

# EconS 501 - Microeconomic Theory I

## Homework #3 - Due date: Tuesday, September 29th, in class.

1. **Constant income elasticity and the Willig approximation to consumer surplus.** A single good has Walrasian demand  $x(p, w)$  that is smooth in  $(p, w)$  and has *constant* income elasticity  $\eta \neq 1$  across all  $(p, w)$ ; i.e.,  $\frac{\partial x(p, w)}{\partial w} \frac{w}{x(p, w)} = \eta$ . Let  $p_0$  be the initial price and  $p$  the new price after some change, both with the same reference wealth  $w$ .

- (a) Show that a constant income elasticity implies the demand takes the form

$$x(p, w) = A(p) w^\eta \quad \text{and, equivalently,} \quad x(p, \hat{w}) = x(p, w) \left( \frac{\hat{w}}{w} \right)^\eta$$

for any two wealth levels  $w$  and  $\hat{w}$ . [*Hint*: integrate  $\frac{d \ln x}{d \ln w} = \eta$  in  $w$  at fixed  $p$ .]

- (b) Let  $u_0 = v(p_0, w)$  and define  $E(p) \equiv e(p, u_0)$ , the expenditure needed at the new price  $p$  to reach the initial utility. Use Shephard's lemma together with part (a) to derive the ordinary differential equation  $E'(p) = x(p, w) \left( \frac{E(p)}{w} \right)^\eta$  with  $E(p_0) = w$ . Solving it, show

$$e(p, v(p_0, w)) = w \left[ 1 + \frac{1 - \eta}{w} \int_{p_0}^p x(s, w) ds \right]^{1/(1-\eta)}.$$

- (c) Denote the consumer-surplus loss by  $A = \int_{p_0}^p x(s, w) ds$  and the equivalent variation by  $EV = E(p) - w$ . Expand the closed-form expression from part (b) to second order in  $A/w$  and prove the *Willig approximation*

$$\frac{EV - A}{A} \approx \frac{\eta}{2} \cdot \frac{A}{w}.$$

Interpret the sign and magnitude: for what values of  $\eta$  (normal, luxury, inferior good) does consumer surplus under- or over-estimate the true welfare loss, and how does the size of the error scale with the budget share  $A/w$ ?

- (d) Take  $\eta = \frac{1}{2}$ ,  $p_0 = 1$ ,  $p = 2$ ,  $x(p, w) = A(p) w^{1/2}$  with  $A(p) = \frac{5}{\sqrt{p}}$ , and  $w = 100$ . Compute  $A$ ,  $EV$  (using the exact formula), and the Willig-approximation right-hand side, and check numerically that the two agree to within a few percent.

2. **Gorman aggregation and the representative consumer.** An economy has  $I$  consumers, indexed  $i = 1, \dots, I$ , each with wealth  $w_i$  and Walrasian demand  $x_i(p, w_i)$ . Aggregate demand is  $X(p, w_1, \dots, w_I) = \sum_i x_i(p, w_i)$ . We say the aggregate demand can be written as if it were the demand of a *representative consumer* if there exists a function  $\tilde{X}(p, W)$ , with  $W = \sum_i w_i$ , such that  $X(p, w_1, \dots, w_I) = \tilde{X}(p, W)$  for every distribution of wealth.

- (a) Show that if every consumer's indirect utility takes the *Gorman form*  $v_i(p, w_i) = a_i(p) + b(p) w_i$  (with a common slope  $b(p)$  across consumers), then Roy's identity yields Walrasian demands linear in own wealth,  $x_{ki}(p, w_i) = \alpha_{ki}(p) + \beta_k(p) w_i$ , and aggregate demand depends only on  $W = \sum_i w_i$ . Identify explicitly the functions  $\alpha_{ki}(p)$  and  $\beta_k(p)$ , and comment on which is consumer-specific and which is not.
- (b) Explain intuitively why the common-slope requirement is essential. If consumers' marginal propensities to consume good  $k$  differed across consumers, so that demands took the form  $x_{ki}(p, w_i) = \alpha_{ki}(p) + \beta_{ki}(p) w_i$  with a consumer-specific slope  $\beta_{ki}$ , show that a wealth transfer of one dollar from consumer  $i$  to consumer  $j$  (leaving  $W = \sum_h w_h$  fixed) would change aggregate demand for good  $k$  by exactly  $\beta_{kj} - \beta_{ki}$ , violating the representative-consumer requirement. Conclude that only demand systems with a common wealth slope aggregate exactly through the sum of wealth.
- (c) Test the Gorman form on two workhorse specifications and draw the policy implication.
  - i. Cobb–Douglas indirect utility. Show that the standard Cobb–Douglas indirect utility  $v_i(p, w_i) = w_i \prod_k p_k^{-\alpha_{k,i}} \prod_k \alpha_{k,i}^{\alpha_{k,i}}$ , with consumer-specific expenditure shares  $\alpha_{k,i}$  summing to one, is Gorman-consistent only in the knife-edge case in which the shares  $\alpha_{k,i}$  are identical across consumers; otherwise the implicit slope  $b(p) = \prod_k p_k^{-\alpha_{k,i}}$  varies across consumers and exact aggregation fails.
  - ii. Quasi-linear indirect utility. Show that quasi-linear indirect utility  $v_i(p, w_i) = w_i + \phi_i(p)$  is Gorman with  $b(p) \equiv 1$  and  $a_i(p) = \phi_i(p)$ . Which of the two utility families therefore permits exact aggregation across heterogeneous consumers? Explain why applied welfare analysis (public finance, environmental economics, cost–benefit analysis) so often assumes quasi-linear preferences.

3. **Non-linear pricing and non-convex budget sets.** A consumer has Cobb–Douglas preferences  $u(x_1, x_2) = x_1 x_2$  over coffee ( $x_1$ ) and rice ( $x_2$ ), with wealth  $w$ . Rice sells at unit price  $p_2$ . Coffee is sold under an all-units quantity discount: each liter costs  $p_1$  if the consumer buys *less* than  $\gamma$  liters and  $q < p_1$  if she buys *at least*  $\gamma$ . Assume  $\gamma < \frac{w}{q}$  so that the discount threshold is inside the budget.

- (a) Sketch the non-linear budget set  $B_{NL}$  in the  $(x_1, x_2)$  plane for the parameter values  $p_1 = 2, q = 1, p_2 = 1, w = 6, \gamma = 2$ . Show that at  $x_1 = \gamma$  the frontier *jumps up*: at the point just above the threshold, total spending on coffee falls from  $p_1 \gamma$  to  $q \gamma$ , freeing  $(p_1 - q) \gamma$  for rice. Conclude that  $B_{NL}$  is non-convex.
- (b) Let  $\hat{x}(\tilde{p}_1, p_2, w)$  denote the Cobb–Douglas Walrasian demand at the linear budget  $B(\tilde{p}_1, p_2, w)$  where coffee is charged at the constant price  $\tilde{p}_1$ . Write down the closed-form of  $\hat{x}$ , and show that  $B_{NL} \subset B(q, p_2, w)$ , so that the low-price optimum  $\hat{x}(q, p_2, w)$  is an upper bound on utility over  $B_{NL}$ .
- (c) Show that when  $\frac{w}{2q} \geq \gamma$  (the low-price optimum already exceeds the discount threshold), the consumer's demand is exactly  $\hat{x}(q, p_2, w)$  and the discount is captured by choosing the unconstrained interior optimum.
- (d) Now suppose  $\frac{w}{2q} < \gamma$  (the low-price optimum falls short of the discount threshold), so the consumer must *overspend* on coffee to unlock the discount. Compare the utility of two candidates: **(i)** the interior optimum at price  $p_1$ , which yields  $\frac{w^2}{4p_1 p_2}$ ; and **(ii)** the bundle  $(x_1, x_2) = (\gamma, \frac{w - q\gamma}{p_2})$  that just satisfies the threshold and spends the residual on rice, yielding  $\frac{\gamma(w - q\gamma)}{p_2}$ . Show that the discount bundle is preferred if and only if

$$\frac{w}{2q} < \gamma < \frac{w}{2q} \left( 1 + \sqrt{\frac{p_1 - q}{p_1}} \right) \equiv \gamma_+.$$

Interpret  $\gamma_+$  as the largest threshold the discount can still overcome.

- (e) Assemble the optimal coffee demand as a function of  $\gamma$ ,

$$x_1^*(\gamma) = \begin{cases} \frac{w}{2q}, & \gamma \leq \frac{w}{2q}, \\ \gamma, & \frac{w}{2q} < \gamma < \gamma_+, \\ \frac{w}{2p_1}, & \gamma \geq \gamma_+, \end{cases}$$

and sketch it. Note the *downward jump* at  $\gamma = \gamma_+$ : coffee demand collapses from  $\gamma_+$  to  $\frac{w}{2p_1} < \gamma_+$  as the threshold rises past  $\gamma_+$ . Interpret this jump as an optimizing response to a non-convex budget set, and contrast with the smooth demand curves one obtains under linear pricing.