Microeconomic Theory II - UNI, Lima
Homework #1 - Answer key

1. **(28 points) Exercise 1 - Dominance, rationalizability and the order of elimination.** Consider the following simultaneous-move game between player 1, who chooses rows, and player 2, who chooses columns. In every cell, the first number denotes player 1's payoff and the second number denotes player 2's payoff.

		Player 2			
		L	C	R	X
Player 1	U	$8,3$	$4,3$	$0,1$	$0,0$
	M	$6,3$	$6,2$	$6,3$	$0,0$
	D	$8,3$	$0,1$	$4,3$	$0,0$
	B	$7,3$	$1,2$	$1,2$	$20,0$

   (a) (8 points) Show that exactly one strategy is strictly dominated by a pure strategy in this game, and eliminate it. Show, in addition, that no strategy of player 1 is strictly dominated in the original game, either by a pure or by a mixed strategy.

   - *Player 2's strategies.* Column X yields payoff 0 to player 2 in every row, whereas column L yields 3 in every row,

$$\begin{aligned}\text{payoffs from } L &= (3,3,3,3)\\ \text{payoffs from } X &= (0,0,0,0)\end{aligned}$$

     where each vector lists player 2's payoff against rows U, M, D and B, respectively. Since $3 > 0$ in all four rows, L strictly dominates X, and we delete X.
   - Columns L, C and R are undominated. Strategy L yields 3 in every row, which is the largest payoff available to player 2 in each of them, so no strategy can improve upon it. Strictly dominating C would require a payoff above 3 against U, and 3 is the highest payoff player 2 can earn in that row. Strictly dominating R would require a payoff above 3 against both M and D, and 3 is again the highest payoff available in each of these rows.
   - *Player 1's strategies.* No strategy of player 1 is strictly dominated while X is still in the game. Strategies U, M and D all yield 0 against X, so every mixture of them also yields 0 against X, which falls short of the 20 that B delivers. Therefore, no mixed strategy strictly dominates B.
   - Regarding the remaining rows, strictly dominating U would require a payoff above 8 against L, and the only strategy attaining 8 in that column is D, which yields $0 < 4$ against C. A symmetric argument applies to D, which would require a payoff above 8 against L while U yields $0 < 4$ against R. Strictly dominating M would require a payoff above 6 against C, and 6 is the largest payoff available to player 1 in that column.

- Therefore, the first round of elimination deletes column X alone.

(b) (6 points) In the game that remains after the first round of elimination, show that one of player 1's strategies is strictly dominated by a mixed strategy even though no pure strategy of player 1 strictly dominates it. Find such a mixed strategy. Then show that no further strategy can be strictly eliminated.

- *Player 1's payoffs after deleting X.* Listing player 1's payoffs against columns L, C and R, we obtain

$$\begin{aligned} U &= (8,4,0), \quad M = (6,6,6) \\ D &= (8,0,4), \quad B = (7,1,1). \end{aligned}$$

- *No pure strategy dominates B.* Strategy B yields $1 > 0$ against R, so U does not dominate it; it yields $7 > 6$ against L, so M does not dominate it; and it yields $1 > 0$ against C, so D does not dominate it. Note that B is not the worst row in any single column, which is precisely why the standard row-by-row comparison misses the dominance.
- *A dominating mixture.* Consider the mixed strategy σ_1 that assigns probability p to U and $1-p$ to D. Its expected payoffs are

$$\begin{aligned} EU_1(\sigma_1, L) &= 8p + 8(1-p) = 8 \\ EU_1(\sigma_1, C) &= 4p + 0(1-p) = 4p \\ EU_1(\sigma_1, R) &= 0p + 4(1-p) = 4(1-p) \end{aligned}$$

so that σ_1 strictly dominates B if and only if $8 > 7$, which holds by definition, $4p > 1$, and $4(1-p) > 1$. Solving for p, we obtain

$$\frac{1}{4} < p < \frac{3}{4}.$$

- Taking $p = \frac{1}{2}$, the mixed strategy $\sigma_1 = \left(\frac{1}{2}U, \frac{1}{2}D\right)$ yields $(8,2,2)$, which lies strictly above $B = (7,1,1)$ in every column. Therefore, B is strictly dominated and we delete it in the second round.
- *No further elimination.* In the remaining game, no strategy of player 1 is strictly dominated: dominating M requires a payoff above 6 against C, dominating U requires a payoff above 8 against L while D yields $0 < 4$ against C, and symmetrically for D. For player 2, dominating C requires a payoff above 3 against U, and dominating R requires a payoff above 3 against both M and D, while L attains exactly 3 in each of these rows. Therefore, the iterated deletion of strictly dominated strategies stops after two rounds.
- The surviving game is

Player 2

Player 1		L	C	R
	U	$8,3$	$4,3$	$0,1$
	M	$6,3$	$6,2$	$6,3$
	D	$8,3$	$0,1$	$4,3$

(c) (7 points) Find every pure-strategy Nash equilibrium (psNE) of the game.

- *Player 1's best responses.* Against L, both U and D yield 8, above the 6 from M and the 7 from B, so $BR_1(L) = \{U, D\}$. Against C, M yields 6, so $BR_1(C) = M$. Against R, M yields 6, so $BR_1(R) = M$. Against X, B yields 20, so $BR_1(X) = B$.
- *Player 2's best responses.* Against U, both L and C yield 3, so $BR_2(U) = \{L, C\}$. Against M, both L and R yield 3, so $BR_2(M) = \{L, R\}$. Against D, both L and R yield 3, so $BR_2(D) = \{L, R\}$. Against B, only L yields 3, so $BR_2(B) = L$.
- Underlining the best response payoffs of each player, we obtain

		Player 2			
		L	C	R	X
Player 1	U	$\underline{8}, \underline{3}$	$4, \underline{3}$	$0, 1$	$0, 0$
	M	$6, \underline{3}$	$\underline{6}, 2$	$\underline{6}, \underline{3}$	$0, 0$
	D	$\underline{8}, \underline{3}$	$0, 1$	$4, \underline{3}$	$0, 0$
	B	$7, \underline{3}$	$1, 2$	$1, 2$	$\underline{20}, 0$

so the psNEs of this game are

$$(U, L)\,, \quad (D, L) \quad \text{and} \quad (M, R)\,.$$

(d) (7 points) Show that player 2's strategy L weakly dominates both C and R, but strictly dominates neither. Starting from the reduced game of part (c), apply iterated elimination of weakly dominated strategies twice, first deleting C before R and then deleting R before C. Do the two orders of elimination lead to the same prediction? Which psNEs survive each of them?

- *Weak dominance.* Listing player 2's payoffs against rows U, M and D in the reduced game,

$$L = (3, 3, 3), \quad C = (3, 2, 1) \quad \text{and} \quad R = (1, 3, 3).$$

  Strategy L weakly dominates C because $3 \geq 3$, $3 > 2$ and $3 > 1$, and the tie in the first row prevents strict dominance. Similarly, L weakly dominates R because $3 > 1$, $3 \geq 3$ and $3 \geq 3$, and the ties in the last two rows again prevent strict dominance.
- *First order of elimination: delete C, then R.* Deleting C leaves player 1 with the payoff vectors $U = (8, 0)$, $M = (6, 6)$ and $D = (8, 4)$ over columns L and R. Since $8 \geq 8$ and $4 > 0$, strategy D weakly dominates U, and we delete U.
- In the remaining game, rows M and D yield $M = (6, 6)$ and $D = (8, 4)$, so neither dominates the other, and columns L and R yield player 2 a payoff of 3 against both surviving rows, so neither dominates the other either. The procedure stops, delivering

$$\{M, D\} \times \{L, R\}\,.$$

  This set contains the psNEs (D, L) and (M, R), but the psNE (U, L) has been destroyed.

- *Second order of elimination: delete R, then C.* Deleting R leaves player 1 with $U = (8,4)$, $M = (6,6)$ and $D = (8,0)$ over columns L and C. Since $8 \geq 8$ and $4 > 0$, strategy U now weakly dominates D, and we delete D.
- In the game $\{U, M\} \times \{L, C\}$, player 2's payoffs are $L = (3,3)$ and $C = (3,2)$, so L weakly dominates C, and we delete C. Once only L remains, player 1 earns 8 from U and 6 from M, so U strictly dominates M, and we delete M. The procedure delivers the single profile

$$(U, L).$$

The psNEs (D, L) and (M, R) have been destroyed.
- *Summary.* The two orders of elimination lead to different predictions, $\{M, D\} \times \{L, R\}$ against (U, L), and each of them destroys at least one psNE. Intuitively, a weakly dominated strategy is a best response against some of the rival's strategies, so deleting it removes a column against which two rows were tied and thereby breaks a tie that a Nash equilibrium relied upon. In contrast, the iterated deletion of strictly dominated strategies never deletes a strategy played in a Nash equilibrium, and its outcome does not depend on the order of elimination.

2. **(25 points) Exercise 2 - Cournot competition with production spillovers among n firms.** Consider $n \geq 2$ firms competing à la Cournot, facing inverse demand function $p(Q) = a - Q$, where $Q = \sum_{j=1}^{n} q_j$ denotes aggregate output and $a > c > 0$. Every firm i's marginal cost is, instead of c, given by $c - \alpha \sum_{j \neq i} q_j$, so that the production of all its rivals shifts firm i's marginal cost. Parameter α measures the strength of these production spillovers. When $\alpha > 0$, rivals' output decreases firm i's marginal cost; when $\alpha < 0$, rivals' output increases firm i's marginal cost. Denote by $Q_{-i} = \sum_{j \neq i} q_j$ the aggregate output of firm i's rivals.

   (a) (4 points) Find every firm i's best response function, $q_i(Q_{-i})$. How is its intercept and its slope affected by parameter α? For which values of α do firms regard their outputs as strategic substitutes, and for which values as strategic complements?

   - *Best response function.* Firm i's profit maximization problem is

   $$\max_{q_i \geq 0} \pi_i = (a - q_i - Q_{-i})q_i - (c - \alpha Q_{-i})q_i$$

   Differentiating with respect to q_i, yields

   $$a - 2q_i - Q_{-i} - c + \alpha Q_{-i} = 0$$

   Solving for q_i, we obtain firm i's best response function

   $$q_i(Q_{-i}) = \frac{a-c}{2} - \frac{1-\alpha}{2} Q_{-i}$$

   - *Effect of α.* The vertical intercept, $\frac{a-c}{2}$, is unaffected by parameter α. The slope, in contrast, increases in α since

   $$\frac{\partial}{\partial \alpha}\left(-\frac{1-\alpha}{2}\right) = \frac{1}{2} > 0$$

so that stronger spillovers make the best response function flatter. Intuitively, when a rival expands its output, firm i faces a lower price but also a lower marginal cost, and the second force offsets part of the first.

- *Strategic substitutes or complements.* The slope $-\frac{1-\alpha}{2}$ is negative if and only if $\alpha < 1$, so outputs are strategic substitutes when spillovers are weak, $\alpha < 1$, and strategic complements when they are strong, $\alpha > 1$. At the knife-edge $\alpha = 1$ the best response is flat, and firm i's output becomes independent of its rivals' production, since the cost saving exactly offsets the price reduction.

(b) (5 points) Find every firm's equilibrium output, aggregate output, price, and profit in the symmetric equilibrium.

- *Equilibrium output.* In a symmetric equilibrium all firms produce the same output, $q_j = q$ for every j, so that $Q_{-i} = (n-1)q$. Inserting this property into the best response function, yields

$$q = \frac{a-c}{2} - \frac{1-\alpha}{2}(n-1)q$$

which simplifies to $2q + (1-\alpha)(n-1)q = a - c$. Solving for q, we obtain

$$q^* = \frac{a-c}{(n+1)-(n-1)\alpha}$$

For compactness, define $D \equiv (n+1)-(n-1)\alpha$ throughout, so that $q^* = \frac{a-c}{D}$.

- *Aggregate output and price.* Aggregate output is

$$Q^* = nq^* = \frac{n(a-c)}{D}$$

and the equilibrium price is

$$\begin{aligned} p^* &= a - Q^* = a - \frac{n(a-c)}{D} \\ &= \frac{a\left[1-(n-1)\alpha\right]+nc}{D} \end{aligned}$$

- *Equilibrium profit.* Firm i's effective marginal cost in equilibrium is $c-\alpha(n-1)q^*$, so its price-cost margin is

$$\begin{aligned} p^* - [c-\alpha(n-1)q^*] &= \frac{a\left[1-(n-1)\alpha\right]+nc}{D} - c + \alpha(n-1)\frac{a-c}{D} \\ &= \frac{a-c}{D} \end{aligned}$$

which coincides with equilibrium output. Therefore, equilibrium profit is

$$\pi^* = \frac{(a-c)^2}{D^2}$$

(c) (4 points) Check the second-order condition of firm i's problem. Then find the restriction on α that guarantees that the symmetric equilibrium you found in part (b) is interior and positive, and the (tighter) restriction that guarantees that the best response dynamics converge to it. Is the binding restriction the second-order condition?

- *Second-order condition.* Differentiating the first-order condition once more, $\dfrac{\partial^2 \pi_i}{\partial q_i^2} = -2 < 0$, which holds for every value of α. The spillover term is linear in Q_{-i} and does not enter the own second derivative, so concavity is never at stake.
- *Interior and positive equilibrium.* Since $a > c$, output $q^* = \dfrac{a-c}{D}$ is positive if and only if $D > 0$, that is,

$$(n+1) - (n-1)\alpha > 0 \text{ or, equivalently, } \alpha < \overline{\alpha} \equiv \frac{n+1}{n-1}$$

  Note that $\overline{\alpha}$ decreases in n, so the more firms operate in the industry, the smaller the spillover the model can accommodate.
- *Stability of the best response dynamics.* Suppose every firm adjusts its output along its best response function. A common deviation of size ε by the $n-1$ rivals changes firm i's output by $-\dfrac{1-\alpha}{2}(n-1)\varepsilon$, so the adjustment converges if and only if

$$\left| \frac{(n-1)(1-\alpha)}{2} \right| < 1$$

  Solving for α, we obtain

$$\frac{n-3}{n-1} < \alpha < \frac{n+1}{n-1}$$

  The upper bound coincides with $\overline{\alpha}$, whereas the lower bound is new: spillovers that are strongly negative make best responses too steep, and outputs oscillate away from the equilibrium. For $n = 2$ the condition reads $\alpha \in (-1, 3)$, and it tightens as n grows.
- *Is the second-order condition binding?* It is not. The second-order condition holds for every α, so the restrictions that students must impose come entirely from the equilibrium conditions rather than from concavity of the objective function.

(d) (4 points) *Comparative statics.* How are equilibrium output, price and profit affected by a marginal increase in α? Interpret.

- Differentiating with respect to α, and using $\dfrac{\partial D}{\partial \alpha} = -(n-1)$, yields

$$\frac{\partial q^*}{\partial \alpha} = \frac{(a-c)(n-1)}{D^2} > 0$$

$$\frac{\partial Q^*}{\partial \alpha} = \frac{n(a-c)(n-1)}{D^2} > 0$$

$$\frac{\partial p^*}{\partial \alpha} = -\frac{n(a-c)(n-1)}{D^2} < 0$$
$$\frac{\partial \pi^*}{\partial \alpha} = \frac{2(a-c)^2(n-1)}{D^3} > 0$$

where every sign is unambiguous once $D > 0$ is imposed.

- Therefore, stronger spillovers unambiguously increase individual and aggregate output and individual profit, while decreasing the equilibrium price. Intuitively, when rivals' production lowers firm i's marginal cost, competition becomes less intense at the margin, every firm expands, and the industry as a whole becomes more efficient. Consumers gain through a lower price and firms gain through a larger margin, since the price-cost margin $\frac{a-c}{D}$ increases in α as well.

(e) (4 points) *Comparative statics.* How are equilibrium output per firm, aggregate output and profit affected by a marginal increase in the number of firms n? Can output per firm *increase* in the number of firms? If so, find the exact condition on α under which this occurs, and interpret it.

- Differentiating with respect to n, and using $\frac{\partial D}{\partial n} = 1 - \alpha$, we obtain

$$\frac{\partial q^*}{\partial n} = -\frac{(a-c)(1-\alpha)}{D^2}$$
$$\frac{\partial Q^*}{\partial n} = \frac{(a-c)(1+\alpha)}{D^2}$$
$$\frac{\partial \pi^*}{\partial n} = -\frac{2(a-c)^2(1-\alpha)}{D^3}$$

- *Aggregate output.* Since $D^2 > 0$, aggregate output increases in n if and only if $\alpha > -1$. Intuitively, entry expands the industry unless the negative spillover is so severe that each additional firm raises its rivals' marginal cost by more than the extra unit it brings to the market.
- *Output per firm.* The sign of $\frac{\partial q^*}{\partial n}$ is the sign of $\alpha - 1$, so output per firm increases in the number of firms if and only if

$$\alpha > 1$$

and the same condition governs the sign of $\frac{\partial \pi^*}{\partial n}$. At the knife-edge $\alpha = 1$ we obtain $q^* = \frac{a-c}{2}$, which is independent of n.
- Intuitively, entry operates through two channels. The first is the familiar business-stealing effect, which shrinks each firm's residual demand. The second is the spillover effect, since an additional rival lowers every incumbent's marginal cost by αq. When $\alpha > 1$ the second force dominates, and each firm produces more as the industry becomes more crowded, so that individual profits rise with the number of competitors as well. Note that this requires $1 < \alpha < \frac{n+1}{n-1}$, an interval that narrows as n grows, so the effect operates only in relatively concentrated industries.

(f) (4 points) *Numerical example.* Evaluate your results at $a = 10$ and $c = 4$ with $n = 3$ firms, for spillover levels $\alpha = 0$, $\alpha = \frac{1}{2}$ and $\alpha = 1$. Then set $a = 10$, $c = 9$ and $\alpha = \frac{3}{2}$, and evaluate equilibrium output, aggregate output, price and profit for $n = 2$, $n = 3$ and $n = 4$. What is the largest number of firms that this industry can sustain in an interior equilibrium?

- *First panel.* With $a = 10$, $c = 4$ and $n = 3$, we obtain $D = 4 - 2\alpha$ and $a - c = 6$, so that

$$\begin{aligned}
\alpha = 0 \quad &: \quad D = 4,\ q^* = \frac{3}{2},\ Q^* = \frac{9}{2},\ p^* = \frac{11}{2},\ \pi^* = \frac{9}{4} \\
\alpha = \frac{1}{2} \quad &: \quad D = 3,\ q^* = 2,\ Q^* = 6,\ p^* = 4,\ \pi^* = 4 \\
\alpha = 1 \quad &: \quad D = 2,\ q^* = 3,\ Q^* = 9,\ p^* = 1,\ \pi^* = 9
\end{aligned}$$

  confirming the comparative statics of part (d): output and profit rise while the price falls. At $\alpha = \frac{1}{2}$ the effective marginal cost is $c - \alpha(n-1)q^* = 4 - \frac{1}{2} \cdot 2 \cdot 2 = 2$, so the margin is $4 - 2 = 2 = \frac{a-c}{D}$, as claimed in part (b).

- *Second panel.* With $a = 10$, $c = 9$ and $\alpha = \frac{3}{2}$, we obtain $a - c = 1$ and $D = (n+1) - \frac{3}{2}(n-1) = \frac{5-n}{2}$, so that

$$\begin{aligned}
n \quad &= \quad 2 : \ D = \frac{3}{2},\ q^* = \frac{2}{3},\ Q^* = \frac{4}{3},\ p^* = \frac{26}{3},\ \pi^* = \frac{4}{9} \\
n \quad &= \quad 3 : \ D = 1,\ q^* = 1,\ Q^* = 3,\ p^* = 7,\ \pi^* = 1 \\
n \quad &= \quad 4 : \ D = \frac{1}{2},\ q^* = 2,\ Q^* = 8,\ p^* = 2,\ \pi^* = 4
\end{aligned}$$

  so that output per firm rises from $\frac{2}{3}$ to 1 and then to 2 as the industry becomes more crowded, and individual profit rises alongside it. This illustrates the non-standard comparative static of part (e), which arises because $\alpha = \frac{3}{2} > 1$.

- *Largest sustainable industry.* An interior equilibrium requires $\alpha < \frac{n+1}{n-1}$, that is, $\frac{3}{2} < \frac{n+1}{n-1}$, which simplifies to $3(n-1) < 2(n+1)$, or $n < 5$. Therefore, the largest number of firms this industry can sustain is $n = 4$. At $n = 5$ we obtain $D = 0$ and the symmetric equilibrium ceases to exist, since the spillover is strong enough to make each firm's best response fully offset its rivals' expansion.

3. **(22 points) Exercise 3 - Bertrand competition with differentiated products and asymmetric costs.** Consider two firms competing in prices and selling differentiated products. Firm i's demand function is $q_i(p_i, p_j) = a - p_i + \gamma p_j$, where $a > 0$ and parameter $\gamma \in (0, 1]$ measures the degree of product substitutability. When γ

approaches zero the goods are essentially unrelated, whereas when $\gamma = 1$ they are close substitutes. Firms face constant marginal costs c_1 and c_2, which may differ, so firm i's profit is $\pi_i = (p_i - c_i)q_i(p_i, p_j)$.

(a) (3 points) Find every firm i's best response function, $p_i(p_j)$, and check its second-order condition. Do firms regard prices as strategic substitutes or as strategic complements? How is the slope of the best response function affected by γ?

- *Best response function.* Every firm i chooses p_i to solve

$$\max_{p_i \geq 0} \pi_i(p_i, p_j) = (p_i - c_i)(a - p_i + \gamma p_j)$$

Differentiating with respect to p_i, and assuming an interior solution, we find

$$a - 2p_i + \gamma p_j + c_i = 0$$

Solving for p_i, we obtain firm i's best response function

$$p_i(p_j) = \frac{a + c_i + \gamma p_j}{2}$$

The second-order condition holds since $\dfrac{\partial^2 \pi_i}{\partial p_i^2} = -2 < 0$.

- *Strategic complements.* The slope of the best response function is $\dfrac{\gamma}{2} > 0$ for every $\gamma \in (0, 1]$, so firms regard prices as strategic complements: when a rival raises its price, firm i finds it optimal to raise its own price as well. Differentiating the slope with respect to γ yields $\dfrac{1}{2} > 0$, so the best response function becomes steeper as products become closer substitutes, indicating that each firm reacts more aggressively to its rival's pricing.

- Note, in addition, that the vertical intercept $\dfrac{a + c_i}{2}$ increases in firm i's own marginal cost, at a rate of $\dfrac{1}{2}$, but is unaffected by its rival's cost.

(b) (4 points) Find equilibrium prices p_1^* and p_2^*, allowing for $c_1 \neq c_2$.

- Simultaneously solving the system of best response functions,

$$\begin{aligned} p_1 &= \frac{a + c_1 + \gamma p_2}{2} \\ p_2 &= \frac{a + c_2 + \gamma p_1}{2} \end{aligned}$$

we insert the second expression into the first, obtaining

$$p_1 = \frac{a + c_1}{2} + \frac{\gamma}{2}\left(\frac{a + c_2 + \gamma p_1}{2}\right)$$

which simplifies to $(4 - \gamma^2)\, p_1 = a(2 + \gamma) + 2c_1 + \gamma c_2$.

- Solving for p_1, and repeating the argument for firm 2, we obtain equilibrium prices

$$p_1^* = \frac{a(2+\gamma) + 2c_1 + \gamma c_2}{4-\gamma^2} \quad \text{and} \quad p_2^* = \frac{a(2+\gamma) + 2c_2 + \gamma c_1}{4-\gamma^2}$$

which are well defined since $\gamma \in (0,1]$ entails $4-\gamma^2 \geq 3 > 0$.

(c) (4 points) Show that the low-cost firm charges a lower price than its rival. Then show that *both* equilibrium prices increase when *either* marginal cost increases, and compare the size of the own-cost effect against that of the cross-cost effect. Interpret.

- *Price ranking.* Subtracting equilibrium prices, we obtain

$$\begin{aligned} p_1^* - p_2^* &= \frac{2c_1 + \gamma c_2 - 2c_2 - \gamma c_1}{4-\gamma^2} \\ &= \frac{(2-\gamma)(c_1-c_2)}{(2-\gamma)(2+\gamma)} = \frac{c_1-c_2}{2+\gamma} \end{aligned}$$

so that $p_1^* < p_2^*$ if and only if $c_1 < c_2$. Firm 1 (firm 2, respectively) charges the lower price whenever it is the more efficient producer.

- Note that the price gap $\dfrac{c_1-c_2}{2+\gamma}$ is smaller in absolute value than the cost gap $c_1 - c_2$, and shrinks further as γ increases. Intuitively, since prices are strategic complements, each firm partially matches its rival's price, and the cost asymmetry is only imperfectly passed through into the price ranking.
- *Cost effects.* Differentiating p_1^* with respect to each marginal cost yields

$$\frac{\partial p_1^*}{\partial c_1} = \frac{2}{4-\gamma^2} > 0 \quad \text{and} \quad \frac{\partial p_1^*}{\partial c_2} = \frac{\gamma}{4-\gamma^2} > 0$$

and, by symmetry, the same expressions apply to p_2^* with the roles of c_1 and c_2 reversed. Therefore, both equilibrium prices unambiguously increase when either marginal cost increases.

- *Comparing the two effects.* Subtracting the cross effect from the own effect, we obtain

$$\frac{\partial p_1^*}{\partial c_1} - \frac{\partial p_1^*}{\partial c_2} = \frac{2-\gamma}{4-\gamma^2} = \frac{1}{2+\gamma} > 0$$

so the own-cost effect exceeds the cross-cost effect for every $\gamma \in (0,1]$. Intuitively, an increase in c_1 raises p_1^* directly, through firm 1's own margin, and again indirectly, once firm 2 responds by raising p_2^* and firm 1 matches part of that increase. An increase in c_2, in contrast, reaches firm 1 only through the second, indirect channel. This cross effect is the feature that separates differentiated Bertrand competition from the homogeneous-good version, where a rival's cost increase is a pure gain for the low-cost firm rather than a reason to raise its own price.

(d) (5 points) Find equilibrium output and profit for every firm. Then find the cutoff $\bar{c}_2$ such that firm 2 is priced out of the market, selling no units in equilibrium, if and only if $c_2 \geq \bar{c}_2$.

- *Equilibrium output.* The first-order condition of part (a), $a - 2p_i + \gamma p_j + c_i = 0$, allows us to write firm i's demand compactly as its own margin, since

$$q_i = a - p_i + \gamma p_j = (a - 2p_i + \gamma p_j + c_i) + p_i - c_i = p_i - c_i$$

  Therefore, equilibrium output equals the equilibrium margin,

$$\begin{aligned} q_1^* &= p_1^* - c_1 \\ &= \frac{a(2+\gamma) - c_1(2-\gamma^2) + \gamma c_2}{4-\gamma^2} \end{aligned}$$

  and symmetrically for firm 2.
- *Equilibrium profit.* Since $\pi_i^* = (p_i^* - c_i)\, q_i^*$ and $q_i^* = p_i^* - c_i$, we obtain

$$\pi_i^* = (q_i^*)^2$$

  so profit is the square of equilibrium sales.
- *Cutoff.* Firm 2 sells nothing when $q_2^* \leq 0$, that is,

$$a(2+\gamma) - c_2\left(2-\gamma^2\right) + \gamma c_1 \leq 0$$

  Solving for c_2, and noting that $2 - \gamma^2 \geq 1 > 0$ for every $\gamma \in (0, 1]$, we obtain

$$c_2 \geq \bar{c}_2 \equiv \frac{a(2+\gamma) + \gamma c_1}{2-\gamma^2}$$

  Note that $\bar{c}_2$ increases in c_1: the less efficient firm 1 is, the more inefficient firm 2 can afford to be while still selling a positive quantity. Intuitively, product differentiation shelters firm 2, which is why $\bar{c}_2$ exceeds a by a wide margin rather than collapsing to c_1 as it would under homogeneous Bertrand competition.

(e) (3 points) Evaluate equilibrium prices in the two limiting cases $\gamma \to 0$ and $\gamma \to 1$. Interpret each of them.

- *Independent products,* $\gamma \to 0$. Equilibrium prices become

$$p_i^* = \frac{2a + 2c_i}{4} = \frac{a + c_i}{2}$$

  which is the monopoly price of a firm facing demand $a - p_i$ and marginal cost c_i. Intuitively, when the goods are unrelated, each firm's sales are unaffected by its rival's price, so each of them prices as a monopolist in its own market.
- *Close substitutes,* $\gamma \to 1$. Equilibrium prices become

$$p_1^* = a + \frac{2c_1 + c_2}{3} \quad \text{and} \quad p_2^* = a + \frac{2c_2 + c_1}{3}$$

  so each price is a weighted average of the two marginal costs, with weight $\frac{2}{3}$ on the firm's own cost and $\frac{1}{3}$ on its rival's. The price gap narrows to $p_1^* - p_2^* = \dfrac{c_1 - c_2}{3}$.

- Note that prices do not collapse to marginal cost as $\gamma \to 1$, in contrast to the homogeneous-good Bertrand model. The reason is that the demand system $q_i = a - p_i + \gamma p_j$ keeps each firm's own-price effect strictly larger than the cross-price effect even at $\gamma = 1$, so that some market power survives.

(f) (3 points) *Numerical example.* Evaluate your results at $a = 10$, $\gamma = \frac{1}{2}$, $c_1 = 1$ and $c_2 = 6$. Report equilibrium prices, quantities and profits, the two cost effects on p_1^*, and the cutoff $\bar{c}_2$.

- *Equilibrium prices.* With $4 - \gamma^2 = \frac{15}{4}$ and $2 + \gamma = \frac{5}{2}$, we obtain

$$p_1^* = \frac{10 \cdot \frac{5}{2} + 2 \cdot 1 + \frac{1}{2} \cdot 6}{\frac{15}{4}} = \frac{30}{\frac{15}{4}} = 8$$

$$p_2^* = \frac{10 \cdot \frac{5}{2} + 2 \cdot 6 + \frac{1}{2} \cdot 1}{\frac{15}{4}} = \frac{\frac{75}{2}}{\frac{15}{4}} = 10$$

so the low-cost firm charges $p_1^* - p_2^* = -2$ relative to its rival, which coincides with $\frac{c_1 - c_2}{2+\gamma} = \frac{1-6}{\frac{5}{2}} = -2$, as required.

- *Quantities and profits.* Using $q_i^* = p_i^* - c_i$,

$$q_1^* = 8 - 1 = 7 \quad \text{and} \quad q_2^* = 10 - 6 = 4$$

which we can verify directly, $q_1^* = 10 - 8 + \frac{1}{2} \cdot 10 = 7$ and $q_2^* = 10 - 10 + \frac{1}{2} \cdot 8 = 4$. Equilibrium profits are

$$\pi_1^* = 7^2 = 49 \quad \text{and} \quad \pi_2^* = 4^2 = 16$$

so the more efficient firm earns roughly three times the profit of its rival, despite a cost advantage of only 5 monetary units.

- *Cost effects.* Evaluating the two derivatives of part (c),

$$\frac{\partial p_1^*}{\partial c_1} = \frac{2}{\frac{15}{4}} = \frac{8}{15} \quad \text{and} \quad \frac{\partial p_1^*}{\partial c_2} = \frac{\frac{1}{2}}{\frac{15}{4}} = \frac{2}{15}$$

so the own-cost effect is four times the cross-cost effect, and their difference is $\frac{6}{15} = \frac{1}{2+\gamma} = \frac{2}{5}$, as predicted.

- *Cutoff.* Firm 2 is priced out of the market if and only if

$$c_2 \geq \bar{c}_2 = \frac{10 \cdot \frac{5}{2} + \frac{1}{2} \cdot 1}{2 - \frac{1}{4}} = \frac{\frac{51}{2}}{\frac{7}{4}} = \frac{102}{7} \simeq 14.57$$

Since $c_2 = 6$ falls well below this cutoff, firm 2 remains active, as our computations confirm.

4. **(25 points) Exercise 4 - Tragedy of the commons with heterogeneous appropriators.** Consider $n \geq 2$ herders who share a common pasture. Herder i sends $g_i \geq 0$ goats to graze, and $G = \sum_{j=1}^n g_j$ denotes the total number of goats in the pasture. The value of a goat decreases in congestion, and is given by $a - G$, where $a > 0$. Herder i's cost of sending g_i goats is $\frac{c_i}{2} g_i^2$, where $c_i > 0$, so that a herder with a low c_i is more efficient than a herder with a high c_i. Herder i's payoff is, therefore,

$$\pi_i(g_i, g_{-i}) = (a - G)g_i - \frac{c_i}{2} g_i^2.$$

Throughout, denote $K \equiv \sum_{j=1}^n \frac{1}{1+c_j}$ and $S \equiv \sum_{j=1}^n \frac{1}{c_j}$.

(a) (3 points) Find herder i's best response function, $g_i(G_{-i})$, where $G_{-i} = \sum_{j \neq i} g_j$, and check its second-order condition. Do herders regard their grazing decisions as strategic substitutes or as strategic complements? How is the slope of the best response function affected by c_i?

- *Best response function.* Herder i solves

$$\max_{g_i \geq 0} \pi_i = (a - g_i - G_{-i})\, g_i - \frac{c_i}{2} g_i^2$$

Differentiating with respect to g_i, yields

$$a - 2g_i - G_{-i} - c_i g_i = 0$$

Solving for g_i, we obtain herder i's best response function

$$g_i(G_{-i}) = \frac{a - G_{-i}}{2 + c_i}$$

The second-order condition holds since $\frac{\partial^2 \pi_i}{\partial g_i^2} = -(2 + c_i) < 0$.

- *Strategic substitutes.* The slope of the best response function is $-\frac{1}{2+c_i} < 0$, so grazing decisions are strategic substitutes: when rivals send more goats to the pasture, herder i responds by sending fewer. Intuitively, congestion lowers the value of every additional goat, which is exactly the externality that drives the tragedy of the commons.
- *Effect of c_i.* Differentiating the slope with respect to c_i yields

$$\frac{\partial}{\partial c_i}\left(-\frac{1}{2+c_i}\right) = \frac{1}{(2+c_i)^2} > 0$$

so the best response function of a less efficient herder is both lower (its intercept $\frac{a}{2+c_i}$ decreases in c_i) and flatter. Intuitively, a herder facing steep costs was already grazing little, so it has less to withdraw when the pasture becomes crowded.

(b) (4 points) Find the Nash equilibrium of this game in closed form, g_i^* for every herder i, along with the equilibrium total number of goats, G^*. Confirm that your expression collapses to the familiar symmetric result when $c_i = c$ for every herder.

- *Rewriting the first-order condition.* Using $G_{-i} = G - g_i$ in the first-order condition of part (a), we obtain

$$a - G - (1 + c_i)\, g_i = 0, \text{ so that } g_i = \frac{a - G}{1 + c_i}$$

which expresses every herder's equilibrium grazing as a share of the common residual value $a - G$.

- *Aggregating.* Summing the previous expression across the n herders, yields

$$G = (a - G) \sum_{j=1}^{n} \frac{1}{1 + c_j} = (a - G)\, K$$

which simplifies to $G\,(1 + K) = aK$. Solving for G, we obtain the equilibrium total

$$G^* = \frac{aK}{1 + K}, \text{ so that } a - G^* = \frac{a}{1 + K}$$

- *Individual grazing.* Inserting $a - G^*$ back into $g_i = \dfrac{a - G}{1 + c_i}$, we obtain herder i's equilibrium grazing level

$$g_i^* = \frac{a}{(1 + c_i)\,(1 + K)}$$

which decreases in c_i, so the more efficient herder grazes more, and decreases in K, so every herder grazes less as the group of appropriators grows.

- *Symmetric benchmark.* When $c_j = c$ for every herder, we have $K = \dfrac{n}{1 + c}$, and the previous expressions become

$$g_i^* = \frac{a}{(1 + c)\left(1 + \dfrac{n}{1 + c}\right)} = \frac{a}{1 + c + n} \text{ and } G^* = \frac{an}{1 + c + n}$$

which is the familiar symmetric result.

(c) (4 points) Find the socially optimal grazing profile, g_i^{SO}, and the socially optimal total, G^{SO}.

- *The planner's problem.* A social planner chooses $(g_1, \ldots, g_n)$ to maximize aggregate payoffs,

$$\max_{g_1, \ldots, g_n \geq 0} W = (a - G)\, G - \sum_{j=1}^{n} \frac{c_j}{2} g_j^2$$

Differentiating with respect to g_i, and noting that $\dfrac{\partial G}{\partial g_i} = 1$, yields

$$a - 2G - c_i g_i = 0, \text{ so that } g_i^{SO} = \frac{a - 2G}{c_i}$$

The planner internalizes the congestion externality, which is why the term $2G$ replaces the G appearing in the equilibrium condition of part (b).

- *Aggregating.* Summing across herders, we obtain $G = (a - 2G)\,S$, which simplifies to $G\,(1 + 2S) = aS$. Solving for G, we find

$$G^{SO} = \frac{aS}{1 + 2S}, \text{ so that } a - 2G^{SO} = \frac{a}{1 + 2S}$$

and, inserting this expression back,

$$g_i^{SO} = \frac{a}{c_i\,(1 + 2S)}$$

- Note that the planner does not equalize grazing levels either. Efficiency requires that marginal costs be equated across herders, $c_i g_i^{SO} = c_j g_j^{SO}$, so the efficient herder still grazes more than its less efficient neighbor. What the planner corrects is the level of aggregate grazing, not its dispersion.

(d) (5 points) Show that $G^* > G^{SO}$ for every $n \geq 2$, so the pasture is unambiguously over-exploited, and that $G^* = G^{SO}$ when $n = 1$.

- *Comparing the two totals.* Subtracting, and putting the two expressions over a common denominator, yields

$$\begin{aligned} G^* - G^{SO} &= a\left(\frac{K}{1+K} - \frac{S}{1+2S}\right) \\ &= a\frac{K\,(1+2S) - S\,(1+K)}{(1+K)\,(1+2S)} \\ &= a\frac{K - S + KS}{(1+K)\,(1+2S)} \end{aligned}$$

where the denominator is positive. Therefore, the sign of $G^* - G^{SO}$ coincides with the sign of $K - S + KS$.

- *Rewriting $K - S$.* Term by term,

$$\frac{1}{1 + c_j} - \frac{1}{c_j} = -\frac{1}{c_j\,(1 + c_j)}$$

so that $K - S = -T$, where $T \equiv \sum_{j=1}^{n} \dfrac{1}{c_j\,(1 + c_j)}$. The sign of $G^* - G^{SO}$ is, therefore, the sign of $KS - T$.

- *Comparing KS against T.* Expanding the product,

$$KS = \left(\sum_{i=1}^{n} \frac{1}{1 + c_i}\right)\left(\sum_{j=1}^{n} \frac{1}{c_j}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{(1 + c_i)\,c_j}$$

The n diagonal terms, those with $i = j$, add up exactly to T. Therefore,

$$KS - T = \sum_{i \neq j} \frac{1}{(1 + c_i)\, c_j} > 0$$

whenever $n \geq 2$, since every off-diagonal term is strictly positive. We conclude that $G^* > G^{SO}$ unambiguously, whatever the configuration of costs.

- *The case* $n = 1$. With a single herder there are no off-diagonal terms, so $KS = T$ and $G^* = G^{SO}$. Intuitively, a sole appropriator bears the entire congestion cost of its own grazing, so no externality remains to be corrected. Over-exploitation is driven by the number of appropriators rather than by their heterogeneity.

(e) (3 points) Which herder over-extracts by the most? Answer this question first in proportional terms, comparing the ratio $\frac{g_i^*}{g_i^{SO}}$ across herders, and then in absolute terms, comparing the difference $g_i^* - g_i^{SO}$. Do the two criteria give the same ranking?

- *Proportional over-extraction.* Taking the ratio of the two expressions found in parts (b) and (c),

$$\begin{aligned} \frac{g_i^*}{g_i^{SO}} &= \frac{a}{(1 + c_i)(1 + K)} \cdot \frac{c_i (1 + 2S)}{a} \\ &= \frac{c_i}{1 + c_i} \cdot \frac{1 + 2S}{1 + K} \end{aligned}$$

  where the second factor is common to all herders. Differentiating the first factor with respect to c_i, yields

$$\frac{\partial}{\partial c_i} \left(\frac{c_i}{1 + c_i} \right) = \frac{1}{(1 + c_i)^2} > 0$$

  so the ratio unambiguously increases in c_i. Therefore, the *least* efficient herder over-extracts by the largest proportion.

- Intuitively, an efficient herder grazes heavily in both scenarios, so its private and efficient levels are both large and their ratio stays close to one. An inefficient herder, in contrast, grazes little under the planner, whose rule $c_i g_i^{SO} = c_j g_j^{SO}$ pushes production toward the low-cost herders, so its equilibrium grazing looks large relative to a small benchmark.

- *Absolute over-extraction.* In levels, the comparison is

$$g_i^* - g_i^{SO} = a \left[\frac{1}{(1 + c_i)(1 + K)} - \frac{1}{c_i (1 + 2S)} \right]$$

  which need not be monotonic in c_i. The first term decreases in c_i at rate $\frac{1}{(1 + c_i)^2}$ while the second decreases at rate $\frac{1}{c_i^2}$, and which of the two declines faster depends on the level of c_i. The numerical example of part (g) exhibits a ranking in which the intermediate herder over-extracts by the most in absolute terms, even though the proportional ranking is strictly increasing in cost.

- Therefore, the two criteria do not deliver the same ranking. This distinction matters for policy: a regulator assigning proportional quota cuts would target the inefficient herders, whereas a regulator seeking the largest reduction in the number of goats should target the herders in the middle of the cost distribution.

(f) (3 points) *Comparative statics.* Consider the symmetric case $c_i = c$ for every herder. How are g_i^* and G^* affected by a marginal increase in the number of herders n? What happens to the value of a goat, $a - G^*$, as n grows without bound?

- From part (b), the symmetric equilibrium is $g_i^* = \dfrac{a}{1+c+n}$ and $G^* = \dfrac{an}{1+c+n}$. Differentiating with respect to n, yields

$$\frac{\partial g_i^*}{\partial n} = -\frac{a}{(1+c+n)^2} < 0$$

$$\frac{\partial G^*}{\partial n} = \frac{a(1+c)}{(1+c+n)^2} > 0$$

so each herder grazes fewer goats as the group expands, but the total number of goats unambiguously increases. Intuitively, entry crowds out each incumbent only partially, so the congestion externality worsens.

- *Full dissipation.* The value of a goat in equilibrium is

$$a - G^* = \frac{a(1+c)}{1+c+n}$$

which decreases in n and converges to zero as n grows without bound, while G^* converges to a. Therefore, in the limit the common resource generates no residual value at all, which is the standard full rent-dissipation result. Note that this occurs even though each herder still covers its own grazing cost, so payoffs converge to zero from above.

(g) (3 points) *Numerical example.* Evaluate your results with $n = 3$ herders, costs $c_1 = 1$, $c_2 = 2$ and $c_3 = 5$, and $a = 132$. Report K and S, the equilibrium and socially optimal grazing levels of every herder, the two totals, and the proportional and absolute over-extraction of each herder.

- *The two summary statistics.* Evaluating K and S at these costs, we obtain

$$\begin{aligned} K &= \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1 \\ S &= 1 + \frac{1}{2} + \frac{1}{5} = \frac{17}{10} \end{aligned}$$

so that $1 + K = 2$ and $1 + 2S = \dfrac{22}{5}$.

- *Nash equilibrium.* Applying $g_i^* = \dfrac{a}{(1+c_i)(1+K)} = \dfrac{132}{2(1+c_i)}$,

$$g_1^* = 33, \; g_2^* = 22, \; g_3^* = 11, \text{ and } G^* = 66$$

so that the value of a goat is $a - G^* = 132 - 66 = 66$.

- *Social optimum.* Applying $g_i^{SO} = \dfrac{a}{c_i\,(1+2S)} = \dfrac{132 \cdot 5}{22 c_i} = \dfrac{30}{c_i}$,

$$g_1^{SO} = 30, \;\; g_2^{SO} = 15, \;\; g_3^{SO} = 6, \;\; \text{and} \;\; G^{SO} = 51$$

  so that the value of a goat rises to $a - G^{SO} = 132 - 51 = 81$. The pasture is over-exploited, $66 > 51$, confirming the result of part (d).
- *Proportional over-extraction.* Taking ratios,

$$\frac{g_1^*}{g_1^{SO}} = \frac{33}{30} = \frac{11}{10}, \;\; \frac{g_2^*}{g_2^{SO}} = \frac{22}{15}, \;\; \frac{g_3^*}{g_3^{SO}} = \frac{11}{6}$$

  or, in percentage terms, 10%, 47% and 83%. The ranking is strictly increasing in cost, as part (e) predicts.
- *Absolute over-extraction.* Taking differences,

$$g_1^* - g_1^{SO} = 3, \;\; g_2^* - g_2^{SO} = 7, \;\; g_3^* - g_3^{SO} = 5$$

  so the ranking is non-monotonic: the intermediate herder grazes the largest number of excess goats, ahead of the least efficient herder and well ahead of the most efficient one. The two criteria disagree, exactly as anticipated in part (e).
- *Payoffs.* Evaluating $\pi_i = (a - G) g_i - \dfrac{c_i}{2} g_i^2$ at each profile, equilibrium payoffs are $\dfrac{3267}{2}$, 968 and $\dfrac{847}{2}$, adding up to 3025, whereas at the social optimum they are 1980, 990 and 396, adding up to 3366. The welfare loss from the common-pool externality amounts to 341, roughly 10% of the efficient aggregate payoff.