

Advanced Microeconomic Theory

Alternatives to Subjective Expected Utility

Outline

- Risk and ambiguity
- Maxmin expected utility
- Choquet expected utility
- The smooth ambiguity model
- Comparing the models

Risk and Ambiguity

- **Risk**: the probabilities of outcomes are known (a fair coin).
- **Ambiguity**: they are not (an urn with an unknown mix of balls).
 - Knight (1921) calls this uncertainty.
- Subjective expected utility (SEU) treats both alike: every event gets one probability.

Risk and Ambiguity

- In the Ellsberg urn, most people choose A over B and C over D.
 - No single prior explains both choices.
- Three ways to accommodate these choices:
 - Many priors, evaluated at the worst one (maxmin).
 - Non-additive probabilities (Choquet).
 - A prior over priors (smooth ambiguity).

Maxmin Expected Utility

Maxmin Expected Utility

- Gilboa and Schmeidler (1989): the individual holds a closed, convex set of priors $\mathcal{C}$ and evaluates each act f by

$$U(f) = \min_{p \in \mathcal{C}} \sum_s p_s u(f(s))$$

- Each act is judged by its worst expected utility in $\mathcal{C}$.
 - If $\mathcal{C}$ has a single prior, MEU coincides with SEU.

Maxmin Expected Utility

- MEU keeps weak order, continuity, and monotonicity, and weakens independence to:
- ***Certainty independence***: for every constant act h and $\alpha \in [0,1]$, $f \succeq g$ if and only if
$$\alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h$$
 - Independence is required only when mixing with an act whose payoff does not depend on the state.

Maxmin Expected Utility

- ***Uncertainty aversion***: for every $\alpha \in [0,1]$,
 $f \sim g$ implies $\alpha f + (1 - \alpha)g \succeq f$
- Mixing two equally good acts can only help: the mixture ***hedges*** ambiguity.
 - An SEU individual is indifferent to such mixtures.

Hedging in the Ellsberg Urn

- Set of priors:

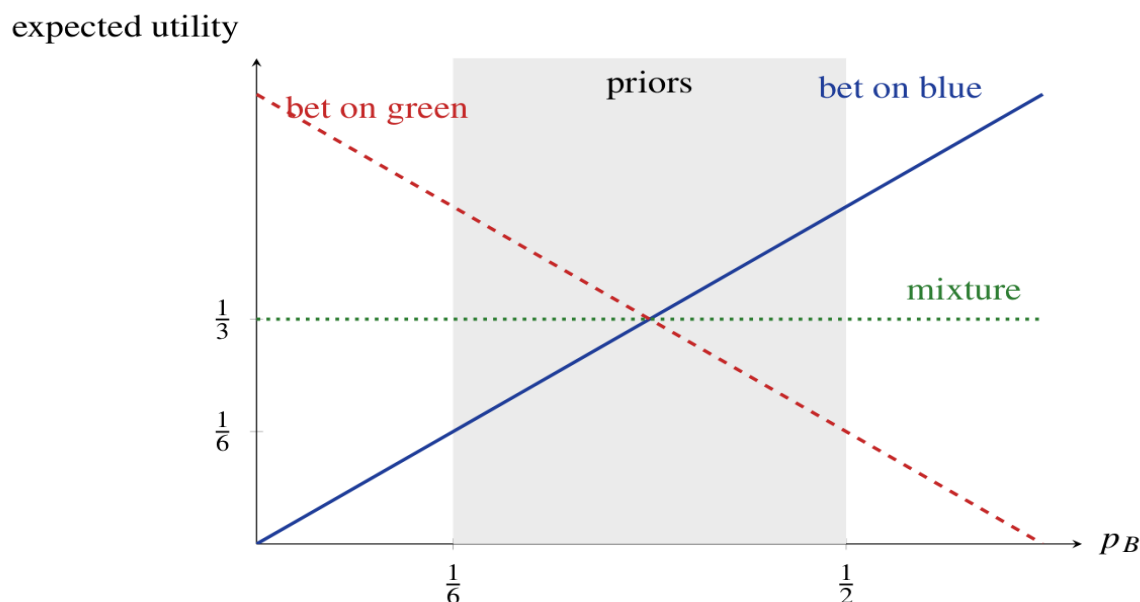
$$p_R = \frac{1}{3}, \quad p_B \in \left[\frac{1}{6}, \frac{1}{2}\right], \quad p_G = \frac{2}{3} - p_B$$

- Bet on blue: worst case $\min_{p_B} p_B = \frac{1}{6}$.
- Bet on green: worst case $\min_{p_B} p_G = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.
- Flip a fair coin to decide which bet to hold:

$$\frac{1}{2}p_B + \frac{1}{2}p_G = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

Hedging in the Ellsberg Urn

- The mixture yields $1/3$ under every prior, more than either bet's worst case, $1/6$.



MEU and the Ellsberg Paradox

- With $u(\$1,000) = 1$, $u(\$0) = 0$ and the same set of priors:

$$U(A) = \frac{1}{3}$$

$$U(B) = \min_{p \in C} p_B = \frac{1}{6}$$

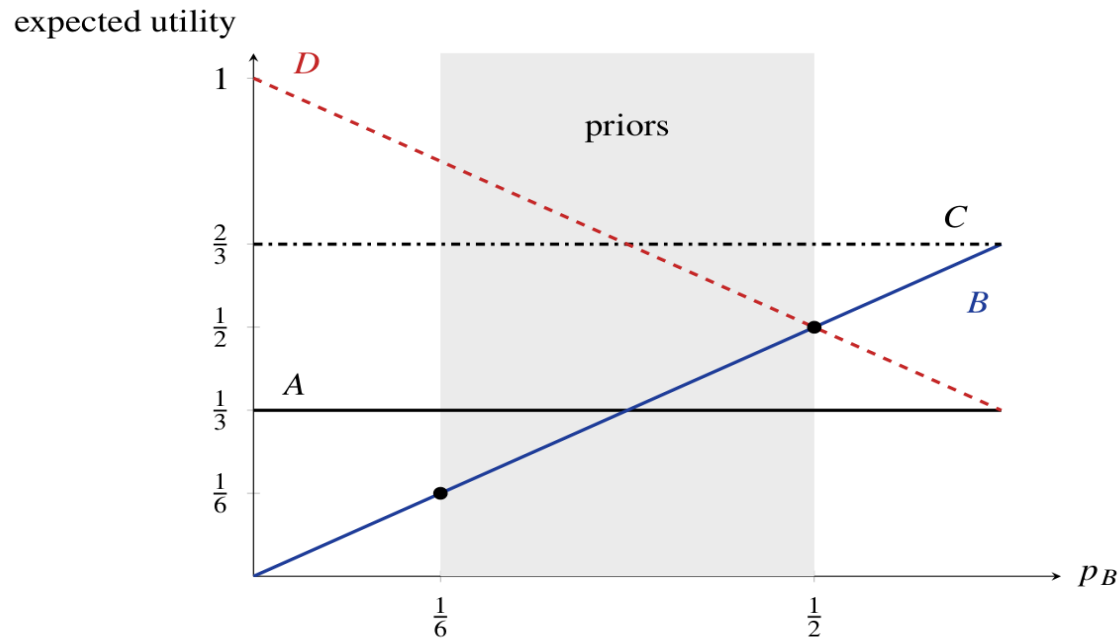
$$U(C) = \frac{2}{3}$$

$$U(D) = \frac{1}{3} + \min_{p \in C} p_G = \frac{1}{2}$$

- So $A \succ B$ and $C \succ D$, as most people choose.

MEU and the Ellsberg Paradox

- Bets B and D are evaluated at opposite ends of the set of priors.



MEU: The Size of the Set of Priors

- Two states, low and high; p is the probability of low; $u(x) = x$.
 - L_A pays \$1 if low and \$100 if high.
 - L_B pays \$3 if low and \$5 if high.
- If $C = [0.3, 0.7]$, the worst prior for both is $p = 0.7$:

$$U(L_A) = 0.7(1) + 0.3(100) = 30.7$$

$$U(L_B) = 0.7(3) + 0.3(5) = 3.6$$

MEU: The Size of the Set of Priors

- A larger set of priors makes the individual more cautious:

Set of priors	$U(L_A)$	$U(L_B)$	Choice
$C = \{0.5\}$ (SEU)	50.5	4.0	L_A
$C = [0.3, 0.7]$	30.7	3.6	L_A
$C = [0, 1]$	1	3	L_B

MEU: The Size of the Set of Priors

- With complete ignorance, $\mathcal{C} = [0, 1]$, MEU ranks acts by their worst outcome (maximin).
- The set of priors measures ambiguity, and the worst-case rule is the attitude toward it.
 - MEU combines both into a single object, $\mathcal{C}$.

MEU and No Trade

- Dow and Werlang (1992): an asset pays \$1 in state H and \$0 otherwise, at price q .
- A risk-neutral individual with priors $p \in [0.4, 0.6]$ on state H.

- Buy one unit, worst case:

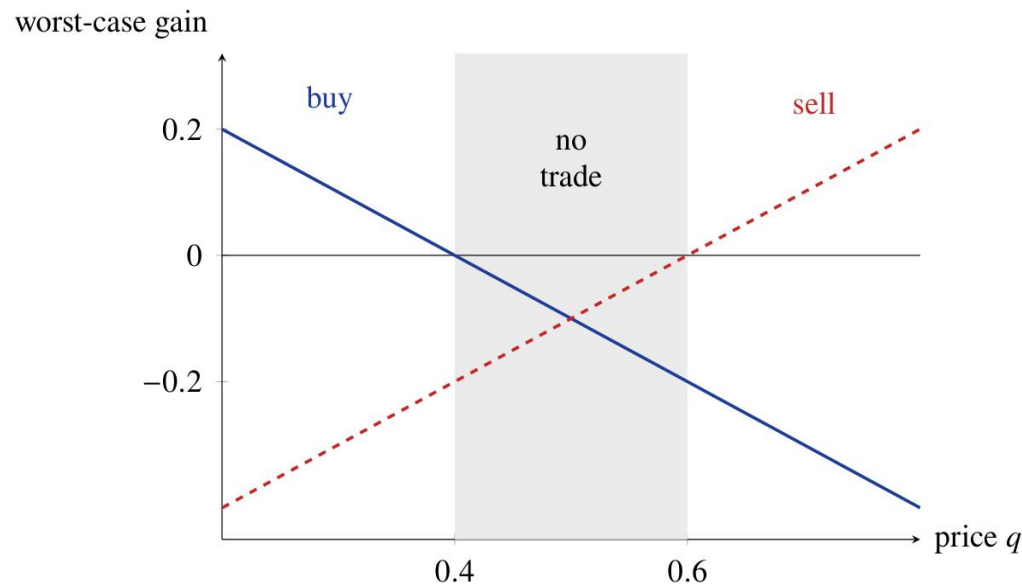
$$\min_p (p - q) = 0.4 - q$$

- Sell one unit, worst case:

$$\min_p (q - p) = q - 0.6$$

MEU and No Trade

- For $0.4 < q < 0.6$, both positions lose in the worst case: the individual neither buys nor sells.
 - An SEU individual trades at every $q \neq p$.



Choquet Expected Utility

Choquet Expected Utility

- Schmeidler (1989) replaces probabilities with a **capacity** v , which assigns a number to each event:
$$v(\emptyset) = 0, \quad v(S) = 1$$
$$A \subseteq B \text{ implies } v(A) \leq v(B)$$
- v need not be additive: for disjoint events, $v(A \cup B)$ can differ from $v(A) + v(B)$.

Choquet Expected Utility

- With two states and $x_a \geq x_b$, the Choquet integral is

$$CEU(f) = v(a)u(x_a) + [1 - v(a)]u(x_b)$$

- The better outcome gets weight $v(a)$; the worse outcome gets the rest.
 - If v is additive, this is SEU.

Choquet Expected Utility

- In general, order outcomes from best to worst,

$$x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(S)}$$

- let E_k be the states with the k best outcomes, and $E_0 = \emptyset$:

$$CEU(f) = \sum_{k=1}^S [v(E_k) - v(E_{k-1})]u(x_{(k)})$$

Choquet Expected Utility

- **Example:** $v(a) = 0.4$, $v(b) = 0.5$, with $u(x) = x$.
 - \$100 in state a, \$0 in b: $CEU = 0.4(100) = 40$.
 - \$100 in state b, \$0 in a: $CEU = 0.5(100) = 50$.
 - \$100 in a, \$60 in b: $CEU = 0.4(100) + 0.6(60) = 76$.
 - \$60 in a, \$100 in b: $CEU = 0.5(100) + 0.5(60) = 80$.
- Since $v(a) + v(b) = 0.9 < 1$, the missing 0.1 always goes to the worse outcome: pessimism.

Choquet Expected Utility

- A capacity is **convex** (supermodular) if, for all events A and B ,

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$$

- Schmeidler (1989): with CEU, uncertainty aversion holds if and only if v is convex.

- In the example,

$$v(a) + v(b) = 0.9 \leq v(S) + v(\emptyset) = 1$$

CEU and the Ellsberg Paradox

- Take the capacity

$$v(R) = \frac{1}{3}, \quad v(B) = v(G) = \frac{1}{6}$$

$$v(B \cup G) = \frac{2}{3}, \quad v(R \cup B) = v(R \cup G) = \frac{1}{2}$$

- It is monotone and convex.

CEU and the Ellsberg Paradox

- Each bet pays \$1,000 on one event, so its CEU is the capacity of that event:

Bet	Pays on	CEU
A	Red	$1/3$
B	Blue	$1/6$
C	Blue or green	$2/3$
D	Red or green	$1/2$

CEU and MEU

- The **core** of a capacity is the set of priors that give every event at least its capacity,
 $\text{core}(v) = \{p : p(E) \geq v(E) \text{ for every event } E\}$
- If v is convex, CEU equals MEU with $C = \text{core}(v)$.

– In the Ellsberg example, the core is

$$\{(\frac{1}{3}, p_B, \frac{2}{3} - p_B) : p_B \in [\frac{1}{6}, \frac{1}{2}]\}$$

CEU and MEU

- This is the set of priors used in the MEU slides, which is why CEU and MEU give the same values for A, B, C, and D.
- Not every MEU set of priors is the core of a capacity, so MEU is more general than convex CEU.

A Simple Capacity

- Start from a prior p and a confidence level $\lambda \in [0, 1]$:

$$v(A) = \lambda p(A) \text{ for } A \neq S, \quad v(S) = 1$$

- Then

$$CEU(f) = \lambda E_p[u(f)] + (1 - \lambda) \min_s u(f(s))$$

– $\lambda = 1$: SEU. $\lambda = 0$: worst outcome only.

A Simple Capacity

- **Example:** $\lambda = 0.8$, $p = 0.5$, and an act paying \$100 if rain and \$60 if sun:
$$0.8(0.5 \cdot 100 + 0.5 \cdot 60) + 0.2 \cdot 60 = 64 + 12 = 76$$
- Below its SEU of 80: the individual shifts weight 0.2 to the worst outcome.

The Smooth Ambiguity Model

Smooth Ambiguity

- Klibanoff, Marinacci, and Mukerji (2005):

$$U(f) = \sum_{\theta} \mu(\theta) \varphi\left(\sum_s p_{\theta}(s) u(f(s))\right)$$

- p_{θ} : possible priors; μ : the individual's belief over them.
- φ : attitude toward ambiguity.

Smooth Ambiguity

- $\varphi(t) = t$: SEU with the average prior.
- φ concave: ambiguity aversion, since spreads in expected utility across priors are penalized.
- As φ becomes infinitely concave, the model approaches MEU over the priors in the support of μ .
- Ambiguity (μ) and the attitude toward it (φ) are separate objects.

Smooth Ambiguity and Ellsberg

- Let p_B be $1/6$ or $1/2$ with equal probability, and
$$\varphi(t) = -e^{-\alpha t}, \quad \alpha > 0$$
- The certainty equivalent of bet B, in units of u , solves

$$\varphi(t_B) = \frac{1}{2} \varphi\left(\frac{1}{6}\right) + \frac{1}{2} \varphi\left(\frac{1}{2}\right)$$

Smooth Ambiguity and Ellsberg

- Solving,

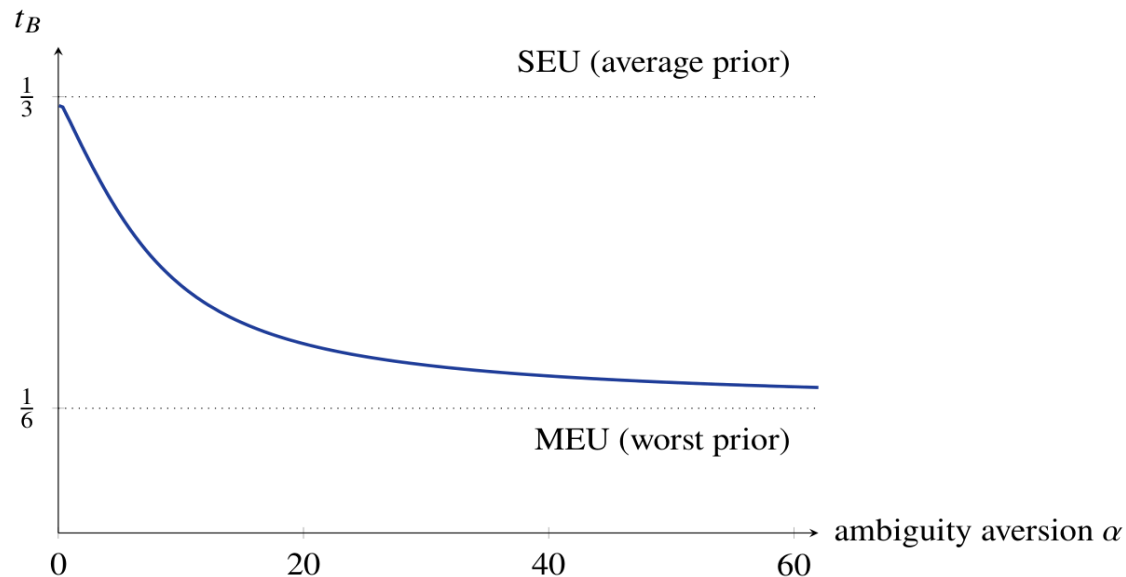
$$t_B = -\frac{1}{\alpha} \ln\left[\frac{1}{2}e^{-\frac{\alpha}{6}} + \frac{1}{2}e^{-\frac{\alpha}{2}}\right]$$

- Bet A has a known value of 1/3; bet C, of 2/3.

α	0.01	1	5	20	100
t_B	0.333	0.320	0.271	0.201	0.174
t_D	0.667	0.653	0.604	0.535	0.507

Smooth Ambiguity and Ellsberg

- More ambiguity aversion moves the value of bet B from the SEU value, $1/3$, to the MEU value, $1/6$.



Smooth Ambiguity: Investment

- Wealth 1; invest x in an asset with gross return R :
$$w = 1 + (R - 1)x, \quad R \in \{0.75, 2\}$$
$$u(w) = 10w - w^2$$
- Two priors on the low return, equally likely:
$$\Pr(R = 0.75) \in \{0.25, 0.9\}$$

Smooth Ambiguity: Investment

- Expected net return per dollar:
 - Good prior:
$$0.25(-0.25) + 0.75(1) = 0.6875$$
 - Bad prior:
$$0.9(-0.25) + 0.1(1) = -0.125$$
- With $\varphi(t) = t^\rho$, $\rho \in (0, 1]$, the individual maximizes

$$V(x) = \frac{1}{2} [E_{0.25} u(w)]^\rho + \frac{1}{2} [E_{0.9} u(w)]^\rho$$

Smooth Ambiguity: Investment

- Relative ambiguity aversion:

$$-\frac{t\varphi''(t)}{\varphi'(t)} = 1 - \rho$$

- Values of x^* above 1 mean the individual borrows to invest.

ρ	1 (SEU)	0.5	0.25	0.1
x^*	2.44	1.92	1.69	1.58

Smooth Ambiguity: Investment

- More ambiguity aversion (smaller ρ) lowers the investment.
- Under MEU with the same two priors, the bad prior has a negative expected return, so $x^* = 0$.
 - The smooth model gives an interior answer between these two extremes.

Comparing the Models

Comparing the Models

- Four ways to evaluate an act:

Model	Beliefs	Evaluation
SEU	One prior	Expected utility
MEU	Set of priors	Worst prior
CEU	Capacity	Choquet integral
Smooth	Prior over priors	Concave ϕ of expected utility

Summary

- Ellsberg choices reveal aversion to unknown probabilities.
- MEU: many priors, worst case.
- CEU: non-additive probabilities; convex capacities give MEU.
- Smooth ambiguity separates the amount of ambiguity from the attitude toward it.

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