Beyond Expected Utility: Weighted Utility, Rank-Dependent Utility, and Prospect Theory

EconS 501, a handout on Sections 5.4 and 5.9

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This handout develops Sections 5.4 (behavioral theories that modify expected utility theory) and 5.9 (prospect theory and reference-dependent utility) of the textbook in more detail. For each theory, we present the formula that evaluates a lottery, explain how it departs from expected utility, and work through parametric examples, figures, and tables. Throughout, a lottery L assigns probability p_i to outcome x_i, for $i = 1, \ldots, n$.

1 Why modify expected utility?

Under expected utility, the value of lottery L is

$$U(L) = \sum_{i=1}^{n} p_i \, u(x_i), \tag{1}$$

which is linear in probabilities. This linearity is a direct consequence of the independence axiom (IA), and it is what experiments such as the Allais paradox call into question.

The Allais paradox. Consider three outcomes, \$2.5 million, \$0.5 million, and \$0, and the two pairs of lotteries

$$L_1 = (0, 1, 0), \quad L_1' = (0.10, 0.89, 0.01), \qquad L_2 = (0, 0.11, 0.89), \quad L_2' = (0.10, 0, 0.90), \tag{2}$$

where each vector lists the probabilities of \$2.5 million, \$0.5 million, and \$0, respectively. Many subjects choose L_1 over L_1' and L_2' over L_2. To see why expected utility cannot rationalize both choices, write $u_{2.5}$, $u_{0.5}$, and u_0 for the utilities of the three outcomes. Then $L_1 \succ L_1'$ requires $u_{0.5} > 0.10\,u_{2.5} + 0.89\,u_{0.5} + 0.01\,u_0$. Subtracting $0.89\,u_{0.5}$ from both sides and adding $0.89\,u_0$ to both sides,

$$0.11\,u_{0.5} + 0.89\,u_0 > 0.10\,u_{2.5} + 0.90\,u_0, \tag{3}$$

which says that $L_2 \succ L_2'$. Hence, any individual with expected utility preferences who chooses L_1 must also choose L_2, whatever her utility function. The theories below relax the IA, so that the way probabilities enter the evaluation of a lottery is no longer linear.

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2 Weighted utility

2.1 The formula

Weighted utility theory (Chew, 1983) evaluates lottery L as

$$V(L) = \sum_{i=1}^{n} w_i \, u(x_i), \qquad \text{where} \qquad w_i = \frac{g(x_i)\, p_i}{\sum_{j=1}^{n} g(x_j)\, p_j}, \tag{4}$$

and $g(\cdot) > 0$ is a weight function defined on outcomes. Each decision weight w_i depends on the probability p_i and on the outcome x_i itself. The denominator ensures that the weights sum to one, $\sum_i w_i = 1$, so $V(L)$ is a weighted average of utilities.

- If g is constant, $g(x_i) = g$ for all i, then $w_i = \frac{g\, p_i}{g \sum_j p_j} = p_i$, and (4) collapses to expected utility (1).
- If g is larger for bad outcomes than for good outcomes, the individual inflates the weight of bad outcomes relative to their probabilities, which captures pessimism.

With two outcomes, x_1 with probability p and x_2 with probability $1 - p$, (4) becomes

$$V(L) = \frac{g(x_1)\, p}{g(x_1)\, p + g(x_2)(1-p)}\, u(x_1) + \frac{g(x_2)(1-p)}{g(x_1)\, p + g(x_2)(1-p)}\, u(x_2). \tag{5}$$

2.2 The axiom behind it

Weighted utility keeps all expected utility axioms except the IA, which it replaces with the following weaker condition.

Weak independence axiom. If $L_1 \sim L_2$, then for every $\alpha \in (0, 1)$ there is an $\alpha' \in (0, 1)$ such that

$$\alpha L_1 + (1-\alpha) L_3 \sim \alpha' L_2 + (1-\alpha') L_3 \qquad \text{for every lottery } L_3. \tag{6}$$

The IA is the special case $\alpha' = \alpha$. Under weak independence, mixing two indifferent lotteries with a third lottery may require different mixing probabilities to preserve indifference, but the same α' must work for every L_3.

2.3 A parametric example

Consider a lottery that pays \$100 with probability p and \$0 with probability $1 - p$, an individual with linear utility, $u(x) = x$, and the weight function $g(0) = 2$ and $g(100) = 1$, so the bad outcome receives twice the weight of the good outcome. From (5), the decision weight on \$100 is

$$w(100) = \frac{1 \cdot p}{1 \cdot p + 2(1-p)} = \frac{p}{2-p}, \tag{7}$$

and the value of the lottery, which is also its certainty equivalent since u is linear, is $V(L) = 100\, \frac{p}{2-p}$. Table 1 compares it with the expected value, $100p$.

Although the individual is risk neutral in the sense of having a linear u, she values the lottery below its expected value at every $p \in (0, 1)$. Indeed, $\frac{p}{2-p} < p$ if and only if $1 < 2 - p$ (dividing both sides by $p > 0$ and rearranging), that is, $p < 1$. The extra weight on the bad outcome works like risk aversion.

Probability of \$100, p	Decision weight $w(100)$	Value $V(L)$	Expected value
0.10	0.053	5.26	10
0.25	0.143	14.29	25
0.50	0.333	33.33	50
0.75	0.600	60.00	75
0.90	0.818	81.82	90

Table 1. Weighted utility with $u(x) = x$, $g(0) = 2$, and $g(100) = 1$.

3 Rank-dependent utility

3.1 The formula

Rank-dependent utility (Quiggin, 1982) also distorts probabilities, but the distortion depends on the rank of each outcome rather than on its value. First, order the outcomes from worst to best, $x_1 < x_2 < \cdots < x_n$. Second, apply a transformation function $\pi : [0, 1] \to [0, 1]$, increasing with $\pi(0) = 0$ and $\pi(1) = 1$, to the probability of receiving a payoff at least as good as each outcome. The decision weight of outcome x_i is

$$w_i = \pi(p_i + p_{i+1} + \cdots + p_n) - \pi(p_{i+1} + \cdots + p_n) \quad \text{for } i < n, \qquad w_n = \pi(p_n), \tag{8}$$

and the lottery is evaluated as

$$V(L) = \sum_{i=1}^{n} w_i \, u(x_i). \tag{9}$$

In (8), $\pi(p_i+\cdots+p_n)$ is the subjective weight of receiving a payoff at least as good as x_i, and $\pi(p_{i+1}+\cdots+p_n)$ is the subjective weight of receiving a payoff strictly better than x_i. Their difference is the weight of receiving exactly x_i. Two properties follow:

- The weights sum to one. Adding (8) over i, the terms cancel in pairs (a telescoping sum), leaving $\pi(p_1 + \cdots + p_n) = \pi(1) = 1$.
- If $\pi(p) = p$, then $w_i = p_i$, and (9) collapses to expected utility.

With two outcomes, $x_1 < x_2$, where x_1 occurs with probability p, (8) gives $w_2 = \pi(1 - p)$ and $w_1 = \pi(1) - \pi(1 - p) = 1 - \pi(1 - p)$, so

$$V(L) = [1 - \pi(1 - p)] \, u(x_1) + \pi(1 - p) \, u(x_2). \tag{10}$$

The textbook writes this expression as $w(p)u(x_1) + [1 - w(p)] \, u(x_2)$, where $w(p) \equiv 1 - \pi(1 - p)$ is the decision weight on the worse outcome.

3.2 Pessimism and optimism

Since π is applied to the probability of receiving *good* outcomes, its shape determines whether the individual inflates good or bad outcomes:

- Pessimism: a convex π, with $\pi(p) \leq p$, shrinks the weight of good outcomes and shifts it to the worst outcomes.

- Optimism: a concave π, with $\pi(p) \geq p$, inflates the weight of good outcomes.

A three-outcome example. Consider a lottery that pays \$0, \$50, or \$100 with probability $\frac{1}{3}$ each, and let $u(x) = x$. The probability of receiving at least \$100 is $\frac{1}{3}$, and the probability of receiving at least \$50 is $\frac{2}{3}$. With the pessimistic function $\pi(p) = p^2$, (8) gives

$$w_3 = \pi\left(\tfrac{1}{3}\right) = \tfrac{1}{9}, \qquad w_2 = \pi\left(\tfrac{2}{3}\right) - \pi\left(\tfrac{1}{3}\right) = \tfrac{4}{9} - \tfrac{1}{9} = \tfrac{3}{9}, \qquad w_1 = 1 - \pi\left(\tfrac{2}{3}\right) = \tfrac{5}{9}, \tag{11}$$

so $V(L) = \frac{5}{9}(0) + \frac{3}{9}(50) + \frac{1}{9}(100) = \frac{250}{9} \approx 27.8$, well below the expected value of 50. Table 2 repeats the calculation for other transformation functions, including the inverse-S function of Prelec (1998), $\pi(p) = e^{-(-\ln p)^{\alpha}}$, discussed below. Each row requires only $\pi(\frac{1}{3})$ and $\pi(\frac{2}{3})$, since $w_3 = \pi(\frac{1}{3})$, $w_2 = \pi(\frac{2}{3}) - \pi(\frac{1}{3})$, and $w_1 = 1 - \pi(\frac{2}{3})$:

- Optimistic, $\pi(p) = \sqrt{p}$: $\pi(\frac{1}{3}) \approx 0.577$ and $\pi(\frac{2}{3}) \approx 0.816$, so $w_3 \approx 0.577$, $w_2 \approx 0.816 - 0.577 = 0.239$, and $w_1 \approx 1 - 0.816 = 0.184$.
- Prelec with $\alpha = \frac{1}{2}$: $-\ln\frac{1}{3} = \ln 3 \approx 1.099$, so $\pi(\frac{1}{3}) = e^{-\sqrt{1.099}} \approx e^{-1.048} \approx 0.351$; and $-\ln\frac{2}{3} = \ln 1.5 \approx 0.405$, so $\pi(\frac{2}{3}) = e^{-\sqrt{0.405}} \approx e^{-0.637} \approx 0.529$. Hence $w_3 \approx 0.351$, $w_2 \approx 0.529 - 0.351 = 0.178$, and $w_1 \approx 1 - 0.529 = 0.471$.

Transformation $\pi(p)$	w_1 (\$0)	w_2 (\$50)	w_3 (\$100)	$V(L)$
p (expected utility)	0.333	0.333	0.333	50.0
p^2 (pessimistic)	0.556	0.333	0.111	27.8
$\sqrt{p}$ (optimistic)	0.184	0.239	0.577	69.7
$e^{-(-\ln p)^{1/2}}$ (Prelec, $\alpha = \frac{1}{2}$)	0.471	0.178	0.351	44.0

Table 2. Rank-dependent decision weights for a lottery over \$0, \$50, and \$100.

The Prelec row shows the typical pattern of the inverse-S function: both extreme outcomes, \$0 and \$100, receive more weight than their probability $\frac{1}{3}$, while the middle outcome receives less.

3.3 Inverse-S transformation functions

Experimental evidence does not support uniform pessimism or optimism. Instead, individuals tend to overweight small probabilities and underweight large ones, which corresponds to an inverse-S (concave, then convex) transformation function. A common example is the Prelec (1998) function

$$\pi(p) = e^{-(-\ln p)^{\alpha}}, \qquad \alpha \in (0, 1). \tag{12}$$

The minus sign in the exponent matters: since $-\ln p \geq 0$, the exponent is not positive, so $\pi(p) \in [0, 1]$. Besides $\pi(0) = 0$ and $\pi(1) = 1$, the function crosses the 45-degree line at an interior point $\bar{p}$ that does not depend on α. Indeed, taking logs of both sides of $\pi(p) = p$ gives $-(-\ln p)^{\alpha} = \ln p$, that is, $(-\ln p)^{\alpha} = -\ln p$. Writing $z \equiv -\ln p$, which is positive for $p \in (0, 1)$, this condition is $z^{\alpha} = z$, or $z^{\alpha-1} = 1$. For $\alpha \neq 1$, it holds only if $z = 1$, that is, $-\ln p = 1$, which yields

$$\bar{p} = \frac{1}{e} \approx 0.368. \tag{13}$$

Below $\bar{p}$, $\pi(p) > p$ (small probabilities are overweighted), and above $\bar{p}$, $\pi(p) < p$ (large probabilities are underweighted). Figure 1 depicts the transformation functions used in Table 2.

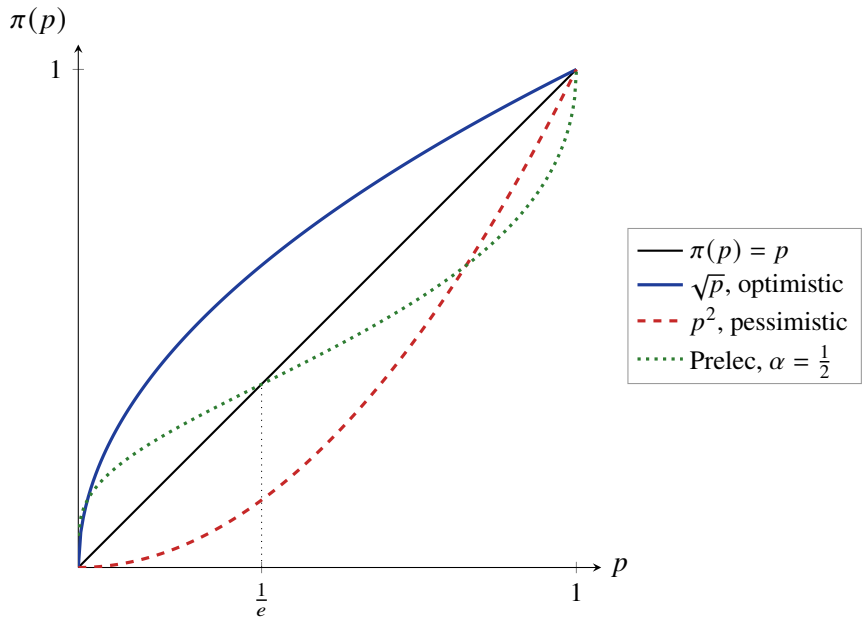


Figure 1. Transformation functions in rank-dependent utility.

Lottery tickets and insurance. Consider a lottery that pays \$100 with probability p and \$0 otherwise, and let $u(x) = x$. In (10), the worse outcome is $x_1 = 0$, which occurs with probability $1 - p$, so the weight on \$100 is $\pi(1 - (1 - p)) = \pi(p)$. Hence the value of the lottery, which equals its certainty equivalent, is $V(L) = [1 - \pi(p)] \cdot 0 + \pi(p) \cdot 100 = 100\,\pi(p)$. For instance, with Prelec's function and $p = 0.01$, $-\ln 0.01 \approx 4.605$, so $\pi(0.01) = e^{-\sqrt{4.605}} \approx e^{-2.146} \approx 0.117$, and $V(L) \approx 11.70$. Table 3 reports this certainty equivalent for several values of p.

p	Expected value	$\pi(p) = p^2$	$\pi(p) = \sqrt{p}$	Prelec, $\alpha = \frac{1}{2}$	TK, $\beta = 0.61$
0.01	1.00	0.01	10.00	11.70	5.53
0.05	5.00	0.25	22.36	17.71	13.16
0.10	10.00	1.00	31.62	21.93	18.63
0.368	36.79	13.53	60.65	36.79	35.37
0.50	50.00	25.00	70.71	43.49	42.06
0.90	90.00	81.00	94.87	72.28	71.17
0.99	99.00	98.01	99.50	90.46	91.16

Table 3. Certainty equivalent of a lottery paying \$100 with probability p.

The last column uses the Tversky–Kahneman weighting function introduced in Section 4. With an inverse-S function, the individual values a long-shot lottery ($p = 0.01$) well above its expected value (more than ten times with the Prelec function), which helps explain the demand for lottery tickets, while she values a likely gain ($p = 0.99$) well below its expected value. Applied to losses, the same overweighting of small probabilities helps explain the demand for insurance against unlikely losses.

3.4 Properties

Rank-dependent utility keeps all expected utility axioms except the IA, which it replaces with a weaker condition called comonotonic independence: the IA only needs to hold for lotteries whose outcomes are ranked in the same order. Because the weights in (8) are computed from the probabilities of receiving *at least* each outcome, shifting probability from a worse outcome to a better one never lowers $V(L)$. Hence rank-dependent utility respects first-order stochastic dominance, an advantage over the original version of prospect theory discussed next, which applies its weighting function to each probability separately and can therefore violate first-order stochastic dominance.

4 Prospect theory

4.1 The three ingredients

Kahneman and Tversky (1979) evaluate a lottery with outcomes x_i and probabilities p_i as

$$V(x, p) = \sum_{i=1}^{n} w(p_i)\, v(x_i). \tag{14}$$

Compared with expected utility, prospect theory departs in three ways:

- Probability weighting: $w(p_i) \neq p_i$. When $w(p_i) > p_i$, the individual overweights outcome x_i; when $w(p_i) < p_i$, she underweights it. The function satisfies $w(0) = 0$ and $w(1) = 1$, so outcomes that are impossible or certain are not distorted.
- Reference dependence: outcomes are evaluated as gains or losses relative to a reference point x_0, which we normalize to $x_0 = 0$.
- The shape of the value function v: concave for gains (risk aversion over gains), convex for losses (risk loving over losses), and steeper for losses than for gains (loss aversion).

4.2 Common functional forms

A common value function is

$$v(x) = \begin{cases} x^{\alpha} & \text{if } x \geq 0, \\ -\lambda(-x)^{\alpha} & \text{if } x < 0, \end{cases} \tag{15}$$

where $\alpha \in (0, 1]$ measures the concavity over gains (and convexity over losses), and $\lambda \geq 1$ measures loss aversion. With $\lambda = 1$, a gain and a loss of the same size have the same absolute effect; with $\lambda > 1$, the loss hurts more. With $\alpha = 1$, the value function consists of two linear segments, with slope 1 for gains and slope λ for losses. Figure 2 depicts (15) for four parameter pairs, including the estimates of Tversky and Kahneman (1992), $\alpha = 0.88$ and $\lambda = 2.25$. For a loss of \$100, $v(-100) = -2.25(100) = -225$ with $\alpha = 1$, and $v(-100) = -2.25(100)^{0.88} \approx -129.5$ with $\alpha = 0.88$; the corresponding gain is worth $v(100) = 100$ and $v(100) = 100^{0.88} \approx 57.5$, respectively.

A common probability weighting function, proposed by Tversky and Kahneman (1992), is

$$w(p) = \frac{p^{\beta}}{\left[p^{\beta} + (1 - p)^{\beta}\right]^{1/\beta}}, \tag{16}$$

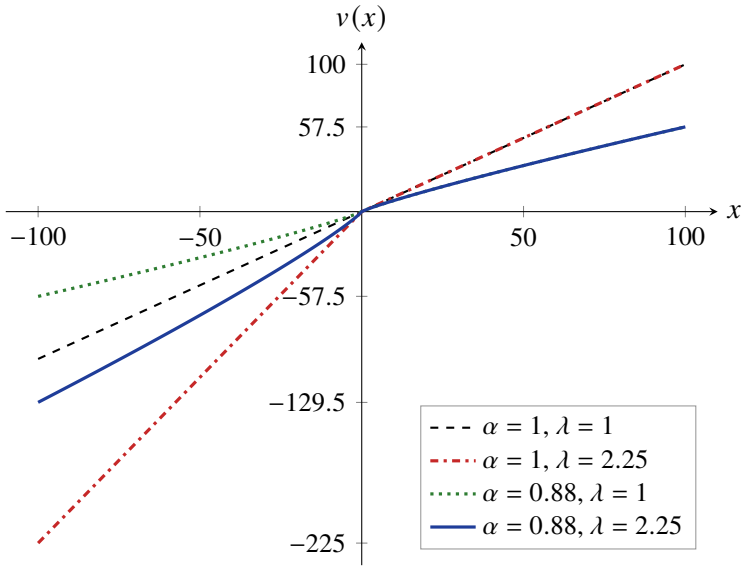


Figure 2. Prospect theory value functions.[1]

where $\beta \in (0, 1]$. With $\beta = 1$, $w(p) = p$; with $\beta < 1$, the function has an inverse-S shape. Three properties are worth knowing:

- The function crosses the 45-degree line below $p = \frac{1}{2}$: the fixed point is $p \approx 0.34$ for $\beta = 0.61$ and $p \approx 0.21$ for $\beta = 0.4$. At $p = \frac{1}{2}$, the denominator is $[2(\frac{1}{2})^\beta]^{1/\beta} = 2^{1/\beta} \cdot \frac{1}{2}$, so $w(\frac{1}{2}) = \frac{(1/2)^\beta}{2^{1/\beta}(1/2)} = 2^{1-\beta-1/\beta}$. Since $\beta + \frac{1}{\beta} > 2$ for every $\beta \neq 1$, the exponent is below -1, and $w(\frac{1}{2}) < \frac{1}{2}$.
- The function is increasing in p only if β is not too small (approximately $\beta \geq 0.28$); for smaller values, $w(p)$ decreases over part of its domain, so it is no longer a sensible weighting function.
- Tversky and Kahneman (1992) estimate $\beta = 0.61$ for gains and $\beta = 0.69$ for losses.

Figure 3 depicts (16) for three values of β, and Table 4 reports some of its values.

p	0.01	0.10	0.25	0.50	0.75	0.90	0.99
$w(p)$, $\beta = 0.61$	0.055	0.186	0.291	0.421	0.568	0.712	0.912
$w(p)$, $\beta = 0.4$	0.111	0.186	0.221	0.268	0.343	0.447	0.695

Table 4. Values of the Tversky–Kahneman weighting function.

4.3 Loss aversion in a mixed gamble

Consider a gamble that wins \$$G$ or loses \$100 with probability $\frac{1}{2}$ each. Assuming the same weighting function for gains and losses, both outcomes receive the same weight $w(\frac{1}{2})$, which cancels out. The individual accepts the gamble if and only if $w(\frac{1}{2})G^\alpha - w(\frac{1}{2})\lambda(100)^\alpha \geq 0$. Dividing by $w(\frac{1}{2}) > 0$ gives $G^\alpha \geq \lambda(100)^\alpha$, and raising both sides to the power $\frac{1}{\alpha}$ gives

$$G \geq 100\,\lambda^{1/\alpha}. \tag{17}$$

With $\alpha = 1$, the gain must be λ times the loss. With $\alpha = 0.88$ and $\lambda = 2.25$, for instance, $\lambda^{1/\alpha} = 2.25^{1.136} \approx 2.513$, so $G \geq 251.3$. Table 5 reports the minimum acceptable gain for several values of λ.

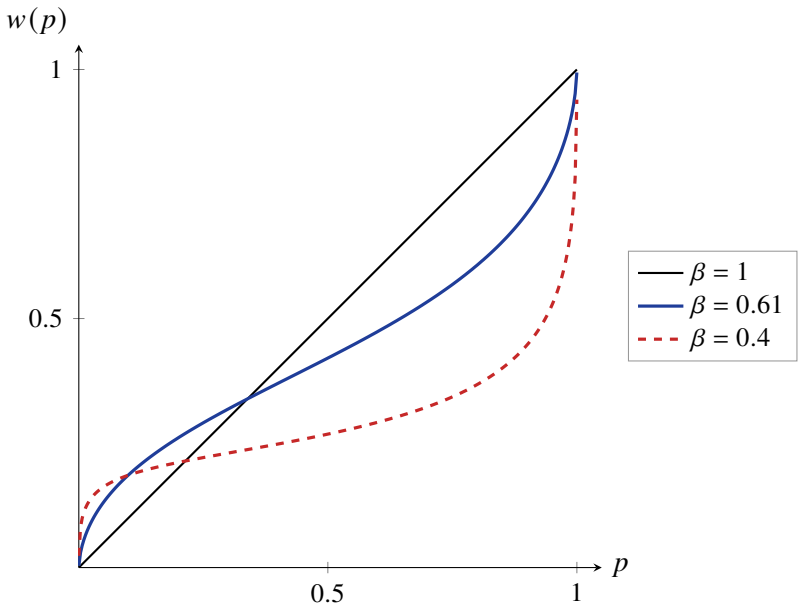


Figure 3. The Tversky–Kahneman probability weighting function.

Loss aversion λ	Minimum G, $\alpha = 1$	Minimum G, $\alpha = 0.88$
1	100	100.0
1.5	150	158.5
2	200	219.8
2.25	225	251.3
2.5	250	283.3
3	300	348.5

Table 5. Minimum gain that makes a 50–50 gamble with a $100 loss acceptable.

An individual with $\lambda = 2.25$ rejects a fair coin flip that wins \$200 or loses \$100, although its expected value is \$50. Loss aversion, not the curvature of v, drives this rejection: with $\lambda = 1$, the minimum gain is \$100 for any α.

5 Prospect theory and the Allais paradox

Applying (14) to the lotteries in (2), with outcomes measured in millions of dollars and the reference point at \$0, so $v(0) = 0$:

$$\begin{aligned} V(L_1) &= v(0.5), & V(L_1') &= w(0.10)\, v(2.5) + w(0.89)\, v(0.5), \\ V(L_2) &= w(0.11)\, v(0.5), & V(L_2') &= w(0.10)\, v(2.5). \end{aligned} \tag{18}$$

Lottery L_1 is certain, so its probability is not distorted ($w(1) = 1$). Hence $L_1 \succ L_1'$ requires $v(0.5) > w(0.10)\, v(2.5) + w(0.89)\, v(0.5)$. Subtracting $w(0.89)\, v(0.5)$ from both sides,

$$[1 - w(0.89)]\, v(0.5) > w(0.10)\, v(2.5). \tag{19}$$

Since $v(2.5) > v(0.5)$, condition (19) requires $1 - w(0.89) > w(0.10)$, that is, $w(0.89) + w(0.10) < 1$: the weights of the two positive outcomes in L_1' must add up to less than one. This property, called subcertainty, makes the certain lottery L_1 relatively attractive, but it is necessary, not sufficient, for (19). In the second pair, both lotteries are uncertain, so both probabilities are distorted, and $L_2' \succ L_2$ requires $w(0.10)\, v(2.5) > w(0.11)\, v(0.5)$. Dividing both sides by $w(0.10)\, v(0.5) > 0$,

$$\frac{v(2.5)}{v(0.5)} > \frac{w(0.11)}{w(0.10)}. \tag{20}$$

The left side measures how much more \$2.5 million is worth than \$0.5 million; with $v(x) = x^\alpha$, it equals $\left(\frac{2.5}{0.5}\right)^\alpha = 5^\alpha$, which is 5 when $\alpha = 1$ and about 1.50 when $\alpha = 0.25$. The right side measures how much more weight probability 0.11 receives than probability 0.10. Since the two probabilities are close, it is close to one: $\frac{0.11}{0.10} = 1.1$ under expected utility ($\beta = 1$), and $\frac{w(0.11)}{w(0.10)} \approx \frac{0.195}{0.186} \approx 1.05$ with $\beta = 0.61$. Hence (20) fails only when the value function is almost flat, as the cutoffs below make precise.

Whether prospect theory reproduces the Allais pattern, however, depends on the parameters. Table 6 evaluates the four lotteries in (18) with $v(x) = x^\alpha$ and the weighting function (16). For instance, with the Tversky–Kahneman estimates $\alpha = 0.88$ and $\beta = 0.61$:

- Values: $v(0.5) = 0.5^{0.88} \approx 0.543$ and $v(2.5) = 2.5^{0.88} \approx 2.240$.
- Weights, from Table 4 and (16): $w(0.10) \approx 0.186$, $w(0.11) \approx 0.195$, and $w(0.89) \approx 0.699$.
- Lotteries: $V(L_1) \approx 0.543$; $V(L_1') \approx 0.186(2.240) + 0.699(0.543) \approx 0.417 + 0.380 = 0.797$; $V(L_2) \approx 0.195(0.543) \approx 0.106$; and $V(L_2') \approx 0.186(2.240) \approx 0.417$.

The first row is expected value maximization. Computing the decision weights with the cumulative (rank-dependent) version of prospect theory instead yields the same answers in the last two columns of every row. With the Tversky–Kahneman estimates (second row), prospect theory predicts $L_1' \succ L_1$ in this example, so it does not explain the first Allais choice. The pattern appears only when both the value function and the weighting function are strongly curved, as in the last three rows, where a small α makes the extra \$2 million in L_1' worth little, and a small β shrinks $w(0.89)$.

α	β	$V(L_1)$	$V(L_1')$	$V(L_2)$	$V(L_2')$	$L_1 \succ L_1'$?	$L_2' \succ L_2$?
1	1	0.500	0.695	0.055	0.250	No	Yes
0.88	0.61	0.543	0.797	0.106	0.417	No	Yes
0.5	0.61	0.707	0.789	0.138	0.295	No	Yes
0.5	0.4	0.707	0.602	0.134	0.294	Yes	Yes
0.25	0.5	0.841	0.736	0.172	0.249	Yes	Yes
0.25	0.4	0.841	0.600	0.159	0.233	Yes	Yes

Table 6. Prospect theory evaluation of the Allais lotteries (in millions of dollars).

Two cutoffs on α

We can make this observation precise by solving each Allais inequality for α, given β. Throughout, $v(x) = x^\alpha$ with $\alpha \in (0, 1]$, $v(0) = 0$, and $w(\cdot)$ is the Tversky–Kahneman function (16) with $\beta \in [0.28, 1]$, so that w is increasing.

First choice, $L_1 \succ L_1'$. From (18), this choice requires $0.5^\alpha > w(0.10)\, 2.5^\alpha + w(0.89)\, 0.5^\alpha$.

- Subtracting $w(0.89)\, 0.5^\alpha$ from both sides, $[1 - w(0.89)]\, 0.5^\alpha > w(0.10)\, 2.5^\alpha$.
- Dividing both sides by $w(0.10)\, 0.5^\alpha > 0$, and using $\frac{2.5^\alpha}{0.5^\alpha} = \left(\frac{2.5}{0.5}\right)^\alpha = 5^\alpha$,

$$\frac{1 - w(0.89)}{w(0.10)} > 5^\alpha.$$

- Taking logs of both sides (the log is increasing, so the inequality keeps its direction), $\ln \frac{1-w(0.89)}{w(0.10)} > \alpha \ln 5$. Since $\ln 5 > 0$, dividing by it yields

$$\alpha < \alpha_A(\beta) \equiv \frac{1}{\ln 5} \ln \frac{1 - w(0.89)}{w(0.10)}. \tag{21}$$

The cutoff $\alpha_A(\beta)$ is positive only if $1 - w(0.89) > w(0.10)$, which is the subcertainty condition derived from (19). The weights depend on β through (16), so the cutoff does too.

Second choice, $L_2' \succ L_2$. From (18), this choice requires $w(0.10)\, 2.5^\alpha > w(0.11)\, 0.5^\alpha$. Dividing both sides by $w(0.10)\, 0.5^\alpha > 0$ gives $5^\alpha > \frac{w(0.11)}{w(0.10)}$, and taking logs,

$$\alpha > \alpha_B(\beta) \equiv \frac{1}{\ln 5} \ln \frac{w(0.11)}{w(0.10)}. \tag{22}$$

Since w is increasing, $w(0.11) > w(0.10)$, so $\alpha_B(\beta) > 0$. However, $w(0.11)$ and $w(0.10)$ are close, so $\alpha_B(\beta)$ is small: the second choice fails only when the value function is almost flat.

Table 7 evaluates both cutoffs. For instance, with $\beta = 0.61$, Table 4 and (16) give $w(0.10) \approx 0.186$, $w(0.11) \approx 0.195$, and $w(0.89) \approx 0.699$, so

$$\alpha_A(0.61) \approx \frac{\ln \frac{0.301}{0.186}}{\ln 5} \approx \frac{0.480}{1.609} \approx 0.298 \qquad \text{and} \qquad \alpha_B(0.61) \approx \frac{\ln \frac{0.195}{0.186}}{\ln 5} \approx \frac{0.047}{1.609} \approx 0.029.$$

Four candidate regions. The two cutoffs divide the (β, α)-quadrant into regions:

β	$w(0.10)$	$w(0.11)$	$w(0.89)$	$\alpha_B(\beta)$	$\alpha_A(\beta)$
0.3	0.139	0.139	0.261	0.002	1.040
0.4	0.186	0.189	0.436	0.011	0.690
0.5	0.198	0.204	0.580	0.020	0.468
0.61	0.186	0.195	0.699	0.029	0.298
0.7	0.168	0.178	0.769	0.036	0.198
0.8	0.144	0.155	0.825	0.044	0.118
1	0.100	0.110	0.890	0.059	0.059

Table 7. Cutoffs for the two Allais choices. Both choices hold if and only if $\alpha_B(\beta) < \alpha < \alpha_A(\beta)$.

- Both Allais choices, $L_1 \succ L_1'$ and $L_2' \succ L_2$: $\alpha_B(\beta) < \alpha < \alpha_A(\beta)$.
- Only $L_2' \succ L_2$ (the individual also chooses L_1'): $\alpha > \alpha_A(\beta)$.
- Only $L_1 \succ L_1'$ (the individual also chooses L_2): $\alpha < \alpha_B(\beta)$.
- Neither choice: $\alpha > \alpha_A(\beta)$ and $\alpha < \alpha_B(\beta)$ at the same time.

The last region is empty, because $\alpha_A(\beta) \geq \alpha_B(\beta)$ for every β. To see why, compare (21) and (22): $\alpha_A(\beta) \geq \alpha_B(\beta)$ if and only if $\frac{1-w(0.89)}{w(0.10)} \geq \frac{w(0.11)}{w(0.10)}$, that is, $w(0.11) + w(0.89) \leq 1$. Since $0.11 = 1 - 0.89$, this condition compares the weights of two complementary probabilities, p and $1 - p$. From (16), the two numerators add up to the term inside the brackets of the denominator, so

$$w(p) + w(1-p) = \frac{p^\beta + (1-p)^\beta}{\left[p^\beta + (1-p)^\beta\right]^{1/\beta}} = \left[p^\beta + (1-p)^\beta\right]^{1-\frac{1}{\beta}}.$$

For $\beta < 1$ and $p \in (0,1)$, we have $p^\beta > p$ and $(1-p)^\beta > 1-p$, so the term in brackets exceeds $p+(1-p) = 1$, while the exponent $1 - \frac{1}{\beta}$ is negative. Hence $w(p) + w(1-p) < 1$, and $\alpha_A(\beta) > \alpha_B(\beta)$ for every $\beta < 1$. With $\beta = 1$, $w(p) = p$, both cutoffs equal $\frac{\ln 1.1}{\ln 5} \approx 0.059$, and exactly one of the two Allais choices holds for every $\alpha \neq 0.059$. This is the expected utility result of Section 1: an individual who chooses L_1 must also choose L_2. Probability weighting ($\beta < 1$) opens a gap between the two cutoffs, and the Allais pattern lives in that gap.

Figure 4 depicts the cutoffs and the six rows of Table 6. The three rows with the Allais pattern, $(\beta, \alpha) = (0.4, 0.5)$, $(0.5, 0.25)$, and $(0.4, 0.25)$, lie in the shaded region between the two cutoffs: for instance, $\alpha = 0.5$ is below $\alpha_A(0.4) \approx 0.690$, and $\alpha = 0.25$ is below $\alpha_A(0.5) \approx 0.468$. The other three rows lie above $\alpha_A(\beta)$: with the Tversky–Kahneman estimates, $\alpha = 0.88$ is far above $\alpha_A(0.61) \approx 0.298$. The Allais pattern requires a small β, which raises the cutoff $\alpha_A(\beta)$, and an α below that cutoff. For instance, with the Tversky–Kahneman curvature $\alpha = 0.88$, the first Allais choice requires β below about 0.34.

6 Prospect theory and the endowment effect

Experiments show that individuals' willingness to accept (WTA) to give up an object, such as a coffee mug, exceeds their willingness to pay (WTP) to acquire it (Kahneman, Knetsch, and Thaler, 1990). Loss aversion explains this gap. Let M denote the value of the mug, and assume $\alpha = 1$ for simplicity.

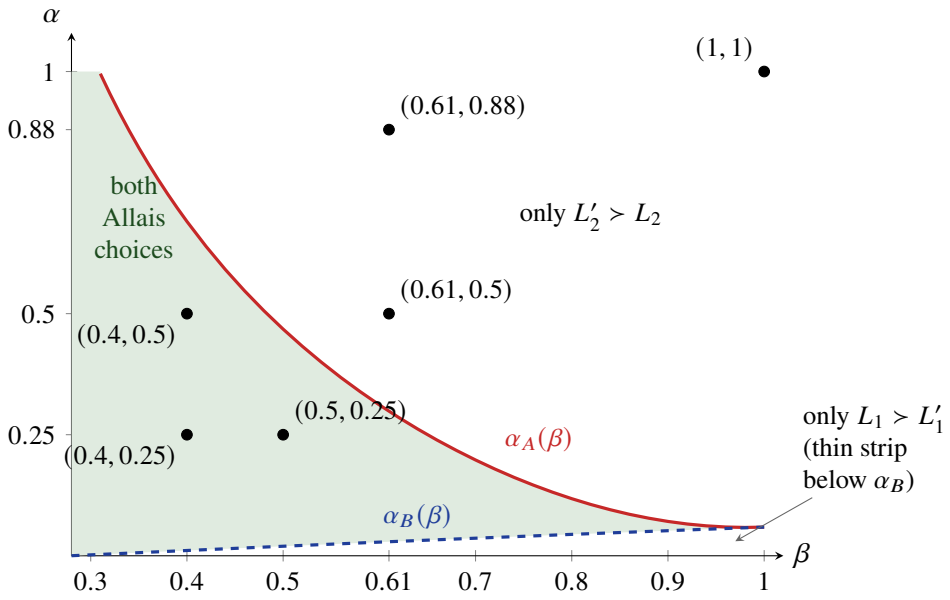


Figure 4. The Allais choices in the (β, α)-quadrant. Each dot is a row of Table 6, labeled (β, α).[2]

- Buying. A subject without the mug gains the mug and loses the price P, a loss weighted by λ. She buys if $M - \lambda P \geq 0$, that is, if $P \leq \frac{M}{\lambda}$, so

$$WTP = \frac{M}{\lambda}. \tag{23}$$

- Selling. A subject who owns the mug gains the price P and loses the mug, now a loss weighted by λ. She sells if $P - \lambda M \geq 0$, that is, if $P \geq \lambda M$, so

$$WTA = \lambda M. \tag{24}$$

In each case, the item given up is coded as a loss. Dividing (24) by (23), $\frac{WTA}{WTP} = \frac{\lambda M}{M/\lambda} = \lambda^2$, so the gap grows with the square of the loss aversion parameter. Table 8 uses $M = \$6$.

Loss aversion λ	WTP	WTA	Ratio $\frac{WTA}{WTP} = \lambda^2$
1	6.00	6.00	1.00
1.5	4.00	9.00	2.25
2	3.00	12.00	4.00
2.25	2.67	13.50	5.06
2.5	2.40	15.00	6.25
3	2.00	18.00	9.00

Table 8. Willingness to pay and to accept for a mug valued at $M = \$6$.

Without loss aversion ($\lambda = 1$), WTP and WTA coincide, as standard consumer theory predicts when income effects are negligible.

7 Reference-dependent preferences

7.1 The model

Kőszegi and Rabin (2006) also let a reference point shape preferences, but they treat the reference point as the individual's expectations. A consumption bundle $x \in \mathbb{R}^N$, evaluated against a reference bundle $r \in \mathbb{R}^N$, yields utility

$$u(x|r) = m(x) + n(x|r), \qquad n(x|r) = \sum_{k=1}^{N} \mu\left(m_k(x_k) - m_k(r_k)\right), \tag{25}$$

where $m(x) = \sum_k m_k(x_k)$ is standard consumption utility, and $n(x|r)$ is gain-loss utility, which applies a prospect-theory value function μ to the change in consumption utility in each good. When outcomes or reference points are uncertain, utility is averaged over both:

$$U(F|r) = \int u(x|r)\, dF(x) \qquad \text{and} \qquad U(F|G) = \int \int u(x|r)\, dG(r)\, dF(x), \tag{26}$$

where F is the distribution of consumption and G is the distribution of the reference point. Kőszegi and Rabin (2006) close the model by requiring that expectations be consistent with behavior, which they call a personal equilibrium.

7.2 A parametric example

Consider one good, with $m(x) = x$ and the piecewise linear gain-loss function $\mu(z) = \eta z$ if $z \geq 0$ and $\mu(z) = \eta\lambda z$ if $z < 0$, with $\eta = 1$ and $\lambda = 2.25$. Table 9 reports $u(x|r)$ for consumption and reference levels of 0 and 10.

	Reference $r = 0$	Reference $r = 10$
Consumption $x = 0$	$0 + 0 = 0$	$0 - 2.25(10) = -22.5$
Consumption $x = 10$	$10 + 10 = 20$	$10 + 0 = 10$

Table 9. Utility $u(x|r)$ with $\eta = 1$ and $\lambda = 2.25$.

Receiving 10 is worth 20 when the individual expected 0, but only 10 when she expected it. Receiving 0 when she expected 10 is very costly, -22.5, because the loss is weighted by λ.

Now suppose the individual receives 10 with probability q and 0 otherwise, and she correctly expects this lottery, so $F = G$. Since consumption and the reference point are drawn independently from the same lottery, there are four combinations:

- $x = 10$ and $r = 10$, with probability q^2: no gain or loss.
- $x = 0$ and $r = 0$, with probability $(1 - q)^2$: no gain or loss.
- $x = 10$ and $r = 0$, with probability $q(1 - q)$: a gain of 10, worth $\eta(10)$.
- $x = 0$ and $r = 10$, with probability $(1 - q)q$: a loss of 10, worth $\eta\lambda(-10)$.

Averaging $u(x|r)$ over these combinations, consumption utility contributes $10q$, and gain-loss utility contributes

$$q(1 - q)\,\eta(10) + (1 - q)q\,\eta\lambda(-10) = -10\,\eta(\lambda - 1)\,q(1 - q), \tag{27}$$

where the equality factors out $10\eta q(1-q)$, leaving $1-\lambda$. With $\eta = 1$, $\lambda = 2.25$, and $q = \frac{1}{2}$, this term is $-10(1.25)(\frac{1}{4}) = -3.125$, and total utility is $5 - 3.125 = 1.875$. Since $\lambda > 1$, expected gain-loss utility is negative whenever the outcome is uncertain. Table 10 reports the results.

Probability q	Consumption utility $10q$	Gain-loss utility	Total $U(F\|F)$
0	0.0	0.000	0.000
0.25	2.5	−2.344	0.156
0.50	5.0	−3.125	1.875
0.75	7.5	−2.344	5.156
1	10.0	0.000	10.000

Table 10. Expectation-based reference dependence with $\eta = 1$ and $\lambda = 2.25$.

The gain-loss penalty is largest when uncertainty is largest ($q = \frac{1}{2}$). A certain outcome generates no gains or losses, which is why an individual with expectation-based reference points may prefer a sure, smaller payoff to a risky lottery with a higher expected value. For instance, receiving \$2 for sure, correctly expected, yields $u(2|2) = 2 + \mu(0) = 2$, which exceeds the 1.875 of the 50–50 lottery over \$10 and \$0, although the lottery's expected value is \$5.

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