EconS 501 – Micro Theory I
Recitation #7 – Production Theory II[1]

1. **MWG 5.C.11.** Show that $\dfrac{\partial z_\ell(w,q)}{\partial q} > 0$ if and only if the marginal cost at q is increasing in w_ℓ.

   Recall. By Shephard's lemma (Proposition 5.C.2(vi)), the conditional factor demand for input ℓ is $z_\ell(w,q) = \partial c(w,q)/\partial w_\ell$. Marginal cost is $MC(w,q) = \partial c(w,q)/\partial q$.

   Answer.

   Step 1. Cross-partial equivalence via Young's theorem. Assume c is twice continuously differentiable. Then

   $$\frac{\partial z_\ell(w,q)}{\partial q} = \frac{\partial}{\partial q}\Big(\frac{\partial c(w,q)}{\partial w_\ell}\Big) = \frac{\partial^2 c(w,q)}{\partial q\,\partial w_\ell} = \frac{\partial^2 c(w,q)}{\partial w_\ell\,\partial q} = \frac{\partial}{\partial w_\ell}\Big(\frac{\partial c(w,q)}{\partial q}\Big) = \frac{\partial MC(w,q)}{\partial w_\ell},$$

   where the middle equality is Young's theorem (equality of mixed partials for a C^2 function).

   Step 2. Read off the equivalence. The chain of equalities in Step 1 shows

   $$\frac{\partial z_\ell(w,q)}{\partial q} > 0 \iff \frac{\partial MC(w,q)}{\partial w_\ell} > 0.$$

   That is, the firm uses more of input ℓ as output rises *iff* making input ℓ more expensive raises the marginal cost of producing q.

   Intuitively, the sign of the cross-partial of the cost function has two equally valid readings. Reading vertically, an extra unit of output pulls in more of input ℓ; reading horizontally, an extra dollar on input ℓ's price makes producing the marginal unit more expensive. Young's theorem says these two readings must agree.

2. **MWG 5.C.13.** A price-taking firm produces q from inputs z_1, z_2 via a differentiable, concave production function $f(z_1, z_2)$. The output price is $p > 0$ and input prices are $(w_1, w_2) \gg 0$. Two unusual features: (i) the firm maximizes *revenue* rather than profit (the manager wants bigger sales than any rival), and (ii) the firm is *cash constrained* to spend at most C dollars on inputs.

   An econometrician tells you that repeated observation of the firm's revenue at various (p, w_1, w_2, C) has pinned down

   $$R(p, w_1, w_2, C) = p\big[\gamma + \ln C - \delta \ln w_1 - (1-\delta)\ln w_2\big],$$

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where γ, δ are known scalars. What is the firm's use of input z_1 at (p, w, C)?

Answer. *Set up the firm's problem and read R as a Roy-style indirect revenue function.*

Step 1. The optimization.

$$\max_{z_1, z_2 \geq 0} \; p\, f(z_1, z_2) \qquad \text{s.t.} \qquad w_1 z_1 + w_2 z_2 \leq C.$$

The value function $R(p, w, C)$ plays the role of the indirect utility function $v(p, w)$ in Section 3.D: the firm's "wealth" is C, the "prices" are (w_1, w_2), and the "utility" is revenue $p\, f(z)$.

Step 2. Apply the Roy-style identity. Proposition 3.G.4 says $x_\ell(p, w) = -(\partial v/\partial p_\ell)/(\partial v/\partial w)$. Transposed to our production problem with C playing the role of wealth and w_ℓ the role of the ℓ-th price,

$$z_\ell(p, w, C) = -\frac{\partial R/\partial w_\ell}{\partial R/\partial C}.$$

Step 3. Compute the two partials. From the given form of R,

$$\frac{\partial R}{\partial w_1} = -\frac{p\,\delta}{w_1}, \qquad \frac{\partial R}{\partial C} = \frac{p}{C}.$$

Step 4. Assemble. Plug into the Roy-style formula:

$$z_1(p, w, C) = -\frac{-p\delta/w_1}{p/C} = \frac{\delta\, C}{w_1}.$$

Analogously, $z_2(p, w, C) = (1 - \delta)\, C/w_2$. The firm's "expenditure share" on input 1 is δ, and on input 2 is $1 - \delta$; both are independent of the output price p, which is exactly what a Cobb-Douglas-in-inputs revenue function should deliver.

Intuitively, the cash-constrained revenue-maximizing firm behaves like a Cobb-Douglas consumer: it splits its "wealth" C across inputs in fixed shares $(\delta, 1 - \delta)$, using less of any input that becomes more expensive. Roy's identity handles the derivation without ever having to reconstruct the underlying f.

3. **MWG 5.D.4.** A firm with M outputs $q = (q_1, \ldots, q_M)$ has cost function $C(q)$. Say $C(\cdot)$ is *subadditive* if for every q there is no way to break production q among several firms (each with cost function C) at a lower total cost — that is, no collection $\{q_j\}_{j=1}^J$ with $\sum_j q_j = q$ and $\sum_j C(q_j) < C(q)$. When C is subadditive the industry is a *natural monopoly*: production is cheapest done by one firm.

   (a) Single output ($M = 1$). Show that if C has *decreasing average cost*, then C is subadditive.

(b) Multiple outputs ($M > 1$). Show by counterexample that the following extension is *not* sufficient: C has *decreasing ray average cost* if for every $q \in \mathbb{R}^M_+$, $C(q)/1 > C(kq)/k$ for every $k > 1$.

Answer. *Part (a): DAC implies subadditivity when $M = 1$.*

Step 1. What DAC gives us. Decreasing average cost with $C(0) = 0$ means: for any $\tilde{q} \leq q$, $\dfrac{C(\tilde{q})}{\tilde{q}} \geq \dfrac{C(q)}{q}$, i.e., $C(\tilde{q}) \geq \dfrac{\tilde{q}}{q} C(q)$.

Step 2. Apply DAC to each split output. Suppose $q = \sum_{j=1}^{J} q_j$ with $q_j \leq q$. Applying Step 1 with $\tilde{q} = q_j$,

$$C(q_j) \geq \frac{q_j}{q} C(q).$$

Step 3. Sum over j. $\sum_j C(q_j) \geq \sum_j \dfrac{q_j}{q} C(q) = \dfrac{\sum_j q_j}{q} C(q) = C(q)$. So splitting production always weakly raises total cost; C is subadditive.

Intuitively, decreasing average cost means one big firm has a lower per-unit cost than each of several smaller firms, so consolidation weakly helps — the definition of a natural monopoly.

Answer. *Part (b): decreasing ray AC does not imply subadditivity.*

Step 1. Candidate counterexample. Take $M = 2$ and

$$C(q_1, q_2) = \sqrt{\min\{q_1, q_2\}}.$$

Step 2. Verify decreasing ray AC. Along any ray $q = t\bar{q}$ (with $\bar{q}$ fixed and $t > 0$), $C(t\bar{q}) = \sqrt{t \min\{\bar{q}_1, \bar{q}_2\}} = \sqrt{t} \cdot \sqrt{\min\{\bar{q}\}}$, so

$$\frac{C(t\bar{q})}{t} = \frac{1}{\sqrt{t}} \sqrt{\min\{\bar{q}\}},$$

which is strictly decreasing in t. So C has decreasing ray average cost.

Step 3. Show subadditivity fails. Take $q^1 = (1, 8)$, $q^2 = (8, 1)$, and $q = q^1 + q^2 = (9, 9)$. Then

$$\begin{aligned} C(q^1) &= \sqrt{\min\{1, 8\}} = \sqrt{1} = 1, \\ C(q^2) &= \sqrt{\min\{8, 1\}} = \sqrt{1} = 1, \\ C(q) &= \sqrt{\min\{9, 9\}} = \sqrt{9} = 3. \end{aligned}$$

So $C(q^1) + C(q^2) = 2 < 3 = C(q)$: splitting the joint order into a firm making $(1, 8)$ and a firm making $(8, 1)$ costs less than one firm making $(9, 9)$. Subadditivity fails.

Intuitively, the "ray" condition compares $C(q)$ only with proportional rescalings of q. The counterexample exploits a *lopsided* split: two firms each producing very unequal outputs pay only for the *smaller* coordinate (the one that enters min), whereas one firm producing the balanced sum pays for the full amount. Ray comparisons miss this because they never sample lopsided allocations.

4. **MWG 5.E.5 – Output distribution between two plants.** A firm owns two plants producing the same good, both with average-cost functions of the form

$$AC_j(q_j) = \alpha + \beta_j\, q_j, \qquad j \in \{1,2\},$$

where α is common but β_j may differ across plants (a plant with smaller $|\beta_j|$ has flatter average cost). Given aggregate output $q = q_1 + q_2$ (assumed to satisfy $q < \alpha / \max_j |\beta_j|$, ensuring interior costs stay positive), find the cost-minimizing split (q_1^*, q_2^*) when

(a) $\beta_j > 0$ for both plants (increasing marginal cost everywhere);

(b) $\beta_j < 0$ for both plants (decreasing marginal cost everywhere);

(c) $\beta_j > 0$ for some plants and $\beta_j < 0$ for others.

Recall. Total cost per plant is $TC_j(q_j) = AC_j(q_j) \cdot q_j = (\alpha + \beta_j q_j)q_j = \alpha q_j + \beta_j q_j^2$, so marginal cost is $MC_j(q_j) = \alpha + 2\beta_j q_j$.

Answer. *Part (a): both* $\beta_j > 0$.

Step 1. Set up. Minimize $TC_1(q_1) + TC_2(q_2)$ subject to $q_1 + q_2 = q$. Equivalently, maximize $\pi_1(q_1) + \pi_2(q_2) = pq_1 - TC_1(q_1) + pq_2 - TC_2(q_2)$ subject to the same constraint.

Step 2. FOCs (Lagrangian). Let $\mathcal{L} = TC_1(q_1) + TC_2(q_2) - \mu(q_1 + q_2 - q)$. Then

$$\frac{\partial \mathcal{L}}{\partial q_1} = MC_1(q_1) - \mu = 0, \qquad \frac{\partial \mathcal{L}}{\partial q_2} = MC_2(q_2) - \mu = 0.$$

Both FOCs pin the multiplier μ to the common marginal cost: $MC_1(q_1^*) = MC_2(q_2^*)$.

Step 3. Solve. Using $MC_j(q_j) = \alpha + 2\beta_j q_j$, the equality of marginal costs gives $2\beta_1 q_1 = 2\beta_2 q_2$, i.e., $q_2 = (\beta_1/\beta_2) q_1$. Substituting into the constraint,

$$q_1\left(1 + \frac{\beta_1}{\beta_2}\right) = q \iff q_1^* = \frac{\beta_2}{\beta_1 + \beta_2}\, q, \qquad q_2^* = \frac{\beta_1}{\beta_1 + \beta_2}\, q.$$

The plant with the *smaller* β_j (flatter MC) produces *more.* The FOC condition "equal marginal cost across plants" generalizes to J plants as $q_j^* = \dfrac{q}{\beta_j \sum_h 1/\beta_h}$.

Answer. *Part (b): both* $\beta_j < 0$.

Step 1. Average cost is decreasing. If $\beta_j < 0$, $AC_j(q_j) = \alpha + \beta_j q_j$ is strictly decreasing on the interior. So concentrating more output at any plant lowers its per-unit cost.

Step 2. Corner solution: concentrate everything on one plant. Cost is minimized by allocating all output to the plant with the *most negative* β_j (the fastest-decreasing AC):

$$q_{j^*}^* = q,\ q_k^* = 0 \text{ for } k \neq j^*, \qquad j^* = \arg\min_j \beta_j.$$

Even if β_1 and β_2 are both very close to 0 (nearly flat AC), whichever is more negative concentrates all production.

Step 3. Sanity check via the interior FOC. The equal-MC rule $\alpha + 2\beta_1 q_1 = \alpha + 2\beta_2 q_2$ would give $q_2 = (\beta_1/\beta_2)q_1$, still positive when both $\beta_j < 0$ (a ratio of negatives is positive). But this interior point is a *maximum* of total cost, not a minimum, since the Hessian of TC is negative when both $\beta_j < 0$. The genuine minimum lies at a corner.

Answer. *Part (c): mixed signs.*

Step 1. Same corner logic. A plant with $\beta_j < 0$ has a decreasing AC and dominates any plant with $\beta_j > 0$ (which has *increasing* AC). Concentrating output at the most negative-β plant strictly lowers total cost.

Step 2. Optimal allocation. Assign all output to the plant with the most negative β_j:

$$q_{j^*}^* = q, \quad q_k^* = 0 \text{ for } k \neq j^*, \qquad j^* = \arg\min_j \beta_j.$$

This is the same corner rule as in part (b): whenever *any* plant has decreasing AC, use it.

Intuitively, the equal-MC rule from part (a) is a first-order condition that only bites when both plants have increasing MC (so total cost is convex and the interior FOC is a genuine minimum). When any plant has decreasing MC/AC, the problem becomes non-convex and the optimum jumps to a corner.

5. **Zero profit under CRS.** A firm has production function $q = f(z_1, \ldots, z_n)$ with constant returns to scale. Show that if the firm pays each input its exact marginal value product, profits are zero.

Answer.

Step 1. CRS ⇒ homogeneity of degree 1. By Exercise 1 of Recitation #6, CRS is equivalent to f being homogeneous of degree 1.

Step 2. Apply Euler's theorem. For a differentiable function f homogeneous of degree 1,

$$\sum_{i=1}^{n} \frac{\partial f}{\partial z_i} z_i = f(z_1, \ldots, z_n). \tag{Euler}$$

Step 3. The marginal-value-product rule. If the firm pays input i its exact marginal value product, then $w_i = p \cdot \partial f/\partial z_i$, equivalently $\partial f/\partial z_i = w_i/p$.

Step 4. Substitute into Euler. Plugging into the Euler identity,

$$\sum_{i=1}^{n} \frac{w_i}{p}\, z_i = f(z_1, \ldots, z_n) \iff \sum_{i=1}^{n} w_i z_i = p\, f(z_1, \ldots, z_n).$$

Step 5. Read off zero profit. The left-hand side is the firm's total factor bill (total cost). The right-hand side is its total revenue. Their difference — profit — is

$$\pi = p\, f(z) - \sum_{i=1}^{n} w_i z_i = 0.$$

Intuitively, under CRS all inputs together explain *all* of output — there is no "residual claimant" role for the firm itself. Paying each input its marginal contribution therefore exhausts revenue, leaving no economic profit. This is the classic Euler-theorem argument behind the "adding-up problem" in neoclassical distribution theory.

6. **Complement (a joint expenditure function).** Exercise 4 above aggregated a single output across two *plants*; this complement aggregates a single expenditure across two *consumers*. Both are "joint minimization" problems that inherit properties from the standard single-agent versions.

   Two agents (call them Ada and Ben) have continuous, locally nonsatiated utilities $u_A, u_B : \mathbb{R}^L_+ \to \mathbb{R}$. Define the *joint expenditure function*

   $$e(p, u, v) \equiv \min_{x \in \mathbb{R}^L_+} \; p \cdot x \qquad \text{s.t.} \qquad u_A(x) \geq u \;\text{ and }\; u_B(x) \geq v.$$

   That is: find the cheapest bundle at prices p that delivers utility at least u to Ada *and* at least v to Ben simultaneously. For each of the properties below, give a proof or a counterexample.

   (a) Is e homogeneous of degree 1 in p?

   (b) Is e non-decreasing in each p_ℓ? Strictly increasing in u?

   (c) Is e concave in p?

   Answer. *Part (a): homogeneity of degree* 1 *in* p. *Yes.*

   Step 1. Feasible set is invariant under p-scaling. The constraints $u_A(x) \geq u$ and $u_B(x) \geq v$ do not depend on prices, so the feasible set of x's is the same whether prices are p or αp (for any $\alpha > 0$).

Step 2. Objective scales linearly. At any x in the feasible set, $(\alpha p) \cdot x = \alpha(p \cdot x)$. So minimizing $(\alpha p) \cdot x$ over the fixed feasible set gives α times the minimum of $p \cdot x$:

$$e(\alpha p, u, v) = \alpha\, e(p, u, v).$$

Standard expenditure-function homogeneity carries over unchanged.

Answer. *Part (b): monotonicity.*

Non-decreasing in p_ℓ: yes.

Step 1. If $p' \geq p$ coordinatewise, then $p' \cdot x \geq p \cdot x$ for every $x \geq 0$; hence the minimum over the fixed feasible set weakly rises. So $e(p', u, v) \geq e(p, u, v)$.

Strictly increasing in u: not necessarily.

Step 2. The catch: one constraint may bind while the other is slack. Suppose at the optimum x^*, Ben's constraint $u_B(x^*) = v$ binds but Ada's is slack, $u_A(x^*) > u$. Then raising u slightly (still keeping $u \leq u_A(x^*)$) does not change the feasible set at the boundary and hence does not change x^* or the minimum cost.

Step 3. Counterexample. Take $L = 1$, $u_A(x) = x$, $u_B(x) = x$ (identical agents), $u = 1$, $v = 2$. The joint constraint reduces to $x \geq \max\{u, v\} = 2$, so $x^* = 2$ and $e = 2p_1$. Now raise u to 1.5: still $\max\{1.5, 2\} = 2$, so x^* and e are unchanged. Hence e is *not* strictly increasing in u; it is merely non-decreasing.

Answer. *Part (c): concavity in p. Yes.*

Step 1. Write e as an infimum of linear functions. For each fixed x in the (fixed) feasible set, the map $p \mapsto p \cdot x$ is linear in p. Then

$$e(p, u, v) = \inf_{x \in F(u,v)} p \cdot x,$$

where $F(u, v) = \{x \geq 0 : u_A(x) \geq u,\ u_B(x) \geq v\}$ is independent of p.

Step 2. Infimum of linear functions is concave. For any p, p' and $\lambda \in [0, 1]$, and any $x \in F$,

$$\big(\lambda p + (1-\lambda)p'\big) \cdot x = \lambda(p \cdot x) + (1-\lambda)(p' \cdot x) \geq \lambda\, e(p, u, v) + (1-\lambda)\, e(p', u, v).$$

Taking the infimum of the left side over $x \in F$,

$$e\big(\lambda p + (1-\lambda)p',\ u,\ v\big) \geq \lambda\, e(p, u, v) + (1-\lambda)\, e(p', u, v).$$

So e is concave in p.

Intuitively, the joint expenditure function inherits the usual expenditure-function properties (HD1, monotone, concave in p) because the two utility floors merely carve out a fixed feasible set — one that does not react to prices. The one caveat is that a slack constraint kills strict monotonicity in that agent's utility target: if Ben's target is the binding one at the optimum, Ada's target could be raised a bit without changing anything. Compare with the multi-plant cost-minimization structure of Exercise 4: there too, joint minimization inherits the properties of the single-firm cost function, with corner-solution phenomena (all output at one plant) arising when a single "binding" technology dominates.