EconS 501 - Microeconomic Theory I
Homework #4 - Answer key

1. **(34 points) LeChatelier with three factors.**

   (a) (12 points) Long-run cost.

   - Minimize $w_1x_1 + w_2x_2 + w_3x_3$ subject to $x_1^a x_2^b x_3^c = y$. First-order conditions give $\dfrac{ax_1}{w_1} = \dfrac{bx_2}{w_2} = \dfrac{cx_3}{w_3}$ (marginal-cost equality). Combining with the constraint and CRS ($a + b + c = 1$),

   $$x_i^{LR}(w, y) = \frac{\alpha_i}{w_i} \cdot y \cdot K, \quad \text{and} \quad c^{LR}(w, y) = K\, y\, w_1^a w_2^b w_3^c,$$

   where $K = \left(\dfrac{1}{a^a b^b c^c}\right) \cdot a^a b^b c^c = 1$ up to the specific normalization (details of the constant depend on how one absorbs the Cobb-Douglas coefficients). Cost is linear in y because CRS means doubling output requires doubling every input, so total cost doubles.

   (b) (14 points) Short-run cost.

   - Fix $x_3 = \overline{x}_3$. Then the constraint reduces to $x_1^a x_2^b = \dfrac{y}{\overline{x}_3^c} \equiv \widetilde{y}$. The Cobb-Douglas sub-problem with two inputs gives $x_i^{SR} = \dfrac{\alpha_i}{w_i} \cdot \widetilde{y}^{1/(a+b)}$ (up to constants) and

   $$c^{SR}(w_1, w_2, y \,|\, \overline{x}_3) = w_3\overline{x}_3 + K_2\, \widetilde{y}^{1/(a+b)} w_1^{a/(a+b)} w_2^{b/(a+b)}.$$

   By the envelope theorem, $c^{SR}(w, y \,|\, \overline{x}_3) \geq c^{LR}(w, y)$ for every y: freezing an input can only raise cost. Equality holds only when $\overline{x}_3 = x_3^{LR}(w, y)$, at which point the constraint on x_3 does not bind.

   (c) (8 points) LeChatelier principle.

   - By the envelope theorem, c^{SR} and c^{LR} are tangent from above at $\overline{x}_3 = x_3^{LR}$, but c^{SR} is more curved (steeper deviations from the LR path). Differentiating twice and applying Shephard's lemma to the demands gives

   $$\left|\frac{\partial \ln x_1^{SR}}{\partial \ln w_1}\right| \leq \left|\frac{\partial \ln x_1^{LR}}{\partial \ln w_1}\right|.$$

   Economically, when x_3 is fixed the firm has one fewer margin along which to substitute, so a rise in w_1 moves x_1 less than when all three inputs can adjust. This is the LeChatelier principle: constrained responses are always smaller (in absolute value) than unconstrained ones.

2. **(34 points) Profit maximization and the convexity of the profit function.**

(a) (12 points) Profit maximization.

- The firm solves $\max_{z_1,z_2\geq 0}\; p\left(\sqrt{z_1}+\sqrt{z_2}\right)-w_1z_1-w_2z_2$. The first-order conditions are $\dfrac{p}{2\sqrt{z_i}}=w_i$, so $\sqrt{z_i}=\dfrac{p}{2w_i}$ and

$$z_1(p,w)=\frac{p^2}{4w_1^2},\qquad z_2(p,w)=\frac{p^2}{4w_2^2}.$$

- Output supply is $y(p,w)=\sqrt{z_1}+\sqrt{z_2}=\dfrac{p}{2w_1}+\dfrac{p}{2w_2}=\dfrac{p}{2}\left(\dfrac{1}{w_1}+\dfrac{1}{w_2}\right)$, and substituting back,

$$\pi(p,w)=p\,y-w_1z_1-w_2z_2=\frac{p^2}{2}\left(\frac{1}{w_1}+\frac{1}{w_2}\right)-\frac{p^2}{4}\left(\frac{1}{w_1}+\frac{1}{w_2}\right)=\frac{p^2}{4}\left(\frac{1}{w_1}+\frac{1}{w_2}\right).$$

The technology is strictly concave in (z_1,z_2), so these conditions identify the unique global maximum.

(b) (10 points) Hotelling's lemma and homogeneity.

- Differentiating π, $\dfrac{\partial\pi}{\partial p}=\dfrac{p}{2}\left(\dfrac{1}{w_1}+\dfrac{1}{w_2}\right)=y(p,w)$ and $\dfrac{\partial\pi}{\partial w_i}=-\dfrac{p^2}{4w_i^2}=-z_i(p,w)$, which is Hotelling's lemma.
- Homogeneity: $\pi(tp,tw)=\dfrac{(tp)^2}{4}\left(\dfrac{1}{tw_1}+\dfrac{1}{tw_2}\right)=t\cdot\dfrac{p^2}{4}\left(\dfrac{1}{w_1}+\dfrac{1}{w_2}\right)=t\,\pi(p,w)$, so π is homogeneous of degree one in (p,w): scaling every price by t scales profit by t.

(c) (12 points) Convexity and the law of supply.

- The second-order partial derivatives of π are

$$\pi_{pp}=\frac{1}{2}\left(\frac{1}{w_1}+\frac{1}{w_2}\right),\quad \pi_{w_iw_i}=\frac{p^2}{2w_i^3},\quad \pi_{pw_i}=-\frac{p}{2w_i^2},\quad \pi_{w_1w_2}=0.$$

The diagonal entries are positive; the leading 2×2 minor is $\pi_{pp}\pi_{w_1w_1}-\pi_{pw_1}^2=\dfrac{p^2}{4w_1^3w_2}>0$; and the determinant of the full 3×3 Hessian is zero. Hence the Hessian is positive semidefinite and π is convex in (p,w). (The zero determinant reflects the degree-one homogeneity of part (b), which places (p,w) in the null space of the Hessian.)

- Convexity in each own argument delivers the comparative statics at once:

$$\frac{\partial y}{\partial p}=\pi_{pp}=\frac{1}{2}\left(\frac{1}{w_1}+\frac{1}{w_2}\right)\geq 0\quad\text{(law of supply)},$$

and $\dfrac{\partial z_i}{\partial w_i}=-\pi_{w_iw_i}=-\dfrac{p^2}{2w_i^3}\leq 0$ (law of factor demand): output rises with its price, and each input falls with its own price.

- Since each $\sqrt{z_i}$ is strictly concave, f is strictly concave and homogeneous of degree $1/2$, so it has decreasing returns to scale and profit is strictly concave in the inputs; the first-order conditions then pin down a finite, interior plan. Under constant returns profit would be linear along rays (zero or unbounded), and under increasing returns unbounded above, so no interior maximum would exist.

3. **(32 points) Returns to scale and the shape of cost.**

   (a) (12 points) Cost minimization.

   - The Lagrangian is $\mathcal{L} = w_1 z_1 + w_2 z_2 - \lambda\left((z_1 z_2)^a - q\right)$. First-order conditions give $w_1 = \lambda\, a\, z_1^{a-1} z_2^a$ and $w_2 = \lambda\, a\, z_1^a z_2^{a-1}$, whose ratio yields the tangency $\dfrac{w_1}{w_2} = \dfrac{z_2}{z_1}$. Combining with the constraint $(z_1 z_2)^a = q$ (equivalently $z_1 z_2 = q^{1/a}$) and solving,

$$z_1^* = q^{1/(2a)}\sqrt{\frac{w_2}{w_1}}, \qquad z_2^* = q^{1/(2a)}\sqrt{\frac{w_1}{w_2}}.$$

     Total cost is $c(q,w) = w_1 z_1^* + w_2 z_2^* = 2\sqrt{w_1 w_2}\cdot q^{1/(2a)}$.

   (b) (12 points) Returns to scale and the shape of cost.

   - Since $f(\lambda z_1, \lambda z_2) = \lambda^{2a} f(z_1, z_2)$, the technology is homogeneous of degree $2a$. Returns to scale are therefore *decreasing* if $a < \dfrac{1}{2}$, *constant* if $a = \dfrac{1}{2}$, and *increasing* if $a > \dfrac{1}{2}$.
   - The cost function is $c(q,w) = 2\sqrt{w_1 w_2}\, q^{1/(2a)}$ with cost elasticity $\dfrac{d\ln c}{d\ln q} = \dfrac{1}{2a}$, which is larger than, equal to, or smaller than 1 exactly when returns are decreasing, constant, or increasing. So c is *convex* in q under decreasing returns (the exponent $1/(2a) > 1$), *linear* under constant returns (exponent $= 1$), and *concave* under increasing returns (exponent < 1): the curvature of cost is the exact mirror image of returns to scale.

   (c) (8 points) Competitive supply.

   - Marginal cost is $c'(q) = \dfrac{\sqrt{w_1 w_2}}{a}\, q^{(1-2a)/(2a)}$. Setting $p = c'(q)$ and solving for $q^*(p)$ is possible only when c' is strictly increasing, i.e., when $(1-2a)/(2a) > 0$, which holds iff $a < \dfrac{1}{2}$ (decreasing returns). For $a = \dfrac{1}{3}$,

$$c(q,w) = 2\sqrt{w_1 w_2}\, q^{3/2}, \qquad c'(q) = 3\sqrt{w_1 w_2}\, q^{1/2}, \qquad q^*(p) = \frac{p^2}{9\, w_1 w_2}.$$

     Under constant returns ($a = 1/2$), c is linear and marginal cost is a constant $2\sqrt{w_1 w_2}$: supply is perfectly elastic at that price and indeterminate at every other. Under increasing returns ($a > 1/2$), marginal cost *falls* with q, so no interior competitive optimum exists (the firm would either shut down or scale

up without bound). Intuitively, only under decreasing returns does doubling output require more than double the inputs, and only then does a price-taking firm face rising marginal cost and a well-defined scale.