

EconS 501 - Microeconomic Theory I

Homework #2 - Answer key

1. (25 points) The linear expenditure system with n goods.

(a) (10 points) Walrasian demand.

- Define $\tilde{x}_i = x_i - \gamma_i$. In these variables the problem is a standard Cobb-Douglas maximization of $\prod_i \tilde{x}_i^{\alpha_i}$ subject to the budget $\sum_i p_i \tilde{x}_i = w - \sum_j p_j \gamma_j \equiv \tilde{w}$, the supernumerary income. The Cobb-Douglas solution is $\tilde{x}_i = \frac{\alpha_i \tilde{w}}{p_i}$, so, returning to original variables,

$$x_i(p, w) = \gamma_i + \frac{\alpha_i}{p_i} \left(w - \sum_j p_j \gamma_j \right).$$

The consumer first covers the subsistence quantities γ_i and then divides what remains in fixed Cobb-Douglas shares α_i : expenditure on each good above subsistence is a constant fraction of supernumerary income.

(b) (10 points) Income elasticity.

- Since only the supernumerary part responds to wealth,

$$\frac{\partial x_i}{\partial w} = \frac{\alpha_i}{p_i}, \quad \text{so} \quad \eta_i = \frac{\partial x_i}{\partial w} \frac{w}{x_i} = \frac{\alpha_i w}{p_i x_i}.$$

Substituting $x_i = \gamma_i + \frac{\alpha_i \tilde{w}}{p_i}$, one obtains $\eta_i = \frac{\alpha_i w}{p_i \gamma_i + \alpha_i \tilde{w}}$. The classification rule is:

- If $\gamma_i > 0$ (positive subsistence), then $p_i \gamma_i$ inflates the denominator but not the numerator, so $\eta_i < 1$: a *necessity*.
- If $\gamma_i = 0$ (no subsistence), the term drops out and $\eta_i = \frac{\alpha_i w}{\alpha_i \tilde{w}} = \frac{w}{\tilde{w}} > 1$ whenever *any* other good has positive subsistence: a *luxury*.
- If all $\gamma_j = 0$, then $\tilde{w} = w$ and $\eta_i = 1$ for all i (Cobb-Douglas benchmark).

(c) (5 points) Numerical example.

- With $n = 3$, $\alpha = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\gamma = (2, 1, 0)$, $p = (1, 1, 1)$, and $w = 15$, the supernumerary income is $\tilde{w} = 15 - (2 + 1 + 0) = 12$. The demands are $x_1 = 2 + \frac{1}{3}(12) = 6$, $x_2 = 1 + \frac{1}{3}(12) = 5$, $x_3 = 0 + \frac{1}{3}(12) = 4$. Income elasticities are $\eta_1 = \frac{(1/3)(15)}{(1)(6)} = \frac{5}{6}$, $\eta_2 = \frac{(1/3)(15)}{(1)(5)} = 1$, $\eta_3 = \frac{(1/3)(15)}{(1)(4)} = \frac{5}{4}$. Good 1 (highest subsistence) is a necessity, good 3 (no subsistence) is a luxury, and good 2 is on the border.

2. (20 points) **WARP holds but SARP fails.**

(a) (4 points) Budgets exhausted.

- Observation 1: $p^1 \cdot x^1 = 4 \cdot 1 + 1 \cdot 4 + 1 \cdot 0 = 8$. Observation 2: $p^2 \cdot x^2 = 1 \cdot 0 + 4 \cdot 1 + 1 \cdot 4 = 8$. Observation 3: $p^3 \cdot x^3 = 1 \cdot 4 + 1 \cdot 0 + 4 \cdot 1 = 8$. Each budget is exhausted (implicitly, $w^i = 8$ in each observation).

(b) (8 points) Direct revealed preferences.

- At $p^1 = (4, 1, 1)$: $p^1 \cdot x^2 = 4(0) + 1(1) + 1(4) = 5 < 8 = w^1$, so x^2 is strictly affordable; $p^1 \cdot x^3 = 4(4) + 1(0) + 1(1) = 17 > 8$, unaffordable. Since x^2 is strictly cheaper than the chosen bundle x^1 , we obtain the *strict* revealed preference $x^1 \succ_R x^2$.
- Proceeding analogously for the other two observations:

$$\mid p^1 \cdot x^2 = 5 < 8 \quad p^2 \cdot x^3 = 5 < 8 \quad p^3 \cdot x^1 = 5 < 8$$

where $p^2 \cdot x^3 = 1(4) + 4(0) + 1(1) = 5$ and $p^3 \cdot x^1 = 1(1) + 1(4) + 4(0) = 5$. In each observation i , the alternative bundle is strictly affordable, so we obtain the strict revealed preferences $x^1 \succ_R x^2$, $x^2 \succ_R x^3$, and $x^3 \succ_R x^1$.

(c) (6 points) WARP holds but SARP fails.

- *WARP* requires that if $x^i \succsim_R x^j$ and the two bundles differ, then x^j is not *directly* revealed preferred to x^i . Checking each pair: from part (b), $x^1 \succ_R x^2$, but x^2 is not revealed preferred to x^1 , since at p^2 we have $p^2 \cdot x^1 = 1(1) + 4(4) + 1(0) = 17 > 8 = w^2$, so x^1 is unaffordable at p^2 . Analogously, $p^3 \cdot x^2 = 1(0) + 1(1) + 4(4) = 17 > 8$ and $p^1 \cdot x^3 = 17 > 8$, so no reverse revelation holds in any pair. Therefore, WARP is satisfied.
- *SARP* rules out cycles in the transitive closure of the direct relation. Here $x^1 \succ_R x^2 \succ_R x^3 \succ_R x^1$ is a length-3 cycle of strict revealed preferences, so SARP fails.

(d) (2 points) Implication.

- With only two goods, WARP is known to be equivalent to SARP (the Ville theorem). With three or more goods, the equivalence breaks: pairwise consistency (WARP) no longer suffices for the existence of a rationalizing preference, because it does not rule out longer cycles. SARP (or, equivalently, the Generalized Axiom of Revealed Preference for finite data) is the necessary-and-sufficient condition for the data to be generated by preference maximization.

3. (20 points) **Integrability with three goods.**

(a) (8 points) Walras' law and homogeneity.

- *Walras' law*: total expenditure is $p_1x_1 + p_2x_2 + p_3x_3$. By construction x_3 absorbs the residual, so $p_3x_3 = w - p_1x_1 - p_2x_2$ and adding the first two terms recovers w . Walras' law holds by definition of x_3 .

- *Homogeneity*: replacing (p, w) by $(\lambda p, \lambda w)$ leaves $x_1 = \frac{\lambda w}{\lambda p_1 + \lambda p_2} = \frac{w}{p_1 + p_2}$ unchanged, and similarly for x_2 . Then x_3 is also unchanged, since the residual $w - p_1 x_1 - p_2 x_2$ scales as λ and so does p_3 .

(b) (10 points) Slutsky asymmetry.

- Direct calculation gives $\frac{\partial x_1}{\partial p_2} = -\frac{w}{(p_1 + p_2)^2}$ and $\frac{\partial x_1}{\partial w} = \frac{1}{p_1 + p_2}$. Hence $S_{12} = \frac{\partial x_1}{\partial p_2} + \frac{\partial x_1}{\partial w} x_2 = -\frac{w}{(p_1 + p_2)^2} + \frac{1}{p_1 + p_2} \cdot \frac{w}{p_1 + 2p_2}$. Symmetrically, from $\frac{\partial x_2}{\partial p_1} = -\frac{w}{(p_1 + 2p_2)^2}$ and $\frac{\partial x_2}{\partial w} = \frac{1}{p_1 + 2p_2}$, we get $S_{21} = -\frac{w}{(p_1 + 2p_2)^2} + \frac{1}{p_1 + 2p_2} \frac{w}{p_1 + p_2}$. Simplifying,

$$S_{12} - S_{21} = \frac{w}{(p_1 + 2p_2)^2} - \frac{w}{(p_1 + p_2)^2} \neq 0 \text{ for generic prices.}$$

Slutsky symmetry fails, so no utility function generates this demand. The system is Walras-consistent and homogeneous but not integrable.

(c) (7 points) Cobb-Douglas passes the test.

- For $y_i = \frac{\alpha_i w}{p_i}$, one computes $\frac{\partial y_1}{\partial p_2} = 0$ and $\frac{\partial y_1}{\partial w} = \frac{\alpha_1}{p_1}$, so $S_{12} = \frac{\alpha_1}{p_1} \cdot \frac{\alpha_2 w}{p_2} = \frac{\alpha_1 \alpha_2 w}{p_1 p_2}$. Symmetrically $S_{21} = \frac{\alpha_1 \alpha_2 w}{p_1 p_2}$: symmetry holds. The same calculation for the other pairs also delivers symmetric entries, so the Cobb-Douglas system is integrable. The underlying utility is the Cobb-Douglas $u(x) = \prod_i x_i^{\alpha_i}$, which delivers these demands as its Walrasian demand.

4. (25 points) CES demand: Walrasian, Hicksian, and the Slutsky decomposition.

(a) (8 points) Walrasian demand and indirect utility.

- The first-order conditions for maximizing u subject to $p_1 x_1 + p_2 x_2 = w$ give $\frac{x_1}{x_2} = \left(\frac{p_2}{p_1}\right)^\sigma$. Combined with the budget constraint,

$$x_i(p, w) = \frac{p_i^{-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}} w \quad \text{and} \quad v(p, w) = \frac{w}{(p_1^{1-\sigma} + p_2^{1-\sigma})^{1/(1-\sigma)}}.$$

(b) (10 points) Hicksian demand and expenditure function.

- By the standard CES duality (setting $v(p, w) = u$ and inverting), $e(p, u) = u (p_1^{1-\sigma} + p_2^{1-\sigma})^{1/(1-\sigma)}$. Then Shephard's lemma gives the Hicksian demand

$$h_i(p, u) = \frac{\partial e}{\partial p_i} = u p_i^{-\sigma} (p_1^{1-\sigma} + p_2^{1-\sigma})^{\sigma/(1-\sigma)}.$$

Substituting $u = v(p, w) = \frac{w}{(p_1^{1-\sigma} + p_2^{1-\sigma})^{1/(1-\sigma)}}$ into h_i recovers the Walrasian demand from part (a), confirming the identity $x_i(p, e(p, u)) = h_i(p, u)$.

(c) (7 points) Slutsky decomposition and numbers.

- The Slutsky equation splits $\frac{\partial x_1}{\partial p_2}$ into the substitution effect $\frac{\partial h_1}{\partial p_2} > 0$ (a rise in p_2 makes good 1 relatively cheaper, so the compensated demand for good 1 rises) and the income effect $-\frac{\partial x_1}{\partial w}x_2 < 0$ (the price rise cuts real income, reducing demand for the normal good 1). At $\rho = -1$, $\sigma = \frac{1}{2}$; with $p_1 = p_2 = 1$ and $w = 4$, symmetry gives $x_1 = x_2 = 2$. Direct calculation of the closed-form Slutsky terms at these values confirms that the two effects sum to $\frac{\partial x_1}{\partial p_2}$, and that the total Marshallian response is negative, consistent with gross-complement behavior when $\sigma < 1$.

5. (25 points) **Corner solutions with quasi-linear utility.**

(a) (8 points) Interior solution.

- The Lagrangian is $\mathcal{L} = a \ln x_1 + x_2 + \lambda(w - p_1 x_1 - x_2)$. The first-order conditions $\frac{a}{x_1} = \lambda p_1$ and $1 = \lambda$ give $\lambda = 1$, so $x_1^{int} = \frac{a}{p_1}$. From the budget, $x_2^{int} = w - p_1 x_1^{int} = w - a$, which is nonnegative iff $w \geq a$.

(b) (6 points) Corner solution.

- If $w < a$, the interior x_2^{int} would be negative, violating $x_2 \geq 0$. The consumer therefore chooses the corner $x_2 = 0$, and spends the whole budget on good 1: $x_1^{cor} = \frac{w}{p_1}$.

(c) (8 points) Combining the two regions.

- Piecing together,

$$x_1(p_1, w) = \begin{cases} \frac{w}{p_1} & \text{if } w < a, \\ \frac{a}{p_1} & \text{if } w \geq a. \end{cases}$$

Below $w = a$ demand rises linearly in w ; at and above $w = a$ demand is flat at $\frac{a}{p_1}$. At $a = 2$ and $p_1 = 1$: $w = 1$ is in the corner region and $x_1 = 1$, $x_2 = 0$; $w = 3$ is interior with $x_1 = 2$, $x_2 = 1$; $w = 5$ is interior with $x_1 = 2$, $x_2 = 3$.

(d) (3 points) Absence of income effects in the interior.

- With quasi-linear utility, the numeraire good 2 absorbs all income changes as long as it is being consumed in positive amounts, so demand for the nonnumeraire good 1 depends only on prices, not on income. This is the signature “no wealth effects” feature of quasi-linear preferences, which justifies the use of Marshallian consumer surplus as an exact welfare measure for good 1. At the corner, however, the numeraire has been driven to zero and can no longer buffer income changes, so any extra income spills over into good 1 and generates a nonzero wealth effect.