

Measuring Risk Aversion and Prudence

EconS 501, a handout on Sections 5.7 and 5.8

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This handout develops Sections 5.7 (Arrow–Pratt coefficients of absolute and relative risk aversion) and 5.8 (prudence) of the textbook in more detail. Each measure is defined, connected to an economic quantity that it approximates (a risk premium, a relative risk premium, or precautionary saving), and evaluated for the utility functions used most in applications. Throughout, $u(x)$ is a Bernoulli utility function over wealth $x > 0$, with $u'(x) > 0$ and $u''(x) < 0$.

1 Why not use $u''(x)$ alone?

A risk-averse individual has a concave utility function, so $u''(x)$ is a natural first candidate to measure risk aversion. However, $u''(x)$ is not invariant to positive linear transformations of u , which represent the same preferences over lotteries. If $v(x) = \alpha + \beta u(x)$ with $\beta > 0$, then

$$v''(x) = \beta u''(x), \tag{1}$$

so we could make an individual look “twice as risk averse” by doubling β , although her choices among lotteries do not change. Dividing by $u'(x)$ removes the scale factor β , which is the idea behind the Arrow–Pratt coefficient.

2 Absolute risk aversion

2.1 Definition and invariance

The Arrow–Pratt coefficient of absolute risk aversion is

$$r_A(x) = -\frac{u''(x)}{u'(x)}. \tag{2}$$

For $v(x) = \alpha + \beta u(x)$, we obtain $r_A(x, v) = -\frac{\beta u''(x)}{\beta u'(x)} = r_A(x, u)$, so the coefficient does not depend on the utility representation we choose. Since $u'(x) > 0$ and $u''(x) < 0$, the coefficient is positive for every risk-averse individual; it equals zero for a risk-neutral individual, whose utility function is linear.

Computing $r_A(x)$ step by step. The recipe always has three steps: differentiate u once, differentiate it twice, and take the negative of the ratio. Two examples:

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- CARA, $u(x) = -e^{-ax}$ with $a > 0$: $u'(x) = ae^{-ax}$ and $u''(x) = -a^2e^{-ax}$, so $r_A(x) = -\frac{a^2e^{-ax}}{ae^{-ax}} = a$. The exponential terms cancel, which is why the coefficient does not depend on wealth.
- Log utility, $u(x) = \ln x$: $u'(x) = \frac{1}{x}$ and $u''(x) = -\frac{1}{x^2}$, so $r_A(x) = -\frac{-1/x^2}{1/x} = \frac{1}{x}$, which decreases in wealth.

2.2 What $r_A(x)$ measures: the risk premium of a small risk

Consider an individual with wealth x who faces a zero-mean risk $\tilde{\varepsilon}$, with $\mathbb{E}[\tilde{\varepsilon}] = 0$ and variance σ^2 . Her risk premium RP is the amount of money she would pay to eliminate the risk, that is,

$$\mathbb{E}[u(x + \tilde{\varepsilon})] = u(x - RP). \quad (3)$$

We approximate both sides of (3) when the risk is small.

- Left side. A second-order Taylor expansion of $u(x + \tilde{\varepsilon})$ around x , followed by taking expectations, yields

$$\mathbb{E}[u(x + \tilde{\varepsilon})] \approx u(x) + u'(x) \mathbb{E}[\tilde{\varepsilon}] + \frac{1}{2} u''(x) \mathbb{E}[\tilde{\varepsilon}^2] = u(x) + \frac{1}{2} u''(x) \sigma^2, \quad (4)$$

where we used $\mathbb{E}[\tilde{\varepsilon}] = 0$ and $\mathbb{E}[\tilde{\varepsilon}^2] = \sigma^2$. We need a second-order expansion here: the first-order term vanishes because the risk has zero mean, so the curvature term $u''(x)$ is the first one that matters.

- Right side. Since RP is a sure amount, a first-order expansion suffices,

$$u(x - RP) \approx u(x) - u'(x) RP. \quad (5)$$

Equating (4) and (5), the term $u(x)$ cancels, and we obtain $\frac{1}{2} u''(x) \sigma^2 = -u'(x) RP$. Solving for RP (Pratt, 1964),

$$RP \approx -\frac{1}{2} \frac{u''(x)}{u'(x)} \sigma^2 = \frac{1}{2} r_A(x) \sigma^2. \quad (6)$$

Therefore, $r_A(x)$ is (twice) the risk premium per unit of variance. An individual with a higher $r_A(x)$ is willing to pay more to avoid the same small risk, and the premium grows proportionally with the variance of the risk.

A numerical example. Consider $u(x) = \ln x$, so $r_A(x) = \frac{1}{x}$, and a fair bet that adds or subtracts \$10 with probability $\frac{1}{2}$ each, so $\sigma^2 = \frac{1}{2}(10)^2 + \frac{1}{2}(-10)^2 = 100$.

- Exact premium. Condition (3) becomes $\frac{1}{2} \ln(x + 10) + \frac{1}{2} \ln(x - 10) = \ln(x - RP)$. The left side equals $\ln \sqrt{(x + 10)(x - 10)} = \ln \sqrt{x^2 - 100}$, so $x - RP = \sqrt{x^2 - 100}$, that is, $RP = x - \sqrt{x^2 - 100}$.
- Approximation. From (6), $RP \approx \frac{1}{2} \cdot \frac{1}{x} \cdot 100 = \frac{50}{x}$.
- CARA benchmark. For $u(x) = -e^{-ax}$, condition (3) becomes $-\frac{1}{2}e^{-a(x+10)} - \frac{1}{2}e^{-a(x-10)} = -e^{-a(x-RP)}$. Multiplying both sides by -1 and factoring e^{-ax} out of each term,

$$e^{-ax} \frac{e^{-10a} + e^{10a}}{2} = e^{-ax} e^{aRP}.$$

Dividing both sides by e^{-ax} , and using $\cosh(z) \equiv \frac{e^z + e^{-z}}{2}$, gives $\cosh(10a) = e^{aRP}$. Taking logs, $aRP = \ln \cosh(10a)$, so $RP = \frac{\ln \cosh(10a)}{a}$, which does not depend on x . With $a = 0.02$, $\cosh(0.2) \approx 1.0201$, so $RP \approx \frac{0.0199}{0.02} \approx 0.993$, close to the approximation $\frac{1}{2}(0.02)(100) = 1$.

Table 1 compares the three columns at six wealth levels and shows three patterns:

Wealth x	$RP, u = \ln x$ (exact)	$RP, u = \ln x$ (approximation)	$RP, \text{CARA with } a = 0.02$ (exact)
25	2.087	2.000	0.993
50	1.010	1.000	0.993
100	0.501	0.500	0.993
200	0.250	0.250	0.993
400	0.125	0.125	0.993
800	0.063	0.063	0.993

Table 1. Risk premium of a fair \$10 bet.

- Approximation (6) is very accurate when the bet is small relative to wealth, and it loses precision as the bet becomes large relative to wealth (at $x = 25$, the bet is 40 percent of wealth, and the approximation is off by about 4 percent).
- The log-utility individual pays less to avoid the same bet as she becomes wealthier: doubling wealth roughly halves the premium, since the approximation $\frac{50}{x}$ is inversely proportional to x .
- The CARA individual pays the same amount at every wealth level. Section 3 studies this difference.

3 Constant, decreasing, and increasing absolute risk aversion

3.1 Classification

Since $r_A(x)$ is a function of wealth, we can ask how it changes as the individual becomes wealthier:

- Constant absolute risk aversion (CARA): $r'_A(x) = 0$ for all x .
- Decreasing absolute risk aversion (DARA): $r'_A(x) < 0$, so wealthier individuals pay a smaller premium to avoid the same risk.
- Increasing absolute risk aversion (IARA): $r'_A(x) > 0$, so wealthier individuals pay a larger premium to avoid the same risk.

Empirical evidence tends to support DARA: wealthier individuals hold more risky assets in absolute terms.

3.2 DARA requires a positive third derivative

Differentiating (2) with respect to x , and using the quotient rule on $\frac{u''(x)}{u'(x)}$ (numerator u'' , with derivative u''' ; denominator u' , with derivative u''),

$$r'_A(x) = -\frac{u'''(x)u'(x) - [u''(x)]^2}{[u'(x)]^2} = -\frac{u'''(x)}{u'(x)} + \left[\frac{u''(x)}{u'(x)}\right]^2. \quad (7)$$

Then $r'_A(x) < 0$ if and only if $\frac{u'''(x)}{u'(x)} > \left[\frac{u''(x)}{u'(x)}\right]^2$. Multiplying both sides by $u'(x) > 0$, this condition becomes

$$u'''(x) > \frac{[u''(x)]^2}{u'(x)} > 0. \quad (8)$$

Result 1 (DARA and the third derivative). If u exhibits DARA, then $u'''(x) > 0$. The converse does not hold: $u'''(x) > 0$ is necessary but not sufficient for DARA, since $u'''(x)$ must exceed the positive term $\frac{[u''(x)]^2}{u'(x)}$.

CARA utility illustrates why $u'''(x) > 0$ is not sufficient. With $u(x) = -e^{-ax}$, we have $u'''(x) = a^3 e^{-ax} > 0$, but this third derivative exactly equals $\frac{[u''(x)]^2}{u'(x)} = \frac{a^4 e^{-2ax}}{ae^{-ax}} = a^3 e^{-ax}$, so condition (8) holds with equality and $r'_A(x) = 0$.

3.3 Parametric examples

Table 2 reports $r_A(x)$ for the utility functions used most in applications, and Figure 1 plots them for specific parameter values.

Utility function	$r_A(x)$	Sign of $r'_A(x)$	Type
$u(x) = -e^{-ax}, a > 0$	a	0	CARA
$u(x) = \ln x$	$\frac{1}{x}$	–	DARA
$u(x) = \frac{x^{1-\gamma}}{1-\gamma}, \gamma > 0, \gamma \neq 1$	$\frac{\gamma}{x}$	–	DARA
$u(x) = (x + \alpha)^\beta, \alpha \geq 0, \beta \in (0, 1)$	$\frac{1-\beta}{x+\alpha}$	–	DARA
$u(x) = \ln(x + \alpha), \alpha \geq 0$	$\frac{1}{x+\alpha}$	–	DARA
$u(x) = \frac{1-\gamma}{\gamma} \left(\frac{ax}{1-\gamma} + b \right)^\gamma, \gamma < 1, \gamma \neq 0$	$\frac{a}{\frac{ax}{1-\gamma} + b}$	–	DARA
$u(x) = \alpha + \beta x - \frac{c}{2}x^2, x < \frac{\beta}{c}$	$\frac{c}{\beta - cx}$	+	IARA

Table 2. Absolute risk aversion of common utility functions.

Each entry follows the three-step recipe above. For instance, for $u(x) = (x + \alpha)^\beta$, we have $u'(x) = \beta(x + \alpha)^{\beta-1}$ and $u''(x) = \beta(\beta - 1)(x + \alpha)^{\beta-2}$, so

$$r_A(x) = -\frac{\beta(\beta - 1)(x + \alpha)^{\beta-2}}{\beta(x + \alpha)^{\beta-1}} = \frac{1 - \beta}{x + \alpha},$$

which decreases in x because the denominator increases. The sixth row is the hyperbolic absolute risk aversion (HARA) family, with $a > 0$ and $\frac{ax}{1-\gamma} + b > 0$. Writing $D \equiv \frac{ax}{1-\gamma} + b$, so that $u(x) = \frac{1-\gamma}{\gamma} D^\gamma$ and $\frac{dD}{dx} = \frac{a}{1-\gamma}$, the chain rule gives

$$u'(x) = \frac{1-\gamma}{\gamma} \gamma D^{\gamma-1} \frac{a}{1-\gamma} = aD^{\gamma-1}, \quad u''(x) = a(\gamma - 1)D^{\gamma-2} \frac{a}{1-\gamma} = -a^2 D^{\gamma-2},$$

so $r_A(x) = \frac{a^2 D^{\gamma-2}}{aD^{\gamma-1}} = \frac{a}{D}$. Its name comes from the fact that $\frac{1}{r_A(x)} = \frac{x}{1-\gamma} + \frac{b}{a}$ is linear in wealth, so $r_A(x)$ is a hyperbola. HARA nests several of the other rows:

- With $b = 0$, $u(x)$ is a positive multiple of $\frac{x^\gamma}{\gamma}$, a CRRA function with $r_R(x) = 1 - \gamma$ (see Section 4).
- Outside the range $\gamma < 1$, setting $\gamma = 2$ yields $u(x) = -\frac{1}{2}(b - ax)^2$, a quadratic utility function, and $r_A(x) = \frac{a}{b - ax}$ increases in wealth.
- As γ grows large in absolute value, $r_A(x)$ approaches $\frac{a}{b}$, a constant, as under CARA.

For the quadratic utility function in the last row of Table 2, $u'(x) = \beta - cx$ and $u''(x) = -c$, so $r_A(x) = \frac{c}{\beta - cx}$ and $r'_A(x) = \frac{c^2}{(\beta - cx)^2} > 0$. This is consistent with Result 1, since $u'''(x) = 0$ rules out DARA. Quadratic

utility is convenient, because expected utility then depends only on the mean and variance of the lottery, but it implies the counterfactual prediction that wealthier individuals are more reluctant to bear a given risk.

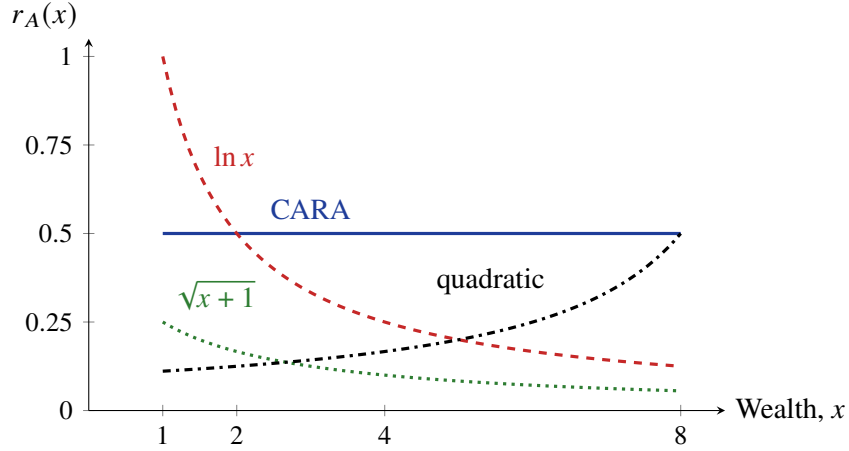


Figure 1. Absolute risk aversion as wealth increases.¹

4 Relative risk aversion

4.1 Definition and interpretation

The coefficient of relative risk aversion is

$$r_R(x) = -x \frac{u''(x)}{u'(x)} = x r_A(x). \quad (9)$$

To interpret $r_R(x)$, note that $\frac{d \ln u'(x)}{d \ln x} = \frac{d \ln u'(x)}{dx} \cdot \frac{dx}{d \ln x} = \frac{u''(x)}{u'(x)} \cdot x$, so

$$r_R(x) = -\frac{d \ln u'(x)}{d \ln x}. \quad (10)$$

Hence, $r_R(x)$ is the elasticity of marginal utility with respect to wealth (in absolute value): it measures the percentage decrease in marginal utility when wealth increases by 1 percent. Unlike $r_A(x)$, which is measured per dollar, $r_R(x)$ is a unit-free number, so it does not change if wealth is measured in dollars or in thousands of dollars.

While $r_A(x)$ measures attitudes towards risks of a given dollar size, $r_R(x)$ measures attitudes towards risks that are proportional to wealth. Consider an individual whose wealth becomes $x(1 + \tilde{\varepsilon})$, where $\tilde{\varepsilon}$ has zero mean and variance σ^2 ; for instance, $\tilde{\varepsilon} = \pm 0.2$ means that she gains or loses 20 percent of her wealth. The dollar risk she faces is $x\tilde{\varepsilon}$, which has zero mean and variance $x^2\sigma^2$. Applying approximation (6) to this dollar risk,

$$RP \approx \frac{1}{2} r_A(x) x^2 \sigma^2. \quad (11)$$

Dividing both sides by wealth x , and using $r_R(x) = x r_A(x)$, the risk premium as a share of wealth is

$$\frac{RP}{x} \approx \frac{1}{2} r_R(x) \sigma^2. \quad (12)$$

4.2 Constant relative risk aversion

The constant relative risk aversion (CRRA) utility function is

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma} \text{ if } \gamma \neq 1, \quad u(x) = \ln x \text{ if } \gamma = 1, \quad (13)$$

where $\gamma > 0$. Differentiating, $u'(x) = x^{-\gamma}$ and $u''(x) = -\gamma x^{-\gamma-1}$, so

$$r_A(x) = -\frac{-\gamma x^{-\gamma-1}}{x^{-\gamma}} = \frac{\gamma}{x} \quad \text{and} \quad r_R(x) = x \cdot \frac{\gamma}{x} = \gamma \quad (14)$$

for every x . The log case also fits: for $u(x) = \ln x$, $u'(x) = \frac{1}{x} = x^{-1}$, which is the CRRA marginal utility with $\gamma = 1$. The textbook's Example 5.10 uses $u(x) = x^b$ with $b \in (0, 1)$, which is a positive multiple of (13) with $\gamma = 1 - b$, so $r_R(x) = 1 - b$.

Figure 2 plots $u(x) = x^b$ for $b = 1, \frac{1}{2}, \frac{1}{3}$, and $\frac{1}{4}$. The linear utility function ($b = 1$) has $r_R(x) = 0$, and the coefficient increases to $\frac{1}{2}, \frac{2}{3}$, and $\frac{3}{4}$, respectively, as the utility function becomes more concave.

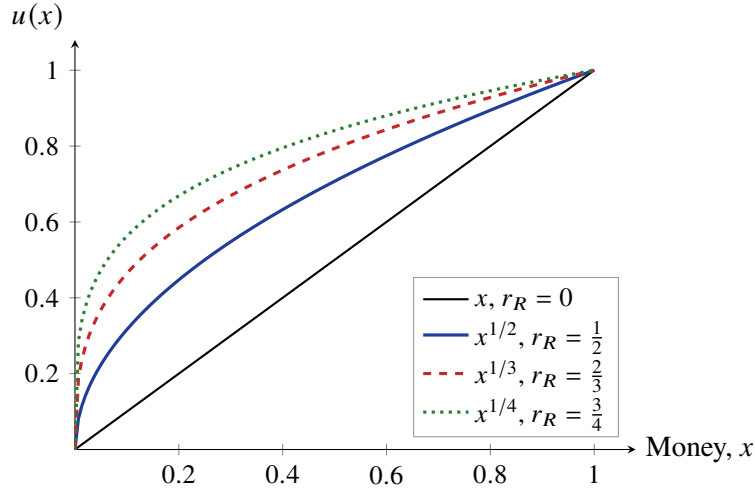


Figure 2. CRRA utility functions.

A numerical example. Consider a CRRA individual with wealth $x = 100$ who faces a bet that raises or lowers her wealth by 20 percent with probability $\frac{1}{2}$ each, so $\sigma^2 = \frac{1}{2}(0.2)^2 + \frac{1}{2}(-0.2)^2 = 0.04$.

- **Exact premium.** Her certainty equivalent c solves $u(c) = \frac{1}{2}u(1.2x) + \frac{1}{2}u(0.8x)$. With $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, the factor $\frac{1}{1-\gamma}$ cancels on both sides, so $c^{1-\gamma} = \frac{1}{2}[(1.2)^{1-\gamma} + (0.8)^{1-\gamma}]x^{1-\gamma}$, which yields

$$c = \left[\frac{(1.2)^{1-\gamma} + (0.8)^{1-\gamma}}{2} \right]^{\frac{1}{1-\gamma}} x \quad \text{and} \quad RP = x - c. \quad (15)$$

For $\gamma = 1$ (log utility), the same steps give $c = \sqrt{(1.2)(0.8)}x$. Since c is proportional to x , the premium as a share of wealth, $\frac{RP}{x}$, does not depend on wealth, as expected under CRRA.

- **Approximation.** From (12), $RP \approx \frac{1}{2}\gamma(0.04)(100) = 2\gamma$.

The approximation works well for moderate values of γ , but it overstates the premium when γ is large. At $\gamma = 10$ the exact premium is \$13.84, not \$20. The approximation is derived for small risks, and a 20 percent bet is not small for a very risk-averse individual: the premium can never exceed the \$20 she would lose in the bad outcome, while the approximation 2γ grows without bound.

$\gamma = r_R$	RP (exact)	RP (approximation)
0.5	1.01	1.00
1	2.02	2.00
2	4.00	4.00
3	5.86	6.00
4	7.56	8.00
10	13.84	20.00

Table 3. Risk premium of a ± 20 percent bet at wealth $x = 100$.

4.3 How relative and absolute risk aversion are related

Differentiating $r_R(x) = x r_A(x)$ with respect to x yields

$$r'_R(x) = r_A(x) + x r'_A(x). \quad (16)$$

Result 2 (DRRA implies DARA). If $r'_R(x) < 0$ (decreasing relative risk aversion, DRRA), then $r'_A(x) < 0$. Indeed, since $r_A(x) > 0$, equation (16) can only be negative if $x r'_A(x) < -r_A(x) < 0$. The converse does not hold:

- For $u(x) = (x+\alpha)^\beta$ with $\alpha > 0$, we have $r_A(x) = \frac{1-\beta}{x+\alpha}$, which decreases in x (DARA), but $r_R(x) = \frac{(1-\beta)x}{x+\alpha}$, which increases in x (increasing relative risk aversion).
- As this individual becomes wealthier, she pays a smaller dollar premium to avoid a \$10 bet, but a larger share of her wealth to avoid a bet proportional to her wealth.

To see the numbers, let $\alpha = 1$ and $\beta = \frac{1}{2}$. At $x = 1$, $r_A(1) = \frac{1/2}{2} = 0.25$ and $r_R(1) = 0.25$. At $x = 9$, $r_A(9) = \frac{1/2}{10} = 0.05$, which is lower, while $r_R(9) = 9(0.05) = 0.45$, which is higher. To see why, differentiate $r_A(x) = (1-\beta)(x+\alpha)^{-1}$ to obtain $r'_A(x) = -\frac{1-\beta}{(x+\alpha)^2}$, and substitute both terms into (16):

$$r'_R(x) = \frac{1-\beta}{x+\alpha} - \frac{(1-\beta)x}{(x+\alpha)^2} = \frac{(1-\beta)(x+\alpha) - (1-\beta)x}{(x+\alpha)^2} = \frac{(1-\beta)\alpha}{(x+\alpha)^2} > 0.$$

The positive term $r_A(x)$ dominates the negative term $x r'_A(x)$, so relative risk aversion increases although absolute risk aversion decreases.

5 Comparing the risk aversion of two individuals

5.1 Four equivalent comparisons

Consider two individuals with utility functions u_1 and u_2 , where u_2 is a concave transformation of u_1 , that is, $u_2(x) = \varphi(u_1(x))$ for an increasing and concave function φ . Differentiating $u_2(x) = \varphi(u_1(x))$ once with the chain rule gives $u'_2 = \varphi'(u_1) u'_1$. Differentiating again, now with the product rule (the derivative of $\varphi'(u_1(x))$ is $\varphi''(u_1) u'_1$), gives $u''_2 = \varphi''(u_1)(u'_1)^2 + \varphi'(u_1) u''_1$. Dividing,

$$-\frac{u''_2}{u'_2} = -\frac{\varphi''(u_1)(u'_1)^2 + \varphi'(u_1) u''_1}{\varphi'(u_1) u'_1} = -\frac{\varphi''(u_1)}{\varphi'(u_1)} u'_1 - \frac{u''_1}{u'_1},$$

so

$$r_A(x, u_2) = -\frac{u_2''(x)}{u_2'(x)} = r_A(x, u_1) - \frac{\varphi''(u_1(x))}{\varphi'(u_1(x))} u_1'(x) \geq r_A(x, u_1), \quad (17)$$

where the inequality holds because $\varphi'' \leq 0$, $\varphi' > 0$, and $u_1' > 0$. The certainty equivalents can be compared with Jensen's inequality. Since $u_2 = \varphi(u_1)$ and φ is concave,

$$u_2(c(F, u_2)) = \mathbb{E}[\varphi(u_1(\tilde{x}))] \leq \varphi(\mathbb{E}[u_1(\tilde{x})]) = \varphi(u_1(c(F, u_1))) = u_2(c(F, u_1)), \quad (18)$$

and since u_2 is increasing, $c(F, u_2) \leq c(F, u_1)$. In words, the more concave individual values every lottery at a lower certainty equivalent. The following statements are equivalent:

- $r_A(x, u_2) \geq r_A(x, u_1)$ for every x .
- The certainty equivalent of individual 2 is lower, $c(F, u_2) \leq c(F, u_1)$, for every lottery F .
- The probability premium of individual 2 is higher, $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$, for every x and ε .
- Whenever individual 2 finds a lottery F at least as good as a riskless outcome $\bar{x}$, individual 1 finds it at least as good as $\bar{x}$ too.

A parametric example. Let $u_1(x) = \sqrt{x}$ and $u_2(x) = \ln x$. Since $\ln x = 2 \ln \sqrt{x}$, we have $u_2 = \varphi(u_1)$ with $\varphi(t) = 2 \ln t$, which is increasing and concave. Indeed, $r_A(x, u_1) = \frac{1}{2x} < \frac{1}{x} = r_A(x, u_2)$. Consider the lottery F that pays \$1 or \$3 with probability $\frac{1}{2}$ each, so its expected value is \$2. The certainty equivalents solve $\sqrt{c_1} = \frac{1}{2}\sqrt{1} + \frac{1}{2}\sqrt{3} = \frac{1+\sqrt{3}}{2}$ and $\ln c_2 = \frac{1}{2} \ln 1 + \frac{1}{2} \ln 3 = \ln \sqrt{3}$. Squaring the first equation and exponentiating the second,

$$c(F, u_1) = \left(\frac{1+\sqrt{3}}{2}\right)^2 \approx 1.866 \quad \text{and} \quad c(F, u_2) = \sqrt{3} \approx 1.732. \quad (19)$$

The probability premium at $x = 2$ and $\varepsilon = 1$ is the extra probability π on the good outcome that makes the individual indifferent between \$2 for sure and the lottery, that is, $u(2) = \left(\frac{1}{2} + \pi\right)u(3) + \left(\frac{1}{2} - \pi\right)u(1)$. Collecting the terms in π , the right side equals $\frac{1}{2}[u(1) + u(3)] + \pi[u(3) - u(1)]$. Solving for π ,

$$\pi = \frac{u(2) - \frac{1}{2}[u(1) + u(3)]}{u(3) - u(1)}. \quad (20)$$

The numerator is the utility lost by accepting the lottery instead of its expected value, and it is positive for a risk-averse individual.

To see the ranking across more than two individuals, Table 4 adds two CRRA individuals to the example: individual 0, with $u_0(x) = x^{3/4}$ ($\gamma = \frac{1}{4}$), who is less risk averse than individual 1, and individual 3, with $u_3(x) = -\frac{1}{x}$ ($\gamma = 2$), who is more risk averse than individual 2. Since $r_A(x) = \frac{\gamma}{x}$ under CRRA, each row has a higher r_A than the row above it at every wealth level, so each utility function is a concave transformation of the one above it. For individual 3, the certainty equivalent solves $-\frac{1}{c} = \frac{1}{2}\left(-\frac{1}{1}\right) + \frac{1}{2}\left(-\frac{1}{3}\right) = -\frac{2}{3}$, so $c = \frac{3}{2} = 1.5$, the harmonic mean of 1 and 3. Her probability premium, from (20), is $\pi = \frac{-\frac{1}{2} + \frac{2}{3}}{-\frac{1}{3} + 1} = \frac{1/6}{2/3} = 0.25$.

Moving down the table, absolute risk aversion rises, the certainty equivalent falls, and both the risk premium and the probability premium rise, exactly as the four equivalent statements predict.

Figure 3 depicts both utility functions after normalizing them so that $u(1) = 0$ and $u(3) = 1$ (a positive linear transformation, which does not change preferences). With this normalization, both individuals assign the same expected utility, $\frac{1}{2}$, to the lottery, and the chord connecting $(1, 0)$ and $(3, 1)$ is common to both. The certainty equivalent is the wealth at which each utility function reaches the height $\frac{1}{2}$. Since u_2 is more concave, it reaches that height at a lower wealth level.

Individual	$r_A(2)$	Certainty equivalent	Risk premium	Probability premium
0, $u_0 = x^{3/4}$	0.125	1.934	0.066	0.033
1, $u_1 = \sqrt{x}$	0.25	1.866	0.134	0.066
2, $u_2 = \ln x$	0.50	1.732	0.268	0.131
3, $u_3 = -\frac{1}{x}$	1.00	1.500	0.500	0.250

Table 4. Comparing individuals facing the same lottery.

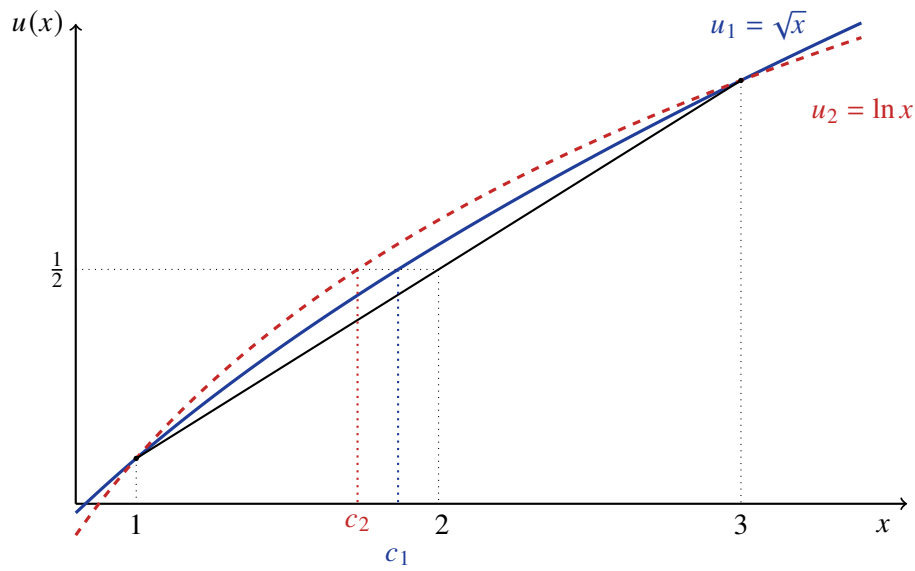


Figure 3. Certainty equivalents of two individuals.²

5.2 Strong risk aversion

Ross (1981) proposed a more demanding comparison. Utility function u_A exhibits stronger risk aversion than u_B if there is a constant $\lambda > 0$ such that

$$\frac{u''_A(x_1)}{u''_B(x_1)} \geq \lambda \geq \frac{u'_A(x_2)}{u'_B(x_2)} \quad \text{for all } x_1 \text{ and } x_2. \quad (21)$$

Evaluating (21) at $x_1 = x_2 = x$ gives $\frac{u''_A(x)}{u''_B(x)} \geq \frac{u'_A(x)}{u'_B(x)}$. Multiplying both sides by $\frac{u'_B(x)}{u'_A(x)} > 0$ yields $\frac{u''_A(x)/u'_A(x)}{u''_B(x)/u'_B(x)} \geq 1$. Since both the numerator and the denominator are negative, this means

$$-\frac{u''_A(x)}{u'_A(x)} \geq -\frac{u''_B(x)}{u'_B(x)}, \quad (22)$$

that is, $r_A(x, u_A) \geq r_A(x, u_B)$. Stronger risk aversion therefore implies higher absolute risk aversion.

Condition (21) compares the two utility functions at different wealth levels, x_1 and x_2 , while the Arrow–Pratt comparison only looks at one wealth level at a time. This is why the stronger notion is useful when the individuals' wealth is itself uncertain: a risk can then hit them at many different wealth levels.

The converse fails. Consider two CARA individuals, $u_i(x) = -e^{-a_i x}$ with $a_A > a_B$, so u_A has higher absolute risk aversion at every wealth level. Since $u'_i(x) = a_i e^{-a_i x}$ and $u''_i(x) = -a_i^2 e^{-a_i x}$,

$$\frac{u''_A(x_1)}{u''_B(x_1)} = \frac{-a_A^2 e^{-a_A x_1}}{-a_B^2 e^{-a_B x_1}} = \left(\frac{a_A}{a_B}\right)^2 e^{-(a_A - a_B)x_1} \quad \text{and} \quad \frac{u'_A(x_2)}{u'_B(x_2)} = \frac{a_A}{a_B} e^{-(a_A - a_B)x_2}. \quad (23)$$

The first ratio falls as x_1 increases and the second rises as x_2 decreases, so for x_1 large enough relative to x_2 the first ratio drops below the second, and no λ fits between them. With $a_A = 2$ and $a_B = 1$, at $x_1 = 1$ and $x_2 = 0$,

$$\frac{u''_A(1)}{u''_B(1)} = \frac{4e^{-2}}{e^{-1}} = 4e^{-1} \approx 1.47 \quad \text{and} \quad \frac{u'_A(0)}{u'_B(0)} = \frac{2}{1} = 2. \quad (24)$$

No constant λ satisfies $1.47 \geq \lambda \geq 2$, so u_A does not exhibit stronger risk aversion than u_B .

6 Prudence

6.1 Precautionary saving

Risk aversion is about the curvature of u . Prudence is about the curvature of the marginal utility u' , and it governs how an individual responds to future risk when she can save. Consider an individual with wealth w who chooses savings s today; she consumes $w - s$ today and $s + \tilde{\varepsilon}$ tomorrow, where $\tilde{\varepsilon}$ is a zero-mean income risk (there is no interest rate or discounting, for simplicity). She solves $\max_s u(w - s) + \mathbb{E}[u(s + \tilde{\varepsilon})]$, with first-order condition

$$u'(w - s) = \mathbb{E}[u'(s + \tilde{\varepsilon})]. \quad (25)$$

The left side is the marginal cost of saving one more dollar (the utility of consumption given up today), which increases in s because u is concave. The right side is the expected marginal benefit (the utility of the extra dollar tomorrow), which decreases in s . Without risk, the right side of (25) is $u'(s)$, so $u'(w - s) = u'(s)$, and the individual smooths consumption by saving $s = \frac{w}{2}$. With risk, the right side depends on the shape of u' :

- If $u'''(x) > 0$, then u' is convex and Jensen's inequality gives $\mathbb{E}[u'(s + \tilde{\varepsilon})] > u'(s)$: the future risk raises the expected marginal utility of saving, so the individual saves more. This additional saving is called precautionary saving.
- If $u'''(x) = 0$ (quadratic utility), then $\mathbb{E}[u'(s + \tilde{\varepsilon})] = u'(s)$ and the risk does not affect savings.
- If $u'''(x) < 0$, the individual saves less in response to the risk.

Kimball (1990) calls an individual prudent if $u'''(x) > 0$. Figure 4 illustrates the first case with $u'(c) = \frac{1}{c}$.

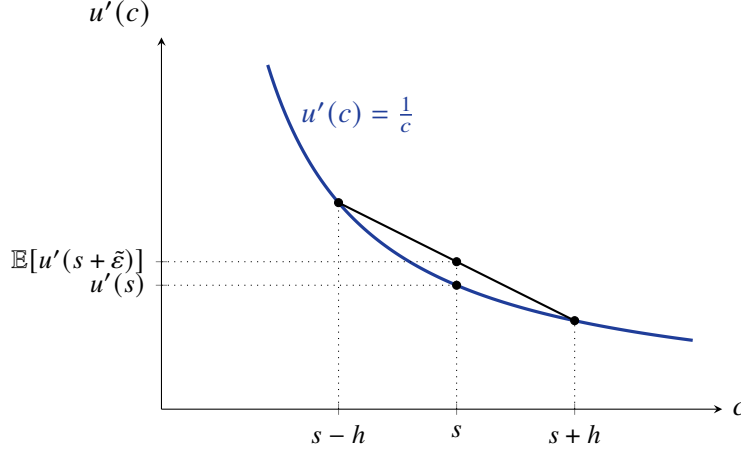


Figure 4. A convex marginal utility and precautionary saving.³

6.2 Absolute and relative prudence

Following Kimball (1990), the coefficients of absolute and relative prudence are

$$K_A(x) = -\frac{u'''(x)}{u''(x)} \quad \text{and} \quad K_R(x) = -x \frac{u'''(x)}{u''(x)} = x K_A(x). \quad (26)$$

They play the same role for u' that r_A and r_R play for u : replacing u by u' in (2) gives exactly $K_A(x)$. Since $u''(x) < 0$, absolute prudence is positive if and only if $u'''(x) > 0$.

To see what $K_A(x)$ measures, define the precautionary premium ψ by

$$\mathbb{E}[u'(x + \tilde{\varepsilon})] = u'(x - \psi), \quad (27)$$

the reduction in sure wealth that has the same effect on marginal utility as the risk. We repeat the steps that led to (6), now applied to u' instead of u :

- Left side: $\mathbb{E}[u'(x + \tilde{\varepsilon})] \approx u'(x) + u''(x)\mathbb{E}[\tilde{\varepsilon}] + \frac{1}{2}u'''(x)\sigma^2 = u'(x) + \frac{1}{2}u'''(x)\sigma^2$.
- Right side: $u'(x - \psi) \approx u'(x) - u''(x)\psi$.

Equating both sides and solving for ψ ,

$$\psi \approx -\frac{1}{2} \frac{u'''(x)}{u''(x)} \sigma^2 = \frac{1}{2} K_A(x) \sigma^2. \quad (28)$$

A more prudent individual (higher K_A) behaves as if the risk reduced her future wealth by a larger amount, and therefore saves more.

A numerical example. Let $w = 10$ and let $\tilde{\varepsilon}$ equal 2 or -2 with probability $\frac{1}{2}$ each. Without risk, every individual saves $s = 5$. With risk, condition (25) becomes $u'(10 - s) = \frac{1}{2}u'(s + 2) + \frac{1}{2}u'(s - 2)$, which we solve for each utility function:

- Quadratic utility, $u(c) = c - \frac{k}{2}c^2$: since $u'(c) = 1 - kc$ is linear, $\frac{1}{2}u'(s + 2) + \frac{1}{2}u'(s - 2) = u'(s)$, and (25) gives $s = 5$, so precautionary saving is zero.
- Square-root utility, $u(c) = \sqrt{c}$ (CRRA with $\gamma = \frac{1}{2}$): the condition $\frac{1}{\sqrt{10-s}} = \frac{1}{2} \left(\frac{1}{\sqrt{s+2}} + \frac{1}{\sqrt{s-2}} \right)$ has no simple closed form; solving numerically gives $s \approx 5.291$.
- Log utility: since $u'(c) = \frac{1}{c}$, (25) becomes $\frac{1}{w-s} = \frac{1}{2} \left(\frac{1}{s+2} + \frac{1}{s-2} \right)$. Adding the two fractions on the right, $\frac{1}{s+2} + \frac{1}{s-2} = \frac{(s-2)+(s+2)}{(s+2)(s-2)} = \frac{2s}{s^2-4}$, so the condition is $\frac{1}{w-s} = \frac{s}{s^2-4}$. Cross-multiplying, $s^2 - 4 = s(w - s)$, that is, $2s^2 - ws - 4 = 0$. The quadratic formula gives $s = \frac{w \pm \sqrt{w^2 + 4(2)(4)}}{2(2)}$, and the positive root is $s = \frac{w + \sqrt{w^2 + 32}}{4}$. With $w = 10$, $s = \frac{10 + \sqrt{132}}{4} \approx \frac{10 + 11.489}{4} \approx 5.372$.
- CARA utility, $u(c) = -e^{-ac}$: since $u'(c) = ae^{-ac}$, (25) becomes $ae^{-a(w-s)} = \frac{1}{2}ae^{-a(s+2)} + \frac{1}{2}ae^{-a(s-2)}$. Factoring ae^{-as} out of the right side, it equals $ae^{-as} \frac{e^{-2a} + e^{2a}}{2} = ae^{-as} \cosh(2a)$. Dividing both sides by a and taking logs, $-a(w - s) = -as + \ln \cosh(2a)$, that is, $2as = aw + \ln \cosh(2a)$, so $s = \frac{w}{2} + \frac{\ln \cosh(2a)}{2a}$. For $a = \frac{1}{2}$, $\ln \cosh(1) \approx 0.434$, so $s \approx 5 + 0.434 = 5.434$. Precautionary saving, $\frac{\ln \cosh(2a)}{2a}$, does not depend on w .
- CRRA utility with $\gamma = 2$, $u(c) = -\frac{1}{c}$: since $u'(c) = c^{-2}$, the condition $\frac{1}{(10-s)^2} = \frac{1}{2} \left[\frac{1}{(s+2)^2} + \frac{1}{(s-2)^2} \right]$ gives $s \approx 5.507$ numerically.
- CRRA utility with $\gamma = 3$, $u(c) = -\frac{1}{2c^2}$: since $u'(c) = c^{-3}$, solving numerically gives $s \approx 5.607$.

Utility function	$K_A(5)$	Savings with risk	Precautionary saving
Quadratic	0	5.000	0.000
$\sqrt{c}$ (CRRA, $\gamma = \frac{1}{2}$)	0.30	5.291	0.291
$\ln c$	0.40	5.372	0.372
$-e^{-c/2}$ (CARA, $a = \frac{1}{2}$)	0.50	5.434	0.434
$-\frac{1}{c}$ (CRRA, $\gamma = 2$)	0.60	5.507	0.507
$-\frac{1}{2c^2}$ (CRRA, $\gamma = 3$)	0.80	5.607	0.607

Table 5. Precautionary saving with $w = 10$ and a ± 2 income risk.

The column $K_A(5)$ uses $K_A(c) = \frac{\gamma+1}{c}$ for CRRA and $K_A(c) = a$ for CARA. Precautionary saving increases with absolute prudence evaluated at the riskless savings level $s = 5$, as predicted by (28): the rows are ordered by $K_A(5)$, and precautionary saving rises from each row to the next.

6.3 How prudence and risk aversion are related

Dividing (7) by $r_A(x) = -\frac{u''(x)}{u'(x)}$, that is, multiplying each term by $-\frac{u'(x)}{u''(x)}$,

$$\frac{r'_A(x)}{r_A(x)} = \left(-\frac{u'''}{u'} \right) \left(-\frac{u'}{u''} \right) + \left(\frac{u''}{u'} \right)^2 \left(-\frac{u'}{u''} \right),$$

which simplifies to

$$\frac{r'_A(x)}{r_A(x)} = \frac{u'''(x)}{u''(x)} - \frac{u''(x)}{u'(x)} = -K_A(x) + r_A(x). \quad (29)$$

Solving for $K_A(x)$, and multiplying both sides by x ,

$$K_A(x) = r_A(x) - \frac{r'_A(x)}{r_A(x)} \quad \text{and} \quad K_R(x) = r_R(x) - x \frac{r'_A(x)}{r_A(x)}. \quad (30)$$

Result 3 (DARA and prudence). Absolute prudence exceeds absolute risk aversion, $K_A(x) > r_A(x)$, if and only if the individual exhibits DARA, $r'_A(x) < 0$. Similarly, $K_R(x) > r_R(x)$ if and only if DARA holds. Indeed, from (30), $K_A(x) - r_A(x) = -\frac{r'_A(x)}{r_A(x)}$, and since $r_A(x) > 0$, this difference is positive exactly when $r'_A(x) < 0$. In words, DARA requires the third derivative of u to be large relative to its curvature, so that prudence exceeds risk aversion. In particular:

- CARA, $u(x) = -e^{-ax}$: $r'_A(x) = 0$, so $K_A(x) = r_A(x) = a$ and $K_R(x) = r_R(x) = ax$.
- CRRA, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$: $r_A(x) = \frac{\gamma}{x}$ and $r'_A(x) = -\frac{\gamma}{x^2}$, so $\frac{r'_A(x)}{r_A(x)} = \frac{-\gamma/x^2}{\gamma/x} = -\frac{1}{x}$, which yields $K_A(x) = \frac{\gamma}{x} + \frac{1}{x} = \frac{\gamma+1}{x}$ and $K_R(x) = \gamma + 1 > \gamma = r_R(x)$. As a check, computing directly, $u''(x) = -\gamma x^{-\gamma-1}$ and $u'''(x) = \gamma(\gamma+1)x^{-\gamma-2}$, so $K_A(x) = \frac{\gamma(\gamma+1)x^{-\gamma-2}}{\gamma x^{-\gamma-1}} = \frac{\gamma+1}{x}$.
- In the textbook's notation, $u(x) = x^b$, relative prudence is $K_R(x) = 2 - b$, which exceeds $r_R(x) = 1 - b$.

7 Cautiousness and temperance

The ratio of relative prudence to relative risk aversion,

$$C(x) = \frac{K_R(x)}{r_R(x)} = \frac{K_A(x)}{r_A(x)} = 1 - \frac{r'_A(x)}{[r_A(x)]^2}, \quad (31)$$

is known as cautiousness. The second equality holds because both K_R and r_R multiply their absolute counterparts by x , and the last equality divides (30) by $r_A(x)$. From (31), $C(x) > 1$ if and only if $K_R(x) > r_R(x)$, which holds if and only if the individual exhibits DARA. For instance:

- CARA: $r'_A(x) = 0$, so $C(x) = 1$.
- CRRA: $C(x) = \frac{(\gamma+1)/x}{\gamma/x} = \frac{\gamma+1}{\gamma} > 1$, which equals 2 under log utility ($\gamma = 1$).
- HARA: differentiating $u''(x) = -a^2 D^{\gamma-2}$ once more gives $u'''(x) = -a^2(\gamma-2)D^{\gamma-3} \frac{a}{1-\gamma} = \frac{2-\gamma}{1-\gamma} a^3 D^{\gamma-3}$, so $K_A(x) = \frac{2-\gamma}{1-\gamma} \frac{a}{D}$ and $C(x) = \frac{K_A(x)}{r_A(x)} = \frac{2-\gamma}{1-\gamma}$, a constant that exceeds 1 when $\gamma < 1$.

Finally, the coefficient of temperance is

$$T(x) = -\frac{u''''(x)}{u'''(x)}, \quad (32)$$

which applies the same recipe one derivative further: it is to u'' what K_A is to u' and r_A is to u . An individual is temperate if $u''''(x) < 0$. Temperance captures the desire to reduce the total exposure to independent risks: a temperate individual who already bears one unavoidable risk becomes less willing to accept a second, independent risk. For CRRA utility, $u'''(x) = \gamma(\gamma+1)x^{-\gamma-2}$ and $u''''(x) = -\gamma(\gamma+1)(\gamma+2)x^{-\gamma-3}$,

Utility function	$r_A(x)$	$r_R(x)$	$K_A(x)$	$K_R(x)$	$C(x)$	$T(x)$
$-e^{-ax}$ (CARA)	a	ax	a	ax	1	a
$\ln x$	$\frac{1}{x}$	1	$\frac{2}{x}$	2	2	$\frac{3}{x}$
$\frac{x^{1-\gamma}}{1-\gamma}$ (CRRA)	$\frac{\gamma}{x}$	γ	$\frac{\gamma+1}{x}$	$\gamma+1$	$\frac{\gamma+1}{\gamma}$	$\frac{\gamma+2}{x}$
$(x+\alpha)^\beta$	$\frac{1-\beta}{x+\alpha}$	$\frac{(1-\beta)x}{x+\alpha}$	$\frac{2-\beta}{x+\alpha}$	$\frac{(2-\beta)x}{x+\alpha}$	$\frac{2-\beta}{1-\beta}$	$\frac{3-\beta}{x+\alpha}$
$\ln(x+\alpha)$	$\frac{1}{x+\alpha}$	$\frac{x}{x+\alpha}$	$\frac{2}{x+\alpha}$	$\frac{2x}{x+\alpha}$	2	$\frac{3}{x+\alpha}$
HARA	$\frac{a}{D}$	$\frac{ax}{D}$	$\frac{2-\gamma}{1-\gamma} \frac{a}{D}$	$\frac{2-\gamma}{1-\gamma} \frac{ax}{D}$	$\frac{2-\gamma}{1-\gamma}$	$\frac{3-\gamma}{1-\gamma} \frac{a}{D}$
$\alpha + \beta x - \frac{c}{2}x^2$	$\frac{c}{\beta - cx}$	$\frac{cx}{\beta - cx}$	0	0	0	not defined

Table 6. Summary of the measures for common utility functions.

so $T(x) = \frac{\gamma+2}{x}$. CARA and CRRA utility functions alternate the signs of their derivatives, $u' > 0$, $u'' < 0$, $u''' > 0$, and $u'''' < 0$, so they are simultaneously risk averse, prudent, and temperate.

In the HARA row, as in Section 3, $D \equiv \frac{ax}{1-\gamma} + b$, so that $r_A(x) = \frac{a}{D}$. Since every prudence and temperance coefficient of HARA is a constant multiple of $r_A(x)$, this family keeps the ratios $C(x)$ and $\frac{T(x)}{r_A(x)}$ fixed while allowing $r_A(x)$ to decrease (if $\gamma < 1$) or increase (if $\gamma > 1$) with wealth.

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