

Production Theory: Worked Examples

EconS 501, a handout on Chapter 4

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This handout complements the slides of Chapter 4 with worked examples. We first evaluate the main properties of common production functions (marginal products, the marginal rate of technical substitution, returns to scale, and the elasticity of substitution). We then solve the profit-maximization problem (PMP) and the cost-minimization problem (CMP) for these technologies, decompose the effect of an input price change into substitution and output effects, and close with duality: how the cost and profit functions recover factor demands, supply, and even the production function itself. Throughout, the firm uses two inputs, z_1 and z_2 , bought at prices $w_1 > 0$ and $w_2 > 0$, to produce a single output $q = f(z_1, z_2)$ sold at price $p > 0$. We use the same parameter values in several sections, so you can check that the different approaches give the same answer.

1 Production functions

1.1 Marginal products and the MRTS

The marginal product of input k is $MP_k = \frac{\partial f(z)}{\partial z_k}$, and the marginal rate of technical substitution between inputs 1 and 2 is the ratio of marginal products,

$$MRTS_{1,2}(z) = \frac{MP_1}{MP_2}, \quad (1)$$

which measures how many units of input 2 the firm can give up when it uses one more unit of input 1, keeping output unchanged. Graphically, $MRTS_{1,2}$ is the absolute value of the slope of the isoquant $\{(z_1, z_2) : f(z_1, z_2) = q\}$.

Example 1 (Cobb–Douglas). Consider $f(z_1, z_2) = z_1^{1/2} z_2^{1/4}$.

- Step 1, marginal products. Differentiating with respect to each input,

$$MP_1 = \frac{1}{2} z_1^{-1/2} z_2^{1/4} \quad \text{and} \quad MP_2 = \frac{1}{4} z_1^{1/2} z_2^{-3/4}.$$

Both are positive, and both decrease in their own input, since $\frac{\partial MP_1}{\partial z_1} = -\frac{1}{4} z_1^{-3/2} z_2^{1/4} < 0$ and $\frac{\partial MP_2}{\partial z_2} = -\frac{3}{16} z_1^{1/2} z_2^{-7/4} < 0$.

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- Step 2, MRTS. Dividing MP_1 by MP_2 , the powers of each input combine as $z_1^{-1/2-1/2} = z_1^{-1}$ and $z_2^{1/4+3/4} = z_2$, so

$$MRTS_{1,2} = \frac{\frac{1}{2} z_1^{-1/2} z_2^{1/4}}{\frac{1}{4} z_1^{1/2} z_2^{-3/4}} = \frac{2z_2}{z_1}.$$

- Step 3, a numerical point. At $z = (4, 16)$, output is $q = 4^{1/2} \cdot 16^{1/4} = 2 \cdot 2 = 4$, the marginal products are $MP_1 = \frac{1}{2} \cdot \frac{1}{2} \cdot 2 = 0.5$ and $MP_2 = \frac{1}{4} \cdot 2 \cdot \frac{1}{8} = \frac{1}{16}$, and $MRTS_{1,2} = \frac{0.5}{1/16} = 8 = \frac{2(16)}{4}$. One more unit of input 1 replaces about eight units of input 2.

Since $MRTS_{1,2} = \frac{2z_2}{z_1}$ falls as z_1 increases and z_2 decreases along the isoquant, the MRTS is diminishing and isoquants are convex.

1.2 Returns to scale

Multiplying all inputs by $t > 1$, the production function exhibits increasing, constant, or decreasing returns to scale if output increases more than, exactly, or less than proportionally, that is, if $f(tz) > tf(z)$, $f(tz) = tf(z)$, or $f(tz) < tf(z)$. In Example 1,

$$f(tz_1, tz_2) = (tz_1)^{1/2} (tz_2)^{1/4} = t^{1/2+1/4} z_1^{1/2} z_2^{1/4} = t^{3/4} f(z_1, z_2),$$

and $t^{3/4} < t$ for every $t > 1$, so the technology exhibits decreasing returns to scale: doubling both inputs multiplies output by $2^{3/4} \approx 1.68$. In general, $z_1^\alpha z_2^\beta$ is homogeneous of degree $\alpha + \beta$, so it exhibits increasing, constant, or decreasing returns when $\alpha + \beta$ is above, equal to, or below one. The scale elasticity, the percentage increase in output from a 1 percent increase in all inputs, equals $\alpha + \beta$ for every input vector; in Example 1, it is $\frac{3}{4}$.

1.3 Elasticity of substitution

The elasticity of substitution measures how much the input ratio responds to a change in the MRTS,

$$\sigma = \frac{\partial \ln(z_2/z_1)}{\partial \ln MRTS_{1,2}}. \quad (2)$$

At a cost-minimizing input choice, $MRTS_{1,2} = \frac{w_1}{w_2}$, so σ also measures the percentage increase in $\frac{z_2}{z_1}$ when input 1 becomes 1 percent more expensive relative to input 2.

Example 2 (CES). Consider $f(z_1, z_2) = (z_1^\rho + z_2^\rho)^{1/\rho}$, where $\rho \leq 1$ and $\rho \neq 0$.

- Step 1, marginal products. By the chain rule, $MP_k = \frac{1}{\rho} (z_1^\rho + z_2^\rho)^{\frac{1}{\rho}-1} \rho z_k^{\rho-1}$ for each input k . The common factor $(z_1^\rho + z_2^\rho)^{\frac{1}{\rho}-1}$ cancels in the ratio, which leaves

$$MRTS_{1,2} = \frac{z_1^{\rho-1}}{z_2^{\rho-1}} = \left(\frac{z_2}{z_1} \right)^{1-\rho}.$$

- Step 2, solve for the input ratio. Taking logs, $\ln MRTS_{1,2} = (1 - \rho) \ln \frac{z_2}{z_1}$, so $\ln \frac{z_2}{z_1} = \frac{1}{1-\rho} \ln MRTS_{1,2}$.
- Step 3, differentiate. From (2),

$$\sigma = \frac{1}{1-\rho}. \quad (3)$$

The elasticity does not depend on the input vector, which is why the function is called “constant elasticity of substitution.” For instance, $\rho = \frac{1}{2}$ gives $\sigma = 2$, while $\rho = -1$ gives $\sigma = \frac{1}{2}$.

The CES function nests the other common technologies. When $\rho = 1$, $f = z_1 + z_2$ is linear and $\sigma = \frac{1}{1-\rho}$ becomes infinite; as $\rho \rightarrow 0$, σ approaches 1, the Cobb–Douglas value (indeed, the weighted version $\left(\frac{1}{2}z_1^\rho + \frac{1}{2}z_2^\rho\right)^{1/\rho}$ converges to $z_1^{1/2}z_2^{1/2}$); and as $\rho \rightarrow -\infty$, f approaches the fixed-proportions function $\min\{z_1, z_2\}$, with $\sigma = 0$. Table 1 summarizes these technologies, using the parameterizations of the examples below, and Figure 1 depicts one isoquant of each.

Production function	$MRTS_{1,2}$	Returns to scale	σ
Linear, $2z_1 + z_2$	2	Constant	$+\infty$
CES, $\left(z_1^{1/2} + z_2^{1/2}\right)^2$	$\left(\frac{z_2}{z_1}\right)^{1/2}$	Constant	2
Cobb–Douglas, $z_1^{1/2}z_2^{1/2}$	$\frac{z_2}{z_1}$	Constant	1
Cobb–Douglas, $z_1^{1/3}z_2^{1/3}$	$\frac{z_2}{z_1}$	Decreasing	1
Cobb–Douglas, $z_1^{1/2}z_2^{1/4}$	$\frac{2z_2}{z_1}$	Decreasing	1
Fixed proportions, $\min\{z_1, 2z_2\}$	not defined at the kink	Constant	0

Table 1. Properties of common production functions.

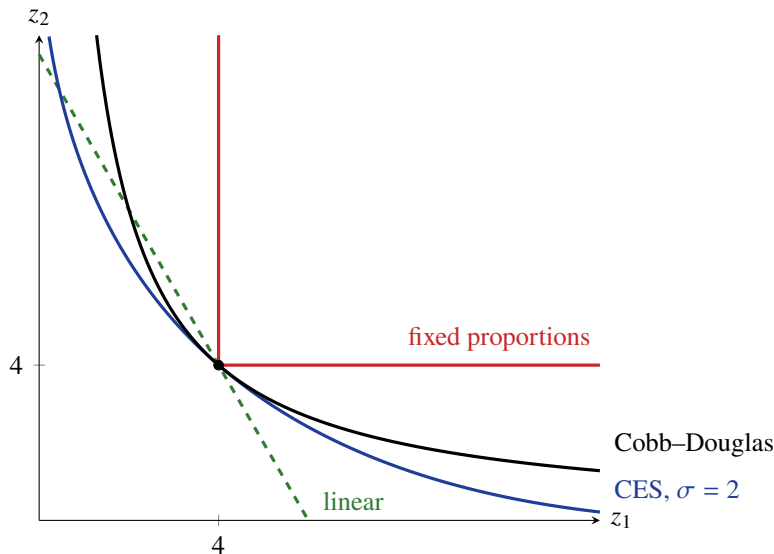


Figure 1: Isoquants through the input vector $(4, 4)$: linear, $2z_1 + z_2 = 12$; CES, $\left(z_1^{1/2} + z_2^{1/2}\right)^2 = 16$; Cobb–Douglas, $z_1^{1/2}z_2^{1/2} = 4$; and fixed proportions, $\min\{z_1, z_2\} = 4$. A larger σ makes the isoquant less curved.

The isoquants in Figure 1 pass through the same input vector, $(4, 4)$, so we can compare their curvature. The linear isoquant has no curvature, and inputs are perfect substitutes. The fixed-proportions isoquant has a right angle, and inputs cannot be substituted at all. The CES and Cobb–Douglas isoquants lie in between, with the CES ($\sigma = 2$) isoquant less curved than the Cobb–Douglas ($\sigma = 1$) one.

2 Profit maximization

With two inputs, the firm solves

$$\max_{z_1, z_2 \geq 0} p f(z_1, z_2) - w_1 z_1 - w_2 z_2. \quad (4)$$

At an interior solution, the first-order conditions are $p MP_k = w_k$ for each input k : the value of the marginal product of each input must equal its price. Dividing the two conditions yields $MRTS_{1,2} = \frac{w_1}{w_2}$. These conditions are sufficient when f is concave, which holds with decreasing (or constant) returns and convex isoquants. The solution gives the unconditional factor demands $z_k(p, w)$, the supply function $q(p, w) = f(z(p, w))$, and the profit function $\pi(p, w)$.

2.1 One input

Example 3. Let $f(z) = z^{1/2}$.

- Step 1, first-order condition. Differentiating $p z^{1/2} - w z$ with respect to z , we obtain $\frac{p}{2} z^{-1/2} - w = 0$. The second-order condition holds, since $-\frac{p}{4} z^{-3/2} < 0$.
- Step 2, factor demand. Rearranging, $z^{-1/2} = \frac{2w}{p}$, that is, $z^{1/2} = \frac{p}{2w}$. Squaring both sides,

$$z(p, w) = \frac{p^2}{4w^2}.$$

- Step 3, supply and profits. Output is $q(p, w) = z^{1/2} = \frac{p}{2w}$, and profits are

$$\pi(p, w) = p \cdot \frac{p}{2w} - w \cdot \frac{p^2}{4w^2} = \frac{p^2}{2w} - \frac{p^2}{4w} = \frac{p^2}{4w}.$$

With $p = 10$ and $w = 1$, the firm hires $z = 25$ units of input, produces $q = 5$ units, and earns $\pi = 10(5) - 25 = 25$, which coincides with $\frac{10^2}{4}$. Figure 2 depicts this solution: the highest isoprofit line, $q = \frac{\pi}{p} + \frac{w}{p} z$, that still touches the production set is tangent to $f(z)$ at $z = 25$.

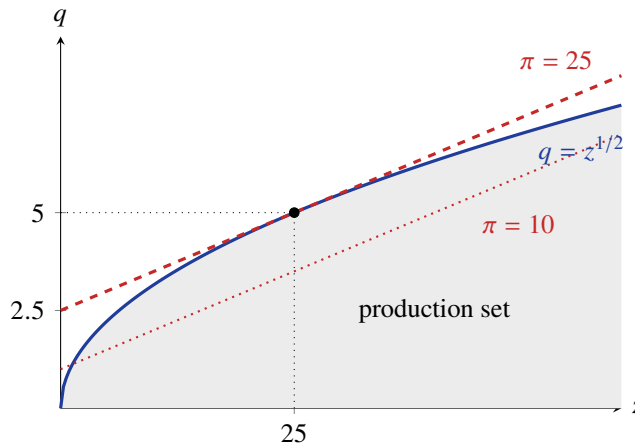


Figure 2: Profit maximization with one input ($p = 10$, $w = 1$). Isoprofit lines have slope $\frac{w}{p} = 0.1$.

2.2 Two inputs with decreasing returns

Example 4. Let $f(z_1, z_2) = z_1^{1/3} z_2^{1/3}$, which exhibits decreasing returns to scale ($\frac{1}{3} + \frac{1}{3} < 1$).

- Step 1, first-order conditions. Differentiating (4) with respect to each input,

$$\frac{p}{3} z_1^{-2/3} z_2^{1/3} = w_1 \quad \text{and} \quad \frac{p}{3} z_1^{1/3} z_2^{-2/3} = w_2. \quad (5)$$

- Step 2, tangency. Dividing the first condition by the second, $\frac{z_2}{z_1} = \frac{w_1}{w_2}$, so $z_2 = \frac{w_1}{w_2} z_1$: the firm spends the same amount on each input, $w_1 z_1 = w_2 z_2$.
- Step 3, solve for z_1 . Substituting z_2 into the first condition in (5),

$$\frac{p}{3} z_1^{-2/3} \left(\frac{w_1}{w_2} \right)^{1/3} z_1^{1/3} = w_1, \quad \text{that is,} \quad \frac{p}{3} \left(\frac{w_1}{w_2} \right)^{1/3} z_1^{-1/3} = w_1.$$

Solving for $z_1^{1/3}$ gives $z_1^{1/3} = \frac{p}{3w_1} \left(\frac{w_1}{w_2} \right)^{1/3}$, and cubing both sides,

$$z_1(p, w) = \frac{p^3}{27w_1^3} \cdot \frac{w_1}{w_2} = \frac{p^3}{27w_1^2 w_2}, \quad z_2(p, w) = \frac{w_1}{w_2} z_1(p, w) = \frac{p^3}{27w_1 w_2^2}. \quad (6)$$

- Step 4, supply. Since $z_1 z_2 = \frac{p^6}{27^2 w_1^3 w_2^3}$, output is $q = (z_1 z_2)^{1/3}$, that is,

$$q(p, w) = \frac{p^2}{9w_1 w_2}. \quad (7)$$

- Step 5, profits. Revenue is $pq = \frac{p^3}{9w_1 w_2} = \frac{3p^3}{27w_1 w_2}$ and each input costs $w_1 z_1 = w_2 z_2 = \frac{p^3}{27w_1 w_2}$, so

$$\pi(p, w) = \frac{3p^3}{27w_1 w_2} - \frac{2p^3}{27w_1 w_2} = \frac{p^3}{27w_1 w_2}. \quad (8)$$

In words, the firm spends one third of its revenue on each input and keeps the remaining third as profit, the shares given by the exponents of the production function.

A numerical example. With $p = 6$, $w_1 = 1$, and $w_2 = 2$, (6)–(8) give $z_1 = \frac{216}{54} = 4$, $z_2 = \frac{216}{108} = 2$, $q = \frac{36}{18} = 2$, which coincides with $(4 \cdot 2)^{1/3} = 2$, and $\pi = \frac{216}{54} = 4$, which coincides with $6(2) - 1(4) - 2(2) = 4$. Table 2 reports the solution at other prices.

p	w_1	w_2	$z_1(p, w)$	$z_2(p, w)$	$q(p, w)$	$\pi(p, w)$
6	1	2	4	2	2	4
9	1	2	13.5	6.75	4.5	13.5
12	1	2	32	16	8	32
6	2	2	1	1	1	2
12	2	4	4	2	2	8

Table 2. PMP with $f(z) = z_1^{1/3} z_2^{1/3}$.

The table illustrates several properties of the PMP:

- Law of supply. A higher output price increases output: $\frac{\partial q}{\partial p} = \frac{2p}{9w_1w_2} > 0$.
- Downward-sloping factor demands. A higher input price reduces the demand for that input: $\frac{\partial z_1}{\partial w_1} = -\frac{2p^3}{27w_1^3w_2} < 0$. In the table, raising w_1 from 1 to 2 cuts z_1 from 4 to 1.
- Homogeneity. Doubling all prices (last row) leaves $z(p, w)$ and $q(p, w)$ unchanged, since they are homogeneous of degree zero, and doubles profits, since $\pi(p, w)$ is homogeneous of degree one: $\frac{(2p)^3}{27(2w_1)(2w_2)} = \frac{8}{4}\pi(p, w) = 2\pi(p, w)$.
- Convexity. Profits are convex in p , since $\frac{\partial^2 \pi}{\partial p^2} = \frac{6p}{27w_1w_2} > 0$. In the table, profits rise from 4 to 13.5 to 32 as p increases from 6 to 9 to 12: each additional \$3 in the output price adds more profit than the previous one.

2.3 Constant returns: no interior solution

Example 5. Let $f(z_1, z_2) = z_1^{1/2}z_2^{1/2}$, which exhibits constant returns. If the firm doubles both inputs, it doubles output, revenue, and costs, so it doubles profits. Hence:

- If producing one unit of output at the lowest cost yields positive profits, scaling up raises profits without bound, and the PMP has no solution.
- If it yields negative profits, the firm produces zero.
- If it yields zero profits, every scale of production is optimal, and profits are zero.

Following the steps of Example 6 below, minimizing $w_1z_1 + w_2z_2$ subject to $z_1^{1/2}z_2^{1/2} = 1$ gives $z_1 = \left(\frac{w_2}{w_1}\right)^{1/2}$ and $z_2 = \left(\frac{w_1}{w_2}\right)^{1/2}$, so the lowest cost of producing one unit is $w_1z_1 + w_2z_2 = 2\sqrt{w_1w_2}$. With $w_1 = 1$ and $w_2 = 4$, this cost is 4, so the firm produces nothing if $p < 4$, is indifferent among all output levels if $p = 4$, and wants to expand without limit if $p > 4$. The first-order conditions cannot pin down the scale of production: they only determine the ratio $\frac{z_2}{z_1} = \frac{w_1}{w_2}$. With increasing returns, for instance $f(z) = z^2$, the problem is worse: $pz^2 - wz$ grows without bound for every $p > 0$, so the point where the first-order condition holds, $z = \frac{w}{2p}$, minimizes profits.

3 Cost minimization

For a given output level q , the firm solves

$$\min_{z_1, z_2 \geq 0} w_1z_1 + w_2z_2 \quad \text{subject to} \quad f(z_1, z_2) \geq q. \quad (9)$$

The solution gives the conditional factor demands $z_k(w, q)$, which depend on output rather than on the output price, and the cost function $c(w, q) = w_1z_1(w, q) + w_2z_2(w, q)$. At an interior solution, the isocost line is tangent to the isoquant, $MRTS_{1,2} = \frac{w_1}{w_2}$, and the Lagrange multiplier λ on the output constraint equals the marginal cost, $\lambda = \frac{\partial c(w, q)}{\partial q}$.

3.1 Cobb–Douglas

Example 6. Return to $f(z_1, z_2) = z_1^{1/3}z_2^{1/3}$ from Example 4.

- Step 1, tangency. Since $MRTS_{1,2} = \frac{z_2}{z_1}$, the condition $MRTS_{1,2} = \frac{w_1}{w_2}$ gives $z_2 = \frac{w_1}{w_2} z_1$, the same relationship as in the PMP.
- Step 2, output constraint. Substituting into $z_1^{1/3} z_2^{1/3} = q$, and cubing both sides, $z_1 \cdot \frac{w_1}{w_2} z_1 = q^3$, that is, $z_1^2 = \frac{w_2}{w_1} q^3$.
- Step 3, conditional factor demands. Taking square roots,

$$z_1(w, q) = q^{3/2} \left(\frac{w_2}{w_1} \right)^{1/2} \quad \text{and} \quad z_2(w, q) = \frac{w_1}{w_2} z_1(w, q) = q^{3/2} \left(\frac{w_1}{w_2} \right)^{1/2}. \quad (10)$$

- Step 4, cost function. Since $w_1 z_1 = q^{3/2} (w_1 w_2)^{1/2}$ and $w_2 z_2 = q^{3/2} (w_1 w_2)^{1/2}$,

$$c(w, q) = 2q^{3/2} (w_1 w_2)^{1/2}. \quad (11)$$

- Step 5, marginal cost. Differentiating with respect to q , $\lambda = \frac{\partial c}{\partial q} = 3q^{1/2} (w_1 w_2)^{1/2}$, which increases in q because of decreasing returns.

With $w_1 = 1$, $w_2 = 2$, and $q = 2$, we obtain $z_1 = 2^{3/2} \cdot 2^{1/2} = 4$, $z_2 = 2^{3/2} \cdot 2^{-1/2} = 2$, $c = 2 \cdot 2^{3/2} \cdot 2^{1/2} = 8$, and marginal cost $3 \cdot 2^{1/2} \cdot 2^{1/2} = 6$. These are the input choices of Example 4 at $p = 6$: the profit-maximizing firm minimizes the cost of the output it chooses, and it chooses the output at which marginal cost equals the price, 6. Figure 3 depicts the tangency between the isoquant $z_1 z_2 = 8$ (that is, $q = 2$) and the isocost line $z_1 + 2z_2 = 8$.

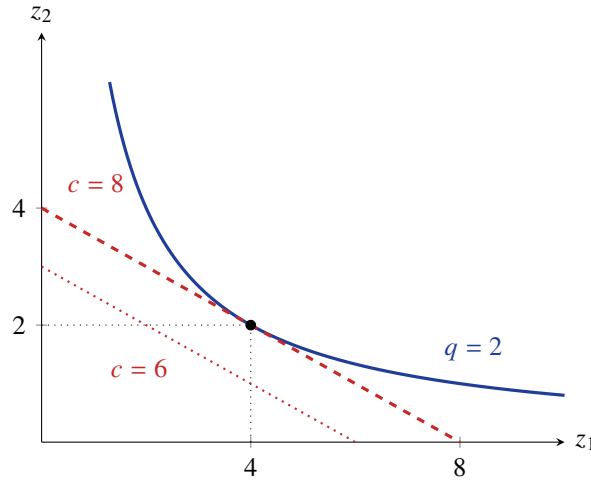


Figure 3: Cost minimization with $f(z) = z_1^{1/3} z_2^{1/3}$, $w_1 = 1$, $w_2 = 2$, and $q = 2$. The isocost line through $(4, 2)$ has slope $-\frac{w_1}{w_2} = -\frac{1}{2}$; a cheaper isocost line, such as $c = 6$, does not reach the isoquant.

The solution satisfies the properties of the CMP:

- Shephard's lemma. $\frac{\partial c}{\partial w_1} = 2q^{3/2} \cdot \frac{1}{2} w_1^{-1/2} w_2^{1/2} = q^{3/2} \left(\frac{w_2}{w_1} \right)^{1/2} = z_1(w, q)$, and similarly for input 2.
- Homogeneity. $c(w, q)$ is homogeneous of degree one in w , since $(tw_1 \cdot tw_2)^{1/2} = t(w_1 w_2)^{1/2}$, while $z(w, q)$ is homogeneous of degree zero in w , since it depends only on the ratio $\frac{w_2}{w_1}$.
- Concavity in w and negative own-price effects. $\frac{\partial z_1(w, q)}{\partial w_1} = -\frac{1}{2} q^{3/2} w_2^{1/2} w_1^{-3/2} < 0$.

- Symmetric cross-price effects. $\frac{\partial z_1(w, q)}{\partial w_2} = \frac{1}{2} q^{3/2} (w_1 w_2)^{-1/2} = \frac{\partial z_2(w, q)}{\partial w_1}$. Both are positive: holding output fixed, inputs are substitutes.

3.2 Other technologies

Example 7 (fixed proportions). Let $f(z_1, z_2) = \min\{z_1, 2z_2\}$. The firm never wastes an input, so it sets $z_1 = 2z_2 = q$. Hence $z_1(w, q) = q$, $z_2(w, q) = \frac{q}{2}$, and

$$c(w, q) = \left(w_1 + \frac{w_2}{2}\right) q.$$

The conditional demands do not respond to input prices: with no substitution possibilities, $\frac{\partial z_k(w, q)}{\partial w_l} = 0$ for all inputs k and l . Shephard's lemma still holds, since $\frac{\partial c}{\partial w_1} = q = z_1(w, q)$.

Example 8 (linear). Let $f(z_1, z_2) = 2z_1 + z_2$. Each unit of input 1 produces two units of output, so producing one unit of output with input 1 alone costs $\frac{w_1}{2}$, while producing it with input 2 alone costs w_2 . The firm uses only the cheaper input:

- If $\frac{w_1}{2} < w_2$, then $z_1(w, q) = \frac{q}{2}$ and $z_2(w, q) = 0$.
- If $\frac{w_1}{2} > w_2$, then $z_1(w, q) = 0$ and $z_2(w, q) = q$.
- If $\frac{w_1}{2} = w_2$, every input vector with $2z_1 + z_2 = q$ is optimal.

In all cases, $c(w, q) = q \min\left\{\frac{w_1}{2}, w_2\right\}$. This cost function has a kink where $w_1 = 2w_2$, the price ratio at which the conditional demand correspondence contains many input vectors and Shephard's lemma does not apply.

Example 9 (CES). Let $f(z_1, z_2) = \left(z_1^{1/2} + z_2^{1/2}\right)^2$, so $\rho = \frac{1}{2}$ and $\sigma = 2$.

- Step 1, tangency. From Example 2, $MRTS_{1,2} = \left(\frac{z_2}{z_1}\right)^{1/2} = \frac{w_1}{w_2}$, so $z_2^{1/2} = \frac{w_1}{w_2} z_1^{1/2}$.
- Step 2, output constraint. Substituting, $q = \left(z_1^{1/2} + \frac{w_1}{w_2} z_1^{1/2}\right)^2 = z_1 \left(\frac{w_1 + w_2}{w_2}\right)^2$.
- Step 3, conditional factor demands. Solving for z_1 , and using $z_2 = \left(\frac{w_1}{w_2}\right)^2 z_1$,

$$z_1(w, q) = \frac{w_2^2}{(w_1 + w_2)^2} q \quad \text{and} \quad z_2(w, q) = \frac{w_1^2}{(w_1 + w_2)^2} q.$$

- Step 4, cost function. $c(w, q) = \frac{w_1 w_2^2 + w_2 w_1^2}{(w_1 + w_2)^2} q = \frac{w_1 w_2 (w_1 + w_2)}{(w_1 + w_2)^2} q = \frac{w_1 w_2}{w_1 + w_2} q$.

With $w_1 = 1$, $w_2 = 3$, and $q = 16$: $z_1 = \frac{9}{16} \cdot 16 = 9$, $z_2 = \frac{1}{16} \cdot 16 = 1$, output is $(3 + 1)^2 = 16$, and $c = 9 + 3 = 12 = \frac{3}{4} \cdot 16$.

3.3 Comparing the technologies

The elasticity of substitution determines how strongly the cost-minimizing input mix responds to input prices. Table 3 reports the ratio $\frac{z_1}{z_2}$ chosen by each constant-returns technology at different price ratios. (With constant returns, the ratio does not depend on q .)

Technology	σ	$\frac{w_1}{w_2} = \frac{1}{2}$	$\frac{w_1}{w_2} = 1$	$\frac{w_1}{w_2} = \frac{3}{2}$	$\frac{w_1}{w_2} = 3$
Linear, $2z_1 + z_2$	$+\infty$	only z_1	only z_1	only z_1	only z_2
CES, $\left(z_1^{1/2} + z_2^{1/2}\right)^2$	2	4	1	0.44	0.11
Cobb–Douglas, $z_1^{1/2} z_2^{1/2}$	1	2	1	0.67	0.33
Fixed proportions, $\min\{z_1, 2z_2\}$	0	2	2	2	2

Table 3. Cost-minimizing input ratio $\frac{z_1}{z_2}$.

For the CES technology, $\frac{z_1}{z_2} = \left(\frac{w_2}{w_1}\right)^2$, so multiplying the price ratio by 6 (from $\frac{1}{2}$ to 3) divides the input ratio by $6^2 = 36$ (from 4 to 0.11). For the Cobb–Douglas technology, $\frac{z_1}{z_2} = \frac{w_2}{w_1}$, so the same price change divides the input ratio by 6 (from 2 to 0.33). In general, the input ratio is divided by 6^σ .

3.4 Substitution and output effects

When w_1 increases, a profit-maximizing firm changes its use of input 1 for two reasons. Holding output fixed, it substitutes away from input 1 (substitution effect). Since its marginal cost increases, it also produces less (output effect). Formally, since $z_1(p, w) = z_1(w, q(p, w))$,

$$\frac{\partial z_1(p, w)}{\partial w_1} = \underbrace{\frac{\partial z_1(w, q)}{\partial w_1}}_{\text{substitution effect}} + \underbrace{\frac{\partial z_1(w, q)}{\partial q} \cdot \frac{\partial q(p, w)}{\partial w_1}}_{\text{output effect}}. \quad (12)$$

Example 10. Take the technology of Examples 4 and 6, with $p = 6$ and $w_2 = 2$, and let w_1 increase from 1 to 2.

- Initial choice, point A. From Table 2, $z_1 = 4$, $z_2 = 2$, and $q = 2$.
- Substitution, point B. Holding output at $q = 2$, the conditional demands (10) at $w_1 = 2$ are $z_1(w, 2) = 2^{3/2} \left(\frac{2}{2}\right)^{1/2} \approx 2.83$ and $z_2(w, 2) \approx 2.83$.
- Final choice, point C. From (6) and (7) at $w_1 = 2$, $z_1 = \frac{216}{27 \cdot 4 \cdot 2} = 1$, $z_2 = \frac{216}{27 \cdot 2 \cdot 4} = 1$, and $q = \frac{36}{9 \cdot 4} = 1$.

Table 4 reports the decomposition, and Figure 4 depicts it.

Input	Substitution effect (A to B)	Output effect (B to C)	Total effect (A to C)
z_1	$2.83 - 4 = -1.17$	$1 - 2.83 = -1.83$	-3
z_2	$2.83 - 2 = +0.83$	$1 - 2.83 = -1.83$	-1

Table 4. Effects of an increase in w_1 from 1 to 2 ($p = 6$, $w_2 = 2$).

Both effects reduce the demand for input 1, so the unconditional demand for input 1 responds more to its price than the conditional demand. For input 2, the two effects have opposite signs: the firm substitutes toward input 2, but it also produces less. In this example, the output effect dominates, so the firm ends up using less of input 2 even though input 1 became relatively more expensive.

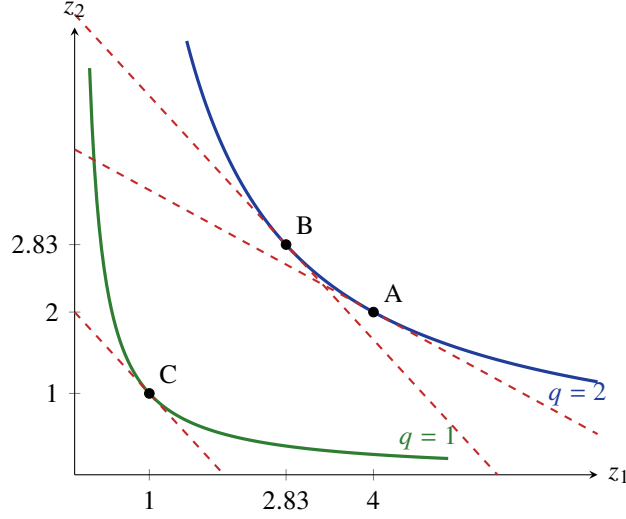


Figure 4: Substitution effect (A to B, along the isoquant $q = 2$) and output effect (B to C) of an increase in w_1 from 1 to 2. Dashed lines are isocost lines, with slope $-\frac{1}{2}$ at A and -1 at B and C.

4 Duality

The cost and profit functions contain all the information about the technology that matters for a price-taking firm. This section shows how to move from one object to another.

4.1 From the cost function to the factor demands and the technology

Example 11. Suppose we only know the cost function $c(w, q) = 2q^{3/2}(w_1w_2)^{1/2}$.

- Step 1, conditional factor demands from Shephard's lemma. $z_1(w, q) = \frac{\partial c}{\partial w_1} = q^{3/2} \left(\frac{w_2}{w_1} \right)^{1/2}$ and $z_2(w, q) = \frac{\partial c}{\partial w_2} = q^{3/2} \left(\frac{w_1}{w_2} \right)^{1/2}$.
- Step 2, eliminate prices. Multiplying the two demands, the price ratios cancel: $z_1z_2 = q^3$.
- Step 3, recover the technology. Solving for q , $q = (z_1z_2)^{1/3}$, the production function of Example 4.

A second route uses the definition of the cost function: an input vector z produces at least q only if, at every price vector w , it costs at least the minimum cost of q , that is, $w_1z_1 + w_2z_2 \geq c(w, q)$. Hence $f(z)$ is the largest q that satisfies this inequality for every w . For $z = (4, 2)$, the inequality becomes $4w_1 + 2w_2 \geq 2q^{3/2}(w_1w_2)^{1/2}$. Dividing both sides by $2(w_1w_2)^{1/2}$ and writing $r \equiv \frac{w_1}{w_2}$, it reads $2r^{1/2} + r^{-1/2} \geq q^{3/2}$. The left side is smallest at $r = \frac{1}{2}$, where its derivative, $r^{-1/2} - \frac{1}{2}r^{-3/2}$, equals zero, and where it equals $2 \cdot \frac{1}{\sqrt{2}} + \sqrt{2} = 2\sqrt{2}$. Hence $q^{3/2} \leq 2\sqrt{2} = 2^{3/2}$, so $q \leq 2$, and $f(4, 2) = 2$, as $(4 \cdot 2)^{1/3} = 2$ confirms.

4.2 From the cost function to supply and profits

Example 12. With the cost function, the PMP becomes a problem in a single variable, $\max_{q \geq 0} pq - c(w, q)$.

- Step 1, price equals marginal cost. With $c(w, q) = 2q^{3/2}(w_1 w_2)^{1/2}$, the first-order condition is $p = 3q^{1/2}(w_1 w_2)^{1/2}$. Marginal cost increases in q , so the second-order condition holds.
- Step 2, supply. Squaring both sides, $p^2 = 9q w_1 w_2$, so $q(p, w) = \frac{p^2}{9w_1 w_2}$, as in (7).
- Step 3, profits. Substituting $q(p, w)$, we have $q^{3/2} = \frac{p^3}{27(w_1 w_2)^{3/2}}$, so $c = \frac{2p^3}{27w_1 w_2}$ and $\pi = \frac{p^3}{9w_1 w_2} - \frac{2p^3}{27w_1 w_2} = \frac{p^3}{27w_1 w_2}$, as in (8).
- Step 4, unconditional factor demands. Substituting $q(p, w)$ into the conditional demands, $z_1(p, w) = z_1(w, q(p, w)) = \frac{p^3}{27(w_1 w_2)^{3/2}} \cdot \frac{w_2^{1/2}}{w_1^{1/2}} = \frac{p^3}{27w_1^2 w_2}$, as in (6).

4.3 From the profit function to supply and factor demands

Hotelling's lemma states that the derivatives of the profit function give supply and (minus) the factor demands: $\frac{\partial \pi}{\partial p} = q(p, w)$ and $\frac{\partial \pi}{\partial w_k} = -z_k(p, w)$.

Example 13. With $\pi(p, w) = \frac{p^3}{27w_1 w_2}$,

- $\frac{\partial \pi}{\partial p} = \frac{3p^2}{27w_1 w_2} = \frac{p^2}{9w_1 w_2} = q(p, w)$;
- $\frac{\partial \pi}{\partial w_1} = -\frac{p^3}{27w_1^2 w_2} = -z_1(p, w)$, and $\frac{\partial \pi}{\partial w_2} = -\frac{p^3}{27w_1 w_2^2} = -z_2(p, w)$.

The cross-price effects are symmetric, $\frac{\partial z_1(p, w)}{\partial w_2} = -\frac{p^3}{27w_1^2 w_2^2} = \frac{\partial z_2(p, w)}{\partial w_1}$, since both equal $-\frac{\partial^2 \pi}{\partial w_1 \partial w_2}$. Both are negative: once we account for the output effect, a higher price of either input reduces the demand for the other one (Table 4 shows the same pattern). For Example 3, $\pi(p, w) = \frac{p^2}{4w}$ gives $\frac{\partial \pi}{\partial p} = \frac{p}{2w} = q$ and $\frac{\partial \pi}{\partial w} = -\frac{p^2}{4w^2} = -z$.

Table 5 collects the dual relationships.

	PMP	CMP
Choice	$z_k(p, w), q(p, w)$	$z_k(w, q)$
Value function	$\pi(p, w)$	$c(w, q)$
Derivative lemma	Hotelling: $\frac{\partial \pi}{\partial p} = q, \frac{\partial \pi}{\partial w_k} = -z_k(p, w)$	Shephard: $\frac{\partial c}{\partial w_k} = z_k(w, q)$
Homogeneity	π degree 1 in (p, w) ; z, q degree 0	c degree 1 in w ; z degree 0 in w
Curvature	π convex in (p, w)	c concave in w
Link	$\pi(p, w) = \max_q pq - c(w, q)$ and $z_k(p, w) = z_k(w, q(p, w))$	

Table 5. Duality between the PMP and the CMP.

4.4 Is it a cost function?

A function $c(w, q)$ can be the cost function of some technology only if it is weakly increasing in w , homogeneous of degree one in w , and concave in w . These properties provide a quick test:

- $c(w, q) = q(w_1 + w_2 + \sqrt{w_1 w_2})$ passes the test: it increases in each price, $c(tw, q) = t c(w, q)$, and $\sqrt{w_1 w_2}$ is concave. Shephard's lemma gives $z_1(w, q) = q \left(1 + \frac{1}{2} \sqrt{\frac{w_2}{w_1}}\right)$, which decreases in w_1 , as it should.
- $c(w, q) = q \frac{w_1^2}{w_2}$ is homogeneous of degree one, $\frac{(tw_1)^2}{tw_2} = t \frac{w_1^2}{w_2}$, but it decreases in w_2 , which would mean $z_2(w, q) = -q \frac{w_1^2}{w_2^2} < 0$. It is not a cost function.
- $c(w, q) = q \sqrt{w_1^2 + w_2^2}$ increases in prices and is homogeneous of degree one, but it is convex in w : Shephard's lemma would give $z_1(w, q) = \frac{q w_1}{\sqrt{w_1^2 + w_2^2}}$, which increases in w_1 . A firm would never use more of an input when its price rises and output is held fixed, so this is not a cost function either.

5 Cost curves and supply

Fixing input prices, we can write the cost function as $c(q)$, with average cost $AC(q) = \frac{c(q)}{q}$ and marginal cost $MC(q) = c'(q)$.

Example 14 (decreasing returns). With $w_1 = 1$ and $w_2 = 2$, the cost function of Example 6 becomes $c(q) = 2\sqrt{2} q^{3/2}$. Then $AC(q) = 2\sqrt{2} q^{1/2}$ and $MC(q) = 3\sqrt{2} q^{1/2}$, so $MC(q) > AC(q)$ for every $q > 0$ and both curves increase. The firm supplies the output at which $p = MC(q)$, that is, $q = \frac{p^2}{18}$, which coincides with (7) at $w_1 w_2 = 2$.

Example 15 (fixed cost). Let $c(q) = 50 + 2q^2$ for $q > 0$, where the fixed cost of 50 can be avoided by producing zero. Then:

- $AC(q) = \frac{50}{q} + 2q$ and $MC(q) = 4q$.
- Minimum average cost. Setting $AC'(q) = -\frac{50}{q^2} + 2 = 0$ gives $q = 5$, where $AC(5) = 10 + 10 = 20 = MC(5)$: the marginal cost curve crosses the average cost curve at its minimum.
- Supply. For $p \geq 20$, the firm produces where $p = 4q$, that is, $q = \frac{p}{4}$, and its profits are $\frac{p^2}{4} - 50 - \frac{p^2}{8} = \frac{p^2}{8} - 50 \geq 0$. For $p < 20$, profits from any positive output are negative, and the firm shuts down. At $p = 20$, the firm is indifferent between $q = 0$ and $q = 5$.

If the fixed cost were sunk, the firm would pay it anyway, so it would produce $q = \frac{p}{4}$ at every price $p > 0$. Figure 5 depicts both curves and the supply curve.

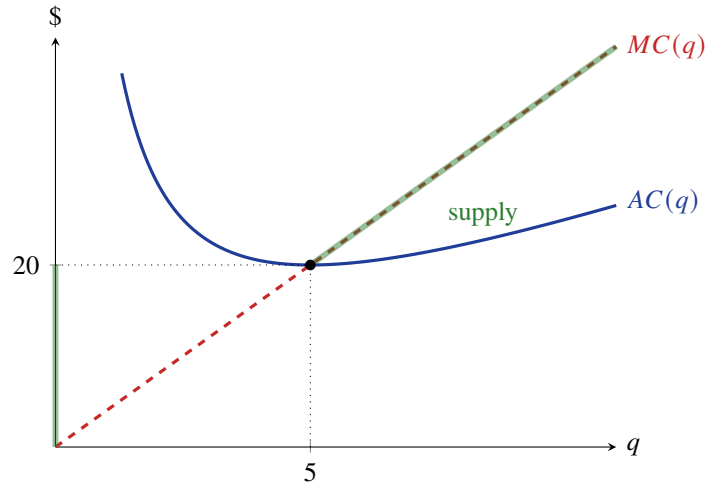


Figure 5: Cost curves for $c(q) = 50 + 2q^2$. With an avoidable fixed cost, the supply curve (shaded) is the vertical axis for $p < 20$ and the marginal cost curve above the minimum of AC .

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