EconS 501 – Micro Theory I
Recitation #6 – Production Theory I[1]

1. **MWG 5.B.2.** Let $f : \mathbb{R}_+^{L-1} \to \mathbb{R}_+$ be the production function of a single-output technology, and let $Y = \{(-z, q) \in \mathbb{R}^L : z \in \mathbb{R}_+^{L-1},\ q \le f(z)\}$ be the associated production set. Show that

   $$Y \text{ exhibits constant returns to scale (CRS)} \iff f \text{ is homogeneous of degree 1.}$$

   Recall. Y exhibits CRS if $(-z, q) \in Y$ and $\alpha \ge 0$ imply $(-\alpha z, \alpha q) \in Y$. Homogeneity of degree 1 means $f(\alpha z) = \alpha f(z)$ for every $z \ge 0$ and every $\alpha \ge 0$.

   Answer. We prove the two directions in turn.

   Direction 1: CRS $\Rightarrow$ homogeneity of degree 1.

   Step 1. Feasibility of $(-z, f(z))$. By definition of Y, $(-z, f(z)) \in Y$ for every $z \ge 0$.

   Step 2. Scale by $\alpha > 0$. CRS gives $(-\alpha z, \alpha f(z)) \in Y$, which means $\alpha f(z) \le f(\alpha z)$.

   Step 3. Reverse the inequality. Apply the same argument to the bundle αz with scaling factor $1/\alpha$:

   $$\frac{1}{\alpha} f(\alpha z) \le f\Big(\frac{1}{\alpha} \cdot \alpha z\Big) = f(z) \iff f(\alpha z) \le \alpha f(z).$$

   Combining Steps 2 and 3, $f(\alpha z) = \alpha f(z)$.

   Direction 2: homogeneity of degree 1 $\Rightarrow$ CRS.

   Step 1. Take an arbitrary feasible point. Let $(-z, q) \in Y$ and $\alpha \ge 0$. By definition of Y, $q \le f(z)$.

   Step 2. Scale outputs and use homogeneity. $\alpha q \le \alpha f(z) = f(\alpha z)$, where the equality uses degree-1 homogeneity.

   Step 3. Read off feasibility. Since $\alpha q \le f(\alpha z)$, the pair $(-\alpha z, \alpha q)$ satisfies the defining inequality of Y; hence $(-\alpha z, \alpha q) \in Y$. So Y satisfies CRS.

   Intuitively, degree-1 homogeneity of f is exactly the statement that "doubling inputs doubles output." That is the same as CRS on the level of the production set: a firm with these primitives can freely rescale any feasible plan.

[1] Felix Munoz-Garcia, School of Economic Sciences, Washington State University, 103H Hulbert Hall, Pullman, WA. Tel. 509-335-8402. Email: `fmunoz@wsu.edu`.

2. **MWG 5.B.3.** For the same single-output setting, show that Y is convex if and only if f is concave.

Answer. Again both directions.

Direction 1: Y convex $\Rightarrow$ f concave.

Step 1. Take two input vectors. Pick $z, z' \in \mathbb{R}_+^{L-1}$ and $\lambda \in [0,1]$. Then $(-z,\ f(z)) \in Y$ and $(-z',\ f(z')) \in Y$.

Step 2. Convex-combine in Y. By convexity of Y,

$$\lambda(-z,\ f(z)) + (1-\lambda)(-z',\ f(z')) = \big(-[\lambda z + (1-\lambda)z'],\ \lambda f(z) + (1-\lambda)f(z')\big)\ \in\ Y.$$

Step 3. Read off concavity. Membership in Y requires the second coordinate to be $\leq f$ of the first coordinate. Writing $z'' = \lambda z + (1-\lambda)z'$,

$$\lambda f(z) + (1-\lambda)f(z')\ \leq\ f(z'').$$

This is the definition of concavity of f.

Direction 2: f concave $\Rightarrow$ Y convex.

Step 1. Take two feasible plans. Let $(-z, q), (-z', q') \in Y$ and $\lambda \in [0,1]$. Then $q \leq f(z)$ and $q' \leq f(z')$.

Step 2. Convex-combine outputs and apply concavity. $\lambda q + (1-\lambda)q' \leq \lambda f(z) + (1-\lambda)f(z') \leq f(\lambda z + (1-\lambda)z')$, where the last inequality uses concavity of f.

Step 3. Read off feasibility. The pair $\big(-[\lambda z + (1-\lambda)z'],\ \lambda q + (1-\lambda)q'\big)$ satisfies the defining inequality of Y, so it belongs to Y. Hence Y is convex.

Intuitively, concavity of f (decreasing returns) says any mixture of two feasible input bundles produces at least the mixture of their outputs; that is exactly what makes the "below-the-frontier" set Y closed under mixing.

3. **The CES production function.** Consider

$$q = f(z_1, z_2) = A\big[\rho\, z_1^{\rho} + (1-\rho)\, z_2^{\rho}\big]^{1/\rho}, \qquad A > 0,\ 0 < \rho < 1$$

(using a distribution-parameter convention where the same ρ weights the two inputs and appears in the exponent; the exposition below focuses on the exponent's role).[2]

[2] The Constant Elasticity of Substitution family is often written $f = A\big[\alpha z_1^{\rho} + (1-\alpha)z_2^{\rho}\big]^{1/\rho}$ with the exponent ρ and distribution parameter α as separate primitives; the results below hold verbatim under either convention.

(a) Compute the marginal rate of technical substitution $\mathrm{MRTS}_{2,1}$ and the elasticity of substitution $\sigma \equiv d\ln(z_2/z_1)/d\ln \mathrm{MRTS}$. Show that CES is homogeneous of degree 1.

(b) Using the expressions in (a), show that:

  (i) $\rho \to -\infty$ delivers the Leontief (fixed-proportions) technology;

  (ii) $\rho = 1$ delivers perfect substitutes;

  (iii) $\rho \to 0$ delivers Cobb-Douglas.

Answer. *Part (a): MRTS, elasticity, and homogeneity.*

Step 1. Marginal products. Differentiating f,

$$\frac{\partial q}{\partial z_1} = A \cdot \frac{1}{\rho}\big[\rho z_1^{\rho} + (1-\rho) z_2^{\rho}\big]^{1/\rho - 1} \cdot \rho\, \rho\, z_1^{\rho-1},$$

which simplifies to

$$\frac{\partial q}{\partial z_1} = A\big[\rho z_1^{\rho} + (1-\rho) z_2^{\rho}\big]^{1/\rho - 1} \rho\, z_1^{\rho-1},$$

and analogously $\partial q/\partial z_2 = A\big[\cdot\big]^{1/\rho-1}(1-\rho)z_2^{\rho-1}$.

Step 2. MRTS. The bracket and A cancel in the ratio, leaving

$$\mathrm{MRTS}_{2,1} = \frac{\partial q/\partial z_1}{\partial q/\partial z_2} = \frac{\rho\, z_1^{\rho-1}}{(1-\rho)\, z_2^{\rho-1}} = \frac{\rho}{1-\rho}\left(\frac{z_2}{z_1}\right)^{1-\rho}.$$

Step 3. Elasticity of substitution. Taking logs,

$$\ln \mathrm{MRTS} = \ln\frac{\rho}{1-\rho} + (1-\rho)\ln\frac{z_2}{z_1},$$

and solving for $\ln(z_2/z_1)$,

$$\ln\frac{z_2}{z_1} = \frac{1}{1-\rho}\left[\ln \mathrm{MRTS} - \ln\frac{\rho}{1-\rho}\right] \implies \sigma \equiv \frac{d\ln(z_2/z_1)}{d\ln \mathrm{MRTS}} = \frac{1}{1-\rho}.$$

σ does not depend on inputs or output — hence the name "constant elasticity of substitution."

Step 4. Homogeneity of degree 1. Scale both inputs by $\lambda > 0$:

$$f(\lambda z_1, \lambda z_2) = A\big[\rho(\lambda z_1)^{\rho} + (1-\rho)(\lambda z_2)^{\rho}\big]^{1/\rho} = A\big[\lambda^{\rho}(\rho z_1^{\rho} + (1-\rho) z_2^{\rho})\big]^{1/\rho} = \lambda f(z_1, z_2).$$

The CES production function is homogeneous of degree 1, so by Exercise 1 the technology exhibits CRS.

Answer. *Part (b): the three limiting cases.*

Step 1. (i) Leontief: $\rho \to -\infty$. Reading the MRTS,

$$\text{MRTS}_{2,1} = \frac{\rho}{1-\rho}\Big(\frac{z_2}{z_1}\Big)^{1-\rho}.$$

As $\rho \to -\infty$, the coefficient $\rho/(1-\rho) \to 1$ and the exponent $1-\rho \to +\infty$. So the MRTS goes to ∞ when $z_2 > z_1$ (isoquant vertical: extra z_1 is worthless) and to 0 when $z_2 < z_1$ (isoquant horizontal: extra z_2 is worthless). Also, $\sigma = 1/(1-\rho) \to 0$: inputs cannot substitute at all. This is exactly the Leontief fixed-proportions technology $f = A\min\{z_1, z_2\}$.

Step 2. (ii) Perfect substitutes: $\rho = 1$. Substituting $\rho = 1$ into the CES formula,

$$f(z_1, z_2) = A\big[1 \cdot z_1 + 0 \cdot z_2\big]^1 = A\, z_1,$$

so under this convention $\rho = 1$ collapses the CES to a technology using only z_1. Under the alternative $f = A[\alpha z_1^\rho + (1-\alpha) z_2^\rho]^{1/\rho}$, $\rho = 1$ gives $f = A[\alpha z_1 + (1-\alpha) z_2]$, the linear "perfect substitutes" technology; either way $\sigma = 1/(1-1) = \infty$: the isoquants are straight lines with slope $-\alpha/(1-\alpha)$, and inputs substitute at a constant rate.

Step 3. (iii) Cobb-Douglas: $\rho \to 0$. $\sigma = 1/(1-\rho) \to 1$ as $\rho \to 0$. The MRTS becomes $\dfrac{\rho}{1-\rho} \cdot \big(\frac{z_2}{z_1}\big)^{1-\rho} \to 0 \cdot \big(\frac{z_2}{z_1}\big)^1$; more informatively, an L'Hôpital-style limit of $[\rho z_1^\rho + (1-\rho) z_2^\rho]^{1/\rho}$ at $\rho = 0$ gives $z_1^\rho z_2^{1-\rho}$ with $\rho \to 0$ delivering the Cobb-Douglas $f = A\, z_1^\rho z_2^{1-\rho}$ (under the alternative convention $A z_1^\alpha z_2^{1-\alpha}$). Cobb-Douglas has $\sigma = 1$ everywhere; CES matches it exactly in this limit.

Intuitively, the parameter ρ smoothly interpolates between the two extremes: as ρ moves from $-\infty$ (Leontief, $\sigma = 0$) through 0 (Cobb-Douglas, $\sigma = 1$) to 1 (perfect substitutes, $\sigma = \infty$), the isoquants morph from L-shaped through convex-in-the-usual-way to straight lines. CES is the go-to functional form in applied macro precisely because a single parameter captures the entire spectrum of substitution behavior.

4. **Marginal productivity along a ray.** Consider a production function $q = f(z_1, z_2)$.

   (a) If f has constant returns to scale, are the marginal productivities of the factors constant along a ray from the origin?

   (b) What if f is homogeneous of degree $k \neq 1$?

Answer. *Part (a): CRS.*

Step 1. Marginal products are homogeneous of degree 0. If f is homogeneous of degree 1, differentiating $f(\lambda z) = \lambda f(z)$ with respect to z_i gives $\lambda\, f_i(\lambda z) = \lambda\, f_i(z)$, i.e., $f_i(\lambda z) = f_i(z)$. Marginal products are homogeneous of degree 0.

Step 2. Constancy along a ray. A ray from the origin scales both inputs by a common λ, so the input ratio z_1/z_2 is fixed. By Step 1, every marginal product f_i takes the same value at every point along the ray:

$$f_i(\lambda z) = f_i(z) \quad \text{for every } \lambda > 0.$$

Step 3. Homotheticity. The MRTS is the ratio of marginal products, so it too is constant along every ray:

$$\text{MRTS}(\lambda z) = \frac{f_1(\lambda z)}{f_2(\lambda z)} = \frac{f_1(z)}{f_2(z)} = \text{MRTS}(z).$$

A production function with this property is *homothetic*: isoquants are radial rescalings of each other. Every CRS production function is therefore homothetic.

Answer. *Part (b): homogeneity of degree $k \neq 1$.*

Step 1. Marginal products are homogeneous of degree $k-1$. Differentiating $f(\lambda z) = \lambda^k f(z)$ in z_i gives $\lambda\, f_i(\lambda z) = \lambda^k\, f_i(z)$, so $f_i(\lambda z) = \lambda^{k-1}\, f_i(z)$.

Step 2. Marginal products change along a ray. Along a ray, moving from z to λz multiplies every f_i by $\lambda^{k-1} \neq 1$ (unless $k = 1$). So marginal products are *not* constant along a ray.

Step 3. But the MRTS still is. Because both f_1 and f_2 scale by the same factor λ^{k-1}, the ratio is unchanged:

$$\text{MRTS}(\lambda z) = \frac{\lambda^{k-1} f_1(z)}{\lambda^{k-1} f_2(z)} = \text{MRTS}(z).$$

Hence any homogeneous production function — of any degree k — is homothetic. Only for $k = 1$ (CRS) do the marginal products themselves remain constant along a ray.

Intuitively, homotheticity is a condition on *ratios* of marginal products, which cancels any common scaling factor. Constancy of the marginal products themselves is stronger and locks in $k = 1$.

5. **Cobb-Douglas cost and profit.** For the technology $q = z_1^\alpha z_2^\beta$ with $\alpha, \beta \geq 0$, derive the conditional factor demand $z(w, q)$, the cost function $c(w, q)$, the supply correspondence $q(w, p)$, and the profit function $\pi(w, p)$.

Answer. *Cost minimization first, then profit maximization.*

Cost minimization.

Step 1. Set up the CMP.

$$\min_{z_1,z_2\geq 0} \; w_1 z_1 + w_2 z_2 \qquad \text{s.t.} \qquad z_1^{\alpha} z_2^{\beta} \geq q.$$

Step 2. Tangency condition (MRTS = input-price ratio).

$$\mathrm{MRTS}_{2,1} = \frac{f_1}{f_2} = \frac{\alpha}{\beta}\cdot\frac{z_2}{z_1} = \frac{w_1}{w_2} \iff z_2 = \frac{\beta}{\alpha}\cdot\frac{w_1}{w_2}\, z_1.$$

Step 3. Substitute into the constraint $z_1^{\alpha} z_2^{\beta} = q$.

$$z_1^{\alpha}\left(\frac{\beta}{\alpha}\frac{w_1}{w_2} z_1\right)^{\beta} = q \iff z_1^{\alpha+\beta} = q\cdot\left(\frac{\alpha}{\beta}\frac{w_2}{w_1}\right)^{\beta}.$$

Solving for z_1 and back-substituting for z_2,

$$\boxed{z_1(w,q) = \left(\frac{\alpha}{\beta}\frac{w_2}{w_1}\right)^{\beta/(\alpha+\beta)} q^{1/(\alpha+\beta)}, \qquad z_2(w,q) = \left(\frac{\beta}{\alpha}\frac{w_1}{w_2}\right)^{\alpha/(\alpha+\beta)} q^{1/(\alpha+\beta)}.}$$

Step 4. Cost function. Plug the conditional demands into $c(w,q) = w_1 z_1 + w_2 z_2$. After collecting the coefficient of $w_1^{\alpha/(\alpha+\beta)} w_2^{\beta/(\alpha+\beta)} q^{1/(\alpha+\beta)}$,

$$\boxed{c(w,q) = K \cdot w_1^{\alpha/(\alpha+\beta)}\, w_2^{\beta/(\alpha+\beta)}\, q^{1/(\alpha+\beta)},}$$

where the constant K collects

$$K = \left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)}.$$

The cost function is Cobb-Douglas in the input prices with exponents $\alpha/(\alpha+\beta)$ and $\beta/(\alpha+\beta)$ (the input cost shares) and is q raised to $1/(\alpha+\beta)$.

Profit maximization.

Step 5. Set up the PMP.

$$\max_{q\geq 0} \; p\,q - c(w,q) = p\,q - K\, w_1^{\alpha/(\alpha+\beta)} w_2^{\beta/(\alpha+\beta)} q^{1/(\alpha+\beta)}.$$

Step 6. FOC. Differentiating in q,

$$p = \frac{1}{\alpha+\beta} K\, w_1^{\alpha/(\alpha+\beta)} w_2^{\beta/(\alpha+\beta)}\, q^{1/(\alpha+\beta)-1}.$$

Step 7. Second-order condition. Differentiating again,

$$\frac{\partial^2\pi}{\partial q^2} = \frac{1}{\alpha+\beta}\left(\frac{1}{\alpha+\beta} - 1\right) K\, w_1^{\alpha/(\alpha+\beta)} w_2^{\beta/(\alpha+\beta)}\, q^{1/(\alpha+\beta)-2}.$$

The bracket $\frac{1}{\alpha+\beta} - 1$ is negative iff $\alpha + \beta > 1$. Wait — we need SOC < 0, so we need the leading coefficient $\frac{1}{\alpha+\beta}(\frac{1}{\alpha+\beta} - 1)$ negative, i.e., $\alpha + \beta < 1$. This is exactly the *decreasing-returns-to-scale* case; only then does a well-defined finite optimum exist. When $\alpha + \beta \geq 1$ profit is unbounded (constant or increasing returns push the firm to infinite scale).

Step 8. Supply correspondence. Solving the FOC for q,

$$q(w,p) = \Big(\frac{(\alpha+\beta)\,p}{K\,w_1^{\alpha/(\alpha+\beta)} w_2^{\beta/(\alpha+\beta)}}\Big)^{(\alpha+\beta)/(1-\alpha-\beta)}, \qquad \alpha+\beta < 1.$$

Step 9. Unconditional factor demands and profit function. Plug $q(w,p)$ into the conditional demands $z_i(w, q(w,p))$ and into $\pi(w,p) = p\,q(w,p) - c(w, q(w,p))$; both expressions are homogeneous of degree 0 (unconditional demands) and 1 (profit function) in (w,p), as Propositions 5.C.1 and 5.C.2 require. Explicit forms are algebraically messy; the key qualitative content is:

- Supply $q(w,p)$ is increasing in p and decreasing in each w_ℓ;
- The elasticity of supply with respect to p is $(\alpha+\beta)/(1-\alpha-\beta)$, which is positive and grows as $\alpha + \beta \to 1$ (returns approach CRS $\Rightarrow$ supply becomes infinitely elastic);
- Profit is homogeneous of degree 1 in (w,p): doubling all prices doubles profit.

Intuitively, the Cobb-Douglas structure delivers "share-form" cost formulas: each input's cost share equals its exponent divided by the sum of exponents. Profit maximization is well-defined only under decreasing returns; otherwise the firm faces no natural scale limit.

6. **Complement (a bliss-point consumer and a near-Leontief consumer).** Exercise 3 above showed how the CES production function morphs from Leontief through Cobb-Douglas to perfect substitutes as ρ varies. The min-of-linear structure that appears in the Leontief limit shows up on the consumer side too. This complement examines two utility functions:

$$u_1(x_1,x_2) = -(x_1-2)^2 - (x_2-3)^2 \qquad \text{(satiation, bliss at } (2,3)),$$
$$u_2(x_1,x_2) = \min\{3x_1 + x_2,\ x_1 + 3x_2\} \qquad \text{(min of two linear forms, kinked at } x_1 = x_2).$$

   (a) Sketch representative indifference curves for each.

   (b) Which of local nonsatiation, convexity, and homotheticity does each preference relation satisfy?

   (c) Derive the Walrasian demand for each.

Answer. *Part (a): indifference curves.*

u_1 has concentric *circles* centered on the bliss point $(2,3)$, with utility rising as one moves *toward* the center. u_2 has curves kinked along the ray $x_1 = x_2$: to the left of the ray (small x_1, large x_2) the binding piece is $3x_1 + x_2$, so the isoquant has slope -3; to the right of the ray the binding piece is $x_1 + 3x_2$, so the slope is $-\frac{1}{3}$. Figure 1 illustrates.

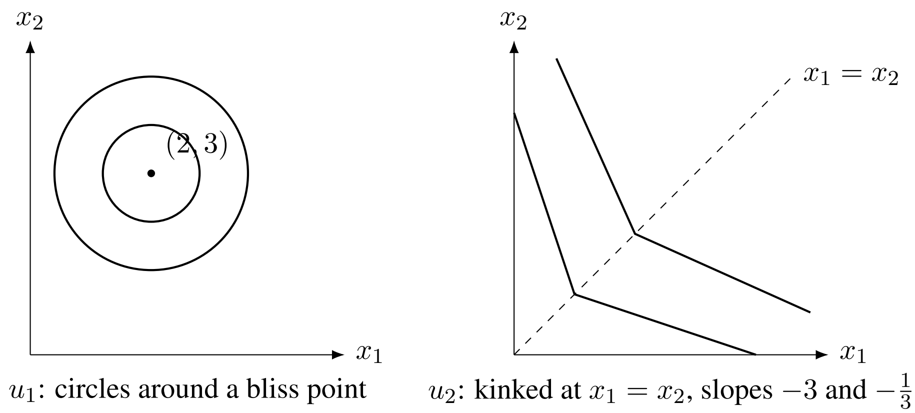

Figure 1: Indifference curves. Left: u_1 has a bliss point at $(2,3)$. Right: u_2 is kinked along $x_1 = x_2$; the piece to the left of the ray has slope -3, the piece to the right has slope $-\frac{1}{3}$.

Answer. *Part (b): LNS, convexity, homotheticity.*

- u_1 *(bliss point).* **Not LNS** — at the bliss point $(2,3)$, no nearby bundle delivers strictly higher utility (the bliss point is the global maximum). **Convex** — the utility is concave, so upper contour sets are convex disks. **Not homothetic** — the concentric circles are not radial rescalings of each other (they translate around $(2,3)$, not around the origin).
- u_2 *(min of linear).* **LNS** — u_2 is strictly increasing in both arguments away from any bundle (add a small amount of either good, both linear expressions rise, and the min rises). **Convex** — a min of linear functions is concave, so upper contour sets are convex. **Homothetic** — u_2 is homogeneous of degree 1, hence homothetic.

Answer. *Part (c): Walrasian demand.*

u_1: bliss-point demand.

Step 1. Case A: bliss point affordable. If $2p_1 + 3p_2 \le w$, the consumer buys $x = (2,3)$ exactly and does *not* spend all wealth (LNS fails). The demand is a single point, but Walras' law holds only weakly.

Step 2. Case B: bliss point unaffordable. If $2p_1 + 3p_2 > w$, the consumer picks the closest point (in the utility metric) to $(2, 3)$ on her budget line. The FOCs from $\max -(x_1 - 2)^2 - (x_2 - 3)^2$ subject to $p \cdot x = w$ give $x_i - \alpha_i = -\lambda p_i/2$; solving,

$$x(p, w) = \Big(2 - \frac{p_1(2p_1 + 3p_2 - w)}{p_1^2 + p_2^2},\ 3 - \frac{p_2(2p_1 + 3p_2 - w)}{p_1^2 + p_2^2}\Big),$$

whenever both coordinates are non-negative (otherwise the relevant coordinate is set to 0).

u_2: near-Leontief demand.

Step 1. Three regimes based on the price ratio. Because $u_2 = \min\{3x_1 + x_2,\ x_1 + 3x_2\}$ is homogeneous of degree 1, the cheapest way to buy one "utility unit" compares per-utility cost at each corner and at the kink.

Step 2. Read off the demand.

$$x(p, w) = \begin{cases} (w/p_1,\ 0) & \text{if } \dfrac{p_1}{p_2} < \dfrac{1}{3}, \\ \Big(\dfrac{w}{p_1 + p_2},\ \dfrac{w}{p_1 + p_2}\Big) & \text{if } \dfrac{1}{3} \le \dfrac{p_1}{p_2} \le 3, \\ (0,\ w/p_2) & \text{if } \dfrac{p_1}{p_2} > 3. \end{cases}$$

For moderate price ratios the consumer sits at the kink $x_1 = x_2$ (a Leontief-like "complements" regime); once one price is more than three times the other, she switches to a corner (a perfect-substitutes-like "switch to the cheap good" regime).

Intuitively, u_2 is the consumer-side twin of the Leontief limit of the CES production function (Exercise 3(b)(i)): the min-of-linear structure delivers "complementary" behavior at the kink and "substitute" behavior at the corners. The transition between regimes is sharp on the consumer side (a kink) just as it is on the producer side (a corner in the isoquant). The bliss-point utility u_1, by contrast, has no natural production analog — production functions are typically monotone in inputs, ruling out any "too much of a good thing."