

Microeconomic Theory II - UNI, Lima

Homework #4 - Answer key

1. **(24 points) Exercise 1 - Cournot competition with two-sided incomplete information.** Consider two firms competing à la Cournot, facing inverse demand function $p(Q) = a - Q$, where $Q = q_1 + q_2$ denotes aggregate output. Every firm i 's marginal cost is privately known, and takes value c_L with probability $\theta \in (0, 1)$ and value $c_H > c_L$ with probability $1 - \theta$. The two draws are independent, the prior θ is common knowledge, and $a > c_H$. Throughout, denote by

$$\bar{c} \equiv \theta c_L + (1 - \theta) c_H \quad \text{and} \quad \Delta \equiv c_H - c_L > 0$$

the expected marginal cost and the cost gap.

- (a) (3 points) Describe this Bayesian game (players, type spaces, prior beliefs, strategies and payoffs). Then find the best response function of every type of firm i , expressed as a function of the expected output of its rival, $\bar{Q} \equiv \theta q_L + (1 - \theta) q_H$, and check its second-order condition.

- *The Bayesian game.* The players are firms 1 and 2. Every firm i has type space $T_i = \{c_L, c_H\}$, and nature draws the two types independently, assigning probability θ to c_L and $1 - \theta$ to c_H in each draw. A strategy of firm i is a pair of outputs, one for each of its types,

$$q_i : T_i \rightarrow \mathbb{R}_+, \quad \text{that is, } (q_L, q_H)$$

and firm i 's payoff, conditional on its own type c , is its expected profit $E[(a - q_i - q_j - c)q_i]$, where the expectation is taken over its rival's type.

- *Best response function.* Since the two draws are independent, firm i 's belief about its rival's type coincides with the prior, so firm i of type c solves

$$\max_{q_i \geq 0} E\pi_i = (a - q_i - \bar{Q} - c)q_i, \quad \text{where } \bar{Q} \equiv \theta q_L + (1 - \theta)q_H$$

Differentiating with respect to q_i , yields

$$a - 2q_i - \bar{Q} - c = 0$$

Solving for q_i , we obtain the best response function of type c ,

$$q_i(\bar{Q}; c) = \frac{a - c - \bar{Q}}{2}$$

The second-order condition holds since $\frac{\partial^2 E\pi_i}{\partial q_i^2} = -2 < 0$.

- Note that the two types share the same slope, $-\frac{1}{2}$, and differ only in their vertical intercept, $\frac{a-c}{2}$, which is larger for the low-cost type. Note, in addition, that a firm's own output enters the rival's problem only through the expectation $\bar{Q}$, which is what makes the four-equation system collapse to a single unknown in part (b).
- (b) (5 points) Solve the resulting four-equation system for the symmetric Bayesian Nash equilibrium (BNE), q_L^* and q_H^* . Report $\bar{Q}^*$ as well, and show that every type's equilibrium profit equals the square of its equilibrium output.
- *The system.* Applying the best response function of part (a) to each type of each firm, and imposing symmetry across firms, the equilibrium solves

$$\begin{aligned} q_L &= \frac{a - c_L - \bar{Q}}{2}, \quad q_H = \frac{a - c_H - \bar{Q}}{2} \\ \bar{Q} &= \theta q_L + (1 - \theta) q_H \end{aligned}$$

which is a system of four equations, two for each firm, in the two unknowns q_L and q_H .

- *Expected output.* Multiplying the first equation by θ , the second by $1 - \theta$, and adding them, yields

$$\bar{Q} = \frac{a - \bar{c} - \bar{Q}}{2}$$

which simplifies to $3\bar{Q} = a - \bar{c}$. Solving for $\bar{Q}$, we obtain

$$\bar{Q}^* = \frac{a - \bar{c}}{3}$$

which coincides with the per-firm output of a symmetric complete-information Cournot duopoly in which both firms produce at the expected cost $\bar{c}$.

- *Equilibrium outputs.* Inserting $\bar{Q}^*$ into each type's best response function, we obtain

$$\begin{aligned} q_L^* &= \frac{1}{2} \left(a - c_L - \frac{a - \bar{c}}{3} \right) = \frac{2a - 3c_L + \bar{c}}{6} \\ q_H^* &= \frac{1}{2} \left(a - c_H - \frac{a - \bar{c}}{3} \right) = \frac{2a - 3c_H + \bar{c}}{6} \end{aligned}$$

Substituting $\bar{c} = \theta c_L + (1 - \theta) c_H$ and rearranging, we find the equivalent decomposition

$$\begin{aligned} q_L^* &= \frac{a - c_L}{3} + \frac{(1 - \theta) \Delta}{6} \\ q_H^* &= \frac{a - c_H}{3} - \frac{\theta \Delta}{6} \end{aligned}$$

so that each type's output equals the output it would produce in a symmetric complete-information duopoly at its own cost, corrected by a term that reflects the rival's uncertainty.

- *Equilibrium profits.* From the first-order condition, $a - 2q - \bar{Q} - c = 0$, we obtain $a - q - \bar{Q} - c = q$, so the price-cost margin of every type coincides with its own output. Therefore,

$$\pi_L^* = (q_L^*)^2 \quad \text{and} \quad \pi_H^* = (q_H^*)^2$$

which is the familiar Cournot property, here holding type by type.

- (c) (5 points) Compare your equilibrium against two benchmarks. First, the complete-information Cournot duopoly in which both costs are commonly known, one firm producing at c_L and the other at c_H . Second, the one-sided-uncertainty model of the lecture, in which firm 1's cost is private while firm 2's cost is commonly known and equal to $\bar{c}$. Does two-sided uncertainty change the behavior of the informed firm relative to the one-sided case?

- *Complete-information benchmark.* When both costs are commonly known, the standard asymmetric-cost Cournot duopoly delivers

$$q_L^{CI} = \frac{a - 2c_L + c_H}{3} \quad \text{and} \quad q_H^{CI} = \frac{a - 2c_H + c_L}{3}$$

Comparing against part (b), the low-cost firm produces *less* under complete information than under two-sided uncertainty whenever θ is low, and the high-cost firm produces *more*, since under complete information its rival knows it is inefficient and expands against it.

- *One-sided benchmark.* Suppose instead that firm 2's cost is commonly known and equal to $\bar{c}$, while firm 1's cost remains private. Firm 2 solves $q_2 = \frac{a - \bar{c} - E[q_1]}{2}$ while each type of firm 1 solves $q_1 = \frac{a - c - q_2}{2}$. Simultaneously solving this system, we obtain

$$q_2^* = \frac{a - \bar{c}}{3}$$

and firm 1's type-contingent outputs coincide, term by term, with q_L^* and q_H^* of part (b).

- Therefore, two-sided uncertainty does not change the behavior of the informed firm at all. Intuitively, a firm's best response depends on its rival's output only through the expectation $\bar{Q}$, and that expectation equals $\frac{a - \bar{c}}{3}$ whether the rival is a single firm producing at the known cost $\bar{c}$ or a firm of unknown cost whose two types average to $\bar{c}$. What matters for a Cournot competitor is the mean of its rival's output, and adding uncertainty on the rival's side leaves that mean untouched.
- (d) (5 points) *Comparative statics.* Find how the spread $q_L^* - q_H^*$ depends on the prior θ and on the cost gap Δ . Compare this spread against the one arising under complete information. Then find how each type's output responds to a marginal increase in θ , and interpret.

- *The spread.* Subtracting the two equilibrium outputs of part (b), we obtain

$$q_L^* - q_H^* = \frac{(2a - 3c_L + \bar{c}) - (2a - 3c_H + \bar{c})}{6} = \frac{\Delta}{2}$$

so the spread is proportional to the cost gap alone and, note, completely independent of the prior,

$$\frac{\partial (q_L^* - q_H^*)}{\partial \theta} = 0$$

- *Comparison against complete information.* Using part (c), the complete-information spread is

$$q_L^{CI} - q_H^{CI} = \frac{3(c_H - c_L)}{3} = \Delta$$

so incomplete information halves the spread, $\frac{\Delta}{2} < \Delta$. Intuitively, under complete information a cost advantage operates through two channels: the firm expands because its own cost is low, and it expands again because its rival, observing that low cost, contracts. Under incomplete information the rival cannot condition its output on the realized cost, so only the direct channel survives and the strategic amplification disappears.

- *Effect of the prior.* Differentiating each equilibrium output with respect to θ , yields

$$\frac{\partial q_L^*}{\partial \theta} = -\frac{\Delta}{6} < 0 \quad \text{and} \quad \frac{\partial q_H^*}{\partial \theta} = -\frac{\Delta}{6} < 0$$

so both types unambiguously contract as the low cost becomes more likely. Intuitively, a higher θ lowers the expected cost $\bar{c}$, which raises the rival's expected output,

$$\frac{\partial \bar{Q}^*}{\partial \theta} = \frac{\Delta}{3} > 0$$

and, since the best response function has slope $-\frac{1}{2}$, every type absorbs exactly half of that increase, contracting by $\frac{\Delta}{6}$.

- Note that this argument also explains why the spread is invariant to θ . The prior enters both types' problems only through the common term $\bar{Q}^*$, so it shifts the two best response functions by the same amount and leaves their vertical gap, which is driven entirely by the cost difference, unchanged.
- (e) (3 points) Find the condition on θ guaranteeing that the equilibrium is interior, so that the high-cost type produces a positive quantity, and state the restriction on the primitives under which the equilibrium is interior for every $\theta \in (0, 1)$.
- *Interiority.* Since $q_L^* > q_H^*$, only the high-cost type can be driven to a corner. Using part (b), $q_H^* > 0$ if and only if

$$\frac{a - c_H}{3} > \frac{\theta \Delta}{6}$$

Solving for θ , we obtain

$$\theta < \bar{\theta} \equiv \frac{2(a - c_H)}{c_H - c_L}$$

- *Interiority for every prior.* The equilibrium is interior for every $\theta \in (0, 1)$ if and only if $\bar{\theta} \geq 1$, that is,

$$2(a - c_H) \geq \Delta, \text{ or equivalently, } 2a \geq 3c_H - c_L$$

which holds whenever the market is large relative to the cost gap. Intuitively, the high-cost type is squeezed out only when its rival is very likely to be efficient and the cost gap is wide, so that the expected output it faces leaves no residual demand worth serving.

- (f) (3 points) *Numerical example.* Evaluate your results at $a = 10$, $c_L = 1$ and $c_H = 4$, first for $\theta = \frac{1}{2}$ and then for $\theta = \frac{1}{3}$. Report $\bar{c}$, $\bar{Q}^*$, both outputs, both profits, and the spread. Compare against the complete-information duopoly with costs $(c_L, c_H) = (1, 4)$.

- *First panel, $\theta = \frac{1}{2}$.* The expected cost is $\bar{c} = \frac{1+4}{2} = \frac{5}{2}$ and $\Delta = 3$, so that

$$\begin{aligned}\bar{Q}^* &= \frac{10 - \frac{5}{2}}{3} = \frac{5}{2} \\ q_L^* &= \frac{10 - 1}{3} + \frac{\frac{1}{2} \cdot 3}{6} = 3 + \frac{1}{4} = \frac{13}{4} \\ q_H^* &= \frac{10 - 4}{3} - \frac{\frac{1}{2} \cdot 3}{6} = 2 - \frac{1}{4} = \frac{7}{4}\end{aligned}$$

As a check, $\frac{1}{2} \cdot \frac{13}{4} + \frac{1}{2} \cdot \frac{7}{4} = \frac{5}{2} = \bar{Q}^*$, as required. Profits are $\pi_L^* = \left(\frac{13}{4}\right)^2 = \frac{169}{16}$ and $\pi_H^* = \left(\frac{7}{4}\right)^2 = \frac{49}{16}$.

- *Second panel, $\theta = \frac{1}{3}$.* Now $\bar{c} = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 4 = 3$, so that $\bar{Q}^* = \frac{7}{3}$ and

$$q_L^* = 3 + \frac{\frac{2}{3} \cdot 3}{6} = \frac{10}{3} \quad \text{and} \quad q_H^* = 2 - \frac{\frac{1}{3} \cdot 3}{6} = \frac{11}{6}$$

Both outputs are larger than in the first panel, consistent with the negative sign of $\frac{\partial q^*}{\partial \theta}$ found in part (d).

- *The spread.* In the first panel the spread is $\frac{13}{4} - \frac{7}{4} = \frac{3}{2}$, and in the second panel it is $\frac{10}{3} - \frac{11}{6} = \frac{3}{2}$ as well, which confirms that the spread equals $\frac{\Delta}{2} = \frac{3}{2}$ regardless of the prior.
- *Complete-information comparison.* With commonly known costs $c_1 = 1$ and $c_2 = 4$, we obtain

$$q_L^{CI} = \frac{10 - 2 + 4}{3} = 4 \quad \text{and} \quad q_H^{CI} = \frac{10 - 8 + 1}{3} = 1$$

so the complete-information spread is $4 - 1 = 3 = \Delta$, twice the spread under incomplete information. Note that the efficient firm produces 4 rather than $\frac{13}{4}$, and the inefficient firm only 1 rather than $\frac{7}{4}$. Intuitively, private information protects the inefficient firm, which is treated by its rival as an average competitor rather than as the weak one it actually is, and it correspondingly harms the efficient one.

- *Interiority.* Here $\bar{\theta} = \frac{2(10-4)}{3} = 4 > 1$, so the equilibrium is interior for every prior, as part (e) anticipates.

2. **(25 points) Exercise 2 - First-price auction with a reserve price.** Consider a seller who owns one indivisible unit of a good, which she values at zero, and who faces $n = 2$ risk-neutral bidders. Every bidder i privately observes his own valuation v_i , independently drawn from a uniform distribution on $[0, 1]$. Before bids are submitted, the seller publicly commits to a reserve price $r \in [0, 1]$: bids below r are rejected, and if no bid is accepted the seller keeps the good. The auction is first-price, so the winner pays his own bid.

(a) (3 points) Find the participation cutoff, that is, the valuation below which a bidder submits no acceptable bid. Then, letting $b(v)$ denote the symmetric equilibrium bidding function of a participating bidder, find the probability that a bidder with valuation v wins the auction.

- *Participation cutoff.* An acceptable bid must satisfy $b \geq r$. A bidder with valuation $v < r$ would therefore have to pay at least $r > v$ upon winning, earning a negative surplus, so he submits no acceptable bid. A bidder with valuation $v > r$ can bid exactly r and earn $v - r > 0$ whenever he wins, so he participates. The cutoff is, therefore,

$$\bar{v} \equiv r$$

and the bidder with $v = r$ is indifferent, earning zero whether he participates or not.

- *Probability of winning.* Suppose $b(v)$ is increasing on $[r, 1]$, a property we confirm in part (c). A bidder with valuation $v \geq r$ wins whenever his rival either stays out, which occurs if $v_j < r$, or participates with a lower bid, which occurs if $r \leq v_j < v$. The two events together amount to $v_j < v$, so

$$\Pr(\text{win} \mid v) = F(v) = v$$

Note that the reserve price does not change the probability of winning of a participating bidder, since a rival who is excluded would have lost anyway.

(b) (4 points) Find the symmetric equilibrium bidding function $b(v)$ for every participating bidder. Then *verify* that your candidate is indeed a symmetric equilibrium: show that a bidder with valuation v who submits the bid of a rival type z finds $z = v$ optimal, checking the first-order condition, the second-order condition, and global optimality. Confirm, in addition, that participating dominates staying out.

- *The envelope argument.* Let $U(v)$ denote the equilibrium expected surplus of a bidder with valuation $v \geq r$,

$$U(v) = [v - b(v)] v$$

By the envelope theorem, $U'(v)$ equals the partial derivative of the objective with respect to v holding the bid fixed, which is the probability of winning found in part (a),

$$U'(v) = v$$

The boundary condition is $U(r) = 0$, since the cutoff bidder is indifferent. Integrating, we obtain

$$U(v) = \int_r^v y dy = \frac{v^2 - r^2}{2}$$

- *The bidding function.* Combining $[v - b(v)] v = \frac{v^2 - r^2}{2}$ and solving for $b(v)$, we obtain

$$b(v) = v - \frac{v^2 - r^2}{2v} = \frac{v^2 + r^2}{2v}$$

which can also be written as $b(v) = \frac{v}{2} + \frac{r^2}{2v}$. Setting $r = 0$ recovers the familiar $b(v) = \frac{v}{2}$ of the auction without a reserve price.

- *Verification, first-order condition.* Consider a bidder with valuation $v \geq r$ who submits the bid of a rival type $z \geq r$, thereby winning with probability z . His expected payoff is

$$\begin{aligned} \pi(z) &= [v - b(z)] z = \left(v - \frac{z^2 + r^2}{2z} \right) z \\ &= vz - \frac{z^2 + r^2}{2} \end{aligned}$$

Differentiating with respect to z , yields $v - z = 0$, so that $z = v$ solves the first-order condition and reporting truthfully is optimal.

- *Second-order and global optimality.* The second-order condition holds since $\frac{\partial^2 \pi}{\partial z^2} = -1 < 0$. For global optimality, subtract the equilibrium payoff,

$$\pi(z) - \pi(v) = vz - \frac{z^2 + r^2}{2} - v^2 + \frac{v^2 + r^2}{2} = -\frac{(z - v)^2}{2} \leq 0$$

with equality if and only if $z = v$. Therefore, no deviation to another participant's bid is profitable, for any z , not only locally.

- *Participation.* A bidder who stays out, or who submits any bid below r , earns zero. A bidder with $v \geq r$ who participates earns $U(v) = \frac{v^2 - r^2}{2} \geq 0$, which is strictly positive for $v > r$. Therefore, participating dominates staying out for every bidder above the cutoff, and the candidate is a symmetric equilibrium.

- (c) (4 points) Show that $b(v)$ is increasing in v on the participation region and that $b(r) = r$. How is a participating bidder's bid affected by a marginal increase in the reserve price? Which bidders raise their bids by the most?

- *Monotonicity.* Differentiating $b(v) = \frac{v^2 + r^2}{2v}$ with respect to v , yields

$$b'(v) = \frac{v^2 - r^2}{2v^2} > 0 \text{ for every } v > r$$

so the bidding function is increasing on the participation region, as assumed in part (a). At the cutoff, $b'(r) = 0$, so the schedule leaves the cutoff flat and then steepens.

- *Value at the cutoff.* Evaluating at $v = r$, we obtain $b(r) = \frac{r^2 + r^2}{2r} = r$, so the marginal participant bids exactly the reserve price and earns zero, which confirms the boundary condition used in part (b). Note, in addition, that $v - b(v) = \frac{(v - r)(v + r)}{2v} > 0$, so bidders still shade their bids below their valuations.
- *Effect of the reserve price.* Differentiating with respect to r , yields

$$\frac{\partial b(v)}{\partial r} = \frac{r}{v} > 0$$

so every participating bidder unambiguously bids more aggressively when the reserve price rises. Since $\frac{r}{v}$ decreases in v , the bidders who raise their bids by the most are those closest to the cutoff. Intuitively, the reserve price truncates the distribution of the rival's bids from below, which matters most for a bidder who would otherwise have submitted a very low bid and who now has to worry that a rival who participates at all is already bidding above r .

- (d) (5 points) Find the seller's expected revenue as a function of the reserve price, $R(r)$. Confirm that $R(0)$ coincides with the expected revenue of the standard first-price auction without a reserve price.

- *Setting up the integral.* A sale occurs if and only if the highest of the two valuations exceeds r , in which case the winner is that highest-valuation bidder and pays $b(v)$. The highest of two independent draws from $U[0, 1]$ has density $2v$, so

$$R(r) = \int_r^1 b(v) 2v \, dv = \int_r^1 \frac{v^2 + r^2}{2v} 2v \, dv = \int_r^1 (v^2 + r^2) \, dv$$

where the density and the denominator of $b(v)$ cancel, which is what makes this integral so simple.

- *Expected revenue.* Integrating, we obtain

$$\begin{aligned} R(r) &= \left[\frac{v^3}{3} + r^2 v \right]_r^1 = \frac{1}{3} + r^2 - \left(\frac{r^3}{3} + r^3 \right) \\ &= \frac{1}{3} + r^2 - \frac{4}{3} r^3 \end{aligned}$$

- *The benchmark without a reserve price.* Setting $r = 0$, we obtain $R(0) = \frac{1}{3}$, which coincides with the expected revenue of the standard first-price auction with two bidders, and with the expected second-highest valuation $E[v_{(2)}] = \frac{1}{3}$, as revenue equivalence requires.
- (e) (3 points) Find the revenue-maximizing reserve price r^* and check its second-order condition. Then repeat the exercise for an arbitrary number of bidders $n \geq 2$, and confirm that r^* is independent of n .
- *Optimal reserve price.* Differentiating $R(r)$ with respect to r , yields

$$R'(r) = 2r - 4r^2 = 2r(1 - 2r)$$

Solving for r , we obtain the interior solution

$$r^* = \frac{1}{2}$$

The second-order condition holds since $R''(r) = 2 - 8r$ evaluated at r^* gives $R''\left(\frac{1}{2}\right) = -2 < 0$. Expected revenue rises from $R(0) = \frac{1}{3}$ to

$$R\left(\frac{1}{2}\right) = \frac{1}{3} + \frac{1}{4} - \frac{4}{3} \cdot \frac{1}{8} = \frac{5}{12}$$

a gain of $\frac{5}{12} - \frac{1}{3} = \frac{1}{12}$.

- *Arbitrary number of bidders.* With n bidders, a participant with valuation v wins with probability v^{n-1} , so the envelope argument of part (b) delivers $U(v) = \int_r^v y^{n-1} dy = \frac{v^n - r^n}{n}$ and, therefore,

$$b_n(v) = v - \frac{v^n - r^n}{n v^{n-1}}$$

Repeating the deviation check of part (b) confirms that truthful bidding solves the bidder's problem for every n .

- *Expected revenue with n bidders.* Since the highest of n draws has density $n v^{n-1}$,

$$\begin{aligned} R_n(r) &= \int_r^1 b_n(v) n v^{n-1} dv = \int_r^1 [n v^n - (v^n - r^n)] dv \\ &= \int_r^1 [(n-1) v^n + r^n] dv = \frac{(n-1)(1 - r^{n+1})}{n+1} + r^n(1 - r) \end{aligned}$$

Differentiating with respect to r , and collecting terms, yields

$$R'_n(r) = n r^{n-1} (1 - 2r)$$

which vanishes at $r^* = \frac{1}{2}$ for every $n \geq 2$. Setting $n = 2$ recovers $2r(1 - 2r)$, as required.

- Therefore, the revenue-maximizing reserve price is $\frac{1}{2}$ regardless of how many bidders compete. Intuitively, the optimal reserve solves the condition that the marginal buyer's *virtual valuation* be zero, $r - \frac{1 - F(r)}{f(r)} = 0$, which for the uniform distribution reads $r - (1 - r) = 0$. That condition involves the seller's own valuation and the distribution of a *single* buyer's valuation, and never the number of buyers, since the reserve price binds only in the event that every rival has already been excluded.
- (f) (3 points) Explain intuitively why a reserve price raises the seller's revenue even though it excludes some buyers. Then compute total surplus as a function of r and decompose, at $r = r^*$, the seller's gain and the bidders' loss.
- *Orders of magnitude.* From part (c), a reserve price raises every participant's bid by $\frac{r^2}{2v}$, a gain of order r^2 that the seller collects in almost every state of the world. The cost of the reserve price, in contrast, materializes only when *both* valuations fall below r , an event of probability r^2 , and in that event the revenue forgone is itself of order r . The loss is, therefore, of order r^3 . Since r^3 vanishes faster than r^2 as r approaches zero, the gain unambiguously dominates for small reserve prices, which is why $r^* > 0$.
 - *Total surplus.* The good is allocated to the highest-valuation bidder whenever that valuation exceeds r , so

$$TS(r) = \int_r^1 v \, 2v \, dv = \frac{2(1 - r^3)}{3}$$

which unambiguously decreases in r , since $\frac{\partial TS}{\partial r} = -2r^2 < 0$. The reserve price is, therefore, purely a redistribution device that destroys surplus along the way.

- *Decomposition at $r^* = \frac{1}{2}$.* Total surplus falls from $TS(0) = \frac{2}{3}$ to $TS\left(\frac{1}{2}\right) = \frac{7}{12}$, an efficiency loss of $\frac{1}{12}$. Bidder surplus, the residual $TS - R$, falls from $\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ to $\frac{7}{12} - \frac{5}{12} = \frac{1}{6}$, a loss of $\frac{1}{6}$. The seller gains $\frac{1}{12}$, so that

$$\underbrace{\frac{1}{6}}_{\text{bidders' loss}} = \underbrace{\frac{1}{12}}_{\text{seller's gain}} + \underbrace{\frac{1}{12}}_{\text{efficiency loss}}$$

Intuitively, half of what the bidders give up reaches the seller and the other half is destroyed, since with probability $r^2 = \frac{1}{4}$ the good stays with a seller who values it at zero while a buyer valued it strictly more.

- (g) (3 points) *Numerical example.* Set $r = \frac{1}{2}$. Report the bid of a bidder with valuation $v = \frac{3}{4}$ and of a bidder with valuation $v = 1$, comparing each against the

bid the same bidder would submit if $r = 0$. Report the seller's expected revenue, the probability that the good goes unsold, total surplus, and the bidders' expected surplus, both at $r = \frac{1}{2}$ and at $r = 0$.

- *Bids.* Applying $b(v) = \frac{v^2 + r^2}{2v}$ with $r = \frac{1}{2}$,

$$b\left(\frac{3}{4}\right) = \frac{\frac{9}{16} + \frac{1}{4}}{\frac{3}{2}} = \frac{\frac{13}{16}}{\frac{3}{2}} = \frac{13}{24} \simeq 0.542$$

$$b(1) = \frac{1 + \frac{1}{4}}{2} = \frac{5}{8}$$

whereas with $r = 0$ the same bidders would submit $\frac{3}{8} = 0.375$ and $\frac{1}{2}$. The bidder with $v = \frac{3}{4}$ raises his bid by $\frac{13}{24} - \frac{9}{24} = \frac{1}{6}$, while the bidder with $v = 1$ raises his by only $\frac{1}{8}$, confirming that the effect is strongest near the cutoff.

- *Revenue and probability of sale.* Expected revenue is $R\left(\frac{1}{2}\right) = \frac{5}{12} \simeq 0.417$, against $R(0) = \frac{1}{3} \simeq 0.333$. The good goes unsold when both valuations fall below $\frac{1}{2}$, which occurs with probability $r^2 = \frac{1}{4}$, whereas with $r = 0$ a sale always takes place.
- *Surplus.* Total surplus is $TS\left(\frac{1}{2}\right) = \frac{7}{12}$ against $TS(0) = \frac{2}{3} = \frac{8}{12}$, and bidder surplus is $\frac{1}{6}$ against $\frac{1}{3}$. Therefore, the seller trades one quarter of the sale probability for a 25% increase in expected revenue, and the bidders pay for all of it.

3. (27 points) **Exercise 3 - Labor-market signaling with three worker types.**

Consider a worker whose productivity takes one of three values, $t_1 < t_2 < t_3$, with prior probabilities p_1 , p_2 and p_3 , which add up to one. The worker privately observes his productivity and then chooses an education level $e \geq 0$, at a cost $c(e, t) = \frac{e}{t}$. After observing e , but not t , two identical firms compete à la Bertrand in wages, so that the worker is paid a wage equal to his expected productivity given the firms' beliefs. The worker's payoff is $w - \frac{e}{t}$.

- (a) (4 points) Define the firms' beliefs and the wage schedule, and define a fully separating perfect Bayesian equilibrium (PBE) of this game. Show that, in every separating PBE, the lowest type acquires no education, $e_1 = 0$.

- *Beliefs and wages.* After observing education e , firms hold a belief $\mu(t_k | e)$ over the three types, with $\sum_{k=1}^3 \mu(t_k | e) = 1$. Bertrand competition in wages

drives each firm's expected profit to zero, so the wage schedule is

$$w(e) = \sum_{k=1}^3 \mu(t_k | e) t_k$$

which lies between t_1 and t_3 for every e .

- *Definition of a separating PBE.* A fully separating PBE is a triple of education levels (e_1, e_2, e_3) , all different from one another, together with a belief system such that: (i) beliefs on the equilibrium path are obtained by Bayes' rule and are, therefore, degenerate, $\mu(t_k | e_k) = 1$, so that $w(e_k) = t_k$; (ii) every type t_k maximizes $w(e) - \frac{e}{t_k}$ given the wage schedule, which requires that no type prefers the education level of another type nor any off-equilibrium education level; and (iii) off-equilibrium beliefs are arbitrary, and we assign $\mu(t_1 | e) = 1$ for every $e \notin \{e_1, e_2, e_3\}$, so that $w(e) = t_1$ off the equilibrium path.
- *The lowest type acquires no education.* Take any separating PBE and suppose, by contradiction, that $e_1 > 0$. On the equilibrium path type t_1 earns $t_1 - \frac{e_1}{t_1}$. Consider a deviation to $e = 0$. Whatever beliefs firms hold at $e = 0$, the wage they offer is at least t_1 , since t_1 is the lowest productivity in the support. Therefore, the deviation yields at least

$$t_1 - \frac{0}{t_1} = t_1 > t_1 - \frac{e_1}{t_1}$$

which is a profitable deviation, a contradiction. Therefore, $e_1 = 0$ in every separating PBE. Intuitively, the lowest type is already identified as the worst worker in the market, so any education it acquires is a pure cost that buys no additional wage.

- (b) (5 points) Write down the six incentive-compatibility constraints of a fully separating PBE, and characterize the entire set of separating education profiles (e_2, e_3) . Then find the least-costly separating PBE, along with the off-equilibrium beliefs that support it.

- *The six constraints.* On the equilibrium path type t_k earns $t_k - \frac{e_k}{t_k}$, with $e_1 = 0$. Requiring that no type prefers another type's education level, and multiplying each inequality by the relevant productivity, we obtain

$$\begin{aligned} t_1 &\geq t_2 - \frac{e_2}{t_1} &\iff e_2 &\geq t_1(t_2 - t_1) \\ t_1 &\geq t_3 - \frac{e_3}{t_1} &\iff e_3 &\geq t_1(t_3 - t_1) \\ t_2 - \frac{e_2}{t_2} &\geq t_1 &\iff e_2 &\leq t_2(t_2 - t_1) \\ t_2 - \frac{e_2}{t_2} &\geq t_3 - \frac{e_3}{t_2} &\iff e_3 - e_2 &\geq t_2(t_3 - t_2) \\ t_3 - \frac{e_3}{t_3} &\geq t_1 &\iff e_3 &\leq t_3(t_3 - t_1) \\ t_3 - \frac{e_3}{t_3} &\geq t_2 - \frac{e_2}{t_3} &\iff e_3 - e_2 &\leq t_3(t_3 - t_2) \end{aligned}$$

where the first four rows collect the constraints of types t_1 and t_2 , and the last two those of type t_3 .

- *The separating set.* Collecting these inequalities, the set of separating profiles is

$$\begin{aligned} t_1(t_2 - t_1) &\leq e_2 \leq t_2(t_2 - t_1) \\ e_2 + t_2(t_3 - t_2) &\leq e_3 \leq e_2 + t_3(t_3 - t_2) \end{aligned}$$

together with $t_1(t_3 - t_1) \leq e_3 \leq t_3(t_3 - t_1)$, which part (c) shows to be slack. Both intervals are non-empty since $t_1 < t_2 < t_3$, so a continuum of separating equilibria exists.

- *The least-costly separating PBE.* Education is a pure cost, so the least-costly separating equilibrium sets each education level at the lower end of its range, which makes the two *upward* adjacent constraints bind,

$$\begin{aligned} e_1^* &= 0 \\ e_2^* &= t_1(t_2 - t_1) \\ e_3^* &= t_1(t_2 - t_1) + t_2(t_3 - t_2) \end{aligned}$$

so that type t_1 is exactly indifferent between 0 and e_2^* , and type t_2 is exactly indifferent between e_2^* and e_3^* . Note that e_3^* is built recursively, adding to e_2^* the increment $t_2(t_3 - t_2)$ needed to deter the type immediately below.

- *Supporting beliefs.* These education levels are sustained by the off-equilibrium beliefs $\mu(t_1 | e) = 1$ for every $e \notin \{0, e_2^*, e_3^*\}$, which deliver the lowest possible wage $w(e) = t_1$ after any deviation. Under these beliefs the most attractive deviation is $e = 0$, since it yields the wage t_1 at no cost, and the third and fifth constraints above are exactly the requirement that types t_2 and t_3 do not take it.

- (c) (5 points) Consider the two *non-adjacent* constraints, namely the one preventing type t_1 from mimicking type t_3 and the one preventing type t_3 from mimicking type t_1 . Evaluate both at the least-costly separating profile and report the slack of each. Are they implied by the adjacent constraints?

- *Type t_1 does not mimic t_3 .* Evaluating $e_3 - t_1(t_3 - t_1)$ at the least-costly profile, and expanding,

$$\begin{aligned} e_3^* - t_1(t_3 - t_1) &= t_1 t_2 - t_1^2 + t_2 t_3 - t_2^2 - t_1 t_3 + t_1^2 \\ &= t_1(t_2 - t_3) - t_2(t_2 - t_3) = (t_2 - t_1)(t_3 - t_2) > 0 \end{aligned}$$

so the constraint holds with slack $(t_2 - t_1)(t_3 - t_2)$.

- *Type t_3 does not mimic t_1 .* Writing $A \equiv t_2 - t_1 > 0$ and $B \equiv t_3 - t_2 > 0$, and evaluating $t_3(t_3 - t_1) - e_3$ at the least-costly profile, we obtain

$$t_3(t_3 - t_1) - e_3^* = A^2 + AB + B^2 > 0$$

so this constraint holds with slack as well, and the slack widens as the type space spreads out.

- *Are they implied?* They are, and not only at the least-costly profile. The smallest admissible e_3 in the whole separating set is $t_1(t_2 - t_1) + t_2(t_3 - t_2)$, which exceeds $t_1(t_3 - t_1)$ by $(t_2 - t_1)(t_3 - t_2)$, and the largest admissible e_3 is $t_2(t_2 - t_1) + t_3(t_3 - t_2)$, which falls short of $t_3(t_3 - t_1)$ by the same amount $(t_2 - t_1)(t_3 - t_2)$. Therefore, the chain of adjacent constraints implies the non-adjacent ones throughout.
 - Intuitively, this is the single-crossing property doing its work. The marginal cost of education, $\frac{1}{t}$, decreases in productivity, so a constraint that stops type t_2 from climbing to e_3 stops the even more reluctant type t_1 by a wider margin. Note, however, that students must *verify* this rather than assume it: with a cost function failing single crossing, a non-adjacent constraint can bind and the recursive construction of e_3^* would no longer be valid.
- (d) (4 points) Consider now a partially pooling PBE in which type t_1 separates at $e_1 = 0$ while types t_2 and t_3 pool at a common education level $e_P > 0$. Find the wage that firms offer at e_P , the range of education levels e_P sustaining this equilibrium, and the off-equilibrium beliefs it requires.

- *The pooling wage.* After observing e_P , Bayes' rule delivers $\mu(t_2 | e_P) = \frac{p_2}{p_2 + p_3}$ and $\mu(t_3 | e_P) = \frac{p_3}{p_2 + p_3}$, so the wage is

$$\hat{t} \equiv \frac{p_2 t_2 + p_3 t_3}{p_2 + p_3} \in (t_2, t_3)$$

while $w(0) = t_1$, since only type t_1 selects $e = 0$.

- *Incentive constraints.* Type t_1 must not join the pool, and types t_2 and t_3 must not abandon it,

$$\begin{aligned} t_1 \geq \hat{t} - \frac{e_P}{t_1} &\iff e_P \geq t_1 (\hat{t} - t_1) \\ \hat{t} - \frac{e_P}{t_2} \geq t_1 &\iff e_P \leq t_2 (\hat{t} - t_1) \\ \hat{t} - \frac{e_P}{t_3} \geq t_1 &\iff e_P \leq t_3 (\hat{t} - t_1) \end{aligned}$$

The third constraint is implied by the second, since their difference is $(t_3 - t_2)(\hat{t} - t_1) > 0$.

- *The range.* Collecting the binding pair, the pooling education level must satisfy

$$t_1 (\hat{t} - t_1) \leq e_P \leq t_2 (\hat{t} - t_1)$$

an interval of length $(t_2 - t_1)(\hat{t} - t_1) > 0$, so a continuum of such equilibria exists.

- *Off-equilibrium beliefs.* These equilibria require $\mu(t_1 | e) = 1$ for every $e \notin \{0, e_P\}$, so that any deviation is met with the lowest wage t_1 . Note that this is the same pessimistic belief that supported the separating equilibria, and it is exactly the belief the Intuitive Criterion questions in part (e).

(e) (5 points) Apply the Cho and Kreps' (1987) Intuitive Criterion. Does the partially pooling PBE of part (d) survive? Which separating PBEs survive? Does the criterion select the least-costly separating PBE, as it does in the two-type model of the lecture? Interpret.

- *The criterion.* For an off-equilibrium education e' , say that e' is equilibrium dominated for type t when even the most favorable wage in the support, t_3 , leaves that type worse off than in equilibrium, $U^*(t) > t_3 - \frac{e'}{t}$. Let $J(e')$ collect those types. The criterion requires firms to place zero probability on $J(e')$, and the equilibrium fails if some remaining type strictly gains from e' under *every* belief supported on the complement, that is, even under the lowest wage that complement can generate.
- *The pooling PBE fails.* Consider e' in the interval

$$e_P + t_2(t_3 - \hat{t}) < e' < e_P + t_3(t_3 - \hat{t})$$

which is non-empty since its length is $(t_3 - t_2)(t_3 - \hat{t}) > 0$. The first inequality says that $\hat{t} - \frac{e_P}{t_2} > t_3 - \frac{e'}{t_2}$, so e' is equilibrium dominated for type t_2 . It is equilibrium dominated for type t_1 as well, since

$$e' > t_1(\hat{t} - t_1) + t_2(t_3 - \hat{t}) = t_1(t_3 - t_1) + (t_2 - t_1)(t_3 - \hat{t}) > t_1(t_3 - t_1)$$

where the first inequality uses $e_P \geq t_1(\hat{t} - t_1)$ from part (d). Therefore, $J(e') = \{t_1, t_2\}$, beliefs must concentrate on t_3 , the wage is t_3 , and the second inequality above delivers

$$t_3 - \frac{e'}{t_3} > \hat{t} - \frac{e_P}{t_3}$$

so type t_3 strictly gains. The partially pooling PBE does not survive the Intuitive Criterion.

- *Separating PBEs, the restriction on e_3 .* Take a separating PBE $(0, e_2, e_3)$ and define

$$E_1 \equiv t_1(t_3 - t_1) \quad \text{and} \quad E_2 \equiv e_2 + t_2(t_3 - t_2)$$

so that type t_1 is pruned at $e' > E_1$ and type t_2 at $e' > E_2$. Note that $E_2 > E_1$ throughout the separating set, since $E_2 \geq t_1(t_2 - t_1) + t_2(t_3 - t_2) = E_1 + (t_2 - t_1)(t_3 - t_2)$. Therefore, for any $e' \in (E_2, e_3)$ only type t_3 survives the pruning, the wage becomes t_3 , and type t_3 strictly gains since $e' < e_3$. Such an e' exists whenever $e_3 > E_2$, so survival requires

$$e_3 = e_2 + t_2(t_3 - t_2)$$

that is, type t_2 's upward constraint must bind.

- *Separating PBEs, the restriction on e_2 .* Consider now $e' \in (E_1, e_2)$. Since $e' < e_2 < e_3 = E_2$, neither t_2 nor t_3 is pruned, and only t_1 is. Beliefs must be supported on $\{t_2, t_3\}$, so the lowest wage they can generate is t_2 , and type t_2 then earns at least $t_2 - \frac{e'}{t_2} > t_2 - \frac{e_2}{t_2}$, a strict gain. Such an e' exists whenever $e_2 > E_1$, so survival additionally requires $e_2 \leq t_1(t_3 - t_1)$.

- *The surviving set.* Combining these restrictions with the separating set of part (b), the equilibria surviving the Intuitive Criterion are those with $e_3 = e_2 + t_2(t_3 - t_2)$ and

$$t_1(t_2 - t_1) \leq e_2 \leq \min\{t_2(t_2 - t_1), t_1(t_3 - t_1)\}$$

- *Does the criterion select the least-costly PBE?* It does not, in general. When $t_1(t_3 - t_1) \geq t_2(t_2 - t_1)$ the second restriction is vacuous and an entire continuum of separating equilibria survives, of which the least-costly one is merely the lower endpoint. The criterion bites on e_2 if and only if

$$t_1(t_3 - t_1) < t_2(t_2 - t_1)$$

which requires the upper gap $t_3 - t_2$ to be small relative to the lower gap $t_2 - t_1$, and even then it only tightens the upper bound rather than pinning e_2 down.

- Intuitively, in the two-type model any education just below e_2 leaves a single candidate in the complement, the high type, so firms are forced to read the deviation favorably and e_2 is driven to its least-costly level. With three types, an education just below e_2 remains attractive to t_3 as well, so the criterion can only rule out t_1 and the wage it forces, t_2 , is exactly the wage type t_2 already earns. In contrast, near the *top* of the education schedule the pruning is sharp, since a high e' is unattractive to both lower types, which is why the criterion pins e_3 down but not e_2 . Refining further requires a stronger criterion, such as D1.
- (f) (4 points) *Numerical example.* Evaluate your results at $t_1 = 1$, $t_2 = 2$ and $t_3 = 4$, with $p_1 = p_2 = p_3 = \frac{1}{3}$. Report the least-costly separating profile, the entire separating set, the slack of every incentive constraint, the payoffs of every type, the pooling equilibrium of part (d) at $e_P = 3$, and an explicit deviation that destroys it.

- *Least-costly separating profile.* Applying part (b),

$$e_1^* = 0, \quad e_2^* = 1(2 - 1) = 1, \quad e_3^* = 1 + 2(4 - 2) = 5$$

with wages $w = 1$, $w = 2$ and $w = 4$, and payoffs

$$U_1 = 1, \quad U_2 = 2 - \frac{1}{2} = \frac{3}{2}, \quad U_3 = 4 - \frac{5}{4} = \frac{11}{4}$$

- *The separating set.* The bounds of part (b) read $1 \leq e_2 \leq 2$ and $e_2 + 4 \leq e_3 \leq e_2 + 8$, together with $3 \leq e_3 \leq 12$.
- *Slack of every constraint at the least-costly profile.* Evaluating the six inequalities of part (b),

$$\begin{aligned} t_1 \text{ vs } t_2 : e_2 - 1 &= 0, & t_2 \text{ vs } t_1 : 2 - e_2 &= 1 \\ t_2 \text{ vs } t_3 : (e_3 - e_2) - 4 &= 0, & t_3 \text{ vs } t_2 : 8 - (e_3 - e_2) &= 4 \\ t_1 \text{ vs } t_3 : e_3 - 3 &= 2, & t_3 \text{ vs } t_1 : 12 - e_3 &= 7 \end{aligned}$$

so the two upward adjacent constraints bind and the remaining four are slack. The two non-adjacent slacks, 2 and 7, match the formulas of part (c), since $(t_2 - t_1)(t_3 - t_2) = 1 \cdot 2 = 2$ and $A^2 + AB + B^2 = 1 + 2 + 4 = 7$.

- *The pooling equilibrium.* The pooling wage is $\hat{t} = \frac{2+4}{2} = 3$, and part (d) requires $e_P \in [1(3-1), 2(3-1)] = [2, 4]$, so $e_P = 3$ is admissible. Payoffs become

$$U_1 = 1, \quad U_2 = 3 - \frac{3}{2} = \frac{3}{2}, \quad U_3 = 3 - \frac{3}{4} = \frac{9}{4}$$

so type t_3 is worse off than in the least-costly separating PBE, $\frac{9}{4} < \frac{11}{4}$, while type t_2 is exactly as well off.

- *A deviation that destroys the pooling PBE.* Part (e) requires $e' \in (3 + 2(4-3), 3 + 4(4-5), 7)$, so take $e' = 6$. Even with the most favorable wage $t_3 = 4$, types t_1 and t_2 lose,

$$4 - \frac{6}{1} = -2 < 1 = U_1 \quad \text{and} \quad 4 - \frac{6}{2} = 1 < \frac{3}{2} = U_2$$

so $e' = 6$ is equilibrium dominated for both. Beliefs must, therefore, place probability one on t_3 , delivering the wage 4, and type t_3 strictly gains,

$$4 - \frac{6}{4} = \frac{5}{2} > \frac{9}{4} = U_3$$

which destroys the pooling equilibrium.

- *The surviving separating equilibria.* Here $t_1(t_3 - t_1) = 3$ exceeds $t_2(t_2 - t_1) = 2$, so the restriction on e_2 derived in part (e) is vacuous, and the surviving set is the whole family

$$e_2 \in [1, 2] \quad \text{with} \quad e_3 = e_2 + 4$$

The Intuitive Criterion pins the increment $e_3 - e_2$ down to 4 but leaves e_2 free, which confirms that it does not select the least-costly separating equilibrium in this three-type model. Had the top gap been narrower, for instance $t_3 = \frac{16}{5}$ with $t_2 = 3$, the bound would have been $t_1(t_3 - t_1) = \frac{11}{5} < t_2(t_2 - t_1) = 6$, and the criterion would have tightened the surviving set.

4. **(24 points) Exercise 4 - Entry deterrence against an incumbent of unknown type, with a noisy signal.** Consider an incumbent firm that is either *tough* with prior probability $\mu \in (0, 1)$ or *weak* with probability $1 - \mu$, and a potential entrant. Before choosing whether to enter, the entrant observes a public signal $s \in \{t, w\}$ about the incumbent's type, which is correct with probability $\Pr(s = t \mid \text{tough}) = \Pr(s = w \mid \text{weak}) = \rho \in \left(\frac{1}{2}, 1\right]$. Fighting is a dominant action for the tough incumbent and accommodating is a dominant action for the weak incumbent, so the entrant earns $-F < 0$ if he is fought, $A > 0$ if he is accommodated, and 0 if he stays out.

- (a) (3 points) Find the entrant's posterior belief that the incumbent is tough after each signal realization, μ_t and μ_w , using Bayes' rule. Show that $\mu_w < \mu < \mu_t$ for every $\rho > \frac{1}{2}$, and confirm that the two posteriors average back to the prior.

- *Posterior after the tough signal.* Applying Bayes' rule,

$$\mu_t \equiv \Pr(\text{tough} \mid s = t) = \frac{\rho\mu}{\rho\mu + (1 - \rho)(1 - \mu)}$$

where the denominator is the unconditional probability of observing the tough signal, $\Pr(s = t) = \rho\mu + (1 - \rho)(1 - \mu)$.

- *Posterior after the weak signal.* Similarly,

$$\mu_w \equiv \Pr(\text{tough} \mid s = w) = \frac{(1 - \rho)\mu}{(1 - \rho)\mu + \rho(1 - \mu)}$$

- *Ranking.* Subtracting the prior from each posterior, we obtain

$$\begin{aligned}\mu_t - \mu &= \frac{\mu(1 - \mu)(2\rho - 1)}{\rho\mu + (1 - \rho)(1 - \mu)} > 0 \\ \mu - \mu_w &= \frac{\mu(1 - \mu)(2\rho - 1)}{(1 - \rho)\mu + \rho(1 - \mu)} > 0\end{aligned}$$

where both signs follow from $\rho > \frac{1}{2}$, so that $\mu_w < \mu < \mu_t$. Intuitively, an informative signal moves the entrant's belief in the direction of the realization, and the movement widens as ρ grows.

- *Consistency.* Weighting each posterior by the probability of the corresponding signal, we obtain

$$\Pr(s = t)\mu_t + \Pr(s = w)\mu_w = \rho\mu + (1 - \rho)\mu = \mu$$

so the posteriors average back to the prior, as the law of total probability requires.

- (b) (5 points) Find the cutoff posterior $\bar{\mu}$ such that the entrant enters if and only if his posterior lies below it. How is $\bar{\mu}$ affected by A and by F ?

- *The entrant's problem.* Since fighting is dominant for the tough incumbent and accommodating is dominant for the weak one, an entrant holding posterior p earns $(1 - p)A - pF$ from entering and 0 from staying out. Therefore, he enters if and only if

$$(1 - p)A - pF > 0$$

Solving for p , we obtain the cutoff

$$p < \bar{\mu} \equiv \frac{A}{A + F}$$

- *Comparative statics.* Differentiating, yields

$$\frac{\partial \bar{\mu}}{\partial A} = \frac{F}{(A + F)^2} > 0 \quad \text{and} \quad \frac{\partial \bar{\mu}}{\partial F} = -\frac{A}{(A + F)^2} < 0$$

so the entrant tolerates a higher probability of meeting a tough incumbent when the prize from accommodation is larger, and a lower one when the punishment from a fight is more severe. Intuitively, $\bar{\mu}$ is the entrant's break-even probability, and it measures how much bad news the entry decision can absorb.

- (c) (5 points) Find the Bayesian Nash equilibrium of this game as a function of μ and ρ . In particular, find the cutoff accuracy $\bar{\rho}$ above which the entrant is deterred after the tough signal, and the second cutoff that applies when the prior alone already deters entry. How is $\bar{\rho}$ affected by the prior μ ?

- *The incumbent's strategy.* Since fighting is dominant for the tough type and accommodating is dominant for the weak type, in every BNE the incumbent fights if and only if it is tough. The strategic content of the game lies entirely in the entrant's signal-contingent decision.
- *Deterrence after the tough signal.* The entrant stays out after $s = t$ if and only if $\mu_t \geq \bar{\mu}$, that is,

$$\rho\mu(1 - \bar{\mu}) \geq \bar{\mu}(1 - \rho)(1 - \mu)$$

Rearranging in terms of the likelihood ratio $\frac{\rho}{1 - \rho}$, and defining

$$\Lambda \equiv \frac{\bar{\mu}(1 - \mu)}{(1 - \bar{\mu})\mu}$$

the condition reads $\frac{\rho}{1 - \rho} \geq \Lambda$. Solving for ρ , we obtain the cutoff

$$\rho \geq \bar{\rho} \equiv \frac{\Lambda}{1 + \Lambda} = \frac{\bar{\mu}(1 - \mu)}{\bar{\mu}(1 - \mu) + (1 - \bar{\mu})\mu}$$

- *Equilibrium when the prior does not deter, $\mu < \bar{\mu}$.* In this case $\Lambda > 1$, so $\bar{\rho} > \frac{1}{2}$ and the cutoff is relevant. The BNE is

$$\begin{aligned} \rho &\in \left(\frac{1}{2}, \bar{\rho}\right) : \text{the entrant enters after both signals} \\ \rho &\geq \bar{\rho} : \text{the entrant stays out after } s = t \text{ and enters after } s = w \end{aligned}$$

and in both cases the incumbent fights if and only if it is tough. Note that the entrant never stays out after $s = w$ here, since $\mu_w < \mu < \bar{\mu}$.

- *Equilibrium when the prior already deters, $\mu \geq \bar{\mu}$.* Now $\Lambda \leq 1$, so $\bar{\rho} \leq \frac{1}{2}$ and the entrant stays out after $s = t$ for every $\rho > \frac{1}{2}$. The relevant cutoff becomes the one governing the weak signal, $\mu_w < \bar{\mu}$, which by the same algebra reads $\frac{1 - \rho}{\rho} < \Lambda$, or

$$\rho > \hat{\rho} \equiv \frac{1}{1 + \Lambda} = 1 - \bar{\rho}$$

Therefore, the entrant stays out after both signals when $\rho \leq \hat{\rho}$, and enters only after $s = w$ when $\rho > \hat{\rho}$. Intuitively, a signal that is too noisy cannot overturn a prior that was already unfavorable to entry.

- *Effect of the prior on the cutoff.* Writing $D \equiv \bar{\mu}(1 - \mu) + (1 - \bar{\mu})\mu$ and differentiating, yields

$$\frac{\partial \bar{\rho}}{\partial \mu} = -\frac{\bar{\mu}(1 - \bar{\mu})}{D^2} < 0 \quad \text{and} \quad \frac{\partial \bar{\rho}}{\partial \bar{\mu}} = \frac{\mu(1 - \mu)}{D^2} > 0$$

so the accuracy required to deter entry unambiguously decreases in the prior probability that the incumbent is tough, and increases in the entrant's tolerance $\bar{\mu}$. Intuitively, the signal has to do less work when the entrant is already inclined to believe that the incumbent is tough.

- (d) (5 points) *Comparative statics.* Is entry deterrence more effective as the signal becomes more accurate? Answer this question in three ways: using the probability that a tough incumbent avoids entry, the probability that a weak incumbent avoids entry, and the unconditional probability that entry does not occur. Do the three measures agree? Then find how the entrant's expected payoff responds to ρ , and interpret.

- *Type-conditional measures.* Consider the interesting region $\mu < \bar{\mu}$ with $\rho \geq \bar{\rho}$, so that entry is deterred if and only if $s = t$. Then

$$\Pr(\text{no entry} \mid \text{tough}) = \rho \quad \text{and} \quad \Pr(\text{no entry} \mid \text{weak}) = 1 - \rho$$

so the tough incumbent is shielded with probability ρ , which unambiguously increases in the accuracy of the signal, while the weak incumbent free-rides with probability $1 - \rho$, which unambiguously decreases in it.

- *Unconditional measure.* Averaging across types,

$$\Pr(\text{no entry}) = \rho\mu + (1 - \rho)(1 - \mu) = (1 - \mu) + \rho(2\mu - 1)$$

Differentiating with respect to ρ , yields

$$\frac{\partial \Pr(\text{no entry})}{\partial \rho} = 2\mu - 1$$

which is *negative* whenever $\mu < \frac{1}{2}$. Therefore, the three measures do not agree.

- Intuitively, a more accurate signal does not deter entry more often, it deters entry more *accurately*. When tough incumbents are rare, $\mu < \frac{1}{2}$, most of the deterrence that a noisy signal produces is undeserved, since it protects weak incumbents who happened to draw a misleading signal. Improving the technology removes that protection faster than it adds protection for the genuinely tough incumbent, so the overall frequency of deterrence falls even though every individual type is being classified correctly more often.

- *The entrant's expected payoff.* In the region $\rho \geq \bar{\rho}$ the entrant enters only after $s = w$, so his ex-ante expected payoff is

$$EU = \rho(1 - \mu)A - (1 - \rho)\mu F$$

Differentiating with respect to ρ , we obtain

$$\frac{\partial EU}{\partial \rho} = (1 - \mu)A + \mu F > 0$$

so the entrant's payoff unambiguously increases in the accuracy of the signal, which is the standard positive value of information. Note that EU is continuous at $\rho = \bar{\rho}$, where it equals the uninformed payoff $(1 - \mu)A - \mu F$, since the entrant is exactly indifferent at the cutoff.

- (e) (3 points) Evaluate the two limiting cases $\rho = \frac{1}{2}$ and $\rho = 1$, and relate each of them to a model we studied in class.

- *Uninformative signal*, $\rho = \frac{1}{2}$. Both posteriors collapse to the prior, $\mu_t = \mu_w = \mu$, so the signal carries no information and the entrant conditions on nothing. The game reduces to the static entry game with incomplete information of the lecture: the entrant enters if and only if $\mu < \bar{\mu}$, earning $(1 - \mu)A - \mu F$. Note that $\bar{\rho} > \frac{1}{2}$ in the region $\mu < \bar{\mu}$, so the entrant indeed enters at $\rho = \frac{1}{2}$, consistent with part (c).
- *Perfect signal*, $\rho = 1$. Now $\mu_t = 1$ and $\mu_w = 0$, so the entrant learns the incumbent's type before moving. The game reduces to the complete-information entry game, played twice: against a tough incumbent the entrant stays out, and against a weak one he enters and is accommodated. Entry occurs with probability $1 - \mu$ and the entrant's expected payoff reaches $(1 - \mu)A$, its upper bound.
- Note that the incumbent's type, and not any action it takes, drives the outcome throughout. The incumbent has no way to influence the signal, so this model isolates the informational side of entry deterrence and eliminates the strategic manipulation that a signaling model, such as limit pricing, would add.

- (f) (3 points) *Numerical example.* Evaluate your results at $A = 4$, $F = 6$ and $\mu = \frac{1}{4}$.

Report $\bar{\mu}$ and $\bar{\rho}$, both posteriors and the equilibrium for $\rho = \frac{3}{5}$, $\rho = \frac{2}{3}$ and $\rho = \frac{3}{4}$, the unconditional probability of no entry, and the value of the signal to the entrant.

- *Cutoffs.* The break-even posterior is $\bar{\mu} = \frac{4}{4 + 6} = \frac{2}{5}$, which exceeds the prior $\mu = \frac{1}{4}$, so the prior alone does not deter entry. The likelihood-ratio statistic is

$$\Lambda = \frac{\frac{2}{5} \cdot \frac{3}{4}}{\frac{3}{5} \cdot \frac{1}{4}} = \frac{\frac{10}{20}}{\frac{3}{20}} = 2, \text{ so that } \bar{\rho} = \frac{2}{3}$$

- *Noisy signal*, $\rho = \frac{3}{5}$. The posterior after the tough signal is

$$\mu_t = \frac{\frac{3}{5} \cdot \frac{1}{4}}{\frac{3}{5} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4}} = \frac{\frac{3}{20}}{\frac{4}{20}} = \frac{3}{4} < \frac{2}{5}$$

so the entrant enters after both signals and the signal is strategically irrelevant, as $\rho < \bar{\rho}$ anticipates.

- *At the cutoff*, $\rho = \frac{2}{3}$. Now $\mu_t = \frac{\frac{2}{3} \cdot \frac{1}{4}}{\frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4}} = \frac{2}{5} = \bar{\mu}$, so the entrant is exactly indifferent after the tough signal, confirming the cutoff.

- *Accurate signal*, $\rho = \frac{3}{4}$. The two posteriors are

$$\mu_t = \frac{\frac{3}{4} \cdot \frac{1}{4}}{\frac{3}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4}} = \frac{3}{5} > \frac{2}{5} \quad \text{and} \quad \mu_w = \frac{\frac{1}{4} \cdot \frac{3}{4}}{\frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4}} = \frac{3}{5} < \frac{2}{5}$$

so the entrant stays out after $s = t$ and enters after $s = w$, which is the partial-deterrence equilibrium of part (c).

- *Probability of no entry*. At $\rho = \frac{3}{4}$,

$$\Pr(\text{no entry}) = \frac{3}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{8}$$

whereas at $\rho = 1$ it falls to $\mu = \frac{1}{4}$. Deterrence becomes *less* frequent as the signal improves, exactly as part (d) anticipates for $\mu < \frac{1}{2}$, even though the tough incumbent is shielded with probability $\frac{3}{4}$ rather than $\frac{1}{4}$ of the time it deserves.

- *Value of the signal*. The entrant's expected payoff is

$$\begin{aligned} \rho \leq \frac{2}{3} : EU &= \frac{3}{4} \cdot 4 - \frac{1}{4} \cdot 6 = \frac{3}{2} \\ \rho = \frac{3}{4} : EU &= \frac{3}{4} \cdot \frac{3}{4} \cdot 4 - \frac{1}{4} \cdot \frac{1}{4} \cdot 6 = \frac{15}{8} \\ \rho = 1 : EU &= \frac{3}{4} \cdot 4 = 3 \end{aligned}$$

so the signal is worth $\frac{15}{8} - \frac{3}{2} = \frac{3}{8}$ at $\rho = \frac{3}{4}$, and $\frac{3}{2}$ at $\rho = 1$. The entrant gains from the signal while the weak incumbent loses from it, which is the sense in which a more open industry protects only the incumbents that genuinely deserve protection.