

EconS 501 - Microeconomic Theory I

Homework #3 - Answer key

1. (34 points) Constant income elasticity and the Willig approximation to consumer surplus.

(a) (6 points) Integrating the elasticity.

- Constant income elasticity means $\frac{d \ln x}{d \ln w} \Big|_p = \eta$ at every (p, w) . Integrating this ODE at fixed p ,

$$\ln x(p, w) = \eta \ln w + C(p) \iff x(p, w) = e^{C(p)} w^\eta \equiv A(p) w^\eta.$$

Taking the ratio at two wealth levels w and $\hat{w}$ (with p fixed) eliminates $A(p)$ and yields $x(p, \hat{w}) = x(p, w)(\hat{w}/w)^\eta$ directly.

(b) (10 points) The expenditure ODE and its closed-form solution.

- Fix the initial utility $u_0 = v(p_0, w)$ and let $E(p) = e(p, u_0)$ denote the expenditure needed to sustain u_0 at the new price p . By Shephard's lemma, $E'(p) = h(p, u_0)$, where h is the Hicksian demand. But along the level set of utility u_0 , wealth is $E(p)$, and by duality $h(p, u_0) = x(p, E(p)) = A(p) E(p)^\eta$. Using $A(p) = x(p, w)/w^\eta$ from part (a),

$$E'(p) = x(p, w) \left(\frac{E(p)}{w} \right)^\eta, \quad E(p_0) = w.$$

- Separating variables,

$$E^{-\eta} dE = x(p, w) w^{-\eta} dp.$$

Integrating from p_0 to p (the LHS integrates to $E^{1-\eta}/(1-\eta)$ evaluated between the endpoints),

$$\frac{E(p)^{1-\eta} - w^{1-\eta}}{1-\eta} = w^{-\eta} \int_{p_0}^p x(s, w) ds,$$

so $E(p)^{1-\eta} = w^{1-\eta} \left[1 + \frac{1-\eta}{w} \int_{p_0}^p x(s, w) ds \right]$, and raising both sides to $\frac{1}{1-\eta}$ gives

$$e(p, v(p_0, w)) = w \left[1 + \frac{1-\eta}{w} \int_{p_0}^p x(s, w) ds \right]^{1/(1-\eta)}.$$

(c) (12 points) The Willig approximation.

- Write $E(p) = w(1+t)^a$ with $t \equiv \frac{(1-\eta)A}{w}$ and $a \equiv \frac{1}{1-\eta}$, so $at = \frac{A}{w}$ and $a(a-1)t^2 = \frac{\eta A^2}{w^2(1-\eta)} \cdot (1-\eta) = \frac{\eta A^2}{w^2}$. Expanding to second order in t ,

$$(1+t)^a \approx 1 + at + \frac{1}{2} a(a-1) t^2 = 1 + \frac{A}{w} + \frac{\eta A^2}{2w^2},$$

so $E(p) \approx w + A + \frac{\eta A^2}{2w}$ and therefore $EV = E(p) - w \approx A + \frac{\eta A^2}{2w}$, which rearranges to

$$\frac{EV - A}{A} \approx \frac{\eta}{2} \cdot \frac{A}{w}.$$

- *Interpretation.* The consumer-surplus area A misses the true welfare loss EV only by a term proportional to (i) the income elasticity η and (ii) the size of the expenditure change relative to wealth, A/w . For a normal good ($\eta > 0$) consumer surplus *under-estimates* EV ; for a luxury ($\eta > 1$) the wedge is large; for a Giffen-like inferior good ($\eta < 0$) the sign reverses and A over-estimates EV . When either the price change is small (so A is small relative to w) or income effects vanish ($\eta = 0$, quasilinear), the two measures coincide up to first order, which is why applied cost-benefit analysis is standardly carried out under quasilinear assumptions.

(d) (6 points) Numerical check with $\eta = \frac{1}{2}$.

- Demand: $x(p, w) = \frac{5}{\sqrt{p}} w^{1/2}$. At $w = 100$: $x(p, 100) = \frac{50}{\sqrt{p}}$. The area under the demand curve between $p_0 = 1$ and $p = 2$:

$$A = \int_1^2 \frac{50}{\sqrt{s}} ds = 50 [2\sqrt{s}]_1^2 = 100(\sqrt{2} - 1) \approx 41.42.$$

- Exact EV from the closed form of part (b):

$$E(p) = 100 \left[1 + \frac{0.5 \cdot 41.42}{100} \right]^2 = 100 (1.2071)^2 \approx 145.71,$$

so $EV = 45.71$.

- Willig approximation from part (c):

$$EV \approx A + \frac{\eta A^2}{2w} = 41.42 + \frac{0.5 \cdot (41.42)^2}{2 \cdot 100} = 41.42 + 4.29 = 45.71.$$

The two figures agree to the second decimal: the Willig approximation is essentially exact even for a price doubling, once η is only $\frac{1}{2}$ and $A/w \approx 0.41$.

2. (34 points) Gorman aggregation and the representative consumer.

(a) (14 points) Gorman form implies linear-in-wealth demands that aggregate.

- Apply Roy's identity to the Gorman indirect utility $v_i(p, w_i) = a_i(p) + b(p) w_i$. Denote $a'_{i,k}(p) \equiv \frac{\partial a_i}{\partial p_k}$ and $b'_k(p) \equiv \frac{\partial b}{\partial p_k}$. Then

$$\begin{aligned} x_{ki}(p, w_i) &= -\frac{\partial v_i / \partial p_k}{\partial v_i / \partial w_i} = -\frac{a'_{i,k}(p) + b'_k(p) w_i}{b(p)} \\ &= -\frac{a'_{i,k}(p)}{b(p)} - \frac{b'_k(p)}{b(p)} w_i \equiv \alpha_{ki}(p) + \beta_k(p) w_i, \end{aligned}$$

so $\alpha_{ki}(p) = -\frac{a'_{i,k}(p)}{b(p)}$ is consumer-specific (through a_i), while the slope $\beta_k(p) = -\frac{b'_k(p)}{b(p)}$ inherits the common $b(p)$ and is therefore identical across consumers.

- Summing across consumers,

$$X_k(p, w_1, \dots, w_I) = \sum_{i=1}^I \alpha_{ki}(p) + \beta_k(p) \sum_{i=1}^I w_i = A_k(p) + \beta_k(p) W,$$

where $A_k(p) \equiv \sum_i \alpha_{ki}(p)$. Aggregate demand for good k depends on the wealth distribution only through the total wealth $W = \sum_i w_i$: this is exactly the representative-consumer property.

(b) (10 points) Why the common slope is essential.

- Suppose slopes were consumer-specific, $x_{ki}(p, w_i) = \alpha_{ki}(p) + \beta_{ki}(p) w_i$. Transfer one dollar from consumer i to consumer j , so $w_i \rightarrow w_i - 1$ and $w_j \rightarrow w_j + 1$, holding W fixed. The intercepts α_{ki} and α_{kj} do not move, and the change in aggregate demand collapses to the change in the slope terms,

$$\Delta X_k = \beta_{kj}(p) [(w_j + 1) - w_j] + \beta_{ki}(p) [(w_i - 1) - w_i] = \beta_{kj}(p) - \beta_{ki}(p),$$

which is nonzero unless $\beta_{ki}(p) = \beta_{kj}(p)$ for every pair (i, j) .

- *Conclusion.* Aggregate demand depends on *who* holds the wealth, not just on the total W : the representative-consumer requirement fails. Only preferences that generate a common wealth slope across consumers (the Gorman form, by construction) ensure that W is a sufficient statistic for aggregate demand.

(c) (10 points) Cobb–Douglas vs quasi-linear preferences.

- Cobb–Douglas.* Standard Cobb–Douglas indirect utility factors as $v_i(p, w_i) = w_i \cdot b_i(p) \cdot \kappa_i$ with $b_i(p) = \prod_k p_k^{-\alpha_{k,i}}$ and $\kappa_i = \prod_k \alpha_{k,i}^{\alpha_{k,i}}$. Writing this in the candidate Gorman form, $v_i(p, w_i) = a_i(p) + [b_i(p)\kappa_i] w_i$ with $a_i(p) \equiv 0$, the slope $b_i(p)\kappa_i$ is generically consumer-specific because it depends on the expenditure-share vector $(\alpha_{k,i})_k$. Exact aggregation therefore fails unless every consumer has the same shares $\alpha_{k,i} = \alpha_k$ (the knife-edge homothetic case in which Cobb–Douglas is trivially Gorman with common slope $b(p) = \prod_k p_k^{-\alpha_k}$).
- Quasi-linear.* Quasi-linear indirect utility is $v_i(p, w_i) = w_i + \phi_i(p)$, which is exactly Gorman with $b(p) \equiv 1$ and $a_i(p) = \phi_i(p)$. Because $b(p)$ is trivially common across consumers, the wealth slope for every non-numeraire good is zero (demands for those goods are independent of wealth) and demand for the numeraire absorbs all wealth changes. Aggregation is exact for every distribution of wealth.
- Policy implication.* Between the two families, only quasi-linear preferences permit exact aggregation across heterogeneous consumers, and this is the underlying reason applied welfare analysis — public finance, environmental economics, and cost–benefit analysis — adopts quasi-linear utility so pervasively: it lets a single representative-consumer surplus stand in for the correct

aggregate welfare change, sidesteps the composition-of-wealth issue behind the Sonnenschein–Mantel–Debreu results, and validates the Willig approximation from Exercise 1 as an exact equality rather than a second-order one.

3. (32 points) Non-linear pricing and non-convex budget sets.

(a) (6 points) The non-linear budget set.

- For $x_1 < \gamma$ the frontier is $x_2 = \frac{w - p_1 x_1}{p_2}$ (slope $-p_1/p_2$). At $x_1 = \gamma$, total coffee spending drops from $p_1 \gamma$ to $q\gamma$, so the residual for rice jumps up from $\frac{w - p_1 \gamma}{p_2}$ to $\frac{w - q\gamma}{p_2}$ — an upward jump of $\frac{(p_1 - q)\gamma}{p_2}$. For $x_1 \geq \gamma$ the frontier continues with the flatter slope $-q/p_2$ up to $x_1 = w/q$. With $p_1 = 2$, $q = 1$, $p_2 = 1$, $w = 6$, $\gamma = 2$: the frontier runs $(0, 6) \rightarrow (2, 2)$ at slope -2 , jumps up to $(2, 4)$, then runs $(2, 4) \rightarrow (6, 0)$ at slope -1 . Because the jump makes the frontier concave (the graph fails the “line-segment” test between $(2, 2)$ and $(6, 0)$), the set B_{NL} is non-convex.

(b) (5 points) Linear-budget benchmark and the upper bound.

- Cobb–Douglas $u = x_1 x_2$ on the linear budget $B(\tilde{p}_1, p_2, w)$ gives $\hat{x}(\tilde{p}_1, p_2, w) = \left(\frac{w}{2\tilde{p}_1}, \frac{w}{2p_2} \right)$. Every bundle in B_{NL} costs at least as much as under the uniform price q (coffee is charged $p_1 \geq q$ below the threshold), so $B_{NL} \subset B(q, p_2, w)$. Utility over the smaller set therefore cannot exceed that over the larger set, so $\hat{x}(q, p_2, w)$ is an upper bound.

(c) (5 points) The optimum when $\frac{w}{2q} \geq \gamma$.

- In this case $\hat{x}_1(q, p_2, w) = \frac{w}{2q} \geq \gamma$, so the low-price interior optimum already lies in the discount region, is therefore feasible in B_{NL} , and attains the upper-bound utility. Demand is $\hat{x}(q, p_2, w)$: the consumer just captures the discount at the interior optimum.

(d) (10 points) The interior threshold γ_+ .

- The two candidates are: (i) the interior optimum at price p_1 (without the discount), $(x_1, x_2) = (\frac{w}{2p_1}, \frac{w}{2p_2})$, with utility $\frac{w^2}{4p_1 p_2}$; and (ii) the smallest bundle that unlocks the discount, $(x_1, x_2) = (\gamma, \frac{w - q\gamma}{p_2})$, with utility $\frac{\gamma(w - q\gamma)}{p_2}$. Setting the two utilities equal to find the indifference γ :

$$\frac{\gamma(w - q\gamma)}{p_2} = \frac{w^2}{4p_1 p_2} \implies q\gamma^2 - w\gamma + \frac{w^2}{4p_1} = 0.$$

The quadratic has roots

$$\gamma_{\pm} = \frac{w}{2q} \left(1 \pm \sqrt{\frac{p_1 - q}{p_1}} \right).$$

The candidates trace two parabolas as functions of γ ; the discount utility wins between the two roots. Combined with $\gamma > \frac{w}{2q}$ (which excludes γ_- since $\gamma_- < \frac{w}{2q}$), the range is $\frac{w}{2q} < \gamma < \gamma_+$.

- *Interpretation.* γ_+ is the highest threshold the consumer is willing to stretch her coffee consumption to meet. For any higher threshold, the sacrifice of rice consumption required to reach γ exceeds the benefit of the cheaper marginal price.

(e) (6 points) Optimal coffee demand as a function of γ .

- Piecing together the three cases from parts (c)–(d),

$$x_1^*(\gamma) = \begin{cases} \frac{w}{2q}, & \gamma \leq \frac{w}{2q}, \\ \gamma, & \frac{w}{2q} < \gamma < \gamma_+, \\ \frac{w}{2p_1}, & \gamma \geq \gamma_+, \end{cases}$$

Coffee demand is *flat* at the low-price optimum for small thresholds, *rises along the 45-degree line* while the consumer buys just enough coffee to unlock the discount, and then *drops discontinuously* from γ_+ to the high-price optimum $\frac{w}{2p_1} < \gamma_+$ once the required threshold becomes too costly to reach.

- *Interpretation.* The downward jump is the fingerprint of a non-convex budget set: the consumer switches between two disjoint local optima — one in the discount region and one in the full-price region — so the demand correspondence is discontinuous in the primitives of the pricing schedule. Contrast with linear pricing, where the demand curve is smooth and small changes in the price schedule move consumption smoothly. Real non-linear-pricing schemes (all-units discounts in wholesale supply contracts, block pricing in utilities, membership-fee pricing) share this property, which is one reason why they generate non-obvious welfare comparative statics.