Microeconomic Theory II - UNI, Lima
Homework #3 - Due date: September 3rd, in class.

1. **Alternating-offer bargaining with asymmetric discount factors.** Two players bargain over a pie of size 1. Time is discrete, $t = 1, 2, \ldots$, and the players alternate in making offers:

   - In odd periods $t = 1, 3, 5, \ldots$ player 1 proposes a split $(x_1, 1 - x_1)$, where x_1 denotes the share player 1 keeps for himself, and player 2 either accepts or rejects it.
   - In even periods $t = 2, 4, 6, \ldots$ player 2 proposes a split $(1 - y_2, y_2)$, where y_2 denotes the share player 2 keeps for herself, and player 1 either accepts or rejects it.
   - If an offer is accepted in period t, the game ends and the pie is divided as agreed. If it is rejected, play moves to period $t + 1$. If the game has a last period T and the period-T offer is rejected, both players earn a payoff of zero.

   Unlike in the standard treatment, players do *not* share the same discount factor. Player 1 discounts future payoffs at rate $\delta_1 \in (0, 1)$ and player 2 at rate $\delta_2 \in (0, 1)$, so that a share x agreed upon in period t is worth $\delta_i^{t-1} x$ to player i. Throughout, assume that a player who is exactly indifferent between accepting and rejecting an offer accepts it.

   Answer the following questions.

   (a) Solve the two-round game, $T = 2$, by backward induction. Report the equilibrium shares of both players.

   (b) Solve the three-round game, $T = 3$, by backward induction. Report the equilibrium shares of both players, and show that player 1's share is larger than in part (a).

   (c) Consider now the infinite-horizon game, $T = \infty$, and look for a stationary subgame perfect equilibrium in which player 1 always proposes to keep x_1 for himself whenever it is his turn, and player 2 always proposes to keep y_2 for herself whenever it is her turn. Write the two stationarity conditions that x_1 and y_2 must satisfy, and solve them to obtain

   $$x_1 = \frac{1 - \delta_2}{1 - \delta_1 \delta_2} \quad \text{and} \quad x_2 = \frac{\delta_2 (1 - \delta_1)}{1 - \delta_1 \delta_2}$$

   where $x_2 \equiv 1 - x_1$ denotes the share that player 2 receives along the equilibrium path. [*Hint*: the proposer offers the responder exactly the present value of what the responder would obtain by rejecting and becoming the proposer one period later.]

   (d) Verify that the shares you found in part (c) collapse to the symmetric Rubinstein shares when $\delta_1 = \delta_2 = \delta$.

(e) *Comparative statics.* Show that every player's equilibrium share increases in his or her own discount factor and decreases in the rival's discount factor, so that patience is a bargaining advantage. Then measure the first-mover advantage by the difference between player 1's share when he proposes first and player 1's share when player 2 proposes first, and show that this advantage vanishes as δ_1 and δ_2 approach one.

(f) *Numerical example.* Evaluate your results at $\delta_1 = \dfrac{1}{2}$ and $\delta_2 = \dfrac{3}{4}$, reporting the equilibrium shares of the two-round game, of the three-round game and of the infinite-horizon game. Then reverse the two discount factors, setting $\delta_1 = \dfrac{3}{4}$ and $\delta_2 = \dfrac{1}{2}$, and report the infinite-horizon shares again. Interpret the comparison.

2. **Collusion in an infinitely repeated Cournot oligopoly: grim trigger against finite punishments.** Consider $n \geq 2$ firms competing à la Cournot in every period $t = 1, 2, \ldots$, facing inverse demand function $p(Q) = a - Q$, where $Q = \sum_{j=1}^{n} q_j$ denotes aggregate output, and sharing a common marginal cost c, where $a > c > 0$. All firms discount future profits at the common rate $\delta \in (0, 1)$, and every firm observes the output of all its rivals at the end of each period.

The firms consider sustaining the collusive agreement in which each of them produces $\dfrac{1}{n}$ of the monopoly output.

Answer the following questions.

(a) Find the output and profit of every firm in the Nash equilibrium of the stage game, q^N and π^N. Then find the collusive output and profit of every firm, q^M and π^M.

(b) Find the profit that a firm earns in the period in which it deviates from the collusive agreement, π^D. [*Warning*: the deviating firm chooses its output optimally, taking as given that its $n-1$ rivals produce the collusive output q^M. Do *not* assume that the deviating firm produces the Cournot output q^N, and confirm explicitly that $q^D \neq q^N$.]

(c) Suppose that firms sustain the collusive agreement with a grim-trigger strategy, so that a deviation triggers a reversion to the Cournot output in every subsequent period. Find the critical discount factor $\bar{\delta}$ above which collusion can be sustained. Show that $\bar{\delta}$ increases in the number of firms n, and that it converges to one as n grows without bound.

(d) *The twist.* Suppose now that the punishment is not permanent: after a deviation, firms revert to the Cournot output for exactly T periods, and return to the collusive agreement from period $T+1$ onwards. Show that the incentive compatibility condition can be written as

$$\sum_{t=1}^{T} \delta^t \geq R, \quad \text{where} \quad R \equiv \frac{\pi^D - \pi^M}{\pi^M - \pi^N}$$

and find R in closed form. Show that the resulting critical discount factor $\overline{\delta}_T$ exceeds the grim-trigger cutoff $\overline{\delta}$ of part (c), and that it converges to $\overline{\delta}$ as T grows without bound.

(e) Show that a critical discount factor below one exists if and only if $T > R$. What is the shortest punishment that can sustain collusion in a duopoly? Find the closed-form expression of $\overline{\delta}_T$ for $T = 2$.

(f) *Numerical example.* Evaluate your results at $a = 10$ and $c = 4$ for $n = 2$, $n = 3$ and $n = 5$ firms. For every industry size, report q^N, q^M, q^D and the corresponding profits, the ratio R, the grim-trigger cutoff $\overline{\delta}$, and the cutoffs $\overline{\delta}_T$ for $T = 2$, $T = 3$ and $T = 5$. Interpret the two comparative statics that your table displays.

3. **Collusion in a repeated Bertrand duopoly with asymmetric costs.** Two firms sell a homogeneous good and compete in prices in every period $t = 1, 2, \ldots$, facing market demand $q(p) = a - p$. Firm 1 is the efficient producer, with constant marginal cost c_1, and firm 2 is the inefficient one, with constant marginal cost $c_2 > c_1$. Consumers buy from the cheapest firm, and split their purchases evenly when both firms charge the same price. Denote the cost gap by $\Delta \equiv c_2 - c_1 > 0$, and assume throughout that $a + c_1 > 2c_2$, so that firm 2 could still earn a positive margin at firm 1's monopoly price. Both firms discount future profits at the common rate $\delta \in (0, 1)$.

The firms consider sustaining the following collusive agreement. Both of them charge the monopoly price of the *efficient* firm,

$$p^M = \arg\max_p \,(p - c_1)\,(a - p)$$

and they divide the resulting sales so that firm 1 serves a share $s \in [0, 1]$ of the market and firm 2 serves the remaining share $1 - s$. A deviation is punished with a grim-trigger strategy, so that firms revert to the Nash equilibrium of the stage game in every subsequent period. For compactness, denote the two collusive margins by $m_1 \equiv p^M - c_1$ and $m_2 \equiv p^M - c_2$.

Answer the following questions.

(a) Find the Nash equilibrium of the stage game and the profit that every firm earns in it, π_1^N and π_2^N. Explain why the equilibrium involves limit pricing rather than marginal cost pricing.

(b) Find p^M, the associated market demand Q^M, and the two margins m_1 and m_2. Then find every firm's collusive profit, π_i^C, and every firm's deviation profit, π_i^D. Is the joint collusive profit equal to the monopoly profit? Explain.

(c) Write the incentive compatibility condition of each firm, and solve each of them for the critical discount factor of that firm, $\overline{\delta}_1$ and $\overline{\delta}_2$. Show that the two cutoffs differ, and that collusion is sustainable if and only if $\delta \geq \max\left\{\overline{\delta}_1, \overline{\delta}_2\right\}$.

(d) *The twist.* Find the market share s^* that minimizes the critical discount factor $\max\left\{\overline{\delta}_1, \overline{\delta}_2\right\}$, and report the minimized cutoff $\overline{\delta}^*$. Which firm receives the larger share, and why? Find, in addition, the set of market shares s for which collusion is sustainable for some discount factor below one.

(e) *Comparative statics.* How are s^* and $\overline{\delta}^*$ affected by a marginal increase in the cost gap Δ? Evaluate the two limiting cases $\Delta \to 0$ and $\Delta \to m_1$, and interpret them.

(f) *Numerical example.* Evaluate your results at $a = 10$, $c_1 = 2$ and $c_2 = 4$. Report p^M, the two margins, the stage-game profits, the optimal share s^* and the cutoff $\overline{\delta}^*$. Show, in addition, that an equal split of the market, $s = \frac{1}{2}$, cannot sustain collusion for any discount factor. Then reduce the cost gap by setting $c_2 = 3$, and report s^* and $\overline{\delta}^*$ again.

4. **Finitely repeated games with multiple stage-game equilibria.** Consider the following simultaneous-move game between player 1, who chooses rows, and player 2, who chooses columns. In every cell, the first number denotes player 1's payoff and the second number denotes player 2's payoff.

Player 1 \ Player 2	L	C	R
U	$6,6$	$0,7$	$0,0$
M	$7,0$	$4,4$	$0,0$
D	$0,0$	$0,0$	$1,1$

This stage game is played T times, players observe the outcome of every past period before choosing their period-t actions, and both of them discount future payoffs at the common rate $\delta \in (0,1]$. Payoffs in the repeated game are the discounted sum of stage-game payoffs.

Answer the following questions.

(a) Find every pure-strategy Nash equilibrium of the stage game, and rank them according to the Pareto criterion. Show, in addition, that the profile (U, L) is not a Nash equilibrium of the stage game even though it Pareto dominates both equilibria you found.

(b) Consider the twice-repeated game, $T = 2$. Construct a subgame perfect equilibrium in which players choose the non-Nash profile (U, L) in the first period, using the Pareto-superior stage-game equilibrium as a reward and the Pareto-inferior one as a punishment. Find the exact condition on δ under which this construction works.

(c) Show that the profile (M, L) cannot be sustained in the first period of the twice-repeated game for any discount factor $\delta \leq 1$, even though it delivers a higher payoff to player 1 than (U, L) does.

(d) Suppose now that the stage game were the 2×2 game that remains after deleting row D and column R. Show that this reduced game has a unique Nash equilibrium, and that the construction of part (b) then fails. Explain why unravelling occurs in that case, and identify the feature of the original 3×3 game that prevents it.

(e) Extend the construction of part (b) to the T-times repeated game, in which players choose (U, L) in every period $t = 1, 2, \ldots, T-1$ and the Pareto-superior equilibrium in period T. Write the incentive compatibility condition of period t, show that the binding constraint is the one of period $T-1$, and conclude that the same cutoff on δ found in part (b) supports cooperation in every period. What happens to the average per-period payoff as T grows without bound?

(f) *Numerical example.* Evaluate your results at $T = 4$. First set $\delta = \dfrac{1}{2}$ and report the equilibrium payoff of the construction of part (e), the payoff of the most profitable deviation in every period, and the payoff that players would obtain if the game unravelled to the Pareto-superior stage-game equilibrium in all four periods. Then set $\delta = \dfrac{13}{50}$, show that the construction of part (e) fails, and show that players can nevertheless support (U, L) in the first period alone. Interpret.

Additional practice (not graded). For further practice on this material, see the textbook, Chapters 6-7.