EconS 501 - Microeconomic Theory I
Homework #5 - Due date: Thursday, October 29th, in class.

1. **First- and second-order stochastic dominance.** Consider two lotteries over money, described by their distribution functions F and G on $[0, M]$.

   (a) *FOSD.* Prove that F first-order stochastically dominates G (i.e., $F(x) \leq G(x)$ for all x) if and only if every expected-utility maximizer with a strictly increasing utility function u prefers F to G. [*Hint*: use integration by parts on $\int u(x)\, dF(x) - \int u(x)\, dG(x) = \int u'(x)\, [G(x) - F(x)]\, dx$.]

   (b) *SOSD.* Prove the analogous result for second-order stochastic dominance: F SOSD G (i.e., $\int_0^x F(t)\, dt \leq \int_0^x G(t)\, dt$ for all x, with equality at $x = M$) if and only if every expected-utility maximizer with an increasing *concave* utility prefers F to G.

   (c) *Numerical illustration.* Consider F: pays 10, 20, or 30 with probabilities $(0.2, 0.5, 0.3)$, and G: pays 5, 15, or 25 with the same probabilities. Show that F FOSD G, so every risk-averse (indeed, every monotone) agent prefers F. Now construct a mean-preserving spread of F that does not FOSD F but is second-order dominated by F.

2. **More risk aversion implies less in the risky asset (Arrow).** Rima and Tomas each start with wealth $w = 1$ and invest a fraction $\alpha \in [0, 1]$ in a risky asset that returns $1 + x$ per unit invested, with $x \sim U[-1, 1]$. Terminal wealth is $W(\alpha; x) = 1 + \alpha x$. Both agents are strictly risk-averse expected-utility maximizers, with utilities u_R, u_T satisfying $u_i'(z) > 0$, $u_i''(z) < 0$, and normalized so that $u_R'(0) = u_T'(0) = 1$. Rima is *globally more risk-averse* than Tomas in the Arrow–Pratt sense:

$$-\frac{u_R''(z)}{u_R'(z)} \geq -\frac{u_T''(z)}{u_T'(z)} \qquad \text{for every } z.$$

   Assume both have interior optimal shares $\alpha_R^*, \alpha_T^* \in (0, 1)$.

   (a) Write each agent's problem $\max_\alpha E[u_i(1 + \alpha x)]$, derive the first-order condition $E[u_i'(1 + \alpha x)\, x] = 0$, and explain why the objective is strictly concave in α (so the first-order condition uniquely characterizes the interior optimum).

   (b) Pratt's theorem asserts that Rima can be written as an *increasing concave* transformation of Tomas: there exists a strictly increasing, strictly concave g such that $u_R(z) = g(u_T(z))$ for every z. Use this to prove that, evaluated at Tomas's optimal share α_T^*,

$$E[u_R'(1 + \alpha_T^* x)\, x] \leq 0.$$

   [*Hint*: chain rule gives $u_R' = g'(u_T) \cdot u_T'$; then use that g' is weakly decreasing (because g is concave) to sign the covariance between $g'(u_T(1 + \alpha_T^* x))$ and $u_T'(1 + \alpha_T^* x)\, x$ at Tomas's optimum.]

(c) Combine parts (a) and (b) with the concavity of Rima's objective to conclude that Rima's optimal share is weakly smaller than Tomas's, $\alpha_R^* \leq \alpha_T^*$. Interpret this as the *Arrow–Pratt monotone comparative static*: greater risk aversion, holding wealth and the return distribution fixed, cannot raise the fraction of wealth held in the risky asset.

3. **Efficient risk sharing and the mutuality principle.** A group of n agents ($i = 1, \ldots, n$) with strictly increasing, strictly concave utilities u_i face random wealth endowments $w_i \sim F_i$ with joint distribution F. A *sharing rule* is a set of measurable functions $s_i : \mathbb{R}^n \to \mathbb{R}$ that assign each state $w = (w_1, \ldots, w_n)$ a division satisfying $\sum_i s_i(w) \leq \sum_i w_i$ almost surely. Each agent's expected utility is $U_i(s) = \int u_i(s_i(w))\, dF(w)$.

   (a) Show that Pareto efficiency requires the resource constraint to bind: $\sum_i s_i(w) = \sum_i w_i$ for (almost) every state w. [Argue by contradiction: if some wealth were unshared, redistribute it to any single agent for a Pareto improvement.]

   (b) Show that if s^* satisfies *Borch's rule*,

$$u_i'(s_i^*(w)) = k_i\, u_1'(s_1^*(w)) \qquad \text{for constants } k_1 = 1,\ k_2, \ldots, k_n > 0,$$

   and every state w, then s^* is Pareto efficient. [*Hint*: for any feasible perturbation $h = (h_1, \ldots, h_n)$ with $\sum_i h_i = 0$, use concavity to write $U_i(s^* + h) - U_i(s^*) \leq \int u_i'(s_i^*)\, h_i\, dF$, weight the ith inequality by $1/k_i$, sum, and apply Borch's rule.]

   (c) Show that every efficient rule depends on the individual endowments only through the aggregate: $s_i^*(w) = G_i\left(\sum_j w_j\right)$ for some function G_i. This is the *mutuality principle*: efficient sharing reallocates the aggregate risk but ignores its individual composition. [*Hint*: differentiate Borch's rule in w_j and show that $\partial s_i^*/\partial w_j$ is independent of j.]

   (d) Take $n = 2$ with $F_1 = F_2 = \mathcal{N}(0, 1)$ independent, $u_1(x) = -e^{-x}$, $u_2(x) = -e^{-2x}$, and $k_2 = 2$. Show that the efficient shares are $s_1^*(w) = \dfrac{2W}{3}$ and $s_2^*(w) = \dfrac{W}{3}$, where $W \equiv w_1 + w_2$, and verify that they depend only on the aggregate W, as required by the mutuality principle. Show that both agents prefer the pooled arrangement to going alone (i.e., $U_i(s^*) \geq \int u_i(w_i)\, dF_i$ for $i = 1, 2$), so risk pooling makes both agents better off in expected utility than bearing their own risk.