

EconS 501 - Microeconomic Theory I
Homework #4 - Due date: Tuesday, October 13th, in class.

1. **LeChatelier with three factors.** A firm produces output y using three inputs with the Cobb-Douglas technology $y = x_1^a x_2^b x_3^c$, where $a, b, c > 0$ and $a + b + c = 1$ (constant returns to scale). Input prices are (w_1, w_2, w_3) .
- (a) *Long run.* All three inputs are variable. Solve the cost minimization problem and show that the long-run cost function is $c^{LR}(w, y) = Ky \cdot w_1^a w_2^b w_3^c$ for some constant K that depends on (a, b, c) . Explain why the cost is linear in y (a consequence of CRS).
- (b) *Short run.* Input x_3 is fixed at $\bar{x}_3$; only x_1 and x_2 adjust. Solve the constrained cost minimization for the short-run cost function $c^{SR}(w_1, w_2, y | \bar{x}_3)$. Show that $c^{SR} \geq c^{LR}$ for every y , with equality only when $\bar{x}_3$ happens to equal the long-run optimum $x_3^{LR}(w, y)$.
- (c) *LeChatelier principle.* Show that the short-run own-price elasticity of demand for x_1 is smaller (in absolute value) than the long-run elasticity: $\left| \frac{\partial \ln x_1^{SR}}{\partial \ln w_1} \right| \leq \left| \frac{\partial \ln x_1^{LR}}{\partial \ln w_1} \right|$. Explain economically: with x_3 frozen, the firm has one fewer margin of substitution, so it responds less to a price change than when all inputs can adjust.

2. **Profit maximization and the convexity of the profit function.** A price-taking firm produces a single output y from two inputs, z_1 and z_2 , with the technology $f(z_1, z_2) = \sqrt{z_1} + \sqrt{z_2}$. The output price is $p > 0$ and the input prices are $w_1, w_2 > 0$.

- (a) Solve the profit-maximization problem $\max_{z_1, z_2 \geq 0} \{p f(z_1, z_2) - w_1 z_1 - w_2 z_2\}$. Derive the input demands $z_1(p, w)$ and $z_2(p, w)$, the output supply $y(p, w)$, and show that the profit function is

$$\pi(p, w) = \frac{p^2}{4} \left(\frac{1}{w_1} + \frac{1}{w_2} \right).$$

- (b) Verify Hotelling's lemma, $\frac{\partial \pi}{\partial p} = y(p, w)$ and $\frac{\partial \pi}{\partial w_i} = -z_i(p, w)$, and show that π is homogeneous of degree one in (p, w_1, w_2) .
- (c) Show that $\pi(p, w)$ is convex in (p, w_1, w_2) . Using this convexity, establish the law of supply and the law of factor demand, $\frac{\partial y}{\partial p} \geq 0$ and $\frac{\partial z_i}{\partial w_i} \leq 0$. Explain why the decreasing returns to scale of f guarantee an interior profit-maximizing plan, whereas constant or increasing returns would not.

3. **Returns to scale and the shape of cost.** A firm produces one output using two inputs, z_1 and z_2 , with the Cobb-Douglas technology $f(z_1, z_2) = (z_1 z_2)^a$, where $a > 0$. Input prices are $w_1, w_2 > 0$.

- (a) Solve the cost-minimization problem $\min_{z_1, z_2 \geq 0} \{w_1 z_1 + w_2 z_2 : f(z_1, z_2) = q\}$ and show that the conditional factor demands and cost function are

$$z_1^* = q^{1/(2a)} \sqrt{\frac{w_2}{w_1}}, \quad z_2^* = q^{1/(2a)} \sqrt{\frac{w_1}{w_2}}, \quad c(q, w) = 2\sqrt{w_1 w_2} \cdot q^{1/(2a)}.$$

- (b) Determine the returns to scale of f (increasing, constant, or decreasing) as a function of a , and show that the elasticity of cost with respect to output, $\frac{d \ln c}{d \ln q} = \frac{1}{2a}$, is above, equal to, or below unity accordingly. Interpret: cost is convex in q under decreasing returns, linear under constant returns, and concave under increasing returns.
- (c) Derive the competitive supply of a price-taking firm from the first-order condition $p = c'(q)$, and show that a determinate interior supply exists only under decreasing returns ($a < \frac{1}{2}$). As a concrete case, take $a = \frac{1}{3}$: verify that $c(q, w) = 2\sqrt{w_1 w_2} q^{3/2}$ and that competitive supply is $q^*(p) = \frac{p^2}{9 w_1 w_2}$. Explain briefly what happens under constant returns (perfectly elastic supply at a specific price) and under increasing returns (no interior competitive supply).