EconS 501 – Micro Theory I
Recitation #5 – Applications of Demand Theory[1]

1. **MWG 3.I.7 – Linear demand, welfare, and optimal commodity taxes.** Three commodities are available. The third is a numeraire with $p_3 = 1$. Walrasian demands for the first two are

$$x_1(p, w) = a + b\,p_1 + c\,p_2,$$
$$x_2(p, w) = d + e\,p_1 + g\,p_2.$$

Neither demand depends on wealth: $\partial x_k/\partial w = 0$.

(a) What parameter restrictions does utility maximization impose?

(b) Compute the equivalent variation of the price change $(p_1, p_2) = (1, 1) \to (2, 2)$ along two different paths. Verify that path independence requires the Slutsky matrix to be symmetric.

(c) Let EV_1, EV_2, and EV denote the equivalent variations of, respectively, $(1, 1) \to (2, 1)$, $(1, 1) \to (1, 2)$, and $(1, 1) \to (2, 2)$. Compare EV with $EV_1 + EV_2$.

(d) Suppose the price increases come from unit taxes. Denote the deadweight losses as DW_1, DW_2, DW. Compare DW with $DW_1 + DW_2$.

(e) The government must raise a fixed (small) revenue R by taxing the two goods at rates (t_1, t_2). Find the tax rates that minimize the deadweight loss.

Answer. *Part (a): parameter restrictions from utility maximization.*

Step 1. Extract the Slutsky matrix. Since $\partial x_k/\partial w = 0$, the standard Slutsky decomposition $s_{k\ell} = \partial x_k/\partial p_\ell + x_\ell \cdot \partial x_k/\partial w$ collapses to $s_{k\ell} = \partial x_k/\partial p_\ell$. Reading the derivatives off the linear demands,

$$S(p, w) = \begin{bmatrix} \partial x_1/\partial p_1 & \partial x_1/\partial p_2 \\ \partial x_2/\partial p_1 & \partial x_2/\partial p_2 \end{bmatrix} = \begin{bmatrix} b & c \\ e & g \end{bmatrix}.$$

(The third row/column, corresponding to the numeraire, is identically zero; deleting it leaves the 2×2 submatrix above without changing symmetry or semidefiniteness of the full 3×3.)

Step 2. Symmetry. A necessary condition for the demands to come from a well-defined utility maximization problem is that the Slutsky matrix be symmetric:

$$\boxed{c = e.}$$

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Step 3. Negative semidefiniteness. A symmetric 2×2 matrix is negative semidefinite iff both diagonal entries are non-positive and the determinant is non-negative:

$$\boxed{b \le 0, \qquad g \le 0, \qquad bg - c^2 \ge 0.}$$

The first two say own-price effects are non-positive (compensated law of demand, own-price version); the last rules out extreme cross-price responses.

Intuitively, $b < 0$ means good 1's own-price effect points the right way (a higher p_1 pushes x_1 down), and analogously for g. The cross-price coefficient c can be positive (substitutes) or negative (complements), but its magnitude is bounded by $\sqrt{bg}$, so the two goods can't be "extreme" substitutes/complements while own-price effects remain non-degenerate.

Answer. *Part (b): equivalent variation along two paths.*

Step 1. Hicksian = Walrasian here. Because $\partial x_k/\partial w = 0$, the primal–dual link $h_\ell(p,u) = x_\ell(p, e(p,u))$ delivers h_ℓ that does not depend on u: pick any two utility levels u, u' and $x_\ell(p, e(p,u)) = x_\ell(p, e(p,u'))$ (since x_ℓ ignores its wealth argument). Hence $h_\ell(p,u) = x_\ell(p,w)$ for every u, w. The EV integrals below can be written directly in terms of the observed linear demands.

Step 2. Path 1: raise p_1 first, then p_2. Along the path $(1,1) \to (2,1) \to (2,2)$, the equivalent variation is the line integral of the Hicksian across the path:

$$\begin{aligned}
EV^{(1)} &= \int_1^2 h_1(p_1, 1, u)\, dp_1 + \int_1^2 h_2(2, p_2, u)\, dp_2 \\
&= \int_1^2 (a + b\,p_1 + c \cdot 1)\, dp_1 + \int_1^2 (d + e \cdot 2 + g\,p_2)\, dp_2 \\
&= \Big(a + \frac{3}{2}b + c\Big) + \Big(d + 2e + \frac{3}{2}g\Big).
\end{aligned}$$

(In the first integral p_2 is held at 1; in the second p_1 is held at 2.)

Step 3. Path 2: raise p_2 first, then p_1. Along $(1,1) \to (1,2) \to (2,2)$,

$$\begin{aligned}
EV^{(2)} &= \int_1^2 h_2(1, p_2, u)\, dp_2 + \int_1^2 h_1(p_1, 2, u)\, dp_1 \\
&= \int_1^2 (d + e + g\,p_2)\, dp_2 + \int_1^2 (a + b\,p_1 + 2c)\, dp_1 \\
&= \Big(d + e + \frac{3}{2}g\Big) + \Big(a + \frac{3}{2}b + 2c\Big).
\end{aligned}$$

Step 4. Path independence requires $c = e$. Subtracting,

$$EV^{(1)} - EV^{(2)} = \big[c + 2e\big] - \big[e + 2c\big] = e - c.$$

The two paths agree if and only if $c = e$ — precisely the symmetry condition from part (a). When the Slutsky matrix is symmetric, the EV of a joint price change is path independent, and we can write

$$EV = a + \frac{3}{2}b + 3c + d + \frac{3}{2}g.$$

For the rest of the exercise we assume $c = e$ (symmetry holds).

Intuitively, without symmetry the "area under the compensated demand" depends on which price we shift first. Symmetry says that the cross-effect "price of good 1 on demand for good 2" equals "price of good 2 on demand for good 1"; with that equality, the total area is well defined regardless of the integration path.

Answer. *Part (c): EV vs. $EV_1 + EV_2$.*

Step 1. One-good EVs. If only p_1 moves from 1 to 2 (with p_2 fixed at 1),

$$EV_1 = \int_1^2 (a + b\,p_1 + c)\,dp_1 = a + \frac{3}{2}b + c.$$

Analogously, if only p_2 moves,

$$EV_2 = \int_1^2 (d + e + g\,p_2)\,dp_2 = d + e + \frac{3}{2}g.$$

Step 2. Sum. Using $c = e$,

$$EV_1 + EV_2 = a + \frac{3}{2}b + 2c + d + \frac{3}{2}g.$$

Step 3. Compare with EV.

$$EV - (EV_1 + EV_2) = \left(a + \frac{3}{2}b + 3c + d + \frac{3}{2}g\right) - \left(a + \frac{3}{2}b + 2c + d + \frac{3}{2}g\right) = c.$$

Interpretation. The two "one-at-a-time" EVs miss exactly a term c. Why? When only p_1 changes, the calculation uses x_2 at the *initial* price of good 2; but during the joint move, p_2 also rises to 2, which shifts the demand curve for good 1 (and vice versa). The shift is worth exactly c in EV terms. If goods are substitutes ($c > 0$), moving both prices at once is *more* costly than the sum of one-at-a-time moves; if they are complements ($c < 0$), it is less.

Answer. *Part (d): deadweight losses.*

Step 1. Set up the accounting identity. A unit tax raises the buyer price from 1 to 2 on the taxed good. Tax revenue at the post-tax quantity minus consumer welfare loss equals the deadweight loss. Symbolically, since $-EV = T + DW$ (welfare loss splits into transferred revenue plus dead loss), $DW = T - (-EV) = T + EV$? Careful with signs: EV of a price *increase* is negative in the consumer's welfare accounting, and $-EV > 0$ is the money-metric loss. Writing $-EV_k$ as the positive welfare loss, the identity is

$$-EV_k = T_k + DW_k \quad \Longleftrightarrow \quad DW_k = (-EV_k) - T_k.$$

For notational convenience the derivations below track EV with a plus sign (as we computed in parts (b)–(c)) and identify $|EV| = EV_{\text{formula}}$ with $-EV_{\text{consumer welfare}}$. The final DW formulas use the magnitudes.

Step 2. Deadweight loss of taxing good 1 *only.* With $t_1 = 1$ (raising p_1 from 1 to 2), the post-tax quantity is $x_1(2, 1, w) = a + 2b + c$, so tax revenue is

$$T_1 = 1 \cdot x_1(2, 1, w) = a + 2b + c.$$

Combining with $EV_1 = a + \frac{3}{2}b + c$,

$$DW_1 = T_1 - EV_1 = (a + 2b + c) - \left(a + \frac{3}{2}b + c\right) = \frac{b}{2}.$$

Since $b \leq 0$, $DW_1 \leq 0$ in the usual sign convention (i.e., $|DW_1| = |b|/2$).

Step 3. Deadweight loss of taxing good 2 *only.* By the same argument with $x_2(1, 2, w) = d + e + 2g$,

$$T_2 = d + e + 2g, \qquad DW_2 = T_2 - EV_2 = (d + e + 2g) - \left(d + e + \frac{3}{2}g\right) = \frac{g}{2}.$$

Step 4. Deadweight loss when both are taxed. With $t_1 = t_2 = 1$, post-tax quantities are $x_1(2, 2, w) = a + 2b + 2c$ and $x_2(2, 2, w) = d + 2c + 2g$ (using $c = e$). Combined tax revenue is

$$T = (a + 2b + 2c) + (d + 2c + 2g) = a + 2b + 4c + d + 2g.$$

Subtracting $EV = a + \frac{3}{2}b + 3c + d + \frac{3}{2}g$,

$$DW = T - EV = \frac{b}{2} + c + \frac{g}{2}.$$

Step 5. Compare DW with $DW_1 + DW_2$.

$$DW - (DW_1 + DW_2) = \left(\frac{b}{2} + c + \frac{g}{2}\right) - \left(\frac{b}{2} + \frac{g}{2}\right) = c.$$

The excess (or shortfall) equals the same cross-effect c that appeared in part (c). Figure 1 shows one of the one-tax deadweight losses.

Answer. *Part (e): optimal commodity taxes.*

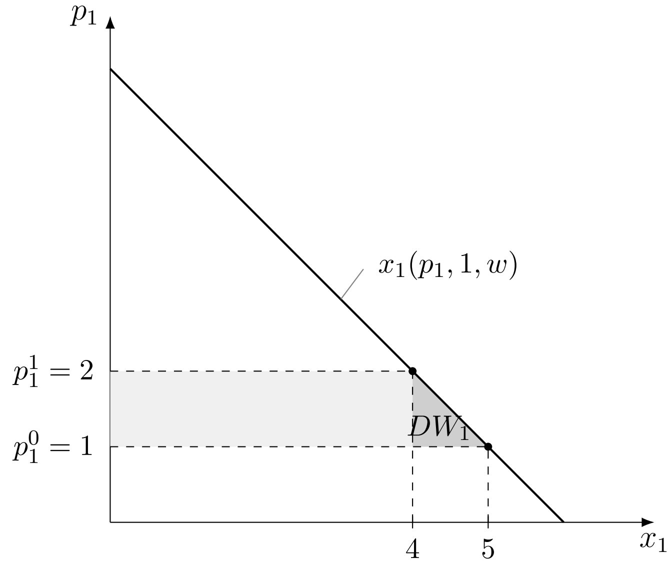

Figure 1: Deadweight loss from a unit tax on good 1 (with p_2 fixed at 1; illustrative parameters $x_1 = 6 - p_1$, so $x_1 = 5$ pre-tax and $x_1 = 4$ post-tax). The full trapezoid is the total welfare loss $|EV_1|$; the rectangle at height p_1^0 (unshaded portion) is the tax revenue T_1; the darker triangle is the deadweight loss $DW_1 = |b|/2$. When p_2 also rises to 2, the demand curve for good 1 shifts as well; that shift is the source of the extra c in $DW - (DW_1 + DW_2)$.

Step 1. Set up the government's problem. The government chooses (t_1, t_2) (small) to minimize the deadweight loss subject to raising tax revenue at least R:

$$\min_{(t_1,t_2)} DW(t_1, t_2) \qquad \text{s.t.} \qquad TR(t_1, t_2) \geq R,$$

where, using the accounting identity from part (d),

$$DW(t_1, t_2) = TR(t_1, t_2) - EV(t_1, t_2) = \sum_{\ell=1}^{2} h_\ell(1+t_1, 1+t_2, u)\, t_\ell - \big[e(1+t_1, 1+t_2, u) - e(1, 1, u)\big].$$

Step 2. First-order conditions. Form the Lagrangian $\mathcal{L} = DW(t) + \mu\big(R - TR(t)\big)$. For each $\ell \in \{1, 2\}$,

$$\frac{\partial DW}{\partial t_\ell} = \mu \frac{\partial TR}{\partial t_\ell}.$$

Step 3. Compute the pieces. Differentiating DW using Shephard's lemma $\partial e/\partial t_\ell = h_\ell$,

$$\frac{\partial DW}{\partial t_\ell} = \sum_{k=1}^{2} \frac{\partial h_k}{\partial t_\ell} t_k + h_\ell - h_\ell = \sum_{k=1}^{2} \frac{\partial h_k}{\partial t_\ell} t_k.$$

Similarly for the revenue side:

$$\frac{\partial TR}{\partial t_\ell} = h_\ell + \sum_{k=1}^{2} \frac{\partial h_k}{\partial t_\ell} t_k.$$

Step 4. Combine. Substituting into the FOC and rearranging,

$$(1+\mu)\sum_{k=1}^{2}\frac{\partial h_k}{\partial t_\ell}\,t_k = \mu\, h_\ell, \qquad \ell = 1, 2.$$

Equivalently, $-\mu = \dfrac{\sum_k h_{k\ell}\, t_k}{h_\ell}$ where $h_{k\ell} = \partial h_k/\partial t_\ell$. Setting $\ell = 1$ and $\ell = 2$ and equating the two expressions gives the *Ramsey rule* for this problem:

$$\frac{b\,t_1 + c\,t_2}{a + b(1+t_1) + c(1+t_2)} = \frac{c\,t_1 + g\,t_2}{d + c(1+t_1) + g(1+t_2)} \quad (= -\mu).$$

Step 5. Close with the revenue constraint. The equation above pins down the *ratio* t_1/t_2 along the optimal-tax locus; the level is determined by the revenue target

$$R = \big[a + b(1+t_1) + c(1+t_2)\big]t_1 + \big[d + c(1+t_1) + g(1+t_2)\big]t_2.$$

Any pair (t_1, t_2) satisfying both equations minimizes DW given the revenue requirement.

Step 6. Numerical illustration. With $R = 4$, $a = c = d = 1$, $b = g = -1$, the revenue constraint reduces to

$$4 = \big[1 - (1+t_1) + (1+t_2)\big]t_1 + \big[1 + (1+t_1) - (1+t_2)\big]t_2 = t_2\,t_1 + t_1\,t_2 = 2t_1t_2,$$

so $t_1t_2 = 2$; any pair on the hyperbola $t_1t_2 = 2$ meeting the ratio equation (which under these symmetric parameters reduces to $t_1 = t_2$) gives the optimum, $t_1 = t_2 = \sqrt{2}$.

Intuitively, the Ramsey rule says the marginal deadweight loss per dollar of revenue should be equalized across goods. When cross-effects are present ($c \neq 0$), this rule is not "tax the more inelastic good more," but a more delicate condition that also weighs how a tax on one good spills into the demand for the other. With symmetric primitives the spillovers wash out and the optimum is symmetric.

2. **Complement (Cobb-Douglas welfare and a policy fight).** Exercise 1 used demands with *no* wealth effects, so Hicksian and Walrasian demands coincided and EV/CV/CS all lined up. This complement runs the parallel exercise for a Cobb-Douglas consumer, where wealth effects matter and EV and CV differ.

   The consumer has $u(x) = \prod_{i=1}^{n} x_i^{\alpha_i}$ with $\sum_i \alpha_i = 1$.

   (a) Derive the indirect utility $v(p, w)$ and the expenditure function $e(p, u)$.

   (b) Compute the equivalent variation of a change $p^0 \to p^1$.

   (c) Compute the compensating variation of the same change.

(d) Take the two-good specialization: $n = 2$, $w = 8$, $\alpha_i = 1/2$ (so $u = \sqrt{x_1 x_2}$), starting prices $p^0 = (16, 1)$. Regulators are debating whether to cut p_1 from 16 down to 1 (a subsidy proposal).

(i) A producer lobby wants to block the cut but is willing to compensate the consumer so she ends up as well off as if the cut had passed. What monetary transfer C makes her indifferent?

(ii) Instead, suppose the cut has just been enacted (prices now $(1, 1)$) and a rival lobby wants to reverse it. What is the largest amount W the consumer would give up to stop the reversal?

Answer. *Part (a): closed forms for v and e.*

Cobb-Douglas Walrasian demand is $x_i = \alpha_i w / p_i$. Plug into u and use $\sum \alpha_i = 1$:

$$v(p, w) = \prod_i \Big(\frac{\alpha_i w}{p_i}\Big)^{\alpha_i} = w \prod_i \Big(\frac{\alpha_i}{p_i}\Big)^{\alpha_i}.$$

Inverting $u = v$ for w (setting $w = e(p, u)$):

$$e(p, u) = u \prod_i \Big(\frac{p_i}{\alpha_i}\Big)^{\alpha_i}.$$

Answer. *Part (b): equivalent variation of $p^0 \to p^1$.*

Step 1. Definition. $EV \equiv e(p^0, u^1) - w$, where $u^1 = v(p^1, w)$ is the post-change utility. It measures the wealth adjustment at *old* prices that would make the consumer as well off as under the change.

Step 2. Compute $e(p^0, v(p^1, w))$. Substituting the formulas for e and v,

$$\begin{aligned} e(p^0, v(p^1, w)) &= v(p^1, w) \prod_i \Big(\frac{p_i^0}{\alpha_i}\Big)^{\alpha_i} \\ &= \Big[w \prod_i \Big(\frac{\alpha_i}{p_i^1}\Big)^{\alpha_i}\Big] \cdot \prod_i \Big(\frac{p_i^0}{\alpha_i}\Big)^{\alpha_i} = w \prod_i \Big(\frac{p_i^0}{p_i^1}\Big)^{\alpha_i}. \end{aligned}$$

The α_i factors cancel, leaving a clean geometric-mean price ratio.

Step 3. Read off EV.

$$\boxed{EV = w\Big[\prod_i \Big(\frac{p_i^0}{p_i^1}\Big)^{\alpha_i} - 1\Big].}$$

Positive when p^1 has (weighted-geometric-mean) lower prices than p^0.

Answer. *Part (c): compensating variation of $p^0 \to p^1$.*

Step 1. Definition. $CV \equiv w - e(p^1, u^0)$, where $u^0 = v(p^0, w)$.

Step 2. Substitute. By the same manipulation as above,

$$e(p^1, v(p^0, w)) = w \prod_i \Big(\frac{p_i^1}{p_i^0}\Big)^{\alpha_i},$$

so

$$\boxed{CV = w\Big[1 - \prod_i \Big(\frac{p_i^1}{p_i^0}\Big)^{\alpha_i}\Big].}$$

EV and CV differ (they evaluate the compensation at different price/wealth points), which is exactly what makes wealth effects show up in Cobb-Douglas welfare measures.

Answer. *Part (d): the policy fight.*

Setup: $n = 2$, $w = 8$, $\alpha_1 = \alpha_2 = 1/2$, so $u = \sqrt{x_1 x_2}$ and $v(p, w) = w/(2\sqrt{p_1 p_2})$.

(i) Producer lobby: block-and-compensate.

Step 1. Identify the transfer. The producer lobby keeps prices at $p^0 = (16, 1)$ and pays the consumer C so that she is as well off as if prices had fallen to $p^1 = (1, 1)$:

$$v\big((16, 1),\ 8 + C\big) = v\big((1, 1),\ 8\big).$$

That is, C is precisely the equivalent variation of the (never-implemented) price cut.

Step 2. Apply the formula from part (b). With $p^0 = (16, 1)$ (the "old" prices) and $p^1 = (1, 1)$ (the "new" prices), and $\alpha_i = 1/2$,

$$EV = w\Big[\Big(\frac{p_1^0}{p_1^1}\Big)^{1/2}\Big(\frac{p_2^0}{p_2^1}\Big)^{1/2} - 1\Big] = 8\big[\sqrt{16}\cdot\sqrt{1} - 1\big] = 8 \cdot 3 = 24.$$

The producer lobby offers $C = \$24$.

(ii) Rival lobby: prevent-the-reversal.

Step 1. Identify the willingness-to-pay. Prices are currently $p^0 = (1, 1)$. The rival lobby wants to raise p_1 back to 16; the consumer would pay up to W out of her wealth to prevent this. The W that makes her exactly indifferent solves

$$v\big((1, 1),\ 8 - W\big) = v\big((16, 1),\ 8\big).$$

This is the compensating variation of the reversal.

Step 2. Plug in and solve. $v((1, 1), 8 - W) = (8 - W)/[2\sqrt{1 \cdot 1}] = (8 - W)/2$; and $v((16, 1), 8) = 8/[2\sqrt{16 \cdot 1}] = 1$. Setting them equal,

$$\frac{8 - W}{2} = 1 \iff W = 6.$$

The consumer would give up at most \$6 to keep prices at $(1, 1)$.

Intuitively, the two numbers ($C = 24$ vs. $W = 6$) differ because EV and CV evaluate the compensation at different price/wealth reference points. In (i), the reference wealth is 8 and reference prices are the *expensive* $(16, 1)$; a dollar there buys little utility, so it takes many dollars to make her whole. In (ii), the reference wealth is 8 and reference prices are the *cheap* $(1, 1)$; a dollar there buys a lot of utility, so it takes fewer dollars to compensate. With Cobb-Douglas preferences the wealth effect is exactly proportional to the geometric-mean price ratio.