Microeconomic Theory II - UNI, Lima
Homework #2 - Due date: August 20th, in class.

1. **Tax audits with a punitive fine and a commitment stage.** Consider a taxpayer and a tax authority (the auditor) who move simultaneously. The taxpayer owes an amount $b > 0$ in taxes, and chooses whether to evade (E) or to comply (C). The auditor chooses whether to audit (A), at a cost $k > 0$, or not to audit (N). If the taxpayer complies, the authority collects b whether or not it audits. If the taxpayer evades and is audited, the authority recovers the unpaid amount b and the taxpayer pays, in addition, a fine $F > 0$. The fine is punitive, and is transferred to the national treasury rather than credited to the audit agency's budget, so that F does not enter the auditor's payoff. If the taxpayer evades and is not audited, the authority collects nothing. Measuring the taxpayer's payoff as the negative of its total payment, the payoff matrix is

		Auditor	
		A	N
Taxpayer	E	$-(b+F), b-k$	$0, 0$
	C	$-b, b-k$	$-b, b$

Answer the following questions.

(a) Show that this game has no pure-strategy Nash equilibrium (psNE) if and only if $k < b$. What is the unique psNE when $k > b$, and how would you interpret it?

(b) Assume $k < b$ throughout the rest of the exercise. Find the mixed-strategy Nash equilibrium (msNE) in closed form. Denote by p the probability that the taxpayer evades and by q the probability that the auditor audits. Report the equilibrium payoff of each player.

(c) *Comparative statics.* How are p^* and q^* affected by a marginal increase in the fine F, in the audit cost k, and in the evaded amount b? Explain why a tougher fine leaves the taxpayer's evasion probability unchanged while reducing the probability of an audit.

(d) Suppose now that the auditor moves *first*, publicly committing to an audit probability $q \in [0, 1]$ that it cannot revise, and that the taxpayer chooses E or C after observing q. Solve this sequential game by backward induction. Find the committed audit probability q^C, the taxpayer's response, and the auditor's payoff. Show that the auditor is strictly better off than in part (b), and that evasion falls. [*Hint*: the auditor's payoff is one function of q on the range of probabilities that induce evasion and a different function on the range that induces compliance. Compare the best point of each range.]

(e) Taxes and fines are transfers, so the only real resource cost in this economy is the audit cost. Define social welfare as $W = -k \Pr(\text{audit})$ and compare W across the two regimes of parts (b) and (d). Does commitment raise welfare? Where does the auditor's gain come from? How is W affected by F?

(f) *Numerical example.* Evaluate your results at $b = 100$, $F = 150$ and $k = 40$. Then raise the fine to $F = 400$, keeping b and k unchanged, and recompute every object. Report p^*, q^*, the auditor's payoff with and without commitment, the gain from commitment, and welfare.

2. **A zero-sum game without a saddle point.** Consider the following two-player zero-sum game. Player 1 chooses rows, A, B, C or D, and player 2 chooses columns, L, M, R or X. Every entry reports the payment that player 2 makes to player 1, so player 1 maximizes and player 2 minimizes it.

		Player 2			
		L	M	R	X
Player 1	A	5	2	1	7
	B	1	4	6	8
	C	4	3	2	6
	D	3	1	0	5

Answer the following questions.

(a) Find player 1's pure maximin value and player 2's pure minimax value, together with the strategies attaining them. Show that the two values differ, and explain why this implies that the game has no saddle point, and therefore no psNE.

(b) Show that exactly one row and exactly one column are strictly dominated, and eliminate them. Show, in addition, that no further strategy is strictly dominated in the 3×3 game that remains.

(c) Find the msNE of the reduced game, $x^* = (x_A, x_B, x_C)$ for player 1 and $y^* = (y_L, y_M, y_R)$ for player 2, and the value of the game v. Verify that both indifference systems deliver the same value.

(d) Verify that v lies strictly between the maximin and the minimax values of part (a). Confirm, in addition, that the two strategies you eliminated in part (b) are not best responses to the equilibrium mixtures, by computing the payoff that row D earns against y^* and the payoff that column X concedes against x^*.

(e) *Comparative statics.* Suppose the entry in row B and column R rises from 6 to $6+\varepsilon$, with $\varepsilon \geq 0$, all other entries unchanged. Find $x^*(\varepsilon)$, $y^*(\varepsilon)$ and $v(\varepsilon)$ in closed form, and show that $\dfrac{\partial v}{\partial \varepsilon}$ evaluated at $\varepsilon = 0$ equals $x_B y_R$, the product of the two equilibrium probabilities attached to that cell. Then evaluate the game exactly at $\varepsilon = 1$. Is the true increase in the value larger or smaller than the one predicted by the derivative at $\varepsilon = 0$? Interpret. For which values of ε does your closed form remain valid?

3. **Stackelberg competition with asymmetric costs: accommodation or deterrence.** Consider a market with inverse demand function $p(Q) = a - Q$, where $Q = q_1 + q_2$. Firm 1 is an established leader that chooses q_1 first, and firm 2 is a follower that observes q_1 and then decides whether to enter the market and, if it does, how much to produce. Firms face constant marginal costs c_1 and c_2, which may differ,

and firm 2 must pay a fixed entry cost $K > 0$ upon entering. Firm 2 stays out, earning zero, if entering is not profitable. Payoffs are

$$\pi_1 = (a - q_1 - q_2 - c_1)q_1 \quad \text{and} \quad \pi_2 = (a - q_1 - q_2 - c_2)q_2 - K.$$

Throughout, assume that a is large enough for every expression below to be positive, and denote $s \equiv \sqrt{K}$ for compactness.

Answer the following questions.

(a) Find firm 2's best response function conditional on entering, check its second-order condition, and find its gross profit as a function of q_1. Then show that firm 2 enters if and only if $q_1 \leq q_1^D \equiv a - c_2 - 2s$, so that q_1^D is the smallest output that makes entry unprofitable.

(b) *Accommodation.* Suppose firm 1 lets firm 2 enter. Find its optimal output q_1^A, the induced follower output q_2^A, and its profit π_1^A. For which values of K does firm 2 indeed enter when firm 1 produces q_1^A?

(c) *Deterrence.* Suppose instead that firm 1 keeps firm 2 out. Find its profit π_1^D as a function of s.

(d) Firm 1 deters entry if and only if $\pi_1^D \geq \pi_1^A$. Show that this comparison reduces to a quadratic in s, and find the cutoff $\overline{K}$ such that firm 1 deters entry if and only if $K \geq \overline{K}$. Show that the discriminant of that quadratic is $\dfrac{(a-c_2)^2}{2} - (c_2 - c_1)^2$, and interpret the requirement that it be positive.

(e) *Comparative statics.* How are q_1^D and π_1^D affected by a marginal increase in K? Show that π_1^D increases in K throughout the relevant range, and find the level of K at which entry becomes blockaded, so that firm 1 can simply produce its monopoly output. Then discuss how an increase in c_2 affects q_1^A, π_1^A, q_1^D and the cutoff $\overline{K}$.

(f) *Numerical example.* Evaluate your results at $a = 12$, $c_1 = 1$ and $c_2 = 2$. Report q_1^A, q_2^A, π_1^A and the cutoff $\overline{K}$. Then compute q_1^D and π_1^D for $K = \dfrac{1}{16}$, $K = \dfrac{1}{4}$, $K = 1$, $K = 4$ and $K = \dfrac{81}{16}$, and report which regime firm 1 chooses in each case.

4. **Entry deterrence through a capacity commitment.** Consider the following three-stage game between an incumbent (firm 1) and a potential entrant (firm 2). Inverse demand is $p(Q) = a - Q$, where Q denotes aggregate output. The timing is as follows.

- In stage 1, the incumbent installs capacity $k \geq 0$ at a unit cost $r > 0$, so that it pays rk up front. This outlay is sunk.
- In stage 2, the entrant observes k and decides whether to enter, paying a fixed cost $F > 0$ if it does.
- In stage 3, if entry occurred both firms compete in quantities simultaneously. If entry did not occur, the incumbent is a monopolist.

Producing one unit of output requires one unit of capacity and one unit of labor, and labor costs $w > 0$ per unit. Since the incumbent has already paid r per unit on its first k units, producing $q_1 \leq k$ costs it only wq_1 at the margin, whereas producing beyond its capacity requires buying extra capacity, so its marginal cost jumps to $w+r$ on units above k. The entrant holds no pre-installed capacity, so its marginal cost is $w+r$ on every unit. Total costs are, therefore,

$$C_1(q_1;k) = \begin{cases} rk + wq_1 & \text{if } q_1 \leq k \\ rk + wq_1 + r(q_1 - k) & \text{if } q_1 > k \end{cases} \quad \text{and} \quad C_2(q_2) = (w+r)q_2 + F.$$

Throughout, write $c \equiv w + r$ for the full marginal cost and $m \equiv a - c$ for the effective size of the market, and assume $m > r > 0$.

Answer the following questions.

(a) *Stage 3.* Suppose entry occurred and the incumbent holds capacity k. Find the entrant's best response function. Then show that the incumbent's best response is kinked, and find the interval $\left[k^{\min}, k^{\max}\right]$ of capacities for which the stage-3 equilibrium has the incumbent producing exactly at capacity, $q_1 = k$, with $q_2 = \dfrac{m-k}{2}$. Interpret $k^{\min}$ and $k^{\max}$.

(b) *Stage 2.* Find the entrant's gross profit as a function of k and show that entry is deterred if and only if the incumbent's stage-3 output is at least $k^D \equiv m - 2\sqrt{F}$. Since the incumbent's output can never exceed $k^{\max}$, no matter how much capacity it installs, show that deterrence is credible if and only if $\sqrt{F} \geq \dfrac{m-r}{3}$.

(c) *Stage 1, accommodation.* Suppose the incumbent lets the entrant in. Find its optimal capacity k^A and profit π_1^A, and show that the outcome coincides with the one that firm 1 would obtain if it were a Stackelberg quantity leader with marginal cost c. Find the condition on r under which k^A is credible, that is, under which the incumbent indeed produces at capacity in stage 3.

(d) *Stage 1, deterrence.* Find the incumbent's profit π_1^D from installing k^D and keeping the entrant out. Compare π_1^D against π_1^A and find the cutoff $\overline{F}$ above which deterrence is more profitable. Then compare $\overline{F}$ against the credibility cutoff of part (b). Which of the two determines the incumbent's behavior, and why?

(e) Define the no-threat benchmark k^{NT} as the capacity the incumbent would install if it were a monopolist facing no entry threat at all. Show that the incumbent over-invests, $k^D > k^{NT}$, whenever it deters entry, and explain why. Show, in addition, that $k^A = k^{NT}$ under linear demand, so that accommodation involves no over-investment relative to the monopoly level, and explain the sense in which the incumbent nonetheless over-invests when we compare k^A against the capacity it would install if the entrant could not observe k.

(f) *Numerical example.* Evaluate your results at $a = 20$, $w = 2$ and $r = 6$. Report c, m, $k^{\min}$, $k^{\max}$, k^A, q_2^A and π_1^A, the two cutoffs of part (d), and then k^D and π_1^D for $F = 1$, $F = 4$ and $F = 9$. State which regime arises in each case.

Additional practice (not graded). For further practice on this material, see the textbook, Chapters 5-7.