Microeconomic Theory II - UNI, Lima
Homework #2 - Answer key

1. **(25 points) Exercise 1 - Tax audits with a punitive fine and a commitment stage.** Consider a taxpayer and a tax authority (the auditor) who move simultaneously. The taxpayer owes an amount $b > 0$ in taxes, and chooses whether to evade (E) or to comply (C). The auditor chooses whether to audit (A), at a cost $k > 0$, or not to audit (N). If the taxpayer complies, the authority collects b whether or not it audits. If the taxpayer evades and is audited, the authority recovers the unpaid amount b and the taxpayer pays, in addition, a fine $F > 0$. The fine is punitive, and is transferred to the national treasury rather than credited to the audit agency's budget, so that F does not enter the auditor's payoff. If the taxpayer evades and is not audited, the authority collects nothing. Measuring the taxpayer's payoff as the negative of its total payment, the payoff matrix is

		Auditor	
		A	N
Taxpayer	E	$-(b+F), b-k$	$0, 0$
	C	$-b, b-k$	$-b, b$

   (a) (4 points) Show that this game has no pure-strategy Nash equilibrium (psNE) if and only if $k < b$. What is the unique psNE when $k > b$, and how would you interpret it?

   - *Checking the four cells.* In profile (E, A) the taxpayer earns $-(b+F)$, and deviating to C yields $-b$. The gain is

$$-b - [-(b+F)] = F > 0$$

     so the taxpayer deviates and (E, A) is not a psNE. In profile (C, A) the auditor earns $b - k$, and deviating to N yields b, a gain of $k > 0$, so (C, A) is not a psNE either. In profile (C, N) the taxpayer earns $-b$, and deviating to E yields 0, a gain of $b > 0$, so (C, N) is never a psNE.
   - *The remaining cell.* In profile (E, N) the auditor earns 0, and deviating to A yields $b - k$, so the deviation is profitable if and only if

$$b - k > 0, \text{ that is, } k < b$$

     The taxpayer, in contrast, never deviates from (E, N), since complying yields $-b < 0$. Therefore, (E, N) fails to be a psNE precisely when $k < b$, and the game has no psNE if and only if $k < b$.
   - *The case $k > b$.* When auditing costs more than the revenue it recovers, (E, N) becomes the unique psNE: the taxpayer evades with certainty and the authority never audits. Intuitively, enforcement is so expensive that the agency abandons it, and the taxpayer's evasion goes unchallenged. Note that a mixed-strategy equilibrium is not needed in this case, since the auditor has a strictly dominant strategy.

(b) (6 points) Assume $k < b$ throughout the rest of the exercise. Find the mixed-strategy Nash equilibrium (msNE) in closed form. Denote by p the probability that the taxpayer evades and by q the probability that the auditor audits. Report the equilibrium payoff of each player.

- *Taxpayer's indifference condition.* Facing an audit probability q, the taxpayer's expected payoffs are

$$\begin{aligned} EU_T(E,q) &= -q(b+F)+(1-q)\cdot 0 = -q(b+F) \\ EU_T(C,q) &= -qb-(1-q)b = -b \end{aligned}$$

  Setting them equal, $-q(b+F) = -b$. Solving for q, we obtain the auditor's equilibrium probability

$$q^* = \frac{b}{b+F}$$

- *Auditor's indifference condition.* Facing an evasion probability p, the auditor's expected payoffs are

$$\begin{aligned} EU_A(A,p) &= p(b-k)+(1-p)(b-k) = b-k \\ EU_A(N,p) &= p\cdot 0+(1-p)b = (1-p)b \end{aligned}$$

  where the first line is independent of p, since the authority collects b and pays k whenever it audits, whatever the taxpayer does. Setting the two expressions equal, $b-k = b-pb$, which simplifies to $pb = k$. Solving for p, we obtain the taxpayer's equilibrium probability

$$p^* = \frac{k}{b}$$

  which lies in $(0,1)$ if and only if $k < b$, exactly the condition of part (a).
- *The equilibrium.* The msNE is, therefore,

$$\sigma_T^* = \left(\frac{k}{b}E, \frac{b-k}{b}C\right) \quad \text{and} \quad \sigma_A^* = \left(\frac{b}{b+F}A, \frac{F}{b+F}N\right)$$

- *Equilibrium payoffs.* Since each player is indifferent, we may evaluate either pure strategy. The taxpayer earns

$$EU_T^* = -b$$

  and the auditor earns

$$EU_A^* = b-k$$

  Note that the taxpayer earns exactly what it would earn by complying with certainty, so evasion delivers no expected gain in equilibrium. Intuitively, the auditor calibrates its audit probability precisely to strip the taxpayer of any expected benefit from cheating.

(c) (5 points) *Comparative statics.* How are p^* and q^* affected by a marginal increase in the fine F, in the audit cost k, and in the evaded amount b? Explain why a tougher fine leaves the taxpayer's evasion probability unchanged while reducing the probability of an audit.

- *Effect of the fine.* Differentiating with respect to F, yields

$$\frac{\partial p^*}{\partial F} = 0 \quad \text{and} \quad \frac{\partial q^*}{\partial F} = -\frac{b}{(b+F)^2} < 0$$

  so a tougher fine leaves evasion unchanged and unambiguously reduces auditing.
- *Effect of the audit cost and of the evaded amount.* Differentiating with respect to k and b, we obtain

$$\frac{\partial p^*}{\partial k} = \frac{1}{b} > 0, \quad \frac{\partial p^*}{\partial b} = -\frac{k}{b^2} < 0 \quad \text{and} \quad \frac{\partial q^*}{\partial b} = \frac{F}{(b+F)^2} > 0$$

  Costlier audits raise evasion, since the auditor must be kept indifferent against a larger cost, whereas a larger evaded amount raises the return to auditing and thereby reduces the evasion the auditor tolerates.
- *Interpreting the fine result.* In a msNE every player's probability is pinned down by the *rival's* indifference condition, not by its own payoffs. The fine F enters only the taxpayer's payoffs, and is absent from the auditor's payoffs by assumption, since it is transferred to the treasury. Therefore F can only move the auditor's probability q^*, and it leaves the taxpayer's probability $p^* = \frac{k}{b}$ untouched.
- Intuitively, a tougher fine does make evasion less attractive at any given audit probability, but the authority responds by auditing less often, and the two forces exactly offset. The taxpayer ends up evading just as frequently as before, while the authority saves on enforcement. Penalty severity and enforcement effort are substitutes in this model, which is the classic Becker point, and it explains why observed evasion rates can be insensitive to statutory penalties.

(d) (4 points) Suppose now that the auditor moves *first*, publicly committing to an audit probability $q \in [0,1]$ that it cannot revise, and that the taxpayer chooses E or C after observing q. Solve this sequential game by backward induction. Find the committed audit probability q^C, the taxpayer's response, and the auditor's payoff. Show that the auditor is strictly better off than in part (b), and that evasion falls.

- *Last mover, the taxpayer.* Observing q, the taxpayer compares $-q(b+F)$ against $-b$, so it evades if and only if

$$-q(b+F) > -b, \quad \text{that is,} \quad q < \overline{q} \equiv \frac{b}{b+F}$$

  and complies if $q > \overline{q}$, being indifferent at $q = \overline{q}$. Note that the cutoff $\overline{q}$ coincides with the equilibrium audit probability q^* of part (b), which is not a coincidence: both are the probability that makes the taxpayer indifferent.
- *First mover, the auditor: the evasion branch.* For $q < \overline{q}$ the taxpayer evades, and the auditor's expected payoff is

$$\pi_A(q) = q(b-k) + (1-q) \cdot 0 = q(b-k)$$

which increases in q since $k < b$. The auditor would like q as large as possible on this branch, and the supremum $\overline{q}(b-k)$ is not attained.

- *First mover, the auditor: the compliance branch.* For $q \geq \overline{q}$ the taxpayer complies, and the auditor's expected payoff is

$$\pi_A(q) = q(b-k) + (1-q)b = b - qk$$

which decreases in q. The best point of this branch is therefore $q = \overline{q}$.

- *Comparing the two branches.* Evaluating both expressions at $q = \overline{q}$ and subtracting, we obtain

$$(b - \overline{q}k) - \overline{q}(b-k) = b(1-\overline{q}) = \frac{bF}{b+F} > 0$$

so the compliance branch strictly dominates. The auditor therefore commits to

$$q^C = \overline{q} = \frac{b}{b+F}$$

and the taxpayer complies with certainty.

- *Summary.* The auditor's payoff under commitment is

$$\begin{aligned}\pi_A^C &= b - \frac{kb}{b+F} \\ &= \frac{b(b+F-k)}{b+F}\end{aligned}$$

and the gain relative to the simultaneous-move game of part (b) is

$$\pi_A^C - (b-k) = k - \frac{kb}{b+F} = \frac{kF}{b+F} > 0$$

so commitment strictly benefits the auditor. Evasion, in turn, falls from $p^* = \dfrac{k}{b}$ to zero.

- Note that $q^C = q^*$: the auditor commits to exactly the same audit probability it randomizes over in the simultaneous game. What commitment changes is not how often the authority audits but the fact that the taxpayer knows it. Intuitively, in the simultaneous game the auditor cannot persuade the taxpayer that it will audit that often, so the taxpayer keeps mixing and evasion survives; moving first makes the very same behavior credible and tips the indifferent taxpayer into compliance.

(e) (3 points) Taxes and fines are transfers, so the only real resource cost in this economy is the audit cost. Define social welfare as $W = -k \Pr(\text{audit})$ and compare W across the two regimes of parts (b) and (d). Does commitment raise welfare? Where does the auditor's gain come from? How is W affected by F?

- *Welfare in each regime.* The audit probability is $q^* = \dfrac{b}{b+F}$ in the simultaneous game and $q^C = \dfrac{b}{b+F}$ under commitment, so

$$W^* = W^C = -\frac{kb}{b+F}$$

and commitment leaves welfare exactly unchanged. Since the audit probability is the same in both regimes, so is the only real cost the economy incurs.

- *Where the gain comes from.* The taxpayer earns $-b$ in both regimes, in part (b) because it is indifferent and in part (d) because it complies. The auditor gains $\frac{kF}{b+F}$. In the simultaneous game the treasury collects fines with probability
$$p^* q^* = \frac{k}{b} \cdot \frac{b}{b+F} = \frac{k}{b+F}$$
so its expected fine revenue is $\frac{kF}{b+F}$, which is precisely the auditor's gain. Under commitment nobody evades, so this revenue disappears. Adding the three parties in either regime,
$$-b + (b-k) + \frac{kF}{b+F} = -b + \frac{b\,(b+F-k)}{b+F} + 0 = -\frac{kb}{b+F}$$
Therefore, commitment is a pure transfer: it converts fine revenue that the agency did not internalize into the agency's own payoff, without moving any real resource.
- *Effect of the fine on welfare.* Differentiating with respect to F, yields
$$\frac{\partial W}{\partial F} = \frac{kb}{(b+F)^2} > 0$$
so a tougher fine unambiguously raises welfare in both regimes. Intuitively, since the fine is a transfer it is socially costless, whereas auditing is not, and part (c) showed that a higher F buys the same deterrence with fewer audits. This is the standard argument for setting the penalty high and the audit rate low.

(f) (3 points) *Numerical example.* Evaluate your results at $b = 100$, $F = 150$ and $k = 40$. Then raise the fine to $F = 400$, keeping b and k unchanged, and recompute every object.

- *First panel, $F = 150$.* Since $k = 40 < 100 = b$, part (a) applies and no psNE exists. The msNE probabilities are
$$p^* = \frac{40}{100} = \frac{2}{5} \quad \text{and} \quad q^* = \frac{100}{250} = \frac{2}{5}$$
with payoffs -100 for the taxpayer and $b - k = 60$ for the auditor.
- Under commitment the auditor sets $q^C = \frac{2}{5}$, the taxpayer complies, and the auditor earns
$$\pi_A^C = 100 - \frac{2}{5} \cdot 40 = 84$$
a gain of $84 - 60 = 24$, which matches $\frac{kF}{b+F} = \frac{40 \cdot 150}{250} = 24$. Welfare is $-\frac{40 \cdot 100}{250} = -16$ in both regimes.

- *Second panel, $F = 400$.* The evasion probability is unchanged, $p^* = \frac{2}{5}$, while the audit probability falls to

$$q^* = \frac{100}{500} = \frac{1}{5}$$

confirming the comparative statics of part (c). The auditor now earns 60 without commitment and

$$\pi_A^C = 100 - \frac{1}{5} \cdot 40 = 92$$

with commitment, a gain of $32 = \frac{40 \cdot 400}{500}$. Welfare improves from -16 to $-\frac{40 \cdot 100}{500} = -8$.
- Therefore, raising the fine from 150 to 400 halves the audit rate and halves the social cost of enforcement, while leaving evasion at 40% in the simultaneous game and at zero under commitment.

2. **(24 points) Exercise 2 - A zero-sum game without a saddle point.** Consider the following two-player zero-sum game. Player 1 chooses rows, A, B, C or D, and player 2 chooses columns, L, M, R or X. Every entry reports the payment that player 2 makes to player 1, so player 1 maximizes and player 2 minimizes it.

		Player 2			
		L	M	R	X
Player 1	A	5	2	1	7
	B	1	4	6	8
	C	4	3	2	6
	D	3	1	0	5

   (a) (5 points) Find player 1's pure maximin value and player 2's pure minimax value, together with the strategies attaining them. Show that the two values differ, and explain why this implies that the game has no saddle point, and therefore no psNE.

   - *Maximin.* Taking the smallest entry of every row, we obtain

   $$\min_j a_{Aj} = 1, \quad \min_j a_{Bj} = 1, \quad \min_j a_{Cj} = 2, \quad \min_j a_{Dj} = 0$$

   so player 1's maximin value is $\underline{v} = 2$, attained by row C. Playing C guarantees player 1 at least 2, whatever player 2 does.
   - *Minimax.* Taking the largest entry of every column, we obtain

   $$\max_i a_{iL} = 5, \quad \max_i a_{iM} = 4, \quad \max_i a_{iR} = 6, \quad \max_i a_{iX} = 8$$

   so player 2's minimax value is $\overline{v} = 4$, attained by column M. Playing M guarantees player 2 that it never pays more than 4.

- *No saddle point.* Since $\underline{v} = 2 < 4 = \overline{v}$, the two values differ. A saddle point is a cell that is simultaneously the smallest entry of its row and the largest entry of its column, and such a cell exists if and only if $\underline{v} = \overline{v}$. In a zero-sum game a psNE is exactly a saddle point, since player 1 maximizing and player 2 minimizing the same number is precisely the requirement that the cell be a row minimum and a column maximum. Therefore, no psNE exists and the equilibrium must be in mixed strategies. Note, in addition, that $\underline{v} \leq \overline{v}$ always holds, so the gap $\overline{v} - \underline{v} = 2$ is the room that randomization has to fill.

(b) (6 points) Show that exactly one row and exactly one column are strictly dominated, and eliminate them. Show, in addition, that no further strategy is strictly dominated in the 3×3 game that remains.

- *Player 1's strategies.* Listing player 1's payoffs against columns L, M, R and X, we obtain

$$\begin{aligned} A &= (5,2,1,7), \quad B = (1,4,6,8) \\ C &= (4,3,2,6), \quad D = (3,1,0,5) \end{aligned}$$

  Comparing entry by entry, $5 > 3$, $2 > 1$, $1 > 0$ and $7 > 5$, so row A strictly dominates row D. Row C strictly dominates it as well, since $4 > 3$, $3 > 1$, $2 > 0$ and $6 > 5$. We therefore delete D.
- No other row of player 1 is strictly dominated. Dominating A would require a payoff above 5 against L, and 5 is the largest payoff available in that column. Dominating B would require a payoff above 6 against R, and 6 is the largest payoff available in that column. Dominating C would require beating 4 against L and 2 against R simultaneously, and the only row reaching 4 or more against L is A, which yields $1 < 2$ against R.
- *Player 2's strategies.* Player 2 minimizes, so a column is strictly dominated when another column concedes strictly less in every row. Listing player 2's payments against rows A, B, C and D,

$$\begin{aligned} L &= (5,1,4,3), \quad M = (2,4,3,1) \\ R &= (1,6,2,0), \quad X = (7,8,6,5) \end{aligned}$$

  Column X concedes more than each of L, M and R in every single row, so it is strictly dominated and we delete it.
- No other column is strictly dominated. Dominating L would require conceding less than 1 against B, and 1 is the smallest payment available in that row. Dominating M would require conceding less than 2 against A, while R concedes 1 there but 6 against B, above the 4 that M concedes. Dominating R would require conceding less than 0 against D, which is impossible.
- *The reduced game.* After deleting row D and column X, we are left with

		Player 2		
		L	M	R
Player 1	A	5	2	1
	B	1	4	6
	C	4	3	2

and no further elimination is possible. The quickest argument is the one that part (c) supplies: the equilibrium of the reduced game attaches strictly positive probability to each of the three rows and to each of the three columns, and a strictly dominated strategy can never be played with positive probability in equilibrium.

(c) (5 points) Find the msNE of the reduced game, $x^* = (x_A, x_B, x_C)$ for player 1 and $y^* = (y_L, y_M, y_R)$ for player 2, and the value of the game v. Verify that both indifference systems deliver the same value.

- *Player 1's mixture.* Player 1 randomizes so as to make player 2 indifferent across its three columns, which requires

$$\begin{aligned} 5x_A + x_B + 4x_C &= v \\ 2x_A + 4x_B + 3x_C &= v \\ x_A + 6x_B + 2x_C &= v \end{aligned}$$

together with $x_A + x_B + x_C = 1$. Subtracting the second equation from the first, and the third from the second, yields

$$3x_A - 3x_B + x_C = 0 \quad \text{and} \quad x_A - 2x_B + x_C = 0$$

Subtracting these two expressions, we obtain $2x_A - x_B = 0$, so that $x_B = 2x_A$, and substituting back into the second one gives $x_C = 3x_A$.

- Imposing that the probabilities add up to one, $x_A(1+2+3) = 1$. Solving for x_A, we obtain

$$x^* = \left(\frac{1}{6}A, \frac{1}{3}B, \frac{1}{2}C\right)$$

and the value follows from any of the three equations,

$$v = 5 \cdot \frac{1}{6} + \frac{1}{3} + 4 \cdot \frac{1}{2} = \frac{5}{6} + \frac{2}{6} + \frac{12}{6} = \frac{19}{6}$$

- *Player 2's mixture.* Player 2 randomizes so as to make player 1 indifferent across its three rows, which requires

$$\begin{aligned} 5y_L + 2y_M + y_R &= v \\ y_L + 4y_M + 6y_R &= v \\ 4y_L + 3y_M + 2y_R &= v \end{aligned}$$

together with $y_L + y_M + y_R = 1$. Subtracting the third equation from the first, yields $y_L - y_M - y_R = 0$, so that $y_L = y_M + y_R$, and combining this with the adding-up condition we immediately obtain $y_L = \frac{1}{2}$.

- Subtracting the second equation from the third gives $3y_L - y_M - 4y_R = 0$. Inserting $y_L = \frac{1}{2}$ and $y_M = \frac{1}{2} - y_R$, this expression simplifies to $1 - 3y_R = 0$. Solving for y_R, we obtain

$$y^* = \left(\frac{1}{2}L, \frac{1}{6}M, \frac{1}{3}R\right)$$

and the value is

$$v = 5 \cdot \frac{1}{2} + 2 \cdot \frac{1}{6} + \frac{1}{3} = \frac{15}{6} + \frac{2}{6} + \frac{2}{6} = \frac{19}{6}$$

which coincides with the value obtained from player 1's system, as the minimax theorem requires.

- *Verification.* Evaluating player 1's mixture against each column and player 2's mixture against each row,

$$\begin{aligned} x^* \text{ against } (L, M, R) &= \left(\frac{19}{6}, \frac{19}{6}, \frac{19}{6}\right) \\ (A, B, C) \text{ against } y^* &= \left(\frac{19}{6}, \frac{19}{6}, \frac{19}{6}\right) \end{aligned}$$

so player 1 guarantees itself $\frac{19}{6}$ and player 2 guarantees that it never pays more than $\frac{19}{6}$. The two security levels coincide, which is exactly the statement that $v = \frac{19}{6}$ is the value of the game.

(d) (5 points) Verify that v lies strictly between the maximin and the minimax values of part (a). Confirm, in addition, that the two strategies you eliminated in part (b) are not best responses to the equilibrium mixtures.

- *Location of the value.* From part (a), $\underline{v} = 2$ and $\overline{v} = 4$, and from part (c), $v = \frac{19}{6} \simeq 3.17$, so that

$$2 < \frac{19}{6} < 4$$

Randomization therefore raises player 1's guarantee from 2 to $\frac{19}{6}$ and lowers player 2's exposure from 4 to $\frac{19}{6}$. Intuitively, a pure maximin strategy must survive the rival's worst reply, whereas a mixture keeps the rival uncertain and so cannot be punished cell by cell.

- *The eliminated row.* Evaluating row D against y^*, we obtain

$$3 \cdot \frac{1}{2} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{3} = \frac{5}{3} < \frac{19}{6}$$

so D is strictly worse than the equilibrium mixture and player 1 would never use it.

- *The eliminated column.* Evaluating column X against x^*, we obtain

$$7 \cdot \frac{1}{6} + 8 \cdot \frac{1}{3} + 6 \cdot \frac{1}{2} = \frac{41}{6} > \frac{19}{6}$$

so playing X would cost player 2 more than twice the value of the game. Therefore, both eliminated strategies remain strictly suboptimal in the full 4×4 game, and the mixed profile (x^*, y^*) extended with zero weight on D and X is an equilibrium of the original game.

(e) (3 points) *Comparative statics.* Suppose the entry in row B and column R rises from 6 to $6+\varepsilon$, with $\varepsilon \geq 0$. Find $x^*(\varepsilon)$, $y^*(\varepsilon)$ and $v(\varepsilon)$ in closed form, and show that $\dfrac{\partial v}{\partial \varepsilon}$ evaluated at $\varepsilon = 0$ equals $x_B y_R$. Then evaluate the game exactly at $\varepsilon = 1$. Is the true increase larger or smaller than the one predicted by the derivative? Interpret. For which values of ε does your closed form remain valid?

- *The perturbed equilibrium.* Repeating the two indifference systems of part (c) with the entry $a_{BR} = 6+\varepsilon$, and solving, we obtain

$$x^*(\varepsilon) = \left(\frac{1-\varepsilon}{2\,(3+\varepsilon)}, \frac{1}{3+\varepsilon}, \frac{3\,(1+\varepsilon)}{2\,(3+\varepsilon)}\right)$$

$$y^*(\varepsilon) = \left(\frac{1}{2}, \frac{1+\varepsilon}{2\,(3+\varepsilon)}, \frac{1}{3+\varepsilon}\right)$$

  and the value of the game

$$v(\varepsilon) = \frac{19+7\varepsilon}{2\,(3+\varepsilon)}$$

  At $\varepsilon = 0$ these expressions collapse to the mixtures and the value of part (c), as required. Note that $y_L = \dfrac{1}{2}$ regardless of ε.

- *The derivative.* Differentiating the value with respect to ε, yields

$$\frac{\partial v}{\partial \varepsilon} = \frac{1}{(3+\varepsilon)^2} > 0$$

  and evaluating it at $\varepsilon = 0$ we obtain $\dfrac{1}{9}$, which coincides with

$$x_B y_R = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

  This is the envelope result for zero-sum games: to a first order the value rises at the rate at which the improved cell is actually reached in equilibrium, holding the two mixtures fixed. Intuitively, a cell that is almost never played can be improved a great deal without moving the value.

- *Second-order behavior.* Differentiating once more,

$$\frac{\partial^2 v}{\partial \varepsilon^2} = -\frac{2}{(3+\varepsilon)^3} < 0$$

  so the value is concave in the entry.

- *The exact evaluation at $\varepsilon = 1$.* Setting $\varepsilon = 1$, so that the entry rises from 6 to 7, we obtain

$$v(1) = \frac{26}{8} = \frac{13}{4}, \quad x^*(1) = \left(0, \frac{1}{4}, \frac{3}{4}\right) \quad \text{and} \quad y^*(1) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

  which we can check directly: against the perturbed matrix, $x^*(1)$ concedes $\dfrac{13}{4}$ in each of the three columns and $y^*(1)$ delivers $\dfrac{13}{4}$ in each of the three rows.

- *Comparing the two magnitudes.* The true increase is

$$\frac{13}{4} - \frac{19}{6} = \frac{39 - 38}{12} = \frac{1}{12} \simeq 0.083$$

which falls short of the $\frac{1}{9} \simeq 0.111$ predicted by the derivative at $\varepsilon = 0$. The derivative treats both mixtures as fixed, whereas in fact player 2 reoptimizes and shifts weight away from the column that has become expensive, since y_R falls from $\frac{1}{3}$ to $\frac{1}{4}$. Intuitively, the opponent can partially insulate itself against the improvement, so the linear prediction overstates the gain, which is the same concavity we found above. The two numbers reconcile exactly, since

$$\int_0^1 \frac{d\varepsilon}{(3+\varepsilon)^2} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

- *Range of validity.* The closed form requires $x_A \geq 0$, that is, $\frac{1-\varepsilon}{2\,(3+\varepsilon)} \geq 0$, which holds if and only if

$$\varepsilon \leq 1$$

At $\varepsilon = 1$ row A leaves the support of the equilibrium mixture, and for $\varepsilon > 1$ the support changes, so the expressions above no longer apply and the equilibrium must be recomputed on the smaller support. Intuitively, as row B becomes stronger, player 1 concentrates on B and C and abandons A altogether.

3. **(26 points) Exercise 3 - Stackelberg competition with asymmetric costs: accommodation or deterrence.** Consider a market with inverse demand function $p(Q) = a - Q$, where $Q = q_1 + q_2$. Firm 1 is an established leader that chooses q_1 first, and firm 2 is a follower that observes q_1 and then decides whether to enter the market and, if it does, how much to produce. Firms face constant marginal costs c_1 and c_2, which may differ, and firm 2 must pay a fixed entry cost $K > 0$ upon entering. Payoffs are $\pi_1 = (a - q_1 - q_2 - c_1)q_1$ and $\pi_2 = (a - q_1 - q_2 - c_2)q_2 - K$. Throughout, denote $s \equiv \sqrt{K}$.

   (a) (5 points) Find firm 2's best response function conditional on entering, check its second-order condition, and find its gross profit as a function of q_1. Then show that firm 2 enters if and only if $q_1 \leq q_1^D \equiv a - c_2 - 2s$.

   - *Last mover, firm 2.* Conditional on entering, firm 2 solves

$$\max_{q_2 \geq 0} \pi_2 = (a - q_1 - q_2 - c_2)q_2 - K$$

Differentiating with respect to q_2, yields $a - q_1 - 2q_2 - c_2 = 0$. Solving for q_2, we obtain firm 2's best response function

$$q_2(q_1) = \frac{a - c_2 - q_1}{2}$$

The second-order condition holds since $\frac{\partial^2 \pi_2}{\partial q_2^2} = -2 < 0$.

- *Gross profit.* Inserting the best response into the residual price-cost margin,

$$a - q_1 - q_2 - c_2 = 2q_2 - q_2 = q_2$$

so the margin equals the quantity, and firm 2's gross profit is

$$\pi_2^{gross}(q_1) = \left(\frac{a - c_2 - q_1}{2}\right)^2$$

which decreases in q_1.

- *Entry condition.* Firm 2 enters if and only if $\pi_2^{gross}(q_1) \geq K$, that is,

$$\frac{a - c_2 - q_1}{2} \geq s, \text{ or equivalently, } q_1 \leq q_1^D \equiv a - c_2 - 2s$$

Therefore, q_1^D is the smallest output that makes entry unprofitable, and any $q_1 > q_1^D$ keeps firm 2 out. Note that q_1^D decreases in both c_2 and K, so a less efficient or more heavily burdened entrant is easier to keep out.

(b) (6 points) *Accommodation.* Suppose firm 1 lets firm 2 enter. Find its optimal output q_1^A, the induced follower output q_2^A, and its profit π_1^A. For which values of K does firm 2 indeed enter when firm 1 produces q_1^A?

- *Second-to-last mover, firm 1.* Anticipating firm 2's response, firm 1's residual margin is

$$\begin{aligned} a - q_1 - q_2(q_1) - c_1 &= a - q_1 - \frac{a - c_2 - q_1}{2} - c_1 \\ &= \frac{a + c_2 - 2c_1 - q_1}{2} \end{aligned}$$

so that firm 1 solves

$$\max_{q_1 \geq 0} \pi_1 = q_1 \frac{a + c_2 - 2c_1 - q_1}{2}$$

- Differentiating with respect to q_1, yields $a + c_2 - 2c_1 - 2q_1 = 0$. Solving for q_1, we obtain the accommodation output

$$q_1^A = \frac{a + c_2 - 2c_1}{2}$$

The second-order condition holds since $\dfrac{\partial^2 \pi_1}{\partial q_1^2} = -1 < 0$.

- *Induced output and profit.* Inserting q_1^A into firm 2's best response,

$$q_2^A = \frac{a - c_2 - q_1^A}{2} = \frac{a + 2c_1 - 3c_2}{4}$$

and firm 1's profit is

$$\pi_1^A = \frac{(a + c_2 - 2c_1)^2}{8}$$

which is the familiar Stackelberg leader profit, now written with asymmetric costs.

- *When does entry actually occur?* Firm 2 enters at q_1^A if and only if $q_1^A \leq q_1^D$, which by part (a) is equivalent to $s \leq q_2^A$, that is,

$$K \leq K_A \equiv \left(\frac{a + 2c_1 - 3c_2}{4}\right)^2$$

Intuitively, the accommodation plan is only meaningful when the entry cost falls below the profit the follower would earn under it, and q_2^A is precisely the square root of that profit.

(c) (5 points) *Deterrence.* Suppose instead that firm 1 keeps firm 2 out. Find its profit π_1^D as a function of s.

- Producing $q_1^D = a - c_2 - 2s$ and facing no rival, firm 1's margin is

$$a - q_1^D - c_1 = a - (a - c_2 - 2s) - c_1 = c_2 - c_1 + 2s$$

so that its deterrence profit is

$$\pi_1^D = (c_2 - c_1 + 2s)\,(a - c_2 - 2s)$$

Note that firm 1 produces more than it would like to as a monopolist, which is the cost of deterrence, but it captures the whole market, which is the benefit.

(d) (5 points) Firm 1 deters entry if and only if $\pi_1^D \geq \pi_1^A$. Show that this comparison reduces to a quadratic in s, and find the cutoff $\overline{K}$ such that firm 1 deters entry if and only if $K \geq \overline{K}$. Show that the discriminant of that quadratic is $\frac{(a-c_2)^2}{2} - (c_2 - c_1)^2$, and interpret the requirement that it be positive.

- *The quadratic.* Writing $u \equiv 2s$ and expanding π_1^D,

$$\begin{aligned} \pi_1^D &= (c_2 - c_1 + u)\,(a - c_2 - u) \\ &= (c_2 - c_1)\,(a - c_2) + u\,(a + c_1 - 2c_2) - u^2 \end{aligned}$$

so the condition $\pi_1^D \geq \pi_1^A$ is equivalent to

$$u^2 - u\,(a + c_1 - 2c_2) + \left[\frac{(a + c_2 - 2c_1)^2}{8} - (c_2 - c_1)\,(a - c_2)\right] \leq 0$$

- *The discriminant.* Computing it, and writing $y \equiv a - c_2$ and $d \equiv c_2 - c_1$ for compactness, so that $a + c_1 - 2c_2 = y - d$ and $a + c_2 - 2c_1 = y + 2d$, we obtain

$$\begin{aligned} D &= (y - d)^2 - 4\left[\frac{(y + 2d)^2}{8} - dy\right] \\ &= y^2 - 2dy + d^2 - \frac{y^2 + 4dy + 4d^2}{2} + 4dy \\ &= \frac{y^2}{2} - d^2 \end{aligned}$$

which, in the original notation, is

$$D = \frac{(a - c_2)^2}{2} - (c_2 - c_1)^2$$

- *The cutoff.* Since the quadratic opens upward, the inequality holds between its two roots, $u = \dfrac{(a + c_1 - 2c_2) \pm \sqrt{D}}{2}$. Recalling that $u = 2s$, the lower root delivers
$$\overline{s} \equiv \frac{(a + c_1 - 2c_2) - \sqrt{D}}{4} \quad \text{and} \quad \overline{K} \equiv \overline{s}^2$$
The upper root is irrelevant on the range where entry would occur. To see this, evaluate the two profits at the largest such value, $s = q_2^A$, where $q_1^D = q_1^A$, so that firm 1 produces the same output in both plans and the only difference is whether firm 2 is present,
$$\pi_1^D - \pi_1^A\big|_{s=q_2^A} = q_1^A q_2^A > 0$$
Therefore, deterrence is strictly preferred at the top of the range, the upper root lies beyond it, and firm 1 deters entry if and only if
$$K \geq \overline{K}$$
- *Interpreting* $D > 0$. In terms of y and d, the condition reads $|d| < \dfrac{y}{\sqrt{2}}$, so the cost asymmetry must be moderate relative to the size of the market. Note that this is not an extra assumption: from part (b), $q_1^A > 0$ requires $d > -\dfrac{y}{2}$ and $q_2^A > 0$ requires $d < \dfrac{y}{2}$, and since $\dfrac{1}{2} < \dfrac{1}{\sqrt{2}}$, both firms being active already guarantees $D > 0$. When the asymmetry exceeds these bounds one of the two firms produces nothing even without an entry cost, and the accommodation-versus-deterrence comparison is vacuous.

(e) (3 points) *Comparative statics.* How are q_1^D and π_1^D affected by a marginal increase in K? Find the level of K at which entry becomes blockaded. Then discuss how an increase in c_2 affects q_1^A, π_1^A, q_1^D and the cutoff $\overline{K}$.

- *Effect of K on the deterrence output.* Differentiating $q_1^D = a - c_2 - 2\sqrt{K}$ with respect to K, yields
$$\frac{\partial q_1^D}{\partial K} = -\frac{1}{\sqrt{K}} < 0$$
so a larger entry cost lets firm 1 deter with less output.
- *Effect of K on the deterrence profit.* Differentiating with respect to s, we obtain
$$\frac{\partial \pi_1^D}{\partial s} = 2\,(a + c_1 - 2c_2 - 4s)$$
which is positive if and only if $s < \overline{s}_B \equiv \dfrac{a + c_1 - 2c_2}{4}$. Since s and K move together, deterrence profit increases in K on that whole range. Intuitively, deterrence is costly because it forces firm 1 above its monopoly output, and a higher entry cost relaxes exactly that constraint.

- *Blockaded entry.* The cutoff $\overline{s}_B$ is precisely the point at which the deterrence output falls to the monopoly output, since solving $a - c_2 - 2s = \dfrac{a - c_1}{2}$ for s yields $s = \dfrac{a + c_1 - 2c_2}{4}$. Therefore, for

$$K \geq K_B \equiv \left(\frac{a + c_1 - 2c_2}{4}\right)^2$$

entry is blockaded: firm 1 simply produces $q_1^M = \dfrac{a - c_1}{2}$ and earns $\pi_1^M = \dfrac{(a - c_1)^2}{4}$, and firm 2 stays out without firm 1 doing anything about it.

- *Effect of c_2.* Differentiating the accommodation objects,

$$\frac{\partial q_1^A}{\partial c_2} = \frac{1}{2} > 0 \;\; \text{and} \;\; \frac{\partial \pi_1^A}{\partial c_2} = \frac{a + c_2 - 2c_1}{4} > 0$$

whereas the deterrence output falls, $\dfrac{\partial q_1^D}{\partial c_2} = -1 < 0$. A weaker rival therefore makes accommodation more attractive in absolute terms, but it also makes deterrence cheaper, and the two effects push the cutoff in opposite directions.

- *Effect of c_2 on the cutoff.* Differentiating $\overline{s}$ with respect to c_2, and using $\dfrac{\partial D}{\partial c_2} = -(a - c_2) - 2(c_2 - c_1)$, yields

$$\frac{\partial \overline{s}}{\partial c_2} = \frac{1}{4}\left[-2 + \frac{(a - c_2) + 2(c_2 - c_1)}{2\sqrt{D}}\right]$$

which is negative if and only if $y + 2d < 4\sqrt{D}$. Squaring and writing $t \equiv \dfrac{d}{y}$, this expression simplifies to

$$20t^2 + 4t - 7 < 0, \;\; \text{that is,} \;\; -\frac{7}{10} < t < \frac{1}{2}$$

and part (d) showed that both firms being active requires $-\dfrac{1}{2} < t < \dfrac{1}{2}$, an interval contained in the previous one. Therefore, $\overline{K}$ unambiguously decreases in c_2: the second force dominates, and a less efficient entrant is deterred at a lower entry cost.

(f) (2 points) *Numerical example.* Evaluate your results at $a = 12$, $c_1 = 1$ and $c_2 = 2$. Report q_1^A, q_2^A, π_1^A and $\overline{K}$, then compute q_1^D and π_1^D for $K = \dfrac{1}{16}$, $K = \dfrac{1}{4}$, $K = 1$, $K = 4$ and $K = \dfrac{81}{16}$.

- *Accommodation.* With $a + c_2 - 2c_1 = 12$ and $a + 2c_1 - 3c_2 = 8$, we obtain

$$q_1^A = 6, \;\; q_2^A = 2 \;\; \text{and} \;\; \pi_1^A = \frac{144}{8} = 18$$

which we can check directly, since $p = 12 - 6 - 2 = 4$ and $\pi_1 = (4 - 1)\,6 = 18$. Entry occurs under accommodation as long as $K \leq K_A = 4$.

- *The cutoff.* The discriminant is

$$D = \frac{10^2}{2} - 1^2 = 49$$

so that

$$\overline{s} = \frac{(12+1-4)-7}{4} = \frac{9-7}{4} = \frac{1}{2} \text{ and } \overline{K} = \frac{1}{4}$$

Entry is blockaded from $K_B = \left(\frac{9}{4}\right)^2 = \frac{81}{16}$ onwards.

- *The five cases.* Applying $q_1^D = 10 - 2s$ and $\pi_1^D = (1+2s)(10-2s)$,

$$\begin{aligned}
K &= \frac{1}{16} : \; q_1^D = \frac{19}{2}, \;\; \pi_1^D = \frac{57}{4} < 18, \;\; \text{accommodate} \\
K &= \frac{1}{4} : \; q_1^D = 9, \;\; \pi_1^D = 18 = 18, \;\; \text{indifferent} \\
K &= 1 : \; q_1^D = 8, \;\; \pi_1^D = 24 > 18, \;\; \text{deter} \\
K &= 4 : \; q_1^D = 6, \;\; \pi_1^D = 30 > 18, \;\; \text{deter} \\
K &= \frac{81}{16} : \; q_1^D = \frac{11}{2}, \;\; \pi_1^D = \frac{121}{4}, \;\; \text{blockaded}
\end{aligned}$$

so the switch occurs exactly at $\overline{K} = \frac{1}{4}$, as predicted.

- Note that π_1^D rises monotonically along the list, from $\frac{57}{4}$ to $\frac{121}{4}$, illustrating the comparative static of part (e). At $K = 4$ the deterrence output has fallen to $6 = q_1^A$, so firm 1 produces the accommodation output and yet firm 2 stays out, and at $K = \frac{81}{16}$ it has fallen further to the monopoly output $\frac{11}{2}$.

4. **(25 points) Exercise 4 - Entry deterrence through a capacity commitment.** Consider a three-stage game between an incumbent (firm 1) and a potential entrant (firm 2), with inverse demand $p(Q) = a - Q$. In stage 1 the incumbent installs capacity $k \geq 0$ at a unit cost $r > 0$, paying rk up front. In stage 2 the entrant observes k and decides whether to enter, paying a fixed cost $F > 0$. In stage 3, if entry occurred both firms compete in quantities simultaneously, and otherwise the incumbent is a monopolist. Producing one unit requires one unit of capacity and one unit of labor at a cost $w > 0$, so that the incumbent's marginal cost is w on units within its capacity and $w + r$ on units beyond it, whereas the entrant's marginal cost is $w + r$ on every unit. Write $c \equiv w + r$ and $m \equiv a - c$, and assume $m > r > 0$.

   (a) (3 points) *Stage 3.* Suppose entry occurred and the incumbent holds capacity k. Find the entrant's best response function. Then show that the incumbent's best response is kinked, and find the interval $\left[k^{\min}, k^{\max}\right]$ of capacities for which the stage-3 equilibrium has the incumbent producing exactly at capacity, $q_1 = k$, with $q_2 = \frac{m-k}{2}$. Interpret $k^{\min}$ and $k^{\max}$.

- *The entrant.* Firm 2 solves $\max_{q_2 \geq 0} (a - q_1 - q_2 - c)q_2 - F$. Differentiating with respect to q_2, and solving, we obtain

$$q_2(q_1) = \frac{m - q_1}{2}$$

with $\dfrac{\partial^2 \pi_2}{\partial q_2^2} = -2 < 0$.

- *The incumbent's kink.* The incumbent's marginal cost jumps from w to $w + r = c$ at $q_1 = k$, so its best response has three branches. It expands beyond capacity only if its marginal profit at $q_1 = k$, computed with the high marginal cost, is still positive, that is, if $a - 2k - q_2 - c > 0$. It contracts below capacity only if its marginal profit at $q_1 = k$, computed with the low marginal cost, is already negative, that is, if $a - 2k - q_2 - w < 0$. Therefore, it produces exactly at capacity whenever

$$\frac{m - q_2}{2} \leq k \leq \frac{a - w - q_2}{2}$$

and the interval is nonempty precisely because $r > 0$ makes the two bounds differ.

- *Solving for the interval.* Imposing $q_2 = \dfrac{m - k}{2}$ in the lower bound, we obtain $k \geq \dfrac{m + k}{4}$, which simplifies to

$$k \geq k^{\min} \equiv \frac{m}{3}$$

Imposing it in the upper bound, and using $a - w = m + r$, we obtain $k \leq \dfrac{m + 2r + k}{4}$, which simplifies to

$$k \leq k^{\max} \equiv \frac{m + 2r}{3}$$

- *Interpretation.* The lower bound $k^{\min} = \dfrac{m}{3}$ is the symmetric Cournot output of two firms with marginal cost c: below it, capacity is not binding and the outcome is the usual Cournot equilibrium, so installing so little capacity buys no commitment. The upper bound $k^{\max} = \dfrac{m + 2r}{3}$ is the Cournot output of a firm with marginal cost w facing a rival with marginal cost c: it is the largest output the incumbent can ever be induced to produce, so capacity installed above $k^{\max}$ is left idle. Intuitively, the incumbent's commitment power is bounded, and the width of the interval, $\dfrac{2r}{3}$, measures how much of it the capacity technology delivers.

(b) (5 points) *Stage 2.* Find the entrant's gross profit as a function of k and show that entry is deterred if and only if the incumbent's stage-3 output is at least $k^D \equiv m - 2\sqrt{F}$. Since the incumbent's output can never exceed $k^{\max}$, show that deterrence is credible if and only if $\sqrt{F} \geq \dfrac{m - r}{3}$.

- *The entrant's gross profit.* As in Exercise 3, the entrant's margin equals its quantity, so against an incumbent output $q_1 = k$ its gross profit is

$$\pi_2^{gross}(k) = \left(\frac{m-k}{2}\right)^2$$

and entry occurs if and only if $\pi_2^{gross}(k) \geq F$, that is, if and only if $k \leq m - 2\sqrt{F}$. Therefore, entry is deterred if and only if the incumbent's stage-3 output is at least

$$k^D \equiv m - 2\sqrt{F}$$

- *Credibility.* By part (a), the incumbent's stage-3 output can never exceed $k^{\max} = \dfrac{m+2r}{3}$, whatever capacity it installs. Deterrence is therefore feasible if and only if $k^D \leq k^{\max}$, that is,

$$m - 2\sqrt{F} \leq \frac{m+2r}{3}$$

which simplifies to $3m - 6\sqrt{F} \leq m + 2r$. Solving for $\sqrt{F}$, we obtain

$$\sqrt{F} \geq \frac{m-r}{3}, \quad \text{or equivalently,} \quad F \geq F_c \equiv \left(\frac{m-r}{3}\right)^2$$

- Intuitively, installing capacity beyond $k^{\max}$ does not intimidate the entrant, because the entrant anticipates that the extra capacity will sit idle once competition begins. A threat that the incumbent would not want to carry out in the stage-3 subgame is not part of any subgame perfect equilibrium, which is precisely what the credibility cutoff records.

(c) (5 points) *Stage 1, accommodation.* Suppose the incumbent lets the entrant in. Find its optimal capacity k^A and profit π_1^A, and show that the outcome coincides with the Stackelberg leader outcome with marginal cost c. Find the condition on r under which k^A is credible.

- *The objective.* When the incumbent produces exactly at capacity, its total cost is $rk + wk = ck$, so its capacity advantage is purely strategic and delivers no aggregate cost saving. Its profit is, therefore,

$$\pi_1 = \left(a - k - \frac{m-k}{2}\right)k - ck = k\frac{m-k}{2}$$

- Differentiating with respect to k, yields $m - 2k = 0$. Solving for k, we obtain

$$k^A = \frac{m}{2}, \quad q_2^A = \frac{m}{4} \quad \text{and} \quad \pi_1^A = \frac{m^2}{8}$$

with $\dfrac{\partial^2 \pi_1}{\partial k^2} = -1 < 0$.

- *The Stackelberg comparison.* These are exactly the outputs and profit of a Stackelberg leader with marginal cost c facing a follower with the same cost. Intuitively, although the two firms choose quantities simultaneously in stage 3, the incumbent's sunk capacity fixes its output in advance, and the entrant is left to best respond to it. The capacity choice of stage 1 therefore does the work that the leader's output choice does in Exercise 3.
- *Credibility.* The capacity k^A is credible if the incumbent does not want to contract below it, that is, if $k^A \leq \dfrac{a-w-q_2^A}{2}$. Using $a-w=m+r$ and $q_2^A = \dfrac{m}{4}$,

$$\frac{a-w-q_2^A}{2} - k^A = \frac{m+r-\dfrac{m}{4}}{2} - \frac{m}{2} = \frac{r}{2} - \frac{m}{8}$$

which is non-negative if and only if

$$r \geq \frac{m}{4}$$

The other bound never binds, since $k^A = \dfrac{m}{2} > \dfrac{m}{3} = k^{\min}$. Intuitively, the capacity cost must be a large enough share of the unit cost; otherwise w is close to c, the incumbent's best response is close to the Cournot one, $\dfrac{m}{3} < \dfrac{m}{2}$, and it would rather leave part of its capacity idle than hold to the Stackelberg output.

(d) (5 points) *Stage 1, deterrence.* Find π_1^D, compare it against π_1^A, and find the cutoff $\overline{F}$. Then compare $\overline{F}$ against the credibility cutoff of part (b). Which of the two determines the incumbent's behavior, and why?

- *Deterrence profit.* Installing k^D and facing no rival, the incumbent earns

$$\begin{aligned}\pi_1^D &= \left(a-k^D\right)k^D - ck^D = k^D\left(m-k^D\right)\\ &= 2\sqrt{F}\left(m-2\sqrt{F}\right)\end{aligned}$$

where we used $k^D = m - 2\sqrt{F}$. Note that the incumbent does produce at capacity, since $\dfrac{m}{2} \leq k^D \leq \dfrac{m+r}{2}$ holds whenever $\dfrac{m-r}{4} \leq \sqrt{F} \leq \dfrac{m}{4}$, and the credibility bound of part (b) is tighter than the lower one.
- *The profit cutoff.* Writing $t \equiv \sqrt{F}$ and setting $\pi_1^D = \pi_1^A$,

$$2t\,(m-2t) = \frac{m^2}{8}, \quad \text{that is,} \quad 4t^2 - 2mt + \frac{m^2}{8} = 0$$

Solving for t, we obtain

$$t = \frac{2m \pm m\sqrt{2}}{8} = \frac{m\left(2\pm\sqrt{2}\right)}{8}$$

Deterrence is only relevant for $t \leq \dfrac{m}{4} = \dfrac{2m}{8}$, since beyond that entry is blockaded, and the upper root exceeds that bound. Therefore, the relevant root is the lower one, and deterrence is more profitable than accommodation if and only if

$$F \geq \overline{F} \equiv \frac{m^2\left(3-2\sqrt{2}\right)}{32}$$

- *Which cutoff binds.* Comparing $F_c = \left(\frac{m-r}{3}\right)^2$ against $\overline{F}$, and taking square roots, $F_c > \overline{F}$ if and only if $\frac{m-r}{3} > \frac{m\left(2-\sqrt{2}\right)}{8}$. Solving for r, we obtain

$$r < \frac{m\left(2+3\sqrt{2}\right)}{8} \simeq 0.78m$$

  which holds for every capacity cost that is not close to the whole market size. Therefore, the credibility cutoff is the binding one, and the incumbent deters entry if and only if $F \geq F_c$, not if and only if $F \geq \overline{F}$.
- *The intermediate range.* For $\overline{F} \leq F < F_c$ the incumbent would strictly prefer to deter, since $\pi_1^D > \pi_1^A$, but it cannot commit to an output large enough to make the entrant's profit fall below F. It therefore accommodates. This is the substantive lesson of the exercise: what determines the outcome is not the ranking of the two profits but whether the capacity technology delivers enough commitment, and the two questions have different answers.

(e) (3 points) Define the no-threat benchmark k^{NT} as the capacity the incumbent would install if it were a monopolist facing no entry threat at all. Show that $k^D > k^{NT}$ whenever it deters entry, and explain why. Show, in addition, that $k^A = k^{NT}$ under linear demand, and explain the sense in which the incumbent nonetheless over-invests under accommodation.

- *The benchmark.* A monopolist facing no entry threat installs exactly the capacity it intends to use, at a total cost of ck, so it solves $\max_{k\geq 0}\ (a-k)\,k - ck$. Differentiating and solving, we obtain

$$k^{NT} = \frac{m}{2} \quad \text{and} \quad \pi_1^{NT} = \frac{m^2}{4}$$

- *Over-investment under deterrence.* Comparing the deterrence capacity against the benchmark,

$$k^D - k^{NT} = m - 2\sqrt{F} - \frac{m}{2} = \frac{m}{2} - 2\sqrt{F}$$

  which is positive if and only if $F < \frac{m^2}{16}$, and that is exactly the range in which entry is not already blockaded. Therefore, whenever the incumbent actively deters, it over-invests. Intuitively, the extra capacity is not bought because it lowers costs, since the total cost of producing at capacity is ck either way, but because it makes a large output credible and pushes the entrant's residual demand below the level that covers F. The over-investment is the price of the commitment.

- *Accommodation and the linear-demand coincidence.* Under accommodation $k^A = \dfrac{m}{2} = k^{NT}$, so the incumbent installs exactly the monopoly capacity. This occurs because with linear demand the Stackelberg leader output happens to coincide with the monopoly output, and it is a knife-edge feature of this demand function rather than a general result.
- *The right yardstick.* The correct comparison is against the capacity the incumbent would install if the entrant could *not* observe k. In that case stage 3 would be an ordinary simultaneous Cournot game between two firms with marginal cost c, and the incumbent would install only $k^{\min} = \dfrac{m}{3}$. Comparing,

$$k^A - k^{\min} = \frac{m}{2} - \frac{m}{3} = \frac{m}{6} > 0$$

so observability itself induces over-investment of $\dfrac{m}{6}$. The strategic motive is therefore present under accommodation as well, and it is only invisible when we measure it against the monopoly yardstick.

(f) (4 points) *Numerical example.* Evaluate your results at $a = 20$, $w = 2$ and $r = 6$, and then compute k^D and π_1^D for $F = 1$, $F = 4$ and $F = 9$.

- *Primitives.* We obtain $c = w + r = 8$ and $m = a - c = 12$, so that

$$k^{\min} = \frac{12}{3} = 4, \quad k^{\max} = \frac{12+12}{3} = 8 \quad \text{and} \quad k^{NT} = 6$$

- *Accommodation.* The accommodation objects are

$$k^A = 6, \quad q_2^A = 3 \quad \text{and} \quad \pi_1^A = \frac{144}{8} = 18$$

which we can check directly, since $p = 20 - 6 - 3 = 11$ and $\pi_1 = (11-8)\,6 = 18$. The capacity k^A is credible, since $r = 6 \geq \dfrac{m}{4} = 3$.

- *The two cutoffs.* The credibility cutoff is $F_c = \left(\dfrac{12-6}{3}\right)^2 = 4$, whereas the profit cutoff is

$$\overline{F} = \frac{144\left(3 - 2\sqrt{2}\right)}{32} = \frac{27}{2} - 9\sqrt{2} \simeq 0.77$$

so credibility binds, as part (d) anticipated. Entry is blockaded from $\dfrac{m^2}{16} = 9$ onwards.

- *The three cases.* Applying $k^D = 12 - 2\sqrt{F}$ and $\pi_1^D = k^D\left(12 - k^D\right)$,

$$\begin{aligned} F &= 1: \; k^D = 10, \;\; \pi_1^D = 20 \\ F &= 4: \; k^D = 8, \;\; \pi_1^D = 32 \\ F &= 9: \; k^D = 6, \;\; \pi_1^D = 36 \end{aligned}$$

- *Reading the three cases.* At $F = 1$ deterrence would be more profitable than accommodation, $20 > 18$, but it requires $k^D = 10 > 8 = k^{\max}$, so it is not credible and the incumbent accommodates, installing $k = 6$ and earning 18. At $F = 4$ we have $k^D = 8 = k^{\max}$, so deterrence becomes just credible, and since $32 > 18$ the incumbent deters, installing $k = 8$ and over-investing by $8 - 6 = 2$ relative to the no-threat benchmark. At $F = 9$ we obtain $k^D = 6 = k^{NT}$, so entry is blockaded and the incumbent simply installs its monopoly capacity, earning $\dfrac{m^2}{4} = 36$.
- Note, finally, the accommodation case of $F = 1$: the incumbent installs 6 rather than the 4 it would install if capacity were unobservable, so even when it lets the entrant in it over-invests by $\dfrac{m}{6} = 2$, as part (e) predicted.