

EconS 501 - Microeconomic Theory I

Homework #1 - Answer key

1. (20 points) Debreu's representation theorem and lexicographic preferences.

(a) (6 points) The “only if” direction of Debreu.

- Suppose $u : X \rightarrow \mathbb{R}$ represents $\succsim$. For every pair of elements $y \prec x$ in the image $u(X) \subseteq \mathbb{R}$, the open interval $(u(y), u(x))$ contains a rational number, since the rationals $\mathbb{Q}$ are dense in $\mathbb{R}$. Consider a bijection $q : \mathbb{Q} \rightarrow \mathbb{N}$, and for each rational r in the image of u select one point $z_r \in X$ with $u(z_r) = r$ (or more generally $u(z_r)$ close to r); let $Z = \{z_r\}$. This set is countable because $\mathbb{Q}$ is countable, and it is order-dense because for any $x \prec y$ in X the interval $(u(x), u(y))$ contains some rational r , hence some $z_r \in Z$ with $x \prec z_r \prec y$. So representability implies the existence of a countable order-dense set.

(b) (9 points) Lexicographic preferences on $\mathbb{R}_+^2$ have no countable order-dense set.

- Consider, for each $t \in \mathbb{R}_+$, the two points $a_t = (t, 0)$ and $b_t = (t, 1)$ in $X = \mathbb{R}_+^2$. By lexicographic order, $a_t \prec b_t$ (they share the same first coordinate and b_t has larger second coordinate).
- If Z is an order-dense set, then for each t there must exist $z_t \in Z$ with $a_t \prec z_t \prec b_t$. But any bundle strictly between a_t and b_t in the lexicographic order must have first coordinate equal to t (else the lexicographic ordering would place it entirely above b_t or below a_t).
- Therefore, different values of t generate different points $z_t \in Z$ (they have different first coordinates). Since $\mathbb{R}_+$ is uncountable, the set $\{z_t : t \in \mathbb{R}_+\}$ is uncountable, so Z itself must be uncountable, contradicting the countability of Z . Hence no countable order-dense set exists, and by Debreu's theorem lexicographic preferences on $\mathbb{R}_+^2$ admit no utility representation.

(c) (5 points) Extension to $\mathbb{R}_+^n$.

- The same argument applies verbatim: for each $t \in \mathbb{R}_+$, consider $(t, 0, 0, \dots, 0) \prec (t, 1, 0, \dots, 0)$ and note that any bundle strictly between them (in the lexicographic order) must have first coordinate t . An uncountable family of pairs (a_t, b_t) still requires an uncountable order-dense set, which by Debreu's theorem rules out any utility representation on $\mathbb{R}_+^n$ for $n \geq 2$. Lexicographic preferences are, in this sense, “too fine” to be captured by any real-valued utility function, no matter how many goods.

2. (20 points) Local non-satiation, monotonicity, and satiation.

(a) (5 points) Local non-satiation holds iff u has no local maximizer.

- ($\Leftarrow$) Suppose x is a local maximizer of u : there is a neighborhood on which $u(x) \geq u(y)$. Then no nearby y has $u(y) > u(x)$, i.e., no nearby y is strictly preferred, so local non-satiation fails at x .

- ($\Rightarrow$) If x is *not* a local maximizer, every neighborhood of x contains some y with $u(y) > u(x)$, i.e., $y \succ x$, so local non-satiation holds at x . Applying this at every point: $\succsim$ is locally non-satiated if and only if u has no local maximizer.
- (b) (5 points) The bliss-point example.
- $u(x) = -(x_1 - a)^2 - (x_2 - b)^2$ is continuous, so $\succsim$ is complete, transitive, and continuous. Its upper contour sets $\{(x_1 - a)^2 + (x_2 - b)^2 \leq -c\}$ are disks, hence convex, so $\succsim$ is convex. The unique global (and only local) maximizer is the interior bliss point (a, b) , where $u = 0$; by part (a), local non-satiation fails there and *only* there.
- (c) (5 points) The monotone region.
- $\nabla u(x) = (-2(x_1 - a), -2(x_2 - b))$, which is strictly positive in both components iff $x_1 < a$ and $x_2 < b$. Hence the monotone region, where both goods are locally desirable, is exactly the rectangle $[0, a) \times [0, b)$. To the right of $x_1 = a$ (or above $x_2 = b$) the corresponding good is locally a “bad.”
- (d) (5 points) Local non-satiation and the budget.
- *Claim: if $\succsim$ is locally non-satiated and x^* is optimal in the budget set, then $p \cdot x^* = w$.* Suppose $p \cdot x^* < w$. By local non-satiation there is y arbitrarily close to x^* with $y \succ x^*$; taking y close enough, continuity of $p \cdot y$ keeps $p \cdot y < w$, so y is affordable and strictly preferred, contradicting optimality. Hence Walras’ law holds with equality. For the bliss-point utility this fails: if $p \cdot (a, b) < w$, the bundle (a, b) is affordable and is the global maximum, so $x^* = (a, b)$ and $p \cdot x^* = p \cdot (a, b) < w$ —the consumer leaves money unspent precisely because local non-satiation does not hold.

3. (20 points) Semiorders and the Scott-Suppes representation.

- (a) (5 points) Strict preference, indifference, and intransitivity.
- With $u(x) = x$, the rule $x \succsim y \Leftrightarrow x \geq y - \delta$ gives $x \succ y \Leftrightarrow x > y + \delta$ and $x \sim y \Leftrightarrow |x - y| \leq \delta$. Strict preference is transitive: $x > y + \delta$ and $y > z + \delta$ imply $x > z + 2\delta > z + \delta$. Indifference is reflexive and symmetric but intransitive: in the chain $0 \sim \delta \sim 2\delta$ each consecutive pair is within δ , yet $2\delta > 0 + \delta$, so $2\delta \succ 0$. (More generally $0 \sim \delta \sim \dots \sim n\delta$ while $n\delta \succ 0$ for $n \geq 2$.)
- (b) (5 points) The Luce semiorder axioms.
- (i) *Strong intervality.* Suppose $x \succ y$ and $z \succ w$, i.e., $x > y + \delta$ and $z > w + \delta$, but, for contradiction, neither $x \succ w$ nor $z \succ y$, i.e., $x \leq w + \delta$ and $z \leq y + \delta$. Then $x > y + \delta \geq z$ and $z > w + \delta \geq x$, so $x > z$ and $z > x$, impossible. Hence $x \succ w$ or $z \succ y$.
 - (ii) *Semitransitivity.* Suppose $x \succ y$ and $y \succ z$ but, for some w , neither $x \succ w$ nor $w \succ z$, i.e., $x \leq w + \delta$ and $w \leq z + \delta$. Adding, $x \leq z + 2\delta$. But $x > y + \delta > z + 2\delta$, a contradiction. Hence $x \succ w$ or $w \succ z$.

(c) (5 points) No standard representation.

- If a standard representation u^* existed, with $x \succsim y \Leftrightarrow u^*(x) \geq u^*(y)$, then $\sim$ would coincide with the relation $u^*(x) = u^*(y)$, which is transitive. This contradicts the intransitive chain of part (a). Therefore no standard (threshold-zero) utility representation exists, even though the Scott-Suppes representation with threshold δ does.

(d) (5 points) Interpretation.

- Transitivity of indifference is an empirical hypothesis, not a logical necessity. A subject who cannot perceive a difference smaller than a just-noticeable threshold δ reports indifference between adjacent alternatives while strictly preferring the extremes of a long chain—the classical “sorites” pattern (Luce, 1956). Semiorders relax the rational-preference model exactly here, keeping strict preference transitive while letting indifference fail transitivity.

4. (20 points) **WARP holds but SARP fails.**

(a) (4 points) Budgets exhausted.

- Observation 1: $p^1 \cdot x^1 = 4 \cdot 1 + 1 \cdot 4 + 1 \cdot 0 = 8$. Observation 2: $p^2 \cdot x^2 = 1 \cdot 0 + 4 \cdot 1 + 1 \cdot 4 = 8$. Observation 3: $p^3 \cdot x^3 = 1 \cdot 4 + 1 \cdot 0 + 4 \cdot 1 = 8$. Each budget is exhausted (implicitly, $w^i = 8$ in each observation).

(b) (8 points) Direct revealed preferences.

- At $p^1 = (4, 1, 1)$: $p^1 \cdot x^2 = 4(0) + 1(1) + 1(4) = 5 < 8 = w^1$, so x^2 is strictly affordable; $p^1 \cdot x^3 = 4(4) + 1(0) + 1(1) = 17 > 8$, unaffordable. Since x^2 is strictly cheaper than the chosen bundle x^1 , we obtain the *strict* revealed preference $x^1 \succ_R x^2$.
- Proceeding analogously for the other two observations:

$$\mid p^1 \cdot x^2 = 5 < 8 \quad p^2 \cdot x^3 = 5 < 8 \quad p^3 \cdot x^1 = 5 < 8$$

where $p^2 \cdot x^3 = 1(4) + 4(0) + 1(1) = 5$ and $p^3 \cdot x^1 = 1(1) + 1(4) + 4(0) = 5$. In each observation i , the alternative bundle is strictly affordable, so we obtain the strict revealed preferences $x^1 \succ_R x^2$, $x^2 \succ_R x^3$, and $x^3 \succ_R x^1$.

(c) (6 points) WARP holds but SARP fails.

- *WARP* requires that if $x^i \succsim_R x^j$ and the two bundles differ, then x^j is not *directly* revealed preferred to x^i . Checking each pair: from part (b), $x^1 \succ_R x^2$, but x^2 is not revealed preferred to x^1 , since at p^2 we have $p^2 \cdot x^1 = 1(1) + 4(4) + 1(0) = 17 > 8 = w^2$, so x^1 is unaffordable at p^2 . Analogously, $p^3 \cdot x^2 = 1(0) + 1(1) + 4(4) = 17 > 8$ and $p^1 \cdot x^3 = 17 > 8$, so no reverse revelation holds in any pair. Therefore, WARP is satisfied.
- *SARP* rules out cycles in the transitive closure of the direct relation. Here $x^1 \succ_R x^2 \succ_R x^3 \succ_R x^1$ is a length-3 cycle of strict revealed preferences, so SARP fails.

(d) (2 points) Implication.

- With only two goods, WARP is known to be equivalent to SARP (the Ville theorem). With three or more goods, the equivalence breaks: pairwise consistency (WARP) no longer suffices for the existence of a rationalizing preference, because it does not rule out longer cycles. SARP (or, equivalently, the Generalized Axiom of Revealed Preference for finite data) is the necessary-and-sufficient condition for the data to be generated by preference maximization.

5. (20 points) **CES preferences: substitution across n goods.**

(a) (5 points) Marginal rate of substitution and homotheticity.

- With $S \equiv \sum_k \alpha_k x_k^\rho$, the marginal utility of good i is $\frac{\partial u}{\partial x_i} = S^{\frac{1}{\rho}-1} \alpha_i x_i^{\rho-1}$. Hence for any pair

$$MRS_{ij} = \frac{\partial u / \partial x_i}{\partial u / \partial x_j} = \frac{\alpha_i}{\alpha_j} \left(\frac{x_i}{x_j} \right)^{\rho-1} = \frac{\alpha_i}{\alpha_j} \left(\frac{x_j}{x_i} \right)^{1-\rho},$$

which depends only on the ratio x_j/x_i . Since every MRS_{ij} is constant along rays from the origin, the preferences are homothetic.

(b) (5 points) Constant elasticity of substitution.

- Taking logs, $\ln MRS_{ij} = \ln \frac{\alpha_i}{\alpha_j} + (1 - \rho) \ln (x_j/x_i)$, so $d \ln MRS_{ij} = (1 - \rho) d \ln (x_j/x_i)$ and

$$\sigma_{ij} = \frac{d \ln (x_j/x_i)}{d \ln MRS_{ij}} = \frac{1}{1 - \rho} = \sigma,$$

the same constant for every pair (i, j) . This constancy—across all pairs and all bundles—is the defining property of CES preferences.

(c) (5 points) Convexity.

- Move along an indifference surface, raising x_i and lowering x_j so as to hold utility fixed. Then x_i/x_j rises, so $(x_j/x_i)^{1-\rho}$ falls (since $1 - \rho > 0$ for $\rho < 1$), and therefore MRS_{ij} falls. A diminishing marginal rate of substitution is exactly the condition that the indifference surfaces are convex to the origin; equivalently the upper contour sets are convex, so $\succsim$ is convex.

(d) (5 points) Limiting cases.

- *Perfect substitutes* ($\rho \rightarrow 1, \sigma \rightarrow \infty$): $MRS_{ij} \rightarrow \alpha_i/\alpha_j$, constant, so the indifference surfaces are hyperplanes. *Cobb-Douglas* ($\rho \rightarrow 0, \sigma \rightarrow 1$): u becomes a monotone transform of $\prod_k x_k^{\alpha_k}$, with unit elasticity. *Leontief* ($\rho \rightarrow -\infty, \sigma \rightarrow 0$): the goods are consumed in fixed proportions, with L-shaped surfaces. Thus σ indexes the curvature of the indifference surfaces: large σ means nearly flat surfaces and easy substitution, small σ means sharply curved surfaces and rigid proportions.