EconS 501 - Microeconomic Theory I
Homework #5 - Answer key

1. **(34 points) First- and second-order stochastic dominance.**

   (a) (12 points) FOSD.

   - *(⇒)* Suppose $F(x) \leq G(x)$ for all x. Integration by parts gives

     $$\int u\,dF - \int u\,dG = [u(x)\,(F-G)]_0^M - \int u'(x)\,(F(x)-G(x))\,dx = \int u'(x)\,(G(x)-F(x)$$

     (the boundary term vanishes because F and G are both distribution functions on $[0, M]$). Since $u' > 0$ and $G - F \geq 0$, the integral is nonnegative, so every monotone u prefers F to G.
   - *(⇐)* By contrapositive: if $F(x_0) > G(x_0)$ at some x_0, choose u to be a step function that jumps up at x_0; then $E_F[u] < E_G[u]$, contradicting the assumption.

   (b) (12 points) SOSD.

   - Under the SOSD condition $\int_0^x [G(t) - F(t)]\,dt \geq 0$ for all x, integrate by parts *twice* against a concave u:

     $$\int u\,dF - \int u\,dG = \int u'(x)\,(G(x)-F(x))\,dx = -\int u''(x) \int_0^x [G(t)-F(t)]\,dt\,dx \geq 0,$$

     where the last inequality uses $u'' \leq 0$ (concavity) and the SOSD condition. So every risk-averse (increasing, concave) u prefers F to G. The converse follows by choosing u to be piecewise linear at appropriate kinks, as in the FOSD direction.

   (c) (10 points) Numerical illustration.

   - F: pays 10, 20, 30 with probabilities $(0.2, 0.5, 0.3)$. G: pays 5, 15, 25 with the same probabilities. Since G is just F shifted down by 5, $G(x) = F(x-5) \geq F(x)$ for all x, so F FOSD G: every monotone u prefers F. Means: $E[F] = 21$, $E[G] = 16$; FOSD implies the ranking.
   - A mean-preserving spread of F that F SOSD-dominates: replace the middle prize of 20 (with prob 0.5) by 10 with prob 0.25 and 30 with prob 0.25. The new lottery H has 10 with prob 0.45, 30 with prob 0.55, mean $E[H] = 10(0.45) + 30(0.55) = 21 = E[F]$, but strictly more spread; every risk-averse agent prefers F to H (SOSD), while F does not FOSD H (both have identical mean, and their CDFs cross).

2. **(34 points) More risk aversion implies less in the risky asset (Arrow).**

   (a) (10 points) The investor's problem and its concavity.

- Each agent chooses $\alpha \in [0,1]$ to solve $\max_\alpha E\left[u_i(1+\alpha x)\right]$. Differentiating in α under the integral yields the first-order condition

$$E\left[u_i'(1+\alpha x)\, x\right] = 0.$$

Differentiating once more,

$$\frac{d^2}{d\alpha^2} E\left[u_i(1+\alpha x)\right] = E\left[u_i''(1+\alpha x)\, x^2\right] < 0,$$

because $u_i'' < 0$ and $x^2 \geq 0$ with strict positivity on a set of positive measure. So the objective is strictly concave in α and the first-order condition uniquely characterizes the interior optimum.

(b) (16 points) Sign of Rima's marginal utility at Tomas's optimum.

- Write $u_R(z) = g(u_T(z))$ with $g' > 0$ and $g'' \leq 0$ (Pratt: greater Arrow–Pratt aversion is equivalent to being an increasing concave transformation). By the chain rule, $u_R'(z) = g'(u_T(z)) \cdot u_T'(z)$. Evaluate at Tomas's optimum α_T^*:

$$E\left[u_R'(1+\alpha_T^* x)\, x\right] = E\left[g'(u_T(1+\alpha_T^* x)) \cdot u_T'(1+\alpha_T^* x)\, x\right].$$

- Split the expectation over $x > 0$ (gains, where $u_T(1+\alpha_T^* x) > u_T(1)$ and hence, by g' weakly decreasing, $g'(u_T(1+\alpha_T^* x)) \leq g'(u_T(1))$) and over $x < 0$ (losses, where the opposite inequality holds). So the multiplier $g'(u_T(\cdot))$ down-weights the positive-x terms and up-weights the negative-x terms of the integrand $u_T'(1+\alpha_T^* x)\, x$, and hence

$$E\left[g'(u_T(1+\alpha_T^* x)) \cdot u_T'(1+\alpha_T^* x)\, x\right] \leq g'(u_T(1)) \cdot E\left[u_T'(1+\alpha_T^* x)\, x\right] = 0,$$

using Tomas's own first-order condition in the last step. Therefore $E\left[u_R'(1+\alpha_T^* x)\, x\right] \leq 0$.

(c) (8 points) Ranking the optimal shares.

- Rima's marginal payoff at α_T^* is nonpositive by part (b). Since her objective is strictly concave in α (part (a)), her optimal share must lie *weakly to the left* of α_T^*: $\alpha_R^* \leq \alpha_T^*$. Intuitively, holding wealth and the return distribution fixed, greater risk aversion cannot raise the fraction of wealth held in the risky asset; this is the classical Arrow–Pratt monotone comparative static (Arrow, 1965). Empirically, the result underpins the standard cross-sectional finding that portfolios of more risk-averse investors tilt toward the risk-free asset.

3. **(32 points) Efficient risk sharing and the mutuality principle.**

(a) (6 points) Efficiency exhausts the resource.

- Suppose $\sum_i s_i(w) < \sum_i w_i$ with positive probability. Redistribute the slack $\sum_i (w_i - s_i(w))$ to any agent j in every such state, and leave other allocations unchanged. Agent j's expected utility strictly rises (utilities are strictly increasing), while everyone else's is unchanged: a Pareto improvement. So an efficient rule must satisfy $\sum_i s_i(w) = \sum_i w_i$ almost surely.

(b) (10 points) Borch's rule is sufficient for efficiency.

- Take any feasible perturbation $h = (h_1, \ldots, h_n)$ with $\sum_i h_i = 0$. Concavity of u_i gives pointwise $u_i(s_i^* + h_i) \le u_i(s_i^*) + u_i'(s_i^*)\, h_i$, hence in expectation,

$$U_i(s^* + h) - U_i(s^*) \le \int u_i'(s_i^*)\, h_i \, dF.$$

Divide the ith inequality by k_i, sum over i, and use Borch's rule $u_i'(s_i^*) = k_i u_1'(s_1^*)$:

$$\sum_i \frac{U_i(s^* + h) - U_i(s^*)}{k_i} \le \int \sum_i \frac{u_i'(s_i^*)\, h_i}{k_i}\, dF = \int u_1'(s_1^*) \underbrace{\sum_i h_i}_{=0}\, dF = 0.$$

So no perturbation makes every agent weakly better off and some strictly better off (that would require the weighted sum to be strictly positive): s^* is Pareto efficient.

(c) (10 points) The mutuality principle.

- Differentiate Borch's rule $u_i'(s_i^*(w)) = k_i\, u_1'(s_1^*(w))$ in w_j:

$$u_i''(s_i^*)\, \frac{\partial s_i^*}{\partial w_j} = k_i\, u_1''(s_1^*)\, \frac{\partial s_1^*}{\partial w_j}, \qquad \text{so} \qquad \frac{\partial s_i^*}{\partial w_j} = \frac{k_i u_1''(s_1^*)}{u_i''(s_i^*)} \cdot \frac{\partial s_1^*}{\partial w_j},$$

a ratio that is independent of j. Combined with the resource constraint $\sum_i \dfrac{\partial s_i^*}{\partial w_j} = 1$ for every j, this forces $\dfrac{\partial s_i^*}{\partial w_j}$ to be the same constant for every j: each share depends on $(w_1, \ldots, w_n)$ only through the aggregate $W = \sum_j w_j$, so $s_i^*(w) = G_i\left(\sum_j w_j\right)$ for some function G_i. This is the mutuality principle: efficient sharing rules pool the aggregate risk and reallocate it, but ignore the individual composition that produced it.

(d) (6 points) A CARA example.

- Borch's rule with $u_1'(x) = e^{-x}$, $u_2'(x) = 2e^{-2x}$, and $k_2 = 2$:

$$2e^{-2s_2} = 2\, e^{-s_1} \quad \Longleftrightarrow \quad s_1 = 2s_2.$$

With $s_1 + s_2 = W$, this yields $s_1^* = \dfrac{2W}{3}$ and $s_2^* = \dfrac{W}{3}$. Both shares depend only on W, as required by mutuality.

- Under the sharing rule, $W \sim \mathcal{N}(0, 2)$ and s_i^* is normal, so a normal-moment calculation gives $U_1(s^*) = -\exp\left(\dfrac{1}{2} \cdot \left(\dfrac{2}{3}\right)^2 \cdot 2\right) = -e^{4/9}$ and analogously $U_2(s^*) = -e^{4/9}$. Going alone, agent 1 gets $\int u_1(w_1)\, dF_1 = -e^{1/2}$ and agent 2 gets $-e^2$. Since $-e^{4/9} \ge -e^{1/2}$ and $-e^{4/9} \ge -e^2$, both agents strictly prefer the pool. Intuitively, pooling two independent normal risks halves the exposure per share, and the more-risk-averse agent 2 benefits especially from that reduction.