EconS 501 - Microeconomic Theory I
Homework #2 - Due date: Tuesday, September 22nd, in class.

1. **The linear expenditure system with n goods.** A consumer has Stone-Geary preferences over n goods,

$$u(x) = \prod_{i=1}^{n} (x_i - \gamma_i)^{\alpha_i},$$

where $\gamma_i \geq 0$ is the subsistence quantity of good i, $\alpha_i > 0$ are budget shares of "supernumerary" income summing to 1, and the consumer faces prices $p \gg 0$ and wealth $w > \sum_i p_i \gamma_i$.

   (a) Derive the Walrasian demand $x_i(p, w)$ for every good i. Show that $x_i = \gamma_i + \dfrac{\alpha_i}{p_i}\left(w - \sum_j p_j \gamma_j\right)$, so the consumer first covers subsistence and then splits the remaining income in fixed Cobb-Douglas shares.

   (b) Compute the income elasticity $\eta_i = \dfrac{\partial x_i}{\partial w}\dfrac{w}{x_i}$, and show that good i is a luxury ($\eta_i > 1$), necessity ($\eta_i < 1$), or unit-elastic ($\eta_i = 1$) depending on the sign of γ_i. State the general classification rule.

   (c) *Numerical example.* Set $n = 3$, $\alpha = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\gamma = (2, 1, 0)$, $p = (1, 1, 1)$, and $w = 15$. Compute the Walrasian demands, the income elasticities, and classify each good as a luxury or a necessity.

2. **Integrability with three goods.** An analyst posits the demand system for three goods,

$$x_1(p,w) = \frac{w}{p_1 + p_2}, \qquad x_2(p,w) = \frac{w}{p_1 + 2p_2}, \qquad x_3(p,w) = \frac{w - p_1 x_1 - p_2 x_2}{p_3}.$$

   (a) Verify that the system satisfies Walras' law and is homogeneous of degree zero in (p,w). (Multiplying (p,w) by any $\lambda > 0$ leaves each demand unchanged.)

   (b) The Slutsky matrix has entries $S_{ij} = \dfrac{\partial x_i}{\partial p_j} + \dfrac{\partial x_i}{\partial w} x_j$. Compute S_{12} and S_{21} and show that they are *not* equal, so this demand system is not generated by utility maximization: it fails Slutsky symmetry.

   (c) Now consider the alternative system

$$y_1(p,w) = \frac{\alpha_1 w}{p_1}, \qquad y_2(p,w) = \frac{\alpha_2 w}{p_2}, \qquad y_3(p,w) = \frac{\alpha_3 w}{p_3},$$

   with $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Show that this system does satisfy Slutsky symmetry (verify one representative entry) and identify the underlying utility function.

3. **CES demand: Walrasian, Hicksian, and the Slutsky decomposition.** A consumer has the CES utility $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}$ with $\rho < 1$, $\rho \neq 0$, and elasticity of substitution $\sigma = \dfrac{1}{1-\rho}$.

   (a) Derive the Walrasian demand $x_i(p, w)$ and the indirect utility $v(p, w)$.

   (b) Derive the Hicksian (compensated) demand $h_i(p, u)$ and the expenditure function $e(p, u)$. Verify the identity $x_i\,(p, e(p, u)) = h_i(p, u)$.

   (c) Use the Slutsky equation $\dfrac{\partial x_i}{\partial p_j} = \dfrac{\partial h_i}{\partial p_j} - \dfrac{\partial x_i}{\partial w} x_j$ to decompose the effect of a rise in p_2 on x_1 into a substitution effect (positive) and an income effect (negative). At $\rho = -1$ (so $\sigma = \frac{1}{2}$), $p_1 = p_2 = 1$ and $w = 4$, evaluate both terms and confirm that they sum to $\dfrac{\partial x_1}{\partial p_2}$.

4. **Corner solutions with quasi-linear utility.** A consumer has the quasi-linear utility $u(x_1, x_2) = a \ln x_1 + x_2$, with $a > 0$, and faces prices (p_1, p_2) and wealth w. Good 2 is the numeraire, $p_2 = 1$.

   (a) Solve the interior utility-maximization problem and show that the interior demand is $x_1^{int} = \dfrac{a}{p_1}$, $x_2^{int} = w - a$, requiring $w \geq a$ for $x_2^{int} \geq 0$.

   (b) For $w < a$, the interior solution violates $x_2 \geq 0$ and the consumer is at a corner. Solve the corner problem and show that then $x_1^{cor} = \dfrac{w}{p_1}$, $x_2^{cor} = 0$.

   (c) Combine the two regions to express the Walrasian demand for x_1 as a piecewise function of w. Sketch qualitatively (in words) how x_1 moves with w: constant in the interior region, linear in w in the corner region. Then evaluate the demand at $a = 2$, $p_1 = 1$, and $w = 1, 3, 5$.

   (d) Explain why the interior demand for x_1 is independent of w in the quasi-linear case (there is no income effect on good 1), and why this independence breaks down at the corner.