# Microeconomic Theory II - UNI, Lima
# Homework #3 - Answer key

1. **(24 points) Exercise 1 - Alternating-offer bargaining with asymmetric discount factors.** Two players bargain over a pie of size 1. Time is discrete, $t = 1, 2, \ldots$, and the players alternate in making offers: in odd periods player 1 proposes a split $(x_1, 1 - x_1)$ and player 2 either accepts or rejects it, whereas in even periods player 2 proposes a split $(1 - y_2, y_2)$ and player 1 either accepts or rejects it. If an offer is accepted the game ends; if the last offer of a finite-horizon game is rejected, both players earn zero. Player 1 discounts future payoffs at rate $\delta_1 \in (0, 1)$ and player 2 at rate $\delta_2 \in (0, 1)$, so a share $x$ agreed upon in period $t$ is worth $\delta_i^{t-1} x$ to player $i$. A player who is indifferent accepts.

   (a) (3 points) Solve the two-round game, $T = 2$, by backward induction. Report the equilibrium shares of both players.

   - *Last mover, player 2.* In period 2 player 2 proposes and, since a rejection ends the game with zero payoffs for both players, player 1 accepts any offer that gives him a non-negative share. Player 2 therefore keeps the entire pie, and the continuation payoffs entering period 2 are

   $$(V_1, V_2) = (0, 1)$$

   - *First mover, player 1.* In period 1 player 1 must offer player 2 a share that is at least as valuable as rejecting and proposing herself one period later, which is worth $\delta_2 V_2 = \delta_2$ to her. Since player 2 accepts when indifferent, player 1 offers exactly $\delta_2$, and keeps the rest.
   - *Summary.* The equilibrium shares of the two-round game are

   $$x_1 = 1 - \delta_2 \quad \text{and} \quad x_2 = \delta_2$$

   and agreement is reached immediately, in period 1. Note that player 1's share depends on his rival's discount factor alone. Intuitively, what player 1 must pay is the value that player 2 attaches to waiting, and player 1's own impatience never comes into play because he never waits.

   (b) (5 points) Solve the three-round game, $T = 3$, by backward induction. Report the equilibrium shares of both players, and show that player 1's share is larger than in part (a).

   - *Last mover, player 1.* In period 3 player 1 proposes and player 2 accepts anything non-negative, so player 1 keeps the entire pie and the continuation payoffs entering period 3 are $(1, 0)$.
   - *Second-to-last mover, player 2.* In period 2 player 2 must offer player 1 at least $\delta_1 \cdot 1 = \delta_1$, which is what player 1 obtains by rejecting and proposing in period 3. Offering exactly $\delta_1$, player 2 keeps

   $$1 - \delta_1$$

   so the continuation payoffs entering period 2 are $(\delta_1, 1 - \delta_1)$.

- *First mover, player 1.* In period 1 player 1 offers player 2 the present value of her period-2 payoff, $\delta_2(1-\delta_1)$. Solving for player 1's share, we obtain

$$\begin{aligned} x_1 &= 1-\delta_2(1-\delta_1) \\ &= 1-\delta_2+\delta_1\delta_2 \text{ and } x_2=\delta_2-\delta_1\delta_2 \end{aligned}$$

- *Comparison against the two-round game.* Subtracting the two expressions, yields

$$(1-\delta_2+\delta_1\delta_2)-(1-\delta_2)=\delta_1\delta_2>0$$

so player 1's share unambiguously increases when the horizon lengthens by one round. Intuitively, an extra round hands the last offer back to player 1, which improves his outside option in period 2 and, through it, lowers the amount he must concede in period 1.

(c) (5 points) Consider now the infinite-horizon game, $T=\infty$, and look for a stationary subgame perfect equilibrium in which player 1 always proposes to keep $x_1$ for himself whenever it is his turn, and player 2 always proposes to keep $y_2$ for herself whenever it is her turn. Write the two stationarity conditions that $x_1$ and $y_2$ must satisfy, and solve them to obtain

$$x_1=\frac{1-\delta_2}{1-\delta_1\delta_2} \text{ and } x_2=\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$$

where $x_2 \equiv 1-x_1$ denotes the share that player 2 receives along the equilibrium path.

- *The two stationarity conditions.* When player 1 proposes, he must leave player 2 exactly the present value of what she would obtain by rejecting, namely proposing herself next period and keeping $y_2$. When player 2 proposes, she must leave player 1 exactly the present value of his own proposer share $x_1$. Therefore,

$$\begin{aligned} x_1 &= 1-\delta_2 y_2 \\ y_2 &= 1-\delta_1 x_1 \end{aligned}$$

where stationarity is what allows us to use the same $x_1$ and $y_2$ in every subgame in which the same player proposes.

- *Solving the system.* Inserting the second condition into the first, yields

$$x_1=1-\delta_2(1-\delta_1 x_1)$$

which simplifies to $x_1(1-\delta_1\delta_2)=1-\delta_2$. Solving for $x_1$, we obtain

$$x_1=\frac{1-\delta_2}{1-\delta_1\delta_2} \text{ and, symmetrically, } y_2=\frac{1-\delta_1}{1-\delta_1\delta_2}$$

where both expressions are well defined since $\delta_1\delta_2<1$.

- *Player 2's equilibrium share.* Along the equilibrium path player 1 proposes in period 1 and player 2 accepts, so player 2 obtains

$$\begin{aligned} x_2 &= 1 - x_1 = \frac{(1-\delta_1\delta_2) - (1-\delta_2)}{1-\delta_1\delta_2} \\ &= \frac{\delta_2 (1-\delta_1)}{1-\delta_1\delta_2} \end{aligned}$$

  and the two shares add up to one, as required. Note that $x_2 = \delta_2 y_2$, so player 2 receives exactly the discounted value of her own proposer share, which is the content of the first stationarity condition.

- *Consistency with the finite-horizon solutions.* Denote by $a_T$ player 1's share in the $T$-round game solved in parts (a) and (b). The backward induction argument used there delivers the recursion $a_T = 1 - \delta_2 b_{T-1}$ and $b_T = 1 - \delta_1 a_{T-1}$, with $a_1 = b_1 = 1$, whose solution for an even horizon $T = 2m$ is

$$a_{2m} = \frac{(1-\delta_2)\left[1-(\delta_1\delta_2)^m\right]}{1-\delta_1\delta_2}$$

  Since $\delta_1\delta_2 < 1$, the bracketed term converges to one as $m$ grows, and $a_{2m}$ converges to $x_1$. Therefore, the stationary solution is the limit of the finite-horizon solutions.

- *Agreement is immediate.* Note that agreement is reached in period 1 for every pair of discount factors, so no delay ever occurs in equilibrium. Intuitively, delay is costly to both players and the proposer can always replicate the outcome of a delayed agreement at a lower cost by offering its present value today.

(d) (5 points) Verify that the shares you found in part (c) collapse to the symmetric Rubinstein shares when $\delta_1 = \delta_2 = \delta$.

- Setting $\delta_1 = \delta_2 = \delta$ and factoring the denominator, $1 - \delta^2 = (1-\delta)(1+\delta)$, we obtain

$$x_1 = \frac{1-\delta}{(1-\delta)(1+\delta)} = \frac{1}{1+\delta} \quad \text{and} \quad x_2 = \frac{\delta(1-\delta)}{(1-\delta)(1+\delta)} = \frac{\delta}{1+\delta}$$

  which are the familiar Rubinstein shares.

- Note that $x_1 > x_2$ for every $\delta < 1$, and that both shares converge to $\frac{1}{2}$ as $\delta$ approaches one. Intuitively, the proposer keeps the larger half only because delay is costly, and the asymmetry disappears once waiting one period becomes free.

(e) (3 points) *Comparative statics.* Show that every player's equilibrium share increases in his or her own discount factor and decreases in the rival's discount factor, so that patience is a bargaining advantage. Then measure the first-mover advantage by the difference between player 1's share when he proposes first and player 1's share when player 2 proposes first, and show that this advantage vanishes as $\delta_1$ and $\delta_2$ approach one.

- *Own patience.* Differentiating $x_1$ with respect to $\delta_1$, yields

$$\frac{\partial x_1}{\partial \delta_1} = \frac{\delta_2 (1 - \delta_2)}{(1 - \delta_1 \delta_2)^2} > 0$$

so player 1's share unambiguously increases in his own discount factor. Intuitively, a more patient player 1 loses less from a rejection, so player 2 must concede more in order to buy his agreement today.

- *The rival's patience.* Differentiating $x_1$ with respect to $\delta_2$, yields

$$\begin{aligned} \frac{\partial x_1}{\partial \delta_2} &= \frac{-(1 - \delta_1 \delta_2) + \delta_1 (1 - \delta_2)}{(1 - \delta_1 \delta_2)^2} \\ &= -\frac{1 - \delta_1}{(1 - \delta_1 \delta_2)^2} < 0 \end{aligned}$$

and, by symmetry, $\frac{\partial x_2}{\partial \delta_2} > 0$ and $\frac{\partial x_2}{\partial \delta_1} < 0$. Therefore, patience is a bargaining advantage for the player who holds it and a bargaining disadvantage for the rival.

- *Measuring the first-mover advantage.* If player 2 proposed first, player 1 would obtain $\delta_1 x_1$, since he would accept in period 1 the discounted value of his own proposer share. The first-mover advantage is, therefore,

$$x_1 - \delta_1 x_1 = (1 - \delta_1) x_1 = \frac{(1 - \delta_1)(1 - \delta_2)}{1 - \delta_1 \delta_2}$$

which is strictly positive for every pair of interior discount factors.

- *The limit.* Setting $\delta_1 = \delta_2 = \delta$ and letting $\delta$ approach one, the advantage becomes

$$\frac{(1 - \delta)^2}{1 - \delta^2} = \frac{1 - \delta}{1 + \delta} \longrightarrow 0$$

so the advantage of moving first vanishes. Intuitively, the proposer's privilege consists of forcing the responder to wait one period in order to counter-offer, and this threat is worthless once a period of delay costs nothing.

- *A useful limiting case.* Writing $\delta_i = e^{-r_i \tau}$, where $r_i$ denotes player $i$'s interest rate and $\tau$ the length of a period, and letting $\tau$ approach zero, we obtain

$$x_1 \longrightarrow \frac{r_2}{r_1 + r_2} \quad \text{and} \quad x_2 \longrightarrow \frac{r_1}{r_1 + r_2}$$

so that shares are inversely proportional to impatience, and the identity of the first proposer becomes irrelevant. Note that this is the sharpest statement of the result that patience, rather than the order of moves, is what determines bargaining power.

(f) (3 points) *Numerical example.* Evaluate your results at $\delta_1 = \frac{1}{2}$ and $\delta_2 = \frac{3}{4}$, reporting the equilibrium shares of the two-round game, of the three-round game and of the infinite-horizon game. Then reverse the two discount factors, setting $\delta_1 = \frac{3}{4}$ and $\delta_2 = \frac{1}{2}$, and report the infinite-horizon shares again. Interpret the comparison.

- *Finite horizons.* At $\delta_1 = \dfrac{1}{2}$ and $\delta_2 = \dfrac{3}{4}$, the two-round game yields

$$x_1 = 1 - \frac{3}{4} = \frac{1}{4} \quad \text{and} \quad x_2 = \frac{3}{4}$$

whereas the three-round game yields

$$x_1 = 1 - \frac{3}{4} + \frac{1}{2} \cdot \frac{3}{4} = \frac{5}{8} \quad \text{and} \quad x_2 = \frac{3}{8}$$

so player 1's share more than doubles when the horizon lengthens by one round, since the extra round returns the last offer to him.

- *Infinite horizon.* With $\delta_1\delta_2 = \dfrac{3}{8}$, so that $1 - \delta_1\delta_2 = \dfrac{5}{8}$, we obtain

$$x_1 = \frac{1 - \frac{3}{4}}{\frac{5}{8}} = \frac{2}{5} \quad \text{and} \quad x_2 = \frac{\frac{3}{4} \cdot \frac{1}{2}}{\frac{5}{8}} = \frac{3}{5}$$

Note that the more patient player 2 secures 60% of the pie even though she never makes the first offer, so patience overturns the first-mover advantage. The first-mover advantage itself is $(1 - \delta_1)\, x_1 = \dfrac{1}{2} \cdot \dfrac{2}{5} = \dfrac{1}{5}$.

- *Reversing the discount factors.* Setting $\delta_1 = \dfrac{3}{4}$ and $\delta_2 = \dfrac{1}{2}$, the product $\delta_1\delta_2$ is unchanged, so

$$x_1 = \frac{1 - \frac{1}{2}}{\frac{5}{8}} = \frac{4}{5} \quad \text{and} \quad x_2 = \frac{\frac{1}{2} \cdot \frac{1}{4}}{\frac{5}{8}} = \frac{1}{5}$$

- *Interpretation.* The same two discount factors deliver $\dfrac{2}{5}$ to player 1 in the first configuration and $\dfrac{4}{5}$ in the second, so which player happens to be the patient one doubles the share of the proposer. Intuitively, the proposer's offer must cover the responder's cost of waiting, and an impatient responder is cheap to buy off, whereas a patient responder can credibly hold out for a counter-offer that the proposer would then have to wait for.

2. **(27 points) Exercise 2 - Collusion in an infinitely repeated Cournot oligopoly: grim trigger against finite punishments.** Consider $n \geq 2$ firms competing à la Cournot in every period $t = 1, 2, \ldots$, facing inverse demand function $p(Q) = a - Q$, where $Q = \sum_{j=1}^{n} q_j$, and sharing a common marginal cost $c$, where $a > c > 0$. All firms discount future profits at the common rate $\delta \in (0, 1)$ and observe the output of all their rivals at the end of each period. The firms consider sustaining the collusive agreement in which each of them produces $\dfrac{1}{n}$ of the monopoly output.

(a) (4 points) Find the output and profit of every firm in the Nash equilibrium of the stage game, $q^N$ and $\pi^N$. Then find the collusive output and profit of every firm, $q^M$ and $\pi^M$.

- *Best response function.* Firm $i$ solves

$$\max_{q_i \geq 0} \pi_i = (a - q_i - Q_{-i} - c)\, q_i$$

Differentiating with respect to $q_i$, yields $a - 2q_i - Q_{-i} - c = 0$. Solving for $q_i$, we obtain

$$q_i(Q_{-i}) = \frac{a - c - Q_{-i}}{2}$$

and the second-order condition holds since $\dfrac{\partial^2 \pi_i}{\partial q_i^2} = -2 < 0$.

- *Stage-game equilibrium.* Imposing symmetry, $Q_{-i} = (n-1)q$, and solving for $q$, we obtain

$$q^N = \frac{a-c}{n+1} \quad \text{and} \quad \pi^N = \left(q^N\right)^2 = \frac{(a-c)^2}{(n+1)^2}$$

where we used that the first-order condition implies $p - c = q_i$ at the optimum, so profit is the square of output.

- *Collusive outcome.* The cartel maximizes $(a - Q - c)\, Q$, which yields $Q^M = \dfrac{a-c}{2}$. Splitting it evenly,

$$q^M = \frac{a-c}{2n} \quad \text{and} \quad \pi^M = \left(a - Q^M - c\right) q^M = \frac{(a-c)^2}{4n}$$

- Note that $\pi^M > \pi^N$ for every $n \geq 2$, since $\dfrac{1}{4n} - \dfrac{1}{(n+1)^2} = \dfrac{(n-1)^2}{4n\,(n+1)^2} > 0$. Intuitively, the cartel internalizes the price effect that each firm imposes on its rivals, which is the externality that Cournot competition leaves uncorrected.

(b) (5 points) Find the profit that a firm earns in the period in which it deviates from the collusive agreement, $\pi^D$.

- *The deviator's problem.* A firm contemplating a deviation takes as given that its $n-1$ rivals produce the collusive output, so that

$$Q_{-i} = (n-1)\, q^M = \frac{(n-1)(a-c)}{2n}$$

and it chooses its own output along the best response function found in part (a). Inserting $Q_{-i}$ into $q_i(Q_{-i})$, we obtain

$$\begin{aligned} q^D &= \frac{1}{2}\left[(a-c) - \frac{(n-1)(a-c)}{2n}\right] \\ &= \frac{(a-c)(n+1)}{4n} \end{aligned}$$

- *Deviation profit.* At this output the price-cost margin again coincides with own output, $p - c = a - c - Q_{-i} - q^D = q^D$, so that

$$\pi^D = \left(q^D\right)^2 = \frac{(a-c)^2 (n+1)^2}{16n^2}$$

- *The deviation output is not the Cournot output.* Comparing $q^D$ against $q^N$, we obtain

$$q^D - q^N = (a-c)\left[\frac{n+1}{4n} - \frac{1}{n+1}\right] = \frac{(a-c)(n-1)^2}{4n(n+1)} > 0$$

  so the deviating firm unambiguously produces *more* than the Cournot output for every $n \geq 2$. Intuitively, the deviator faces a residual demand that its rivals have deliberately restricted, so it expands beyond what it would produce if every rival were also best responding.
- *Ranking of the three profits.* Subtracting,

$$\pi^D - \pi^M = \frac{(a-c)^2 (n-1)^2}{16n^2} > 0 \ \text{ and } \ \pi^M - \pi^N = \frac{(a-c)^2 (n-1)^2}{4n(n+1)^2} > 0$$

  so that $\pi^D > \pi^M > \pi^N$ for every $n \geq 2$, which is the ranking that makes the collusive problem interesting.

(c) (5 points) Suppose that firms sustain the collusive agreement with a grim-trigger strategy, so that a deviation triggers a reversion to the Cournot output in every subsequent period. Find the critical discount factor $\bar{\delta}$ above which collusion can be sustained. Show that $\bar{\delta}$ increases in the number of firms $n$, and that it converges to one as $n$ grows without bound.

- *The incentive compatibility condition.* Colluding forever yields $\dfrac{\pi^M}{1-\delta}$, whereas deviating yields $\pi^D$ today and the Cournot profit thereafter. Collusion is sustainable if and only if

$$\frac{\pi^M}{1-\delta} \geq \pi^D + \frac{\delta \pi^N}{1-\delta}$$

  Multiplying both sides by $1-\delta$ and rearranging, we find $\delta\left(\pi^D - \pi^N\right) \geq \pi^D - \pi^M$. Solving for $\delta$, we obtain

$$\delta \geq \bar{\delta} \equiv \frac{\pi^D - \pi^M}{\pi^D - \pi^N}$$

- *Closed form.* Using the profits of parts (a) and (b), the denominator factors as

$$\pi^D - \pi^N = \frac{(a-c)^2\left[(n+1)^4 - 16n^2\right]}{16n^2 (n+1)^2} = \frac{(a-c)^2 (n-1)^2 \left(n^2+6n+1\right)}{16n^2 (n+1)^2}$$

  where we used $(n+1)^4 - 16n^2 = \left[(n+1)^2 - 4n\right]\left[(n+1)^2 + 4n\right]$. Dividing the numerator $\pi^D - \pi^M$ found in part (b) by this expression, we obtain

$$\bar{\delta} = \frac{(n+1)^2}{n^2+6n+1}$$

- *Effect of the number of firms.* Differentiating with respect to $n$, yields

$$\frac{\partial\overline{\delta}}{\partial n}=\frac{2\left(n+1\right)\left(n^2+6n+1\right)-\left(n+1\right)^2\left(2n+6\right)}{\left(n^2+6n+1\right)^2}=\frac{4\left(n+1\right)\left(n-1\right)}{\left(n^2+6n+1\right)^2}>0$$

for every $n>1$, so the cutoff unambiguously increases in the number of firms. Taking the limit,

$$\lim_{n\to\infty}\overline{\delta}=\lim_{n\to\infty}\frac{n^2+2n+1}{n^2+6n+1}=1$$

- Intuitively, a larger industry hurts collusion through two channels. Each firm's share of the collusive pie shrinks at rate $\frac{1}{n}$, so the reward from colluding falls, while the residual demand left unserved by the cartel grows, so the temptation to deviate rises. Both channels push the cutoff toward one, and collusion becomes essentially impossible in a sufficiently fragmented industry.

(d) (4 points) *The twist.* Suppose now that the punishment is not permanent: after a deviation, firms revert to the Cournot output for exactly $T$ periods, and return to the collusive agreement from period $T+1$ onwards. Show that the incentive compatibility condition can be written as $\sum_{t=1}^{T}\delta^t\geq R$, where $R\equiv\frac{\pi^D-\pi^M}{\pi^M-\pi^N}$, and find $R$ in closed form. Show that the resulting critical discount factor $\overline{\delta}_T$ exceeds the grim-trigger cutoff $\overline{\delta}$ of part (c), and that it converges to $\overline{\delta}$ as $T$ grows without bound.

- *The two streams of profits.* Colluding forever still yields $\frac{\pi^M}{1-\delta}$. Deviating now yields $\pi^D$ today, the Cournot profit during the $T$ punishment periods, and the collusive profit from period $T+1$ onwards,

$$\pi^D+\pi^N\sum_{t=1}^{T}\delta^t+\delta^{T+1}\frac{\pi^M}{1-\delta}$$

- *Rearranging the condition.* Multiplying the inequality $\frac{\pi^M}{1-\delta}\geq\pi^D+\pi^N\sum_{t=1}^{T}\delta^t+\delta^{T+1}\frac{\pi^M}{1-\delta}$ by $1-\delta$, and using $(1-\delta)\sum_{t=1}^{T}\delta^t=\delta-\delta^{T+1}$, we obtain

$$\pi^M\left(1-\delta^{T+1}\right)\geq\left(1-\delta\right)\pi^D+\left(\delta-\delta^{T+1}\right)\pi^N$$

Splitting $1-\delta^{T+1}=(1-\delta)+\left(\delta-\delta^{T+1}\right)$ and collecting terms, which simplifies to

$$\left(\delta-\delta^{T+1}\right)\left(\pi^M-\pi^N\right)\geq\left(1-\delta\right)\left(\pi^D-\pi^M\right)$$

Dividing both sides by $(1-\delta)\left(\pi^M-\pi^N\right)>0$, and noting that $\frac{\delta-\delta^{T+1}}{1-\delta}=\sum_{t=1}^{T}\delta^t$, we find

$$\sum_{t=1}^{T}\delta^t\geq R\equiv\frac{\pi^D-\pi^M}{\pi^M-\pi^N}$$

The left-hand side is the discounted length of the punishment, and the right-hand side is the temptation to deviate measured in units of the per-period gain from colluding.

- *Closed form of $R$.* Using the two differences computed in part (b),

$$R = \frac{(a-c)^2 (n-1)^2}{16n^2} \cdot \frac{4n(n+1)^2}{(a-c)^2 (n-1)^2} = \frac{(n+1)^2}{4n}$$

  which increases in $n$, since $\dfrac{\partial R}{\partial n} = \dfrac{1}{4}\left(1 - \dfrac{1}{n^2}\right) > 0$ for every $n > 1$.
- *Consistency with the grim trigger.* Letting $T$ grow without bound, the left-hand side converges to $\dfrac{\delta}{1-\delta}$, so the condition becomes $\delta \geq \dfrac{R}{1+R}$. Evaluating this expression,

$$\frac{R}{1+R} = \frac{\dfrac{(n+1)^2}{4n}}{\dfrac{4n+(n+1)^2}{4n}} = \frac{(n+1)^2}{n^2+6n+1} = \overline{\delta}$$

  which recovers the grim-trigger cutoff of part (c).
- *The finite punishment is more demanding.* For every $\delta \in (0,1)$ the partial sum falls short of the infinite one,

$$\sum_{t=1}^{T} \delta^t = \frac{\delta}{1-\delta} - \frac{\delta^{T+1}}{1-\delta} < \frac{\delta}{1-\delta}$$

  so the left-hand side of the condition is smaller at every $\delta$, and the value of $\delta$ that solves it with equality must be larger. Therefore, $\overline{\delta}_T > \overline{\delta}$. Since the sum increases in $T$ and converges to $\dfrac{\delta}{1-\delta}$, the cutoff $\overline{\delta}_T$ decreases in $T$ and converges to $\overline{\delta}$.
- Intuitively, a punishment that expires returns the deviator to the collusive path, so part of the loss it inflicts is only postponed rather than permanent. Firms must therefore be more patient in order to be deterred by it, and the shorter the punishment, the more patient they must be.

(e) (5 points) Show that a critical discount factor below one exists if and only if $T > R$. What is the shortest punishment that can sustain collusion in a duopoly? Find the closed-form expression of $\overline{\delta}_T$ for $T = 2$.

- *Existence of the cutoff.* The function $f(\delta) \equiv \sum_{t=1}^{T} \delta^t$ is continuous and strictly increasing on $[0,1]$, with $f(0) = 0$ and $f(1) = T$. Therefore, the equation $f(\delta) = R$ admits a solution $\overline{\delta}_T \in (0,1)$ if and only if

$$T > R = \frac{(n+1)^2}{4n}$$

  and, when $T \leq R$, the collusive agreement cannot be sustained by any discount factor below one.

- *One period of punishment is never enough.* Since $(n+1)^2 - 4n = (n-1)^2 \geq 0$, we obtain $R \geq 1$ for every $n$, with equality only at $n = 1$. Therefore, $T = 1$ fails for every industry with at least two firms. Intuitively, a single Cournot period cannot cost the deviator more than the extra profit it made while deviating, because that gain arrived one period earlier and is therefore discounted less.
- *The duopoly case.* At $n = 2$ we obtain $R = \dfrac{9}{8}$, so $T = 2$ is the shortest punishment that works, since $2 > \dfrac{9}{8} > 1$.
- *Closed form for $T = 2$.* The condition becomes $\delta^2 + \delta = R$, a quadratic whose positive root is

$$\overline{\delta}_2 = \frac{-1 + \sqrt{1 + 4R}}{2} = \frac{-1 + \sqrt{1 + \dfrac{(n+1)^2}{n}}}{2}$$

which is below one if and only if $R < 2$, that is, if and only if $(n+1)^2 < 8n$, or equivalently $n < 3 + 2\sqrt{2} \simeq 5.83$. Therefore, a two-period punishment can sustain collusion in industries of up to five firms, and fails from six firms onwards.

(f) (4 points) *Numerical example.* Evaluate your results at $a = 10$ and $c = 4$ for $n = 2$, $n = 3$ and $n = 5$ firms. For every industry size, report $q^N$, $q^M$, $q^D$ and the corresponding profits, the ratio $R$, the grim-trigger cutoff $\overline{\delta}$, and the cutoffs $\overline{\delta}_T$ for $T = 2$, $T = 3$ and $T = 5$. Interpret the two comparative statics that your table displays.

- *Outputs and profits.* With $a - c = 6$, the expressions of parts (a) and (b) yield

$$\begin{aligned}
n &= 2: \ q^N = 2,\ q^M = \frac{3}{2},\ q^D = \frac{9}{4},\ \pi^N = 4,\ \pi^M = \frac{9}{2},\ \pi^D = \frac{81}{16} \\
n &= 3: \ q^N = \frac{3}{2},\ q^M = 1,\ q^D = 2,\ \pi^N = \frac{9}{4},\ \pi^M = 3,\ \pi^D = 4 \\
n &= 5: \ q^N = 1,\ q^M = \frac{3}{5},\ q^D = \frac{9}{5},\ \pi^N = 1,\ \pi^M = \frac{9}{5},\ \pi^D = \frac{81}{25}
\end{aligned}$$

confirming the ranking $\pi^D > \pi^M > \pi^N$ in every industry, and confirming that $q^D > q^N > q^M$, so the deviator overshoots the Cournot output rather than matching it.

- *Ratios and cutoffs.* Evaluating $R = \dfrac{(n+1)^2}{4n}$ and $\overline{\delta} = \dfrac{(n+1)^2}{n^2 + 6n + 1}$, and solving $\sum_{t=1}^{T} \delta^t = R$ for $T = 2$, $T = 3$ and $T = 5$, we obtain

| $n$ | $R$ | $\overline{\delta}$ | $\overline{\delta}_5$ | $\overline{\delta}_3$ | $\overline{\delta}_2$ | $\overline{\delta}_1$ |
|---|---|---|---|---|---|---|
| 2 | $\frac{9}{8}$ | $\frac{9}{17} \simeq 0.529$ | 0.541 | 0.584 | 0.673 | none |
| 3 | $\frac{4}{3}$ | $\frac{4}{7} \simeq 0.571$ | 0.589 | 0.646 | 0.758 | none |
| 5 | $\frac{9}{5}$ | $\frac{9}{14} \simeq 0.643$ | 0.677 | 0.765 | 0.932 | none |

where the $T = 2$ entries can be checked against the closed form of part (e),

for instance $\bar{\delta}_2 = \dfrac{-1 + \sqrt{\dfrac{11}{2}}}{2} \simeq 0.673$ at $n = 2$.

- *Reading the table across rows.* Moving from $n = 2$ to $n = 5$, every cutoff increases, which is the comparative static of part (c). Intuitively, each firm's slice of the collusive pie shrinks while the residual demand it can capture by deviating grows, so collusion requires more patience in a more fragmented industry.
- *Reading the table across columns.* Moving from left to right, every cutoff increases as the punishment becomes shorter, and the entire row converges to the grim-trigger cutoff as $T$ grows. Note how steep the loss is in a five-firm industry, where a two-period punishment requires $\delta \simeq 0.932$ against the 0.643 demanded by permanent reversion. A one-period punishment sustains nothing in any of the three industries, since $R > 1$ throughout.

3. **(25 points) Exercise 3 - Collusion in a repeated Bertrand duopoly with asymmetric costs.** Two firms sell a homogeneous good and compete in prices in every period $t = 1, 2, \ldots$, facing market demand $q(p) = a - p$. Firm 1 has marginal cost $c_1$ and firm 2 has marginal cost $c_2 > c_1$, consumers buy from the cheapest firm and split their purchases evenly when prices coincide, and the cost gap is denoted $\Delta \equiv c_2 - c_1 > 0$, with $a + c_1 > 2c_2$. Both firms discount at the common rate $\delta \in (0, 1)$. The collusive agreement has both firms charging the monopoly price of the efficient firm, $p^M = \arg\max_p (p - c_1)(a - p)$, with firm 1 serving a share $s \in [0, 1]$ of the market and firm 2 the remaining share $1 - s$, and deviations are punished with a grim trigger. Denote $m_1 \equiv p^M - c_1$ and $m_2 \equiv p^M - c_2$.

   (a) (4 points) Find the Nash equilibrium of the stage game and the profit that every firm earns in it, $\pi_1^N$ and $\pi_2^N$. Explain why the equilibrium involves limit pricing rather than marginal cost pricing.

   - *Equilibrium prices.* Firm 2 cannot profitably charge a price below $c_2$, so the lowest price it is willing to match is $c_2$. Firm 1 therefore sets $p_1 = c_2$, captures the entire market, and no deviation is profitable: raising its price above $c_2$ surrenders all sales to firm 2, while lowering it below $c_2$ reduces the margin without adding rivals to displace. Firm 2 sets $p_2 = c_2$ and sells nothing.
   - *Stage-game profits.* Serving demand $a - c_2$ at a margin of $c_2 - c_1 = \Delta$, we obtain
   $$\pi_1^N = \Delta (a - c_2) \quad \text{and} \quad \pi_2^N = 0$$
   This equilibrium requires that firm 1 prefer limit pricing to its unconstrained monopoly price, which holds when $p^M = \dfrac{a + c_1}{2} \geq c_2$, that is, when $a + c_1 \geq 2c_2$, as assumed.
   - *Why limit pricing rather than marginal cost pricing.* With symmetric costs, undercutting continues until price equals marginal cost, because a firm charging above cost is always undercut by a rival with the same cost. With asymmetric costs the process stops at $c_2$, since firm 2 cannot follow firm 1 any

lower. Intuitively, the efficient firm's market power is bounded by its rival's cost rather than by its own, so its equilibrium margin equals the cost gap.

- Note, in addition, that $a - c_2 = (a - c_1) - \Delta = 2m_1 - \Delta$, so we may write $\pi_1^N = \Delta(2m_1 - \Delta)$, which will be convenient below.

(b) (5 points) Find $p^M$, the associated market demand $Q^M$, and the two margins $m_1$ and $m_2$. Then find every firm's collusive profit, $\pi_i^C$, and every firm's deviation profit, $\pi_i^D$. Is the joint collusive profit equal to the monopoly profit? Explain.

- *The collusive price.* Differentiating $(p - c_1)(a - p)$ with respect to $p$, yields $a + c_1 - 2p = 0$. Solving for $p$, we obtain

$$p^M = \frac{a + c_1}{2}, \quad Q^M = a - p^M = \frac{a - c_1}{2}, \quad m_1 = \frac{a - c_1}{2} \quad \text{and} \quad m_2 = m_1 - \Delta$$

so that $Q^M = m_1$, a coincidence of the linear specification that simplifies every expression below. The assumption $a + c_1 > 2c_2$ is exactly $m_2 > 0$.

- *Collusive profits.* Splitting the collusive sales,

$$\pi_1^C = s m_1 Q^M = s m_1^2 \quad \text{and} \quad \pi_2^C = (1 - s) m_2 Q^M = (1 - s) m_1 m_2$$

- *Deviation profits.* A firm that undercuts the collusive price by an arbitrarily small amount captures the entire market at essentially the collusive price, so that

$$\pi_1^D = m_1 Q^M = m_1^2 \quad \text{and} \quad \pi_2^D = m_2 Q^M = m_1 m_2$$

Note that each firm's deviation profit is its own collusive profit evaluated at a share of one, which is the feature that will make the incentive constraints so clear.

- *Joint profit falls short of the monopoly profit.* Adding the two collusive profits,

$$\begin{aligned} \pi_1^C + \pi_2^C &= m_1 [s m_1 + (1 - s) m_2] \\ &= m_1^2 - (1 - s)\Delta m_1 \end{aligned}$$

which falls short of the monopoly profit $m_1^2$ by $(1 - s)\Delta Q^M$ whenever $s < 1$. Intuitively, the agreement assigns part of the production to the high-cost firm, so the cartel wastes $\Delta$ on every unit that firm 2 supplies. A fully efficient cartel would set $s = 1$ and compensate firm 2 with a side payment, but the shares $s$ are the only instrument available here.

(c) (5 points) Write the incentive compatibility condition of each firm, and solve each of them for the critical discount factor of that firm, $\overline{\delta}_1$ and $\overline{\delta}_2$. Show that the two cutoffs differ, and that collusion is sustainable if and only if $\delta \geq \max\left\{\overline{\delta}_1, \overline{\delta}_2\right\}$.

- *The general condition.* Firm $i$ colludes if and only if

$$\frac{\pi_i^C}{1 - \delta} \geq \pi_i^D + \frac{\delta \pi_i^N}{1 - \delta}$$

which, rearranging as in Exercise 2, delivers $\delta \geq \dfrac{\pi_i^D - \pi_i^C}{\pi_i^D - \pi_i^N}$.

- *Firm 2.* Since $\pi_2^N = 0$, the cutoff simplifies dramatically,

$$\overline{\delta}_2 = \frac{m_1 m_2 - (1-s)\, m_1 m_2}{m_1 m_2} = s$$

so firm 2's cutoff is simply the share it does *not* receive, measured as the fraction of the market that firm 1 takes. Intuitively, firm 2 earns nothing in the punishment phase, so its whole calculation reduces to comparing the flow $(1-s)$ of the collusive pie against the one-shot capture of all of it.

- *Firm 1.* Using $\pi_1^N = \Delta\,(2m_1 - \Delta)$, the denominator factors as

$$\pi_1^D - \pi_1^N = m_1^2 - 2\Delta m_1 + \Delta^2 = (m_1 - \Delta)^2 = m_2^2$$

so that firm 1's cutoff is

$$\overline{\delta}_1 = \frac{(1-s)\, m_1^2}{m_2^2}$$

Note the surprising role of $m_2$: the punishment payoff of the *efficient* firm is governed by the margin of the *inefficient* one, because limit pricing at $c_2$ is what firm 1 falls back on.

- *The two cutoffs differ.* Setting $\overline{\delta}_1 = \overline{\delta}_2$ requires $(1-s)\, m_1^2 = s m_2^2$, which pins down a single value of $s$. At every other share the two firms face different constraints, and since both must be satisfied simultaneously, collusion is sustainable if and only if

$$\delta \geq \max\left\{\overline{\delta}_1, \overline{\delta}_2\right\}$$

Note that $\overline{\delta}_1$ decreases in $s$ while $\overline{\delta}_2$ increases in $s$, so the two constraints pull in opposite directions.

(d) (4 points) *The twist.* Find the market share $s^*$ that minimizes the critical discount factor $\max\left\{\overline{\delta}_1, \overline{\delta}_2\right\}$, and report the minimized cutoff $\overline{\delta}^*$. Which firm receives the larger share, and why? Find, in addition, the set of market shares $s$ for which collusion is sustainable for some discount factor below one.

- *The minimization.* Denote $k \equiv \dfrac{m_1^2}{m_2^2} > 1$, so that $\overline{\delta}_1 = (1-s)\, k$ and $\overline{\delta}_2 = s$. The first function decreases in $s$ from $k$ to zero, the second increases from zero to one, so their maximum is minimized where the two cross. Setting $(1-s)\, k = s$ and solving for $s$, we obtain

$$s^* = \frac{k}{1+k} = \frac{m_1^2}{m_1^2 + m_2^2}$$

and the minimized cutoff is

$$\overline{\delta}^* = s^* = \frac{m_1^2}{m_1^2 + m_2^2}$$

- *Which firm receives the larger share.* Since $m_1 > m_2$, we obtain $s^* > \dfrac{1}{2}$, so the efficient firm receives the larger share of the market. Intuitively, firm 1 is the harder firm to keep in line, because a deviation costs it only the difference between the monopoly margin and the limit-pricing profit it can always secure on its own, whereas firm 2 loses everything. The cartel compensates the firm with the better outside option, and the compensation takes the form of a larger quota.

- *Feasibility.* Firm 2's constraint $\overline{\delta}_2 = s$ never exceeds one, so only firm 1's constraint can be violated. Requiring $\overline{\delta}_1 = (1-s)\,k \leq 1$ and solving for $s$, we obtain
$$s \geq 1 - \frac{1}{k} = 1 - \frac{m_2^2}{m_1^2}$$
so collusion is sustainable for some $\delta < 1$ if and only if $s \in \left[1 - \frac{m_2^2}{m_1^2}, 1\right]$. Note that an equal split does not belong to this interval whenever $m_1^2 > 2m_2^2$, so a symmetric agreement can be infeasible no matter how patient the firms are.

(e) (4 points) *Comparative statics.* How are $s^*$ and $\overline{\delta}^*$ affected by a marginal increase in the cost gap $\Delta$? Evaluate the two limiting cases $\Delta \to 0$ and $\Delta \to m_1$, and interpret them.

- *Effect of the cost gap.* Writing $m_2 = m_1 - \Delta$ and differentiating with respect to $\Delta$, yields
$$\frac{\partial \overline{\delta}^*}{\partial \Delta} = \frac{\partial}{\partial \Delta}\left[\frac{m_1^2}{m_1^2 + (m_1 - \Delta)^2}\right] = \frac{2m_1^2\,(m_1 - \Delta)}{\left(m_1^2 + m_2^2\right)^2} > 0$$
which is positive for every $\Delta < m_1$, that is, whenever firm 2 is still viable at the collusive price. Since $s^* = \overline{\delta}^*$, the optimal share of the efficient firm rises at exactly the same rate.
- Therefore, a wider cost gap unambiguously makes collusion harder to sustain and pushes the collusive quota toward the efficient firm. Intuitively, the gap improves firm 1's punishment payoff, since limit pricing becomes more lucrative, and simultaneously erodes firm 2's collusive profit, since its margin $m_2$ shrinks. Both forces raise the share that firm 1 must be given, and firm 2's constraint $\overline{\delta}_2 = s$ then binds at a higher discount factor.
- *Symmetric limit,* $\Delta \to 0$. We obtain $m_2 \to m_1$, so that
$$s^* \to \frac{1}{2} \quad \text{and} \quad \overline{\delta}^* \to \frac{1}{2}$$
which is the familiar result for a symmetric Bertrand duopoly, where an equal split of the market sustains collusion whenever firms are at least as patient as $\frac{1}{2}$.
- *The other extreme,* $\Delta \to m_1$. We obtain $m_2 \to 0$, so that $s^* \to 1$ and $\overline{\delta}^* \to 1$, and collusion becomes impossible. Intuitively, an inefficient firm that earns nothing at the collusive price has nothing to lose from deviating and nothing to gain from complying, so no quota can keep it inside the agreement. Note that in this limit the stage-game equilibrium already delivers firm 1 the whole market at a price arbitrarily close to $p^M$, so there is little left for the cartel to accomplish.

(f) (3 points) *Numerical example.* Evaluate your results at $a = 10$, $c_1 = 2$ and $c_2 = 4$. Report $p^M$, the two margins, the stage-game profits, the optimal share $s^*$ and the cutoff $\overline{\delta}^*$. Show, in addition, that an equal split of the market, $s = \frac{1}{2}$, cannot sustain collusion for any discount factor. Then reduce the cost gap by setting $c_2 = 3$, and report $s^*$ and $\overline{\delta}^*$ again.

- *Prices, margins and stage-game profits.* We obtain $\Delta = 2$ and

$$p^M = \frac{10+2}{2} = 6, \quad Q^M = 4, \quad m_1 = 4 \quad \text{and} \quad m_2 = 6 - 4 = 2$$

so the stage-game profits are $\pi_1^N = 2\,(10-4) = 12$ and $\pi_2^N = 0$. The condition $a + c_1 > 2c_2$ holds, since $12 > 8$.

- *Optimal share and cutoff.* With $k = \frac{16}{4} = 4$, we obtain

$$s^* = \frac{4}{5} \quad \text{and} \quad \overline{\delta}^* = \frac{4}{5}$$

Checking the two constraints at this share, firm 1 earns $\pi_1^C = \frac{4}{5} \cdot 16 = \frac{64}{5}$ against $\pi_1^D = 16$ and $\pi_1^N = 12$, so that $\overline{\delta}_1 = \frac{16 - \frac{64}{5}}{16-12} = \frac{4}{5}$, whereas firm 2 earns $\pi_2^C = \frac{1}{5} \cdot 8 = \frac{8}{5}$ against $\pi_2^D = 8$ and $\pi_2^N = 0$, so that $\overline{\delta}_2 = \frac{8 - \frac{8}{5}}{8} = \frac{4}{5}$, as required.

- *The equal split fails.* Setting $s = \frac{1}{2}$, firm 1's cutoff becomes

$$\overline{\delta}_1 = \left(1 - \frac{1}{2}\right) \cdot 4 = 2 > 1$$

so firm 1 deviates for every admissible discount factor. Indeed, at $s = \frac{1}{2}$ firm 1 collects 8 per period under the agreement, which is below the 12 it earns by simply competing, so the agreement is worse for firm 1 than the punishment it is supposed to fear. The feasible shares are $s \geq 1 - \frac{4}{16} = \frac{3}{4}$.

- *Joint profit.* At $s^* = \frac{4}{5}$ the two firms collect $\frac{64}{5} + \frac{8}{5} = \frac{72}{5}$, which falls short of the monopoly profit 16 by $\frac{8}{5}$. This shortfall is $(1 - s^*)\,\Delta Q^M = \frac{1}{5} \cdot 2 \cdot 4 = \frac{8}{5}$, exactly as part (b) predicts.

- *A smaller cost gap.* Setting $c_2 = 3$, so that $\Delta = 1$ and $m_2 = 3$, we obtain $k = \frac{16}{9}$ and

$$s^* = \frac{16}{25} = 0.64 \quad \text{and} \quad \overline{\delta}^* = \frac{16}{25}$$

so halving the cost gap lowers the required patience from 0.8 to 0.64 and moves the collusive quota from 80% to 64% of the market. This confirms the comparative static of part (e): asymmetry is the enemy of collusion.

4. **(24 points) Exercise 4 - Finitely repeated games with multiple stage-game equilibria.** Consider the following simultaneous-move game between player 1, who chooses rows, and player 2, who chooses columns.

| | | | Player 2 | |
|---|---|---|---|---|
| | | $L$ | $C$ | $R$ |
| Player 1 | $U$ | $6,6$ | $0,7$ | $0,0$ |
| | $M$ | $7,0$ | $4,4$ | $0,0$ |
| | $D$ | $0,0$ | $0,0$ | $1,1$ |

The stage game is played $T$ times, players observe the outcome of every past period, and both discount future payoffs at the common rate $\delta \in (0,1]$.

(a) (3 points) Find every pure-strategy Nash equilibrium of the stage game, and rank them according to the Pareto criterion. Show, in addition, that the profile $(U,L)$ is not a Nash equilibrium of the stage game even though it Pareto dominates both equilibria you found.

- *Player 1's best responses.* Against $L$, strategy $M$ yields 7, above the 6 from $U$ and the 0 from $D$, so $BR_1(L) = M$. Against $C$, strategy $M$ yields 4, so $BR_1(C) = M$. Against $R$, strategy $D$ yields 1, so $BR_1(R) = D$.
- *Player 2's best responses.* By symmetry of the payoff matrix, $BR_2(U) = C$, $BR_2(M) = C$ and $BR_2(D) = R$.
- Underlining the best response payoffs of each player, we obtain

| | | | Player 2 | |
|---|---|---|---|---|
| | | $L$ | $C$ | $R$ |
| Player 1 | $U$ | $6,6$ | $0,\underline{7}$ | $0,0$ |
| | $M$ | $\underline{7},0$ | $\underline{4},\underline{4}$ | $0,0$ |
| | $D$ | $0,0$ | $0,0$ | $\underline{1},\underline{1}$ |

so the game has exactly two pure-strategy Nash equilibria,

$$(M,C) \quad \text{with payoffs} \quad (4,4) \quad \text{and} \quad (D,R) \quad \text{with payoffs} \quad (1,1)$$

- *Pareto ranking.* Since $4 > 1$ for both players, equilibrium $(M,C)$ Pareto dominates equilibrium $(D,R)$. We refer to them as the *good* and the *bad* equilibrium throughout.
- *The cooperative profile.* Profile $(U,L)$ yields $(6,6)$, which Pareto dominates both equilibria. It is not a Nash equilibrium, however, since player 1 deviates to $M$ and earns $7 > 6$, while player 2 deviates to $C$ and earns $7 > 6$. The deviation gain is $7-6 = 1$ for each player, and the difference between the two equilibrium payoffs is $4-1 = 3$. These two numbers are all that the analysis below requires.

(b) (5 points) Consider the twice-repeated game, $T = 2$. Construct a subgame perfect equilibrium in which players choose the non-Nash profile $(U,L)$ in the first period, using the Pareto-superior stage-game equilibrium as a reward and the Pareto-inferior one as a punishment. Find the exact condition on $\delta$ under which this construction works.

- *The candidate strategy profile.* Each player follows the rule:

  play $(U, L)$ in period 1; in period 2 play $(M, C)$ if the period-1 outcome was $(U, L)$, and

- *Last period.* Whatever happened in period 1, the profile prescribed for period 2 is a Nash equilibrium of the stage game, so no player can gain by departing from it in the final period. Sequential rationality therefore holds in every period-2 subgame, whether it lies on the equilibrium path or off it.
- *First period.* Following the prescription gives player 1 a payoff of 6 today and the good equilibrium tomorrow, whereas his most profitable deviation is $M$, which gives 7 today and the bad equilibrium tomorrow. Comparing the two streams,

  $$6 + 4\delta \geq 7 + 1\delta$$

  which simplifies to $3\delta \geq 1$. Solving for $\delta$, we obtain

  $$\delta \geq \overline{\delta} \equiv \frac{1}{3}$$

  and the same condition applies to player 2, whose deviation to $C$ also pays 7.
- *Reading the cutoff.* Note that the cutoff is the ratio of the deviation gain to the differential between the two stage-game equilibria,

  $$\overline{\delta} = \frac{7-6}{4-1} = \frac{1}{3}$$

  Intuitively, the multiplicity of stage-game equilibria supplies a reward of size 3 that can be conditioned on first-period behaviour, and cooperation survives if and only if the discounted value of that reward covers the one-period temptation of size 1.

(c) (5 points) Show that the profile $(M, L)$ cannot be sustained in the first period of the twice-repeated game for any discount factor $\delta \leq 1$, even though it delivers a higher payoff to player 1 than $(U, L)$ does.

- *Player 2's constraint.* At $(M, L)$ player 2 earns 0, and her most profitable deviation is $C$, which yields 4. Rewarding compliance with the good equilibrium and punishing a deviation with the bad one, her condition is

  $$0 + 4\delta \geq 4 + 1\delta$$

  which simplifies to $3\delta \geq 4$. Solving for $\delta$, we obtain $\delta \geq \dfrac{4}{3} > 1$, so the constraint fails for every admissible discount factor.
- Intuitively, the reward that the stage game can offer is bounded above by the differential $4 - 1 = 3$ between its two equilibria, whereas player 2's temptation at $(M, L)$ equals 4. Note that no strategy profile whose deviation gain exceeds 3 can ever be supported in the first period of a twice-repeated game, whatever the discount factor.

(d) (5 points) Suppose now that the stage game were the $2 \times 2$ game that remains after deleting row $D$ and column $R$. Show that this reduced game has a unique Nash equilibrium, and that the construction of part (b) then fails. Explain why unravelling occurs in that case, and identify the feature of the original $3 \times 3$ game that prevents it.

- *The reduced game.* Deleting $D$ and $R$, we obtain

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $C$ |
| Player 1 | $U$ | $6, 6$ | $0, 7$ |
| | $M$ | $7, 0$ | $4, 4$ |

  which is a prisoner's dilemma. Strategy $M$ strictly dominates $U$ for player 1, since $7 > 6$ and $4 > 0$, and $C$ strictly dominates $L$ for player 2, so the unique Nash equilibrium is $(M, C)$.

- *Unravelling.* In the final period every subgame is a copy of the stage game, so subgame perfection requires the unique equilibrium $(M, C)$ after every history. Period-2 payoffs are therefore $(4, 4)$ regardless of what happened in period 1, so first-period behaviour cannot be rewarded or punished. Facing a constant continuation payoff, each player simply maximizes his period-1 payoff, which again delivers $(M, C)$. The unique subgame perfect equilibrium prescribes $(M, C)$ in both periods.

- *Extending the argument.* The same reasoning applied backwards from period $T$ shows that $(M, C)$ is played in every period of the $T$-times repeated game, for every $T$ and every $\delta$. Intuitively, with a unique stage-game equilibrium the continuation value carries no information about past play, so the repeated game inherits the incentives of the one-shot game.

- *The feature that prevents unravelling.* What breaks the argument in the original $3 \times 3$ game is the *multiplicity* of stage-game equilibria. Two equilibria with different payoffs allow the continuation to vary with the history while remaining sequentially rational, which is exactly the reward-punishment differential of size 3 used in part (b). Note that this is a property of the stage game alone, not of the length of the horizon.

(e) (3 points) Extend the construction of part (b) to the $T$-times repeated game, in which players choose $(U, L)$ in every period $t = 1, 2, \ldots, T-1$ and the Pareto-superior equilibrium in period $T$. Write the incentive compatibility condition of period $t$, show that the binding constraint is the one of period $T-1$, and conclude that the same cutoff on $\delta$ found in part (b) supports cooperation in every period. What happens to the average per-period payoff as $T$ grows without bound?

- *The candidate strategy profile.* Players choose $(U, L)$ in periods 1 through $T-1$ and $(M, C)$ in period $T$, as long as no deviation has occurred. Any deviation triggers $(D, R)$ in every remaining period.

- *The condition of period $t$.* Complying from period $t \leq T-1$ onwards yields 6 in every period through $T-1$ and 4 in period $T$, whereas deviating yields 7 today and 1 in every remaining period. Comparing the two streams, discounted to period $t$,

$$6\sum_{k=0}^{T-1-t}\delta^k + 4\delta^{T-t} \geq 7 + \sum_{k=1}^{T-t}\delta^k$$

which simplifies to

$$5\sum_{k=1}^{T-1-t}\delta^k + 3\delta^{T-t} \geq 1$$

- *The binding constraint.* At $t = T-1$ the first sum is empty and the condition reduces to $3\delta \geq 1$, which is exactly the condition of part (b). For every $t < T-1$ the left-hand side contains the same term $3\delta^{T-t}$, which is smaller, plus the strictly positive terms $5\sum_{k=1}^{T-1-t}\delta^k$, and a direct comparison shows that the latter more than compensates. Therefore, the constraint of the penultimate period is the binding one, and

$$\delta \geq \overline{\delta} = \frac{1}{3}$$

supports cooperation in every period.

- Intuitively, a deviation early in the game triggers a punishment that lasts for many periods, whereas a deviation in period $T-1$ is punished only once. The incentive to cooperate is therefore weakest at the end of the game, which is why the horizon must be truncated one period before the last.
- *Average payoffs.* Normalizing the equilibrium payoff by $\dfrac{1-\delta}{1-\delta^T}$, the average per-period payoff is

$$\frac{1-\delta}{1-\delta^T}\left[6\frac{1-\delta^{T-1}}{1-\delta} + 4\delta^{T-1}\right] = 6 - \frac{2\delta^{T-1}(1-\delta)}{1-\delta^T}$$

which converges to 6 as $T$ grows without bound. Therefore, the loss caused by the finite horizon is confined to a single terminal period, and it vanishes in relative terms. This is the finite-horizon counterpart of the folk theorem: when the stage game has several equilibria, the set of average payoffs supportable in the finitely repeated game expands with $T$ and approaches the set supportable in the infinitely repeated game.

(f) (3 points) *Numerical example.* Evaluate your results at $T = 4$. First set $\delta = \dfrac{1}{2}$ and report the equilibrium payoff of the construction of part (e), the payoff of the most profitable deviation in every period, and the payoff that players would obtain if the game unravelled to the Pareto-superior stage-game equilibrium in all four periods. Then set $\delta = \dfrac{13}{50}$, show that the construction of part (e) fails, and show that players can nevertheless support $(U, L)$ in the first period alone. Interpret.

- *Equilibrium payoff at $\delta = \dfrac{1}{2}$.* Playing $(U, L)$ in periods 1, 2 and 3 and $(M, C)$ in period 4, each player earns

$$6 + 6\cdot\frac{1}{2} + 6\cdot\frac{1}{4} + 4\cdot\frac{1}{8} = 6 + 3 + \frac{3}{2} + \frac{1}{2} = 11$$

- *Deviation payoffs.* Deviating in period $t$ and receiving $(D, R)$ thereafter yields

$$\begin{aligned} t &= 1: \ 7 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{63}{8} = 7.875 \\ t &= 2: \ 6 + \frac{1}{2} \cdot 7 + \frac{1}{4} + \frac{1}{8} = \frac{79}{8} = 9.875 \\ t &= 3: \ 6 + 3 + \frac{1}{4} \cdot 7 + \frac{1}{8} = \frac{87}{8} = 10.875 \end{aligned}$$

  all of which fall below 11, so every incentive constraint is satisfied. Note that the margin shrinks as the deviation moves toward the end of the game, from 3.125 in period 1 to 0.125 in period 3, which confirms that the penultimate period carries the binding constraint.
- *The unravelling benchmark.* Playing the good equilibrium in all four periods yields

$$4\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = 4 \cdot \frac{15}{8} = \frac{15}{2} = 7.5$$

  so the construction of part (e) delivers 11 against 7.5, an improvement of roughly 47%.
- *An impatient case,* $\delta = \frac{13}{50}$. Since $3\delta = \frac{39}{50} < 1$, the constraint of period $T - 1 = 3$ fails, so the construction of part (e) cannot be used. Cooperation in the period immediately before the last is impossible for this discount factor, whatever the length of the game.
- *Cooperation far from the end.* Consider instead the profile that plays $(U, L)$ in period 1 alone and $(M, C)$ in periods 2, 3 and 4, with $(D, R)$ after any deviation. The condition becomes $6 + 4\left(\delta + \delta^2 + \delta^3\right) \geq 7 + \left(\delta + \delta^2 + \delta^3\right)$, or equivalently

$$3\left(\delta + \delta^2 + \delta^3\right) \geq 1$$

  Evaluating the left-hand side at $\delta = \frac{13}{50}$ gives $3(0.26 + 0.0676 + 0.017576) \simeq 1.036 > 1$, so the constraint holds and $(U, L)$ is supported in the first period.
- *Interpretation.* The cutoff of this weaker construction is $\simeq 0.253$ when $T = 4$, against the $\frac{1}{3}$ of part (b), and it decreases in $T$, converging to $\frac{1}{4}$ from above, since $\frac{3\delta}{1-\delta} \geq 1$ if and only if $\delta \geq \frac{1}{4}$. Intuitively, a deviation in period 1 of a long game is punished for many periods, so impatient players can be deterred early even though they cannot be deterred late. Therefore, lengthening the horizon does not relax the constraint of the penultimate period, but it does enlarge the set of early outcomes that can be supported, which is the sense in which the supportable set grows with $T$.