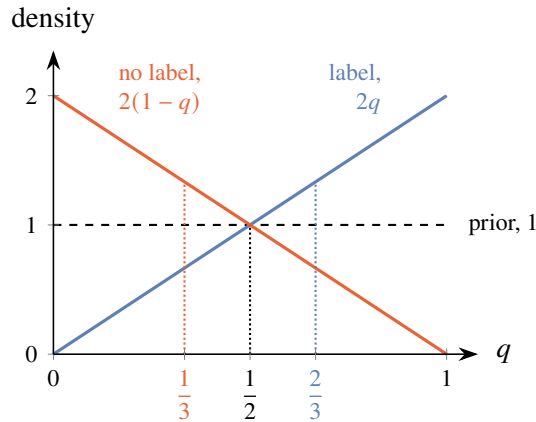


Figure 4: Density of quality before the label decision is observed (prior) and after a label or no label, with uniform quality and a uniform standard.

Consumers also revise their estimate of the standard. Conditional on a label, the density of S is

$$g(s \mid Q \geq S) = \frac{1 - F(s)}{\mu}, \quad (6)$$

which is decreasing in s : easy standards are over-represented among labeled products, because more firms clear them. Integrating,

$$E[S \mid Q \geq S] = \frac{E[Q^2]}{2\mu} = \frac{1}{3}, \quad E[S \mid Q < S] = \frac{1 - E[Q^2]}{2(1 - \mu)} = \frac{2}{3}, \quad (7)$$

where the second equality in each expression uses uniform F . Meeting the standard is good news about quality and bad news about the standard: the estimate of S falls from $\frac{1}{2}$ to $\frac{1}{3}$. Harbaugh et al. (2011) call this downward revision the *Groucho effect*. Failing to meet the standard works the other way: the estimate of S rises to $\frac{2}{3}$, the *reverse Groucho effect*.

Figure 5 depicts the revision of the standard. Conditional on a label, the density of S is $2(1 - s)$, from (6); conditional on no label, it is $F(s)/(1 - \mu) = 2s$, which is increasing in s because hard standards are over-represented among unlabeled products. The means of the two densities are the estimates $\frac{1}{3}$ and $\frac{2}{3}$ in (7).

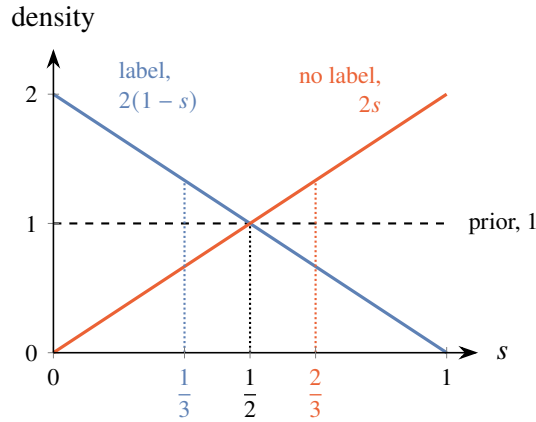


Figure 5: Density of the standard before the label decision is observed (prior) and after a label or no label, with uniform quality and a uniform standard.

Figure 6 depicts the two cases in the unit square, with the standard on the horizontal axis and quality on the vertical axis, for uniform F . In panel (a), with an uncertain standard, the prior point $(\frac{1}{2}, \frac{1}{2})$ moves to the centroid of the upper triangle $\{Q \geq S\}$ (shaded blue) after a label and to the centroid of the lower triangle $\{Q < S\}$ (shaded red) without one; both moves are diagonal, because the label shifts the estimates of Q and S in opposite directions. In panel (b), with a known standard s , the estimate of s cannot move, so the full update falls on quality: the posterior mean jumps to the upper line $\bar{q}(s)$ or the lower line $\underline{q}(s)$ of (2).

Comparing the two panels of Figure 6 with the averages in (3),

$$\underbrace{E[\underline{q}(S)]}_{\frac{1}{4}} < \underbrace{E[Q \mid Q < S]}_{\frac{1}{3}} < \underbrace{\mu}_{\frac{1}{2}} < \underbrace{E[Q \mid Q \geq S]}_{\frac{2}{3}} < \underbrace{E[\bar{q}(S)]}_{\frac{3}{4}}. \quad (8)$$

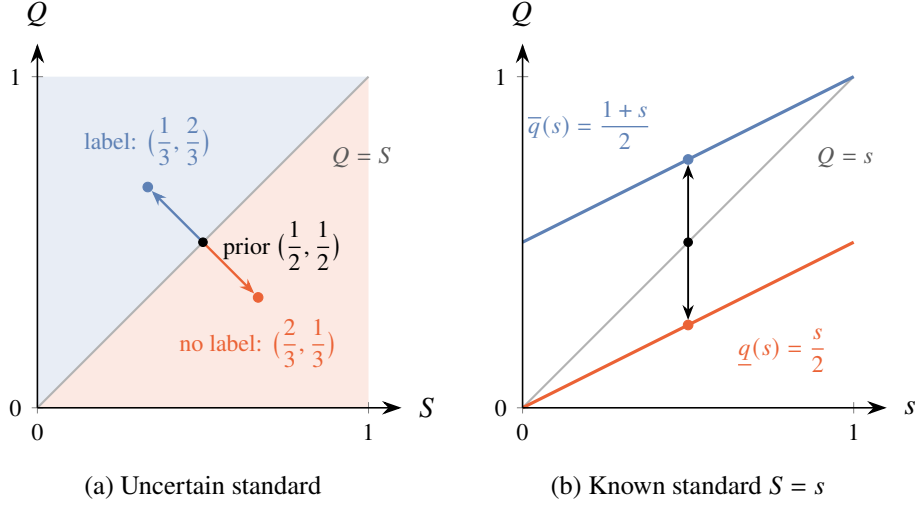


Figure 6: Updated estimates of quality and of the standard after a label and after no label, with uniform quality and a uniform standard.

Result 1 (moderation). Compared with a known standard, with an uncertain standard:

- a label is weaker good news;
- the absence of a label is weaker bad news.

The paper proves the ranking (8) for any F and G (inequality (6) of the paper); the uniform numbers make the size of the gap visible.

Figure 7 places the five numbers of (8) on the unit interval.

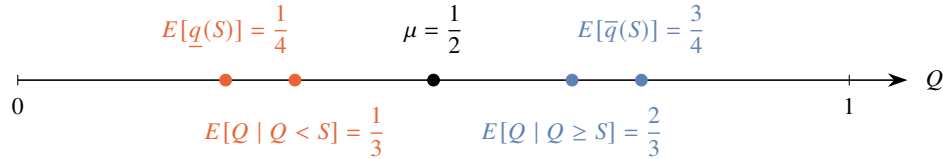


Figure 7: Expected quality after a label or no label, with a known and with an uncertain standard, for uniform quality and a uniform standard.

4. Step (c): How Much Does a Label Reveal?

Measure the information in a label by the share of the prior variance of quality that it removes,

$$R^2 \equiv 1 - \frac{E[(Q - \widehat{Q})^2]}{\sigma^2}, \quad (9)$$

where $\widehat{Q}$ is consumers' posterior mean. The numerator is the expected mean-squared error (MSE) that the paper uses, so a higher R^2 is a lower MSE. Without a label the MSE is $\sigma^2 = \frac{1}{12}$. With a known standard s the MSE equals $\frac{s^3 + (1-s)^3}{12}$, whose average over s is $\frac{1}{24}$, so $R^2 = \frac{1}{2}$. With an uncertain standard the two posterior means are $\frac{2}{3}$ and $\frac{1}{3}$, each reached with probability $\frac{1}{2}$, so the MSE is $\frac{1}{3} - (\frac{1}{2} \cdot \frac{4}{9} + \frac{1}{2} \cdot \frac{1}{9}) = \frac{1}{18}$ and $R^2 = \frac{1}{3}$.

Result 2 (Proposition 1 of the paper). A label is less informative on average when its standard is uncertain. A sum-of-squares identity shows why: the expected MSE with an uncertain standard minus the expected MSE with a known standard equals

$$\int_0^1 \left[F(s)(E[Q | Q < S] - \underline{q}(s))^2 + (1 - F(s))(E[Q | Q \geq S] - \bar{q}(s))^2 \right] dG(s) > 0, \quad (10)$$

because consumers who do not know s must use a single posterior mean in each event, while consumers who know s use the best mean for each s . With uniform F and G the integral equals $\frac{1}{18} - \frac{1}{24} = \frac{1}{72}$. Figure 8 depicts the expected MSE against the standard s for consumers who know s (blue curve) and for consumers who use the posterior means $\frac{2}{3}$ and $\frac{1}{3}$ whatever s is (red curve). The shaded area between the curves is the integral in (10), and the dotted lines mark the two averages $\frac{1}{24}$ and $\frac{1}{18}$. Consumers who know s place the posterior means at the midpoints of the two events, which minimizes the error at every s , so the blue curve lies below the red one everywhere. The red curve rises above the dashed line $\frac{1}{12}$, the MSE without a label, for standards below about 0.09 and above about 0.91, where consumers who do not know s misread the label; on average over s , the label still lowers the MSE from $\frac{1}{12}$ to $\frac{1}{18}$.

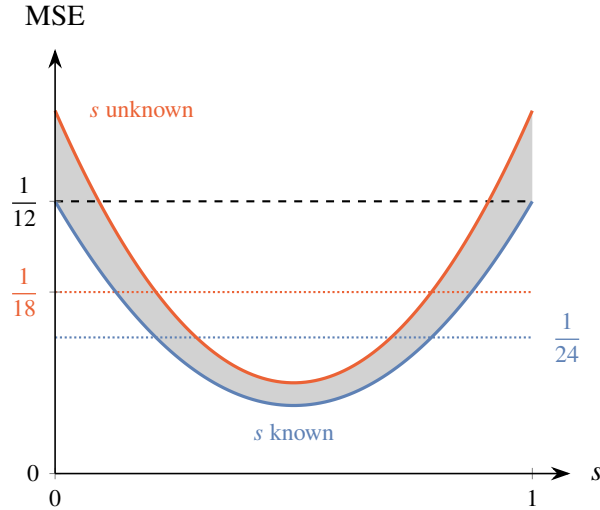


Figure 8: Expected MSE of consumers' estimate of quality against the standard s , with uniform quality; the shaded area is the integral in (10), and the dashed line is the MSE without a label.

Table 1 reports the uniform values.

Standard	$E[Q \text{label}]$	$E[Q \text{no label}]$	$E[S \text{label}]$	$E[S \text{no label}]$	R^2
Known (average over s)	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Uncertain	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$

Table 1: Posterior means and informativeness when every firm able to certify does so, with uniform quality and a uniform standard.

5. Step (d): Labeling and Non-Labeling Equilibria

The equilibrium concept is perfect Bayesian equilibrium with one refinement, described below. In a *labeling equilibrium* every firm that meets the standard certifies, so the lack of a label reveals failure. The firm certifies if the gain is at least the cost,

$$E[Q | Q \geq S] - E[Q | Q < S] \geq c. \quad (11)$$

In a *non-labeling equilibrium* no firm certifies, so the absence of a label is no news and the firm earns μ . A label is then off the equilibrium path, and the paper assumes that consumers attribute an unexpected label to any type that meets the standard with equal likelihood, so a firm that deviates earns $E[Q | Q \geq S]$. No firm deviates if

$$E[Q | Q \geq S] - \mu \leq c. \quad (12)$$

Closed forms. Substituting (4) and (5) into (11),

$$\begin{aligned} \bar{c} &\equiv E[Q | Q \geq S] - E[Q | Q < S] = \frac{E[Q^2]}{\mu} - \frac{\mu - E[Q^2]}{1 - \mu} \\ &= \frac{E[Q^2](1 - \mu) - \mu(\mu - E[Q^2])}{\mu(1 - \mu)} = \frac{E[Q^2] - \mu^2}{\mu(1 - \mu)} = \frac{\sigma^2}{\mu(1 - \mu)}, \end{aligned} \quad (13)$$

and, from (12),

$$\underline{c} \equiv E[Q | Q \geq S] - \mu = \frac{E[Q^2] - \mu^2}{\mu} = \frac{\sigma^2}{\mu} = (1 - \mu) \bar{c}. \quad (14)$$

With a uniform standard, the labeling cutoff is the variance of quality relative to its largest possible value given the mean, $\mu(1 - \mu)$. With uniform F , $\underline{c} = \frac{1}{6}$ and $\bar{c} = \frac{1}{3}$.

Figure 9 depicts the two incentive conditions for uniform F , with the certification cost on the horizontal axis. In panel (a), the labeling equilibrium, a firm that meets the standard compares the payoff from certifying, $E[Q | Q \geq S] - c = \frac{2}{3} - c$ (blue line), with the payoff from staying silent, $E[Q | Q < S] = \frac{1}{3}$ (red line): certifying pays up to $c = \frac{1}{3} = \bar{c}$. In panel (b), the non-labeling equilibrium, a firm compares the payoff from deviating to a label, $\frac{2}{3} - c$, with the equilibrium payoff $\mu = \frac{1}{2}$: deviating does not pay from $c = \frac{1}{6} = \underline{c}$ on.

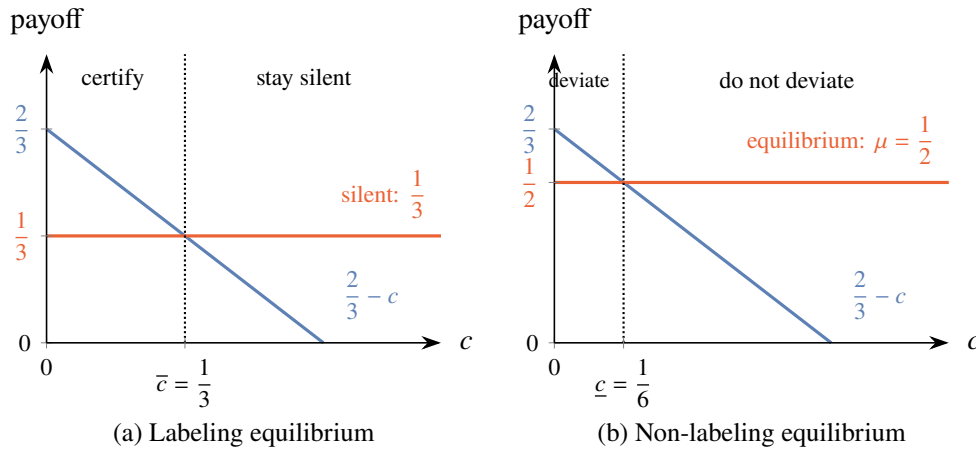


Figure 9: A firm's payoff from each action against the certification cost c , with uniform quality and a uniform standard.

Result 3 (Proposition 2 of the paper).

- A labeling equilibrium exists if and only if $c \leq \bar{c}$.
- A non-labeling equilibrium exists if and only if $c \geq \underline{c}$.
- Both exist if and only if $\underline{c} \leq c \leq \bar{c}$.

Since $\underline{c} = (1 - \mu)\bar{c} < \bar{c}$, the interval of multiple equilibria is never empty. The two cutoffs split the cost axis into three regions, which the figures below name by the equilibria they support:

- L: $c < \underline{c}$, only the labeling equilibrium exists;
- LN: $\underline{c} \leq c \leq \bar{c}$, both equilibria exist;
- N: $c > \bar{c}$, only the non-labeling equilibrium exists.

The known-standard benchmark. With a known standard s and uniform F , (2) gives

$$\bar{c}(s) = \bar{q}(s) - \underline{q}(s) = \frac{1}{2}, \quad \underline{c}(s) = \bar{q}(s) - \mu = \frac{s}{2}, \quad (15)$$

so a labeling equilibrium exists for $c \leq \frac{1}{2}$ whatever the standard, and a non-labeling equilibrium exists for $c \geq \frac{s}{2}$, a cutoff whose average over s is $\frac{1}{4}$. Figure 10 depicts the three regions for a known standard, with the standard s on the horizontal axis and the cost on the vertical axis. The labeling cutoff $\bar{c}(s) = \frac{1}{2}$ is flat, and the non-labeling cutoff $\underline{c}(s) = \frac{s}{2}$ rises with s , so the region of multiple equilibria LN is the wedge between them; the dotted line marks the average $\frac{1}{4}$ of the rising cutoff.

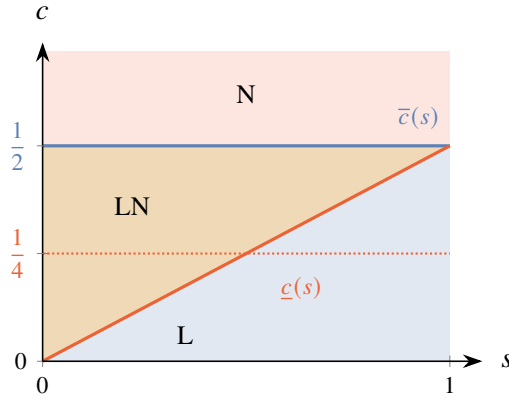


Figure 10: Labeling (L), multiple-equilibria (LN), and non-labeling (N) regions with a known standard s , for uniform quality.

Result 4 (Proposition 3 of the paper). Uncertainty about the standard:

- shrinks the range of costs that supports labeling;
- enlarges the range of costs that supports non-labeling.

Figure 11 depicts the two ranges on a cost axis for uniform quality: labeling survives up to $c = \frac{1}{3}$ instead of $\frac{1}{2}$, and non-labeling starts at $c = \frac{1}{6}$ instead of $\frac{1}{4}$. Intuitively, the Groucho effect makes a label less valuable, which hurts the labeling equilibrium, and the reverse Groucho effect makes the lack of a label less damaging, which helps the non-labeling equilibrium.

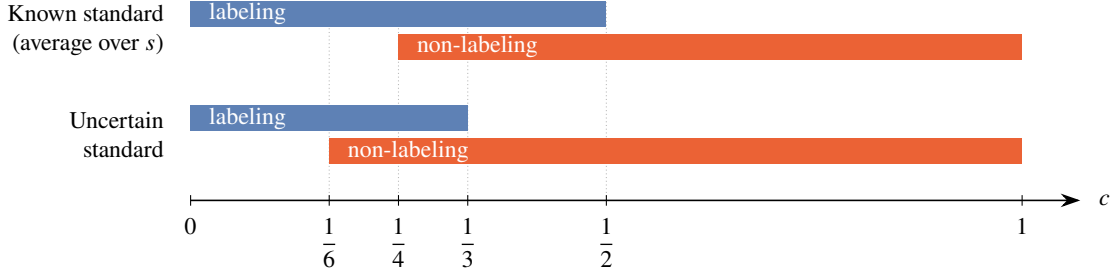


Figure 11: Costs that support a labeling equilibrium and a non-labeling equilibrium, with uniform quality and a uniform standard.

6. Step (e): Reputations

Let quality follow a Beta distribution with parameters (α, β) , so that $\mu = \frac{\alpha}{\alpha+\beta}$ and $\sigma^2 = \frac{\mu(1-\mu)}{\alpha+\beta+1}$. Substituting into (13) and (14),

$$\bar{c} = \frac{1}{\alpha + \beta + 1}, \quad \underline{c} = \frac{\beta}{(\alpha + \beta)(\alpha + \beta + 1)}. \quad (16)$$

The paper models a firm with a *bad reputation* as $Q \sim \text{Beta}(1, b)$ with $b \geq 1$, so that $\mu = \frac{1}{1+b} \leq \frac{1}{2}$, and a firm with a *good reputation* as $Q \sim \text{Beta}(a, 1)$ with $a \geq 1$, so that $\mu \geq \frac{1}{2}$. Writing each family in terms of its mean, (16) becomes

$$\bar{c} = \begin{cases} \frac{\mu}{1+\mu} & \text{if } \mu \leq \frac{1}{2} \text{ (bad reputation),} \\ \frac{1-\mu}{2-\mu} & \text{if } \mu \geq \frac{1}{2} \text{ (good reputation),} \end{cases} \quad \underline{c} = (1-\mu)\bar{c}. \quad (17)$$

Figure 12 depicts the two families for $b = a = 3$, together with the uniform density. A bad reputation piles the density of quality near $q = 0$ (mean $\frac{1}{4}$), and a good reputation piles it near $q = 1$ (mean $\frac{3}{4}$).

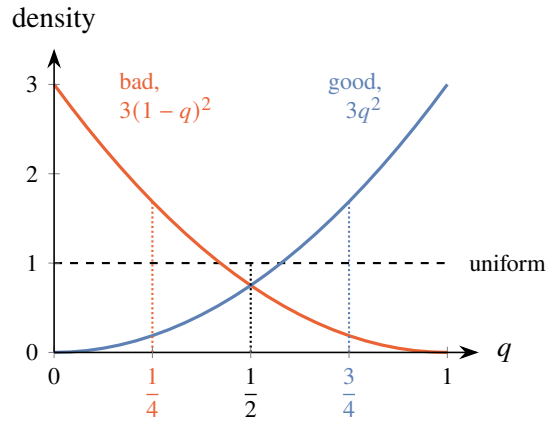


Figure 12: Density of quality for a bad reputation, Beta(1,3), a good reputation, Beta(3,1), and the uniform distribution.

Figure 13 depicts the regions L, LN, and N of Step (d) against the prior mean μ , with the certification cost on the vertical axis: panel (a) for an uncertain standard, from (17), and panel (b) for a known standard, where the cutoffs are averaged over s . In panel (a) the labeling cutoff $\bar{c}$ peaks at $\mu = \frac{1}{2}$, where $\bar{c} = \frac{1}{3}$, and falls to

zero as the reputation becomes very bad or very good; the non-labeling cutoff $\underline{c}$, below which labeling is the only equilibrium, peaks at $\mu = \sqrt{2} - 1$, where $\underline{c} = 3 - 2\sqrt{2}$. Intuitively, consumers who see a label on a product with a bad reputation conclude that the standard must be easy: in the bad-reputation family,

$$E[S | Q \geq S] = \frac{E[Q^2]}{2\mu} = \frac{1}{b+2}, \quad (18)$$

which falls to zero as b grows, so the label is worth little. For a product with a good reputation the reverse Groucho effect dominates: consumers who see no label conclude that the standard must be hard, since in the good-reputation family

$$E[S | Q < S] = \frac{1 - E[Q^2]}{2(1 - \mu)} = \frac{a+1}{a+2}, \quad (19)$$

which rises to one as a grows, so the firm loses little by not certifying. In panel (b), with a known standard, a firm with a bad reputation that is in fact of high quality can prove it, and the labeling cutoff stays above 0.45 for every reputation.

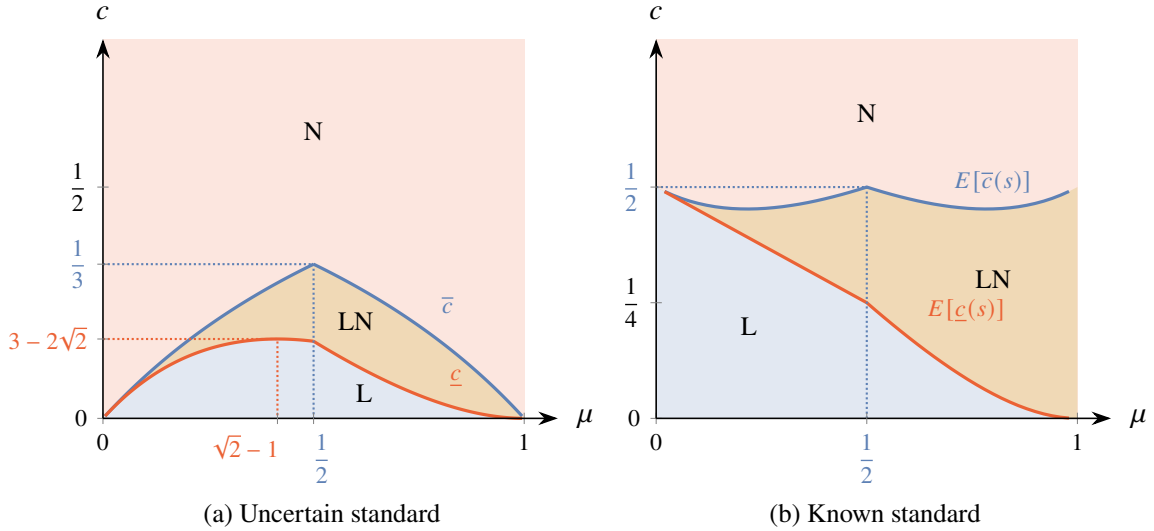


Figure 13: Labeling (L), non-labeling (N), and multiple-equilibria (LN) regions against the prior mean μ , with a uniform standard; $Q \sim \text{Beta}(1, b)$ for $\mu \leq \frac{1}{2}$ and $Q \sim \text{Beta}(a, 1)$ for $\mu \geq \frac{1}{2}$.

Figure 14 depicts the posterior estimates of the standard in (18) and (19) for every mean μ , from the same two families. In terms of μ ,

$$E[S | Q \geq S] = \begin{cases} \frac{\mu}{1+\mu} & \text{if } \mu \leq \frac{1}{2}, \\ \frac{1}{2(2-\mu)} & \text{if } \mu \geq \frac{1}{2}, \end{cases} \quad E[S | Q < S] = \begin{cases} \frac{1+2\mu}{2(1+\mu)} & \text{if } \mu \leq \frac{1}{2}, \\ \frac{1}{2-\mu} & \text{if } \mu \geq \frac{1}{2}. \end{cases} \quad (20)$$

A label lowers the estimate of the standard below the prior value $\frac{1}{2}$ (blue curve), most strongly when the reputation is bad; the lack of a label raises it above $\frac{1}{2}$ (red curve), most strongly when the reputation is good.

What drives the hump. Equation (16) shows that, with a uniform standard, the labeling cutoff $\bar{c} = \frac{1}{\alpha+\beta+1}$ depends only on how concentrated the prior is, measured by $\alpha + \beta$, and not on where it is centered. In the two families above, moving the mean toward 0 or 1 also raises $\alpha + \beta$ (to $1 + b$ or $a + 1$), and that is what shrinks the labeling region at both ends of Figure 13(a). The paper points to this in its footnote 6, where it

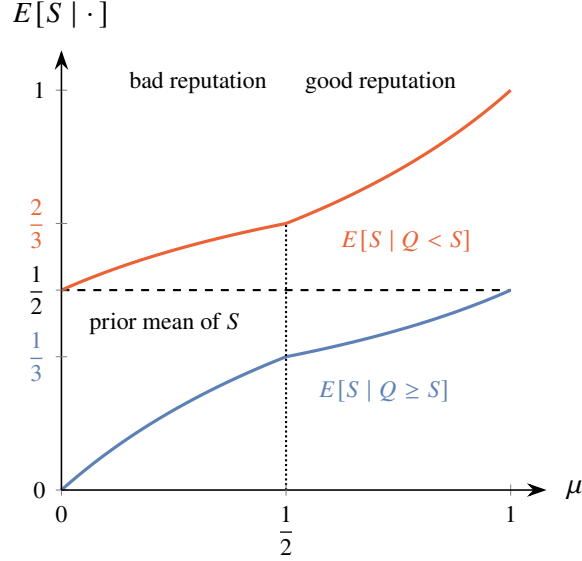


Figure 14: Expected standard after a label and after no label against the prior mean μ , with a uniform standard; $Q \sim \text{Beta}(1, b)$ for $\mu \leq \frac{1}{2}$ and $Q \sim \text{Beta}(a, 1)$ for $\mu \geq \frac{1}{2}$.

notes that these families are most diffuse at $\mu = \frac{1}{2}$. A firm with mean $\frac{1}{4}$ and $\text{Beta}(\frac{1}{2}, \frac{3}{2})$ quality, for instance, has the same $\bar{c} = \frac{1}{3}$ as a firm with uniform quality. The strength of the Groucho effect does depend on the mean: for any Beta prior, $E[S | Q \geq S] = \frac{\alpha+1}{2(\alpha+\beta+1)}$, which, for a given concentration $\alpha + \beta$, is lower the worse the reputation. A label is less valuable for a product with a bad reputation because it signals an easy standard, and also because a concentrated prior leaves little to learn.

7. Step (f): Label Proliferation and Focal Labels

Now let there be $n \geq 1$ labels with standards drawn independently from the uniform distribution and the same cost c , and let the firm display at most one label. Consider the *symmetric* strategy: the firm adopts the toughest label it meets. Consumers cannot tell which label is toughest, and the labels are interchangeable to them, so a label tells consumers only that the firm meets at least the *easiest* standard, $S_{1:n} \equiv \min\{S_1, \dots, S_n\}$. A firm of quality q meets it with probability

$$\Pr(S_{1:n} \leq q) = 1 - (1 - q)^n. \quad (21)$$

Using (21) in place of (1) in the steps behind (4) and (5), and taking uniform F ,

$$E[Q | Q \geq S_{1:n}] = \frac{n+3}{2(n+2)}, \quad E[Q | Q < S_{1:n}] = \frac{1}{n+2}. \quad (22)$$

Figure 15 depicts (21) in panel (a) and (22) in panel (b). Panel (a) shows that, for any given quality, the probability of meeting the easiest standard rises with n , so a label becomes easier to get; panel (b) shows that both posterior means fall with n , toward $\mu = \frac{1}{2}$ and toward 0, so failing every label becomes a sign of very low quality.

The two cutoffs are

$$\bar{c}_n = \frac{n+1}{2(n+2)}, \quad \underline{c}_n = \frac{1}{2(n+2)}. \quad (23)$$

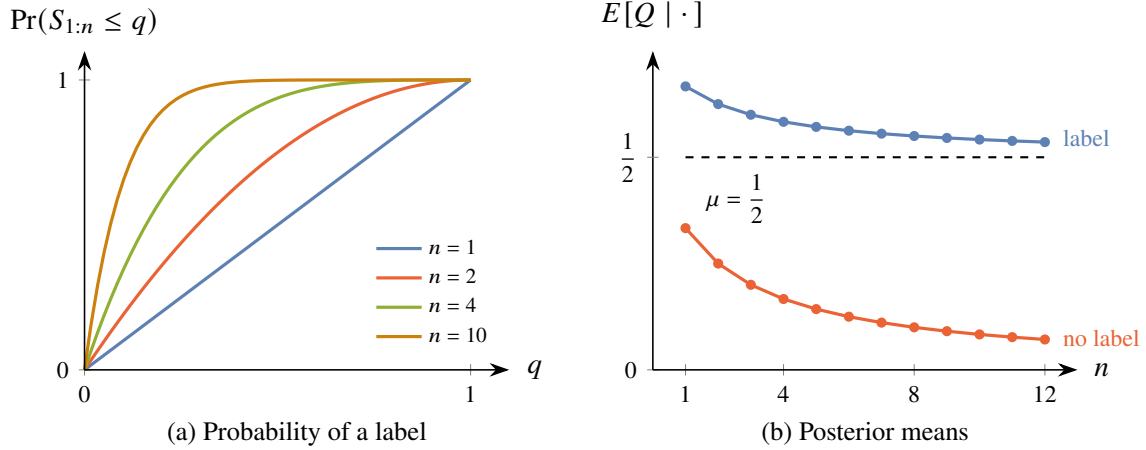


Figure 15: Probability of meeting the easiest of n uniform standards, and expected quality given a label and given no label, with uniform quality.

Result 5 (Proposition 4 of the paper). With uncertain standards, adding labels enlarges the range of costs that supports the non-labeling equilibrium. As n grows:

- non-labeling becomes an equilibrium for every $c > 0$;
- labeling remains an equilibrium if and only if $c \leq \mu$;
- the labeling equilibrium becomes uninformative.

In the uniform case the cutoffs in (23) are exact for every n : for a given cost c , a non-labeling equilibrium exists if and only if $n \geq \frac{1}{2c} - 2$, and a labeling equilibrium exists if and only if $n \geq \frac{4c-1}{1-2c}$ (for $c < \frac{1}{2}$). As n grows, $\underline{c}_n \rightarrow 0$ and $\bar{c}_n \rightarrow \frac{1}{2} = \mu$, so the interval of multiple equilibria widens toward $[0, \frac{1}{2}]$.

Informativeness. Under the symmetric strategy the label splits quality at the random cutoff $S_{1:n}$, and from (9) and (22),

$$R_{\text{sym}}^2(n) = \frac{3n}{(n+2)^2}. \quad (24)$$

With one label $R_{\text{sym}}^2 = \frac{1}{3}$, with two labels it rises to $\frac{3}{8}$, it returns to $\frac{1}{3}$ at $n = 4$, and it falls toward zero after that. Intuitively, a second label makes the lack of any label a sharper signal of low quality, and at first this outweighs the fact that a label becomes easier to get; with many labels, almost every firm meets the easiest standard, so a label proves almost nothing. The paper's Proposition 4 concerns the limit $n \rightarrow \infty$; the uniform example shows that the decline in informativeness need not start at $n = 1$.

Focal labels. Consider now that one label, say label X with standard S_X , is *focal*: consumers expect a firm to adopt it whenever it can, and to adopt another label only if it fails X . The market then sorts firms into three groups:

$$\{Q \geq S_X\}, \quad \{S_{1:n} \leq Q < S_X\}, \quad \{Q < S_{1:n}\}, \quad (25)$$

with posterior means $\frac{2}{3}$, $\frac{n+4}{3(n+2)}$, and $\frac{1}{n+2}$ in the uniform case. Figure 16 depicts the partitions of the quality interval for two labels, X and Y , with realized standards $s_Y < s_X$, so that $s_{1:2} = s_Y$. Under the symmetric strategy consumers see only whether a firm has some label, and they split quality at $s_{1:2}$; under the focal strategy they also see whether the label is X , and the labeled group splits at s_X into two groups.

A firm in the middle group must prefer some label to none, $\frac{n+4}{3(n+2)} - \frac{1}{n+2} \geq c$, and a firm in the top group must prefer label X to no label, $\frac{2}{3} - \frac{1}{n+2} \geq c$ (it also prefers X to any other label, since $\frac{2}{3}$ exceeds the middle-group

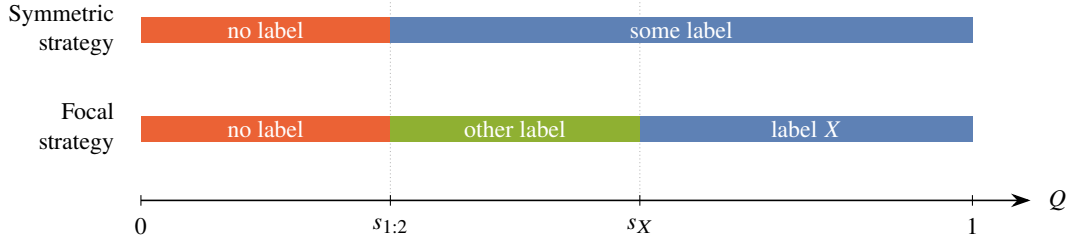


Figure 16: Groups of firms by quality under the symmetric and the focal strategies, for two labels with realized standards $s_{1:2} = 0.3$ and $s_X = 0.6$.

mean); the first condition binds, so for $n \geq 2$ a focal labeling equilibrium exists if and only if

$$c \leq \frac{n+1}{3(n+2)}. \quad (26)$$

The cutoff in (26) equals $\frac{2}{3}\bar{c}_n$, so a focal equilibrium requires a lower cost than the symmetric labeling equilibrium. The partition (25) is finer than the symmetric one, and

$$R_{\text{focal}}^2(n) = \frac{n(n+8)}{3(n+2)^2}, \quad R_{\text{focal}}^2(n) - R_{\text{sym}}^2(n) = \frac{n(n-1)}{3(n+2)^2} \geq 0. \quad (27)$$

Result 6 (Proposition 5 of the paper).

- With uncertain standards, a focal labeling equilibrium exists if the cost satisfies (26), and it is more informative than the symmetric one.
- With known standards, no focal labeling equilibrium exists.

Figure 17 depicts the cutoffs (23) and (26) in panel (a), with the regions L, LN, and N of the symmetric labeling strategy shaded, and the informativeness of the label in (24) and (27) in panel (b), with the number of labels on the horizontal axis. Panel (b) shows R_{sym}^2 peaking at $\frac{3}{8}$ with two labels, and R_{focal}^2 peaking at $\frac{4}{9}$ with four labels and converging to $\frac{1}{3}$, the value for a single uncertain label, because non-focal labels become uninformative as they multiply. With known standards a firm always prefers the toughest label it meets, and under the paper's refinement a deviation to a tougher label is rewarded, so focality cannot be sustained. This is the paper's case for “look for the label” campaigns and mandatory labels: they keep the information that a single label conveys, which proliferation destroys under the symmetric strategy.

Table 2 reports the uniform values.

Known standards. With known standards, more labels let a firm reveal its quality more finely, and in the limit the problem becomes the one Lizzeri (1999) studies, in which every quality level can be certified. The paper shows that as n grows, non-labeling is an equilibrium almost surely if and only if $\mu \geq 1 - c$, and, for log-concave F , labeling is an equilibrium almost surely if and only if $\mu \leq 1 - c$. With uniform quality both conditions turn on whether c is above or below $\frac{1}{2}$, and a label becomes highly informative, instead of uninformative as with uncertain standards.

8. Step (g): Several Firms and the Strategic Choice of Labels

One label, many firms. Let m firms with independent qualities from F face one label. In a labeling equilibrium the share of firms with a label converges to $1 - F(s)$, so consumers learn the standard, and

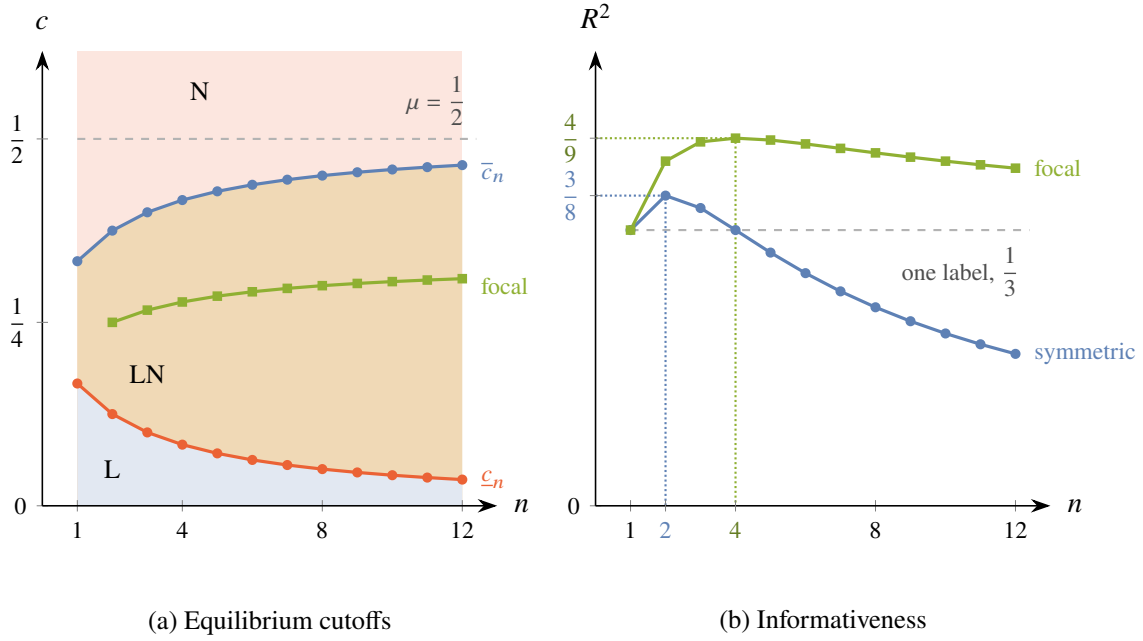


Figure 17: Equilibrium cutoffs and informativeness of a label against the number of labels n , with uniform quality and uniform standards.

Labels n	$\underline{c}_n$	$\bar{c}_n$	Focal cutoff	R^2_{sym}	R^2_{focal}
1	$\frac{1}{6}$	$\frac{1}{3}$	—	$\frac{1}{3}$	$\frac{1}{3}$
2	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{5}{12}$
3	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{4}{15}$	$\frac{9}{25}$	$\frac{11}{25}$
4	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{18}$	$\frac{1}{3}$	$\frac{4}{9}$
6	$\frac{1}{16}$	$\frac{7}{16}$	$\frac{7}{24}$	$\frac{9}{32}$	$\frac{7}{16}$
10	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{36}$	$\frac{5}{24}$	$\frac{5}{12}$
$\rightarrow \infty$	0	$\frac{1}{2}$	$\frac{1}{3}$	0	$\frac{1}{3}$

Table 2: Cutoffs and informativeness under the symmetric and the focal labeling strategies, with uniform quality and uniform standards.

each firm effectively faces a known standard: with uniform quality, labeling is supported for $c \leq \bar{c}(s) = \frac{1}{2}$ whatever the standard. In a non-labeling equilibrium no firm certifies, so consumers have nothing to learn from, and the condition stays $c \geq \underline{c} = \frac{1}{6}$ (Proposition 6 of the paper). As the number of firms grows, the interval of multiple equilibria therefore widens from $[\frac{1}{6}, \frac{1}{3}]$ to $[\frac{1}{6}, \frac{1}{2}]$. Figure 18 depicts the two ranges of costs, as in Figure 11: with many firms, labeling survives up to $c = \frac{1}{2}$ instead of $\frac{1}{3}$, and non-labeling still starts at $c = \frac{1}{6}$.

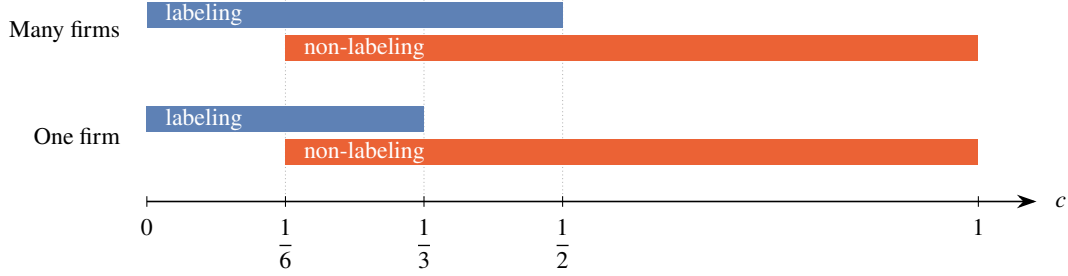


Figure 18: Costs that support a labeling equilibrium and a non-labeling equilibrium with one firm and with many firms, for uniform quality and a uniform standard.

Two firms, two labels. With two firms and two labels, the label one firm adopts carries news about the standards the other firm faces. Under the symmetric strategy, consumers see either both firms with the *same* label or the firms with *different* labels, and in the second case they cannot tell which firm met the tougher standard. Denoting by $S_{1:2} < S_{2:2}$ the easier and the tougher standard, the events are

$$\text{Same} = \{S_{1:2} \leq Q_1 < S_{2:2}, S_{1:2} \leq Q_2 < S_{2:2}\} \cup \{Q_1 \geq S_{2:2}, Q_2 \geq S_{2:2}\}, \quad (28)$$

$$\text{Different} = \{S_{1:2} \leq Q_1 < S_{2:2} \leq Q_2\} \cup \{S_{1:2} \leq Q_2 < S_{2:2} \leq Q_1\}. \quad (29)$$

Figure 19 depicts the two events for given standards, in the plane of the two firms' qualities. The events Same and Different are the shaded cells; in the remaining region at least one firm meets neither standard and has no label.

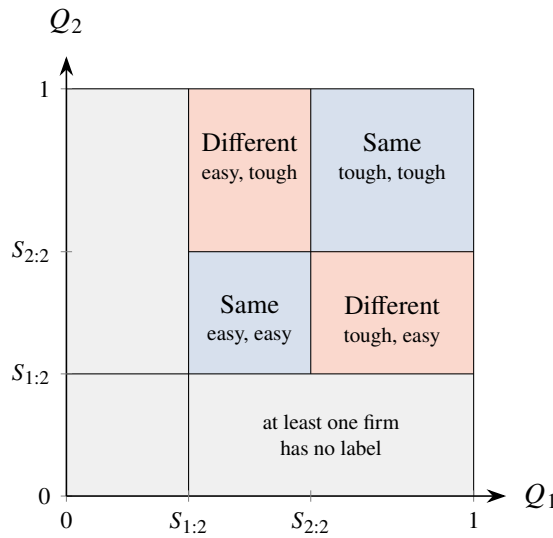


Figure 19: Events Same and Different in the plane of the two qualities, for two labels with given standards $S_{1:2} < S_{2:2}$; each cell lists the label of firm 1 and of firm 2.

A firm that meets both standards while its rival meets only the easier one can pool with the rival by adopting the easier label, and a firm that meets both standards while its rival does too can separate by adopting the easier label. The symmetric strategy survives only if neither deviation pays, which requires $E[Q_i | \text{Same}] = E[Q_i | \text{Different}]$. With uniform quality the two events occur with probabilities $\frac{1}{3}$ and $\frac{1}{6}$, and

$$E[Q_i | \text{Same}] = E[Q_i | \text{Different}] = \frac{3}{5}, \quad (30)$$

so the uniform case is exactly the knife-edge at which the symmetric strategy is an equilibrium.

Figure 20 depicts both conditional means against the prior mean μ , for the two reputation families of Step (e), with uniform standards. The curves cross at $\mu = \frac{1}{2}$, at the value $\frac{3}{5}$ in (30). To the right of the crossing, the same label is better news than different labels: when reputations are good, the label a rival adopts is likely to be the tougher one, so each firm wants to adopt it too. The paper says that a firm with a good reputation *legitimizes* the label it adopts. To the left of the crossing, different labels are better news: the label a rival with a bad reputation adopts is likely to be the easier one, so each firm wants to avoid it, and a firm with a bad reputation *spoils* the label it adopts. Away from $\mu = \frac{1}{2}$ the symmetric labeling equilibrium does not exist, while a focal labeling equilibrium exists under the same cost condition as with a single firm (Proposition 7 of the paper). When firms choose in sequence, these incentives produce labeling cascades: the second firm herds onto the first firm's label if reputations are good and moves away from it if they are bad. The paper offers this spillover as one reason why certifiers recruit reputable firms as early adopters of a new label.

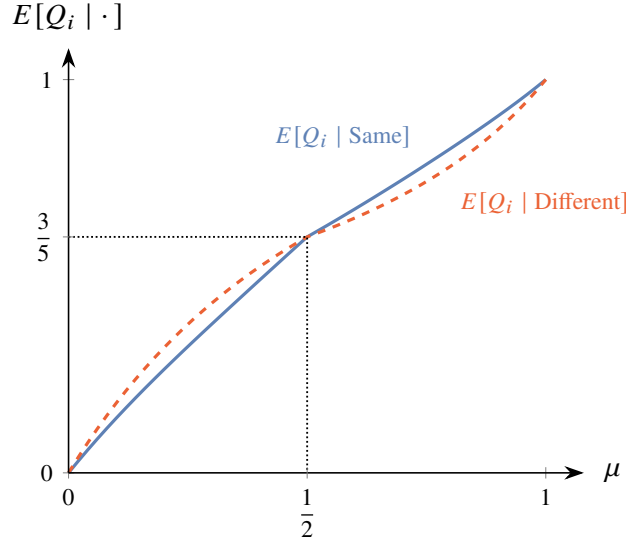


Figure 20: Expected quality of firm i when both firms display the same label and when they display different labels, against the prior mean μ ; two firms, two uniform standards, $Q \sim \text{Beta}(1, b)$ for $\mu \leq \frac{1}{2}$ and $Q \sim \text{Beta}(a, 1)$ for $\mu \geq \frac{1}{2}$.

9. Step (h): What the Toy Model Keeps and Drops

Every one of the paper's main conclusions survives the uniform standard:

1. The Groucho and reverse Groucho effects: a label lowers the estimate of the standard from $\frac{1}{2}$ to $\frac{1}{3}$ and its absence raises it to $\frac{2}{3}$ (Step b).
2. Uncertain standards reveal less: R^2 falls from $\frac{1}{2}$ to $\frac{1}{3}$ (Step c, Proposition 1).

3. Labeling is harder to sustain and non-labeling easier: the cutoffs move from $\frac{1}{2}$ and $\frac{1}{4}$ to $\frac{1}{3}$ and $\frac{1}{6}$, and for any F , $\bar{c} = \frac{\sigma^2}{\mu(1-\mu)}$ and $\underline{c} = \frac{\sigma^2}{\mu}$ (Step d, Propositions 2 and 3).
4. Proliferation makes non-labeling an equilibrium for every $c > 0$ and the labeling equilibrium uninformative, while a focal label restores information (Step f, Propositions 4 and 5).
5. Several firms widen the interval of multiple equilibria, and with several labels firms legitimize or spoil labels (Step g, Propositions 6 and 7).

With a uniform standard, the labeling cutoff depends on the prior only through $\frac{\sigma^2}{\mu(1-\mu)}$, which separates the role of a firm's reputation (its mean) from the precision of consumers' prior (its dispersion); the general analysis leaves this split implicit. In addition, the informativeness of the symmetric labeling equilibrium is not monotone in the number of labels: in the uniform case it peaks at two labels and declines after that.

Relative to Harbaugh et al. (2011), the toy model takes the standard to be uniform, so the pass probability (1) is linear in quality; the paper allows any distribution G , and its propositions hold for any F and G . As in the paper, the firm's payoff is linear in expected quality, without prices or product-market competition between firms, and quality and standards are fixed: firms do not invest in quality, and certifiers do not choose their standards or fees. Each of these features can be relaxed one at a time.

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- Lizzeri, A. (1999). "Information Revelation and Certification Intermediaries." *RAND Journal of Economics*, 30(2), pp. 214–231.