EconS 501 – Micro Theory I
Recitation #8 – Competitive Markets and Optimal Commodity Taxes[1]

1. **Nicholson–Snyder 12.5 – Long-run equilibrium with scarce entrepreneurs.** Demand for stilts is $Q = 1500 - 50P$, and each stilt-making firm has long-run cost $C(q) = 0.5q^2 - 10q$. Entrepreneurial talent is scarce: the supply of entrepreneurs is $Q^S = 0.25w$ where w is the annual wage. Every firm requires exactly one entrepreneur, so the number of firms equals the number of entrepreneurs hired. Long-run per-firm cost is therefore

$$C(q, w) = 0.5\,q^2 - 10\,q + w.$$

   (a) Find the long-run equilibrium: aggregate quantity, per-firm quantity, price, number of firms, and entrepreneurial wage.

   (b) Repeat when demand shifts to $Q = 2428 - 50P$.

   (c) Compute the increase in entrepreneurial rents between (a) and (b), and verify it equals the change in producer surplus measured along the industry supply curve.

Answer. *Part (a): baseline equilibrium.*

Step 1. Entrepreneur market clearing. With $Q^S = 0.25\,w$ entrepreneurs supplied and n hired,

$$n = 0.25\,w \iff w = 4n.$$

Step 2. Long-run zero-profit condition ($MC = AC$). With per-firm cost $C(q, w) = 0.5q^2 - 10q + w$, marginal cost is $MC = q - 10$ and average cost is $AC = 0.5q - 10 + w/q$. Setting $MC = AC$,

$$q - 10 = 0.5q - 10 + \frac{w}{q} \iff 0.5q = \frac{w}{q} \iff q^2 = 2w.$$

Substituting $w = 4n$ from Step 1,

$$q^2 = 8n \iff q = \sqrt{8n}.$$

Step 3. Firm-level supply from $P = MC$. Perfect competition delivers $P = MC$, so $P = q - 10$, i.e., $q = P + 10$. Aggregate supply is then $Q^S = n(P + 10)$.

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Step 4. Consistency: total output equals aggregate supply. Since $Q = nq = n\sqrt{8n}$ and $Q^S = n(P+10)$, equating gives $\sqrt{8n} = P + 10$, i.e.,

$$P = \sqrt{8n} - 10.$$

Step 5. Market clearing. Setting $Q^D = 1500 - 50P$ equal to $Q = n\sqrt{8n}$ and substituting for P,

$$\begin{aligned} n\sqrt{8n} &= 1500 - 50(\sqrt{8n} - 10) \\ n\sqrt{8n} + 50\sqrt{8n} &= 2000 \\ (n+50)\sqrt{8n} &= 2000. \end{aligned}$$

Try $n = 50$: $(50+50)\sqrt{400} = 100 \cdot 20 = 2000$. ✓ So $n = 50$.

Step 6. Everything else. With $n = 50$,

$$q = \sqrt{400} = 20, \quad Q = nq = 1000, \quad P = 20 - 10 = 10, \quad w = 4 \cdot 50 = 200.$$

Answer. *Part (b): the demand shift.* The market-clearing equation with new demand $Q^D = 2428 - 50P$ becomes $(n+50)\sqrt{8n} = 2928$. Try $n = 72$: $(72+50)\sqrt{576} = 122 \cdot 24 = 2928$. ✓ So

$$n = 72, \quad q = 24, \quad Q = 1728, \quad P = 14, \quad w = 288.$$

Every quantity rises: more firms, more output per firm, higher price, higher entrepreneurial wage.

Answer. *Part (c): the increase in rents equals producer surplus.*

Step 1. Change in entrepreneurial rents. Entrepreneur j (with j-th unit of talent along the entrepreneur supply curve) has reservation wage $4j$; hired at market wage w, she earns rent $w - 4j$. Aggregate rent is $\int_0^n (w - 4j)\,dj = nw - 2n^2 = n(w - 2n)$. So aggregate rent is

$$\begin{aligned} &\text{part (a): } n(w-2n) = 50(200-100) = 5000. \\ &\text{part (b): } n(w-2n) = 72(288-144) = 72 \cdot 144 = 10\,368. \end{aligned}$$

Change in rents $= 5\,368$.

Step 2. Producer surplus along the industry supply curve. The long-run industry supply curve is upward-sloping (entry raises AC via higher w). Between $(P, Q) = (10, 1000)$ and $(14, 1728)$, the producer surplus gain is the trapezoidal area under P but above the supply curve:

$$\Delta PS \approx (14-10) \cdot 1000 + \tfrac{1}{2} \cdot (14-10) \cdot (1728 - 1000) = 4000 + 1456 = 5456.$$

The two numbers (5,368 from the entrepreneur side and 5,456 from the industry side) differ by roughly 1% — the discrepancy reflects the linear approximation to a mildly concave supply curve; exact integration matches to the penny.

Intuitively, the reason for the near-equality is that the entire slope of the long-run industry supply comes from the scarcity of entrepreneurs (firms themselves have U-shaped AC that would give a flat supply otherwise). So the "producer surplus" along the industry supply curve is not surplus to firms — it is rent to the scarce factor.

2. **Nicholson–Snyder 12.9 – Ad valorem taxation.** An ad valorem tax at rate t (e.g., $t = 0.05$ for 5%) creates a wedge $P^D = (1+t)P^S$ between demand and supply prices. Let e_D and e_S be the demand and supply elasticities.
   (a) Show that $\dfrac{d\ln P^D}{dt} = \dfrac{e_S}{e_S - e_D}$ and $\dfrac{d\ln P^S}{dt} = \dfrac{e_D}{e_S - e_D}$.
   (b) Show that the deadweight loss of a small ad valorem tax is $DW \approx -0.5\,\dfrac{e_D\,e_S}{e_S - e_D}\,t^2\,P_0\,Q_0$.
   (c) Compare with the unit-tax case.

Answer. *Part (a): pass-through in elasticities.*

Step 1. Differentiate the price identity at $t = 0$. From $P^D = (1+t)P^S$, $dP^D = (1+t)\,dP^S + P^S\,dt$. Evaluating at $t = 0$ (before any tax) gives

$$dP^D = dP^S + P^S\,dt. \tag{$*$}$$

Step 2. Differentiate market clearing $Q^D(P^D) = Q^S(P^S)$.

$$\frac{\partial Q^D}{\partial P^D}\,dP^D = \frac{\partial Q^S}{\partial P^S}\,dP^S.$$

Substitute dP^D from $(*)$:

$$\frac{\partial Q^D}{\partial P^D}(dP^S + P^S\,dt) = \frac{\partial Q^S}{\partial P^S}\,dP^S \iff \frac{\partial Q^D}{\partial P^D}\,P^S\,dt = \Big(\frac{\partial Q^S}{\partial P^S} - \frac{\partial Q^D}{\partial P^D}\Big)dP^S.$$

Step 3. Rewrite in elasticities. Divide both sides by $P^S\,Q_0$ (with Q_0 the common equilibrium quantity) and use $e_D = (\partial Q^D/\partial P^D)(P/Q)$, $e_S = (\partial Q^S/\partial P^S)(P/Q)$:

$$\frac{dP^S}{dt}\cdot\frac{1}{P^S} = \frac{d\ln P^S}{dt} = \frac{e_D}{e_S - e_D}.$$

Using $(*)$ once more,

$$\frac{d\ln P^D}{dt} = \frac{d\ln P^S}{dt} + 1 = \frac{e_D}{e_S - e_D} + 1 = \frac{e_S}{e_S - e_D}.$$

Since $e_D < 0$ and $e_S > 0$, the numerator of each expression has the "correct" sign: the demander's price rises ($d \ln P^D/dt > 0$) and the supplier's price falls ($d \ln P^S/dt < 0$).

Answer. *Part (b): the deadweight loss.*

Step 1. Triangle area. The tax creates a triangular deadweight loss of base $|dQ|$ and height $P_0\,dt$, so

$$DW \approx -\tfrac{1}{2}\,(P_0\,dt)\,(dQ).$$

The negative sign flags a welfare loss.

Step 2. Express dQ in terms of dP^D. Since $e_D = (dQ/Q_0)/(dP^D/P_0)$, we have $dQ = e_D\,(Q_0/P_0)\,dP^D$.

Step 3. Substitute and use the pass-through from (a).

$$DW \approx -\tfrac{1}{2}\,P_0\,dt \cdot e_D\,\frac{Q_0}{P_0}\,dP^D = -\tfrac{1}{2}\,e_D\,Q_0\,dt \cdot dP^D.$$

From (a), $dP^D/dt = P^D \cdot e_S/(e_S - e_D) \approx P_0\,e_S/(e_S - e_D)$ evaluated at $t = 0$. Plugging in and integrating dt from 0 to t (approximating for small t),

$$DW \approx -\tfrac{1}{2}\,\frac{e_D\,e_S}{e_S - e_D}\,t^2\,P_0\,Q_0.$$

The factor $e_D\,e_S/(e_S - e_D)$ is negative (numerator negative, denominator positive), so DW is negative (welfare loss). The magnitude scales with t^2: doubling the tax quadruples the DWL.

Answer. *Part (c): compared with the unit tax.* The unit tax τ is equivalent to the ad valorem tax t evaluated at $\tau = t\,P^S$. Substituting τ/P^S for t in the formulas above recovers exactly the standard unit-tax pass-through and DWL expressions. The two tax instruments therefore differ only in *how the tax base is measured* (per-unit vs. percentage-of-price), not in the underlying welfare arithmetic. In elasticity form, both deliver the same "inverse harmonic mean" triangle formula.

3. **The Ramsey rule.** A three-good economy ($k = 1, 2, 3$). Every consumer has preferences

$$U(x) = x_1 + g(x_2) + h(x_3),$$

with g, h increasing and strictly concave. Each good $k \neq 1$ is produced with constant returns to scale from good 1 (one unit of good 1 per unit of good k). Good 1 is the numeraire ($p_1 = 1$). Let t_k be a specific commodity tax on good k, so the consumer's price is $q_k = 1 + t_k$.

   (a) Consider two tax schemes t and t'. Show that if $(1 + t'_k) = \theta(1 + t_k)$ for every k and some $\theta > 0$, then the two schemes raise the same revenue.

(b) Argue from (a) that the government can always restrict attention to schemes that leave one good untaxed.

(c) Set $t_1 = 0$; suppose the government must raise revenue R. Find the tax rates on goods 2 and 3 that minimize the welfare loss.

(d) Show that the optimal tax rates are inversely proportional to the demand elasticities.

(e) When should the two goods be taxed at the same rate? Which is taxed more?

Answer. *Part (a): revenue-equivalence of proportional rescalings.*

Step 1. Consumer's budget. With $q_k = 1 + t_k$ for every k, the consumer's budget (writing x_1 negatively as "sold numeraire" if you like) is

$$(1 + t_1)x_1 + (1 + t_2)x_2 + (1 + t_3)x_3 = 0.$$

Step 2. Revenue formula. $R = \sum_k t_k x_k = \sum_k (q_k - 1)x_k = \sum_k q_k x_k - \sum_k x_k$. Since the budget forces $\sum_k q_k x_k = 0$, $R = -\sum_k x_k$.

Step 3. Rescaling and HD0 of demand. Under t' with $1 + t'_k = \theta(1 + t_k)$, the price vector is scaled by θ. Walrasian demand is homogeneous of degree 0 in prices (Proposition 3.D.2), so $x_k(q') = x_k(q)$ for every k. Hence $R' = -\sum_k x'_k = -\sum_k x_k = R$.

Answer. *Part (b): free normalization.* From (a), any tax scheme t is revenue-equivalent to t' with $\theta = 1/(1 + t_{k_0})$ for any chosen good k_0; then $1 + t'_{k_0} = 1$, i.e., $t'_{k_0} = 0$. So without loss we may set the tax on *some* good — conventionally the numeraire, good 1 — to zero. This is not a normative statement; it is just an artifact of the fact that only *relative* tax rates affect real allocations.

Answer. *Part (c): the optimal tax problem.*

Step 1. Consumer choice pins x_2, x_3. With $t_1 = 0$, the budget is $x_1 = -(1 + t_2)x_2 - (1 + t_3)x_3$. Substituting into U,

$$U = -(1 + t_2)x_2 - (1 + t_3)x_3 + g(x_2) + h(x_3).$$

FOCs: $g'(x_2) = 1 + t_2$ and $h'(x_3) = 1 + t_3$. These define $x_2(t_2)$ and $x_3(t_3)$, each depending only on its own tax.

Step 2. Government's problem. Choose (t_2, t_3) to minimize the welfare loss (equivalently, maximize U) subject to raising revenue $R = t_2 x_2(t_2) + t_3 x_3(t_3)$. Lagrangian:

$$\mathcal{L} = g(x_2(t_2)) + h(x_3(t_3)) - (1 + t_2)x_2(t_2) - (1 + t_3)x_3(t_3) + \mu\big[t_2 x_2(t_2) + t_3 x_3(t_3) - R\big].$$

Step 3. FOCs in t_2 and t_3. Using $g'(x_2) = 1+t_2$ (so the envelope terms $g'x_2' - (1+t_2)x_2'$ cancel) and analogously for t_3, the FOCs simplify to

$$-x_k + \mu\big[x_k + t_k x_k'\big] = 0, \qquad k = 2, 3.$$

Step 4. Solve for t_k. Collecting terms in x_k,

$$(\mu - 1)\, x_k + \mu\, t_k\, x_k' = 0 \iff t_k = -\frac{\mu - 1}{\mu} \cdot \frac{x_k}{x_k'}.$$

Since $x_k' < 0$, the factor $-x_k/x_k' > 0$; and $(\mu - 1)/\mu > 0$ whenever $\mu > 1$ (the revenue constraint binds strictly, so raising an extra dollar of revenue is worth more to the government than a dollar of private consumption). Both factors positive, so $t_k > 0$.

Answer. *Part (d): the inverse elasticity rule.*

Step 1. Elasticity form. Define the demand elasticity of good k as $\varepsilon_k = q_k\, x_k'/x_k$ (so $\varepsilon_k < 0$). Then $x_k/x_k' = q_k/\varepsilon_k$, and dividing the Step 4 formula by q_k,

$$\frac{t_k}{q_k} = \frac{t_k}{1 + t_k} = -\frac{\mu - 1}{\mu} \cdot \frac{1}{\varepsilon_k} = \frac{\mu - 1}{\mu} \cdot \frac{1}{|\varepsilon_k|}. \tag{IE}$$

The prefactor $(\mu - 1)/\mu \in (0, 1)$ is the same for every good; only $|\varepsilon_k|$ differs across goods. The right-hand side depends on k only through ε_k. Hence

$$\boxed{\frac{t_k}{1 + t_k} \propto \frac{1}{|\varepsilon_k|}.}$$

This is the *inverse elasticity rule*: optimal tax rates are inversely proportional to the (absolute value of the) demand elasticity.

Step 2. Intuition. A high $|\varepsilon_k|$ means the tax base evaporates quickly when the price rises; taxing an elastic good generates a lot of deadweight loss per dollar of revenue. Inelastic goods (like staples) are cheap to tax — consumers keep buying almost the same amount.

Answer. *Part (e): equal taxation and comparison.* From (IE), t_2 and t_3 satisfy $t_k/(1+t_k) = c/|\varepsilon_k|$ for a common constant c (fixed by the revenue constraint). Thus:

- Both goods should be taxed at the *same* rate iff they have the same demand elasticity, $|\varepsilon_2| = |\varepsilon_3|$.
- Otherwise, the good with the *lower* elasticity (i.e., the more *inelastic* good) should be taxed at the higher rate.

Intuitively, the government wants to raise revenue with the smallest possible distortion; the inverse elasticity rule allocates more tax burden where the behavioral response is weakest. This is a first cousin of the Ramsey (1927) result on optimal taxation in a many-good setting.

4. **An application: log utility.** Consider $U = \ln x_1 + \gamma \ln x_2 - \ell$ under the budget $w\ell = q_1 x_1 + q_2 x_2$.

   (a) Show that both demand elasticities equal -1.

   (b) Setting producer prices $p_1 = p_2 = 1$, show that the inverse elasticity rule implies $t_1/t_2 = q_1/q_2$.

   (c) With $w = 100$ and $\gamma = 1$, find the tax rates that raise $R = 10$.

Answer. *Part (a): unit-elastic demands.*

Step 1. Solve the UMP. Substitute the budget into U: $U = \ln x_1 + \gamma \ln x_2 - (q_1 x_1 + q_2 x_2)/w$. FOCs give $1/x_1 = q_1/w$ and $\gamma/x_2 = q_2/w$, so

$$x_1 = \frac{w}{q_1}, \qquad x_2 = \frac{\gamma w}{q_2}.$$

Step 2. Compute elasticities. With $x_k = (k\text{-th constant})\, w/q_k$,

$$\varepsilon_k = \frac{\partial x_k}{\partial q_k} \cdot \frac{q_k}{x_k} = \left(-\frac{\text{const} \cdot w}{q_k^2}\right) \cdot \frac{q_k}{\text{const} \cdot w/q_k} = -1.$$

Both goods have unit-elastic demand.

Answer. *Part (b): $t_1/t_2 = q_1/q_2$.* With $\varepsilon_1 = \varepsilon_2 = -1$, the inverse elasticity rule (IE) gives

$$\frac{t_1}{q_1} = \frac{\mu - 1}{\mu} \cdot \frac{1}{|\varepsilon_1|} = \frac{\mu - 1}{\mu} \cdot \frac{1}{|\varepsilon_2|} = \frac{t_2}{q_2} \iff \frac{t_1}{t_2} = \frac{q_1}{q_2}.$$

Equal elasticities force the tax to be a common proportion of the consumer price on both goods.

Answer. *Part (c): numerical revenue target.*

Step 1. Revenue equation. $R = t_1 x_1 + t_2 x_2 = t_1 \cdot (w/q_1) + t_2 \cdot (\gamma w/q_2)$. With $\gamma = 1$ and $w = 100$,

$$10 = \frac{100\, t_1}{q_1} + \frac{100\, t_2}{q_2}.$$

Step 2. Apply the rule from (b). $t_1/t_2 = q_1/q_2$ implies $t_1 = (q_1/q_2) t_2$. Substituting,

$$10 = \frac{100\, (q_1/q_2) t_2}{q_1} + \frac{100\, t_2}{q_2} = \frac{100\, t_2}{q_2} \cdot 2 = \frac{200\, t_2}{q_2} = \frac{200\, t_2}{1 + t_2}.$$

Solving, $10(1+t_2) = 200\,t_2$, i.e., $t_2 = 1/19$. By symmetry, $t_1 = 1/19$.

Intuitively, when both demands are unit-elastic, the optimal-tax rule collapses to *uniform taxation*: same rate on both goods. This is the special case where the standard "tax the inelastic good more" advice does not apply because no good is more inelastic than the other.

5. **An application: power utility.** Consider $U = x_1^{\alpha_1} + x_2^{\alpha_2} - \ell$.

   (a) Show that Walrasian demand is $x_k = \big(\alpha_k w/q_k\big)^{1/(1-\alpha_k)}$ and that $\varepsilon_k = -1/(1-\alpha_k)$.

   (b) Setting $p_k = 1$, show that the inverse elasticity rule implies

$$\frac{1}{t_2} = \frac{\alpha_2 - \alpha_1}{1-\alpha_2} + \frac{1-\alpha_1}{1-\alpha_2} \cdot \frac{1}{t_1}.$$

   (c) With $w = 100$, $\alpha_1 = 0.75$, $\alpha_2 = 0.5$, find the tax rates that raise (i) $R = 10$ and (ii) $R = 300$.

   (d) Report the proportional reduction in demand for each good relative to the no-tax situation. Comment.

Answer. *Part (a): demands and elasticities.*

Step 1. Substitute budget into utility. $\ell = (q_1 x_1 + q_2 x_2)/w$, so

$$U = x_1^{\alpha_1} + x_2^{\alpha_2} - \frac{q_1 x_1 + q_2 x_2}{w}.$$

Step 2. FOCs. $\alpha_k\, x_k^{\alpha_k - 1} = q_k/w$, so

$$x_k = \left(\frac{\alpha_k w}{q_k}\right)^{1/(1-\alpha_k)}.$$

Step 3. Elasticity. $\ln x_k = (1-\alpha_k)^{-1}\big[\ln(\alpha_k w) - \ln q_k\big]$, so $\partial \ln x_k/\partial \ln q_k = -1/(1-\alpha_k)$. That is, $\varepsilon_k = -1/(1-\alpha_k)$.

Answer. *Part (b): the tax-ratio relationship.* Applying the inverse-elasticity rule (IE),

$$\frac{t_k}{q_k} = -\frac{1/\varepsilon_k}{1+\mu^{-1}} \iff (1-\alpha_k)\cdot\frac{t_k}{1+t_k} \text{ is constant across } k.$$

Writing this constant equal for $k=1$ and $k=2$ and solving for $1/t_2$ delivers the stated relationship.

Answer. *Part (c): numerical solutions.* With $\alpha_1 = 0.75$, $\alpha_2 = 0.5$, the relationship in (b) becomes $1/t_2 = -0.5 + 0.5/t_1$, i.e., $t_2 = 2t_1/(1 - t_1)$. Plugging into the revenue constraint $R = t_1 x_1 + t_2 x_2$ (with the demand formulas from (a)),

$$R = t_1\Big(\frac{75}{1+t_1}\Big)^4 + t_2\Big(\frac{25}{1+t_2}\Big)^2.$$

Substituting t_2 and simplifying,

$$R = 625\Big[\frac{50\,625\,t_1}{(1+t_1)^4} + \frac{16\,t_1^4}{(1-t_1)^2}\Big].$$

Solving numerically:

(i) $R = 10$: $t_1 \approx 0.003$, $t_2 \approx 0.006$.

(ii) $R = 300$: $t_1 \approx 0.181$, $t_2 \approx 0.443$.

For very small revenue targets both taxes are tiny; as R rises, the more inelastic good (good 2, since $\alpha_2 < \alpha_1$ makes $|\varepsilon_2| = 1/(1 - 0.5) = 2 < 4 = |\varepsilon_1|$) is taxed at the higher rate. (The revenue curve has a maximum around $t_1 \approx 0.4$; a Laffer property kicks in beyond that.)

Answer. *Part (d): proportional demand reductions.* Baseline demands (no tax, $q_k = 1$) are $x_k^0 = (\alpha_k w)^{1/(1-\alpha_k)}$. Tax-distorted demands are $x_k(t_k) = (\alpha_k w/(1 + t_k))^{1/(1-\alpha_k)}$. The proportional reduction is

$$\frac{x_k^0 - x_k}{x_k^0} = 1 - (1 + t_k)^{-1/(1-\alpha_k)}.$$

R	x_1^0	x_1 (reduction)	x_2^0	x_2 (reduction)
0	3 164	3 164 (0%)	25	25 (0%)
10	3 164	3 125 (1.23%)	25	24.69 (1.24%)
300	3 164	1 624 (48.7%)	25	12.00 (52.0%)

Intuitively, the optimal Ramsey taxes reduce demand for the two goods by *almost the same* proportion at every revenue target — roughly 1.24% for $R = 10$ and roughly 50% for $R = 300$. That is the essence of the Ramsey rule: even when elasticities differ, the tax structure that minimizes deadweight loss is one that equates the proportional demand reduction across taxed goods.

6. **Complement (restrictions on a candidate profit function).** Exercises 1–5 above analyzed *consumers* facing taxes; this complement shifts to the *firm* side of the competitive market to check when a candidate profit function is even admissible. A price-taking firm produces one output $q \geq 0$ from two inputs $z_1, z_2 \geq 0$ with a strictly

increasing, concave production function f; input prices are r_1, r_2, the output price is p. Suppose an econometrician hands you the estimated profit function

$$\pi(r_1, r_2, p) = \left(\frac{p}{r_1^{\alpha}\, r_2^{\beta}}\right)^{\gamma}, \qquad \alpha, \beta, \gamma > 0.$$

(a) What restriction on (α, β, γ) makes π satisfy price homogeneity?

(b) Derive the implied output supply and state the parameter restrictions it needs.

(c) Derive the implied factor demands and state the parameter restrictions they need.

Answer. *Part (a): homogeneity of degree* 1.

Step 1. Standard property. Every profit function is HD1 in all prices — doubling p, r_1, r_2 doubles π.

Step 2. Check the candidate. $\pi(\lambda r_1, \lambda r_2, \lambda p) = (\lambda p/(\lambda^{\alpha+\beta} r_1^{\alpha} r_2^{\beta}))^{\gamma} = \lambda^{\gamma(1-\alpha-\beta)}\pi(r, p)$. HD1 requires $\gamma(1 - \alpha - \beta) = 1$, i.e.,

$$\boxed{\gamma = \frac{1}{1 - \alpha - \beta},}$$

which is well-defined and positive iff $\alpha + \beta < 1$. Under $\alpha + \beta \in [0, 1)$, $\gamma \geq 1$ automatically.

Answer. *Part (b): output supply via Hotelling's lemma.*

Step 1. Hotelling. $q = \partial\pi/\partial p = \gamma\, p^{\gamma-1}\, r_1^{-\alpha\gamma}\, r_2^{-\beta\gamma}$.

Step 2. Law of supply. Supply should be increasing in own price: $\partial q/\partial p = \gamma(\gamma - 1)\, p^{\gamma-2}\, r_1^{-\alpha\gamma}\, r_2^{-\beta\gamma} \geq 0$ iff $\gamma \geq 1$, which the HD1 restriction from (a) already delivers under $\alpha + \beta < 1$.

Step 3. Homogeneity of supply. Supply is HD0 in prices when the same restriction from (a) holds (each price appears with exponents summing to zero across p, r_1, r_2).

Answer. *Part (c): factor demands via Hotelling's lemma.*

Step 1. Hotelling for inputs. $z_\ell = -\partial\pi/\partial r_\ell$. For good 1,

$$z_1 = -\frac{\partial\pi}{\partial r_1} = \alpha\gamma\, p^{\gamma}\, r_1^{-\alpha\gamma-1}\, r_2^{-\beta\gamma}.$$

Analogously for z_2.

Step 2. Positivity and downward-sloping demand. $z_1 > 0$ iff $\alpha > 0$; $\partial z_1/\partial r_1 = \alpha\gamma(-\alpha\gamma - 1)\, p^{\gamma} r_1^{-\alpha\gamma-2} r_2^{-\beta\gamma} < 0$ automatically (negative coefficient times positive powers).

Step 3. Homogeneity of demand. HD0 in prices holds under the same restriction from (a).

Intuitively, HD1 of the profit function is just the "no money illusion" requirement: rescaling all prices proportionally leaves the firm's real problem unchanged. The single restriction $\gamma = 1/(1-\alpha-\beta)$ then forces supply and factor demands to be HD0 and to slope the right way. Any candidate profit function that fails HD1 cannot possibly come from a well-defined competitive firm — a useful check whenever an econometrician hands you a fitted form.