EconS 501 – Micro Theory I
Recitation #4 – Utility and Demand III[1]

1. **Shephard's lemma from Roy's identity.** Prove that Shephard's lemma is implied by Roy's identity. (Assume we are at an interior optimum, so all derivatives exist.)

   Recall the two statements we will use.

   - *Roy's identity.* At an interior optimum, $x_\ell(p,w) = -\dfrac{\partial v(p,w)/\partial p_\ell}{\partial v(p,w)/\partial w}$ for each good ℓ.
   - *Shephard's lemma.* $h_\ell(p,u) = \dfrac{\partial e(p,u)}{\partial p_\ell}$ for each good ℓ.

   The primal–dual link between v and e is $v\big(p, e(p,u)\big) = u$ (Proposition 3.E.1); the primal–dual link between x and h is $h(p,u) = x\big(p, e(p,u)\big)$. We will use both.

   Answer.

   Step 1. Differentiate the identity $v(p, e(p,u)) = u$ with respect to p_ℓ. The right-hand side does not depend on p, so the total derivative is zero. Apply the chain rule to the left-hand side — p_ℓ enters both directly (first argument) and through $e(p,u)$ (second argument):

   $$\underbrace{\frac{\partial v}{\partial p_\ell}\big(p, e(p,u)\big)}_{\text{direct effect}} + \underbrace{\frac{\partial v}{\partial w}\big(p, e(p,u)\big) \cdot \frac{\partial e(p,u)}{\partial p_\ell}}_{\text{indirect effect}} = 0. \tag{$*$}$$

   Step 2. Rewrite $\partial v/\partial p_\ell$ using Roy's identity. Roy says $\partial v/\partial p_\ell = -x_\ell \cdot \partial v/\partial w$, evaluated at the same (p,w). Applied at wealth $w = e(p,u)$:

   $$\frac{\partial v}{\partial p_\ell}\big(p, e(p,u)\big) = -\, x_\ell\big(p, e(p,u)\big) \cdot \frac{\partial v}{\partial w}\big(p, e(p,u)\big).$$

   Step 3. Substitute into ($*$). The direct-effect term becomes $-x_\ell(p, e(p,u)) \cdot \partial v/\partial w$; adding it to the indirect-effect term,

   $$\frac{\partial v}{\partial w}\big(p, e(p,u)\big) \cdot \left[\frac{\partial e(p,u)}{\partial p_\ell} - x_\ell\big(p, e(p,u)\big)\right] = 0.$$

[1]Felix Munoz-Garcia, School of Economic Sciences, Washington State University, 103H Hulbert Hall, Pullman, WA. Tel. 509-335-8402. Email: `fmunoz@wsu.edu`.

Step 4. Cancel $\partial v/\partial w$. Marginal utility of wealth is strictly positive (utility is monotone, so extra wealth relaxes the budget and strictly raises the optimal utility), so $\partial v/\partial w > 0$. Divide it out:

$$\frac{\partial e(p,u)}{\partial p_\ell} = x_\ell\big(p, e(p,u)\big).$$

Step 5. Recognize $h_\ell(p,u)$ on the right-hand side. The primal–dual identity says $x_\ell(p, e(p,u)) = h_\ell(p,u)$ (Walrasian demand at compensated wealth equals Hicksian demand). Hence

$$\boxed{\frac{\partial e(p,u)}{\partial p_\ell} = h_\ell(p,u),}$$

which is Shephard's lemma.

Intuitively, the chain rule turns the identity "the cheapest way to achieve u, evaluated at the utility level u, is u" into an equation linking how e moves with p_ℓ (Shephard) to how v moves with p_ℓ (Roy) at the same optimum. Roy tells you the price gradient of v in terms of the Walrasian demand; Shephard tells you the price gradient of e in terms of the Hicksian. Because x and h coincide at compensated wealth, the two statements are two faces of the same envelope.

2. **Verifying the demand-side propositions for Cobb-Douglas.** For $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ with $\alpha \in (0,1)$, verify that:

   (a) Shephard's lemma holds: $\partial e/\partial p_\ell = h_\ell$;

   (b) The Slutsky substitution matrix $D_p h(p,u)$ is symmetric, negative semidefinite, and satisfies $D_p h(p,u) \cdot p = 0$;

   (c) Roy's identity holds: $x_\ell(p,w) = -(\partial v/\partial p_\ell)/(\partial v/\partial w)$.

Building blocks. From standard Cobb-Douglas derivations (or Recitation #3, Exercise 2 with a different exponent choice), and writing $K \equiv \alpha^\alpha(1-\alpha)^{1-\alpha}$,

$$\begin{aligned}
x_1(p,w) &= \frac{\alpha w}{p_1}, & x_2(p,w) &= \frac{(1-\alpha)w}{p_2}, \\
v(p,w) &= K\,\frac{w}{p_1^\alpha p_2^{1-\alpha}}, & e(p,u) &= \frac{u\,p_1^\alpha p_2^{1-\alpha}}{K}, \\
h_1(p,u) &= \frac{\alpha}{K}\,u\,p_1^{\alpha-1}p_2^{1-\alpha}, & h_2(p,u) &= \frac{1-\alpha}{K}\,u\,p_1^\alpha p_2^{-\alpha}.
\end{aligned}$$

Answer. *Part (a): Shephard's lemma.* Differentiate e in p_1 and p_2:

$$\begin{aligned}
\frac{\partial e}{\partial p_1} &= \frac{u}{K}\cdot \alpha\, p_1^{\alpha-1}p_2^{1-\alpha} = \frac{\alpha}{K}\,u\,p_1^{\alpha-1}p_2^{1-\alpha} = h_1(p,u), \\
\frac{\partial e}{\partial p_2} &= \frac{u}{K}\cdot (1-\alpha)\,p_1^\alpha p_2^{-\alpha} = \frac{1-\alpha}{K}\,u\,p_1^\alpha p_2^{-\alpha} = h_2(p,u).
\end{aligned}$$

Both match the Hicksian formulas above.

Answer. *Part (b): the Slutsky matrix.*

Step 1. Compute $D_p h(p,u)$. Differentiating h_1 and h_2 in p_1, p_2,

$$D_p h(p,u) = \frac{u}{K}\begin{bmatrix} \alpha(\alpha-1)\, p_1^{\alpha-2} p_2^{1-\alpha} & \alpha(1-\alpha)\, p_1^{\alpha-1} p_2^{-\alpha} \\ (1-\alpha)\alpha\, p_1^{\alpha-1} p_2^{-\alpha} & -(1-\alpha)\alpha\, p_1^{\alpha} p_2^{-\alpha-1} \end{bmatrix}.$$

Step 2. Symmetry. The off-diagonal entries are equal ($\alpha(1-\alpha)\, p_1^{\alpha-1} p_2^{-\alpha}$ from either direction), so $D_p h(p,u)$ is symmetric.

Step 3. Negative semidefiniteness. Check the two leading principal minors.

- First-order: $\partial h_1/\partial p_1 = \dfrac{\alpha(\alpha-1)u\, p_1^{\alpha-2} p_2^{1-\alpha}}{K}$. Since $\alpha \in (0,1)$, $\alpha - 1 < 0$, so this entry is negative (≤ 0).
- Second-order: the determinant is

$$\begin{aligned} \det D_p h &= \left(\frac{u}{K}\right)^2 \Big[\alpha(\alpha-1)\, p_1^{\alpha-2} p_2^{1-\alpha} \cdot \big(-(1-\alpha)\alpha\, p_1^{\alpha} p_2^{-\alpha-1}\big) \\ &\qquad - \big(\alpha(1-\alpha)\, p_1^{\alpha-1} p_2^{-\alpha}\big)^2\Big] \\ &= \left(\frac{u}{K}\right)^2 \Big[\alpha^2(1-\alpha)^2 p_1^{2\alpha-2} p_2^{-2\alpha} - \alpha^2(1-\alpha)^2 p_1^{2\alpha-2} p_2^{-2\alpha}\Big] = 0. \end{aligned}$$

  So the determinant is exactly zero: the second-order minor equals zero, consistent with negative semidefiniteness (and *not* strict — $D_p h$ is rank-deficient).

The signs (≤ 0, $= 0$) are exactly what negative semidefiniteness of a 2×2 symmetric matrix requires.

Step 4. The null-direction $D_p h \cdot p = 0$. Compute the first component of $D_p h(p,u) \cdot p$:

$$\begin{aligned} &\frac{u}{K}\Big[\alpha(\alpha-1)\, p_1^{\alpha-2} p_2^{1-\alpha} \cdot p_1 + \alpha(1-\alpha)\, p_1^{\alpha-1} p_2^{-\alpha} \cdot p_2\Big] \\ &= \frac{u}{K}\Big[\alpha(\alpha-1)\, p_1^{\alpha-1} p_2^{1-\alpha} + \alpha(1-\alpha)\, p_1^{\alpha-1} p_2^{1-\alpha}\Big] = 0, \end{aligned}$$

because $\alpha(\alpha-1) + \alpha(1-\alpha) = 0$. The second component vanishes by the same cancellation.

Intuitively, $D_p h \cdot p = 0$ is the compensated-demand analog of Euler's theorem: Hicksian demands are homogeneous of degree zero in p, and p is the direction in which price scaling has no effect on compensated behavior. Together with symmetry and negative semidefiniteness, this rules out "all-substitutes" or "all-complements" patterns across all pairs of goods.

Answer. *Part (c): Roy's identity.*

Step 1. Compute $\partial v/\partial p_1$ and $\partial v/\partial w$.

$$\frac{\partial v}{\partial p_1} = Kw \cdot \frac{\partial}{\partial p_1}\left(p_1^{-\alpha} p_2^{-(1-\alpha)}\right) = -\alpha Kw\, p_1^{-\alpha-1} p_2^{-(1-\alpha)}, \qquad \frac{\partial v}{\partial w} = \frac{K}{p_1^{\alpha} p_2^{1-\alpha}}.$$

Step 2. Form the Roy ratio.

$$-\frac{\partial v/\partial p_1}{\partial v/\partial w} = -\frac{-\alpha Kw\, p_1^{-\alpha-1} p_2^{-(1-\alpha)}}{K\, p_1^{-\alpha} p_2^{-(1-\alpha)}} = \alpha w \cdot \frac{p_1^{-\alpha-1}}{p_1^{-\alpha}} = \frac{\alpha w}{p_1} = x_1(p, w).$$

The analogous computation for good 2 gives $-(\partial v/\partial p_2)/(\partial v/\partial w) = (1-\alpha)w/p_2 = x_2(p, w)$.

Roy's identity is verified.

3. **Additively separable utility.** A utility function is *additively separable* if $u(x) = \sum_{\ell=1}^{L} u_\ell(x_\ell)$ (three-good example: $u_1(x_1) + u_2(x_2) + u_3(x_3)$). Each u_ℓ may be linear or nonlinear, but the consumer never enjoys "interactions" between goods.

   (a) Show that the induced ordering on any subset of goods is independent of the fixed values attached to the remaining goods.

   (b) Show that the Walrasian and Hicksian demands generated by additively separable, strictly concave, differentiable, interior preferences admit no inferior goods.

Answer. *Part (a): independence of the induced ordering.*

Step 1. Setup. Fix a subset of goods $T \subseteq \{1, \dots, L\}$ (the "goods of interest") and split any bundle $z \in \mathbb{R}^L_+$ into (z_T, z_{-T}), where $z_T \in \mathbb{R}^{|T|}_+$ collects the coordinates in T and $z_{-T} \in \mathbb{R}^{L-|T|}_+$ collects the rest.

We must show: for every $z_T, z'_T \in \mathbb{R}^{|T|}_+$ and every $z_{-T} \in \mathbb{R}^{L-|T|}_+$,

$$(z_T, z_{-T}) \succsim (z'_T, z_{-T}) \iff (z_T, z'_{-T}) \succsim (z'_T, z'_{-T})$$

for every alternative "fixed" background z'_{-T}. In words: how the consumer ranks two bundles that differ only in the T-coordinates does not depend on what she has of the non-T goods.

Step 2. Write both rankings in utility form. Because $\succsim$ is represented by the additively separable u,

$$(z_T, z_{-T}) \succsim (z'_T, z_{-T}) \iff \sum_{\ell \in T} u_\ell(z_\ell) + \sum_{\ell \notin T} u_\ell(z_\ell) \;\geq\; \sum_{\ell \in T} u_\ell(z'_\ell) + \sum_{\ell \notin T} u_\ell(z_\ell).$$

The two “non-T” terms on left and right are *identical* (both use z_{-T}), so they cancel:

$$\sum_{\ell \in T} u_\ell(z_\ell) \;\geq\; \sum_{\ell \in T} u_\ell(z'_\ell). \tag{$\star$}$$

Step 3. Repeat with the alternative background z'_{-T}. By the same cancellation,

$$(z_T, z'_{-T}) \succsim (z'_T, z'_{-T}) \iff \sum_{\ell \in T} u_\ell(z_\ell) \;\geq\; \sum_{\ell \in T} u_\ell(z'_\ell).$$

This is the *same* inequality ($\star$). Hence the two rankings hold or fail together. The induced ordering on the goods in T is independent of the background z_{-T}.

Answer. *Part (b): all goods are normal.*

Step 1. The tangency condition, good by good. At an interior optimum of either the UMP or the EMP, tangency between the indifference surface and the budget hyperplane gives, for every pair of goods (k, ℓ),

$$\frac{u'_k\big(x_k(p,w)\big)}{u'_\ell\big(x_\ell(p,w)\big)} = \frac{p_k}{p_\ell} \quad\iff\quad u'_k\big(x_k(p,w)\big) = \frac{p_k}{p_\ell} \cdot u'_\ell\big(x_\ell(p,w)\big). \tag{$\dagger$}$$

Because u is additively separable, $\partial u/\partial x_k = u'_k(x_k)$: the marginal utility of good k depends *only* on x_k, not on the other coordinates. That is what makes the argument below work.

Step 2. A picture of ($\dagger$). Fix good ℓ (any interior good). Reading ($\dagger$) as an equation for x_k: the LHS, $u'_k(x_k)$, is a decreasing function of x_k (strict concavity of u_k); the RHS, $(p_k/p_\ell) \cdot u'_\ell(x_\ell)$, is a horizontal line (constant in x_k). The intersection determines x_k; see Figure 1.

Step 3. Suppose w rises (with prices fixed). By Walras’ law, wealth must go somewhere: some good, say good ℓ, sees its demand rise, $x_\ell(p,w)$ up. Strict concavity of u_ℓ then gives $u'_\ell(x_\ell(p,w))$ *down*. Consequently the horizontal line in Figure 1 shifts down, and its intersection with $u'_k(x_k)$ moves to a larger value of x_k (since u'_k is decreasing).

Hence $x_k(p,w)$ increases too: good k is normal. Because k was an arbitrary good, *every* good is normal.

Step 4. Why the argument breaks for non-separable u. Without additive separability the marginal utility $\partial u/\partial x_k$ depends on *all* coordinates, not only x_k; in Figure 1 the “curve” also shifts, and one cannot sign the direction of the new intersection. The additive-separability assumption is exactly what confines each marginal utility to its own good and rules out inferiority.

Hicksian demands satisfy the same tangency ($\dagger$) and the same monotonicity argument (with u replaced by “target utility”), so no Hicksian demand can be inferior either.

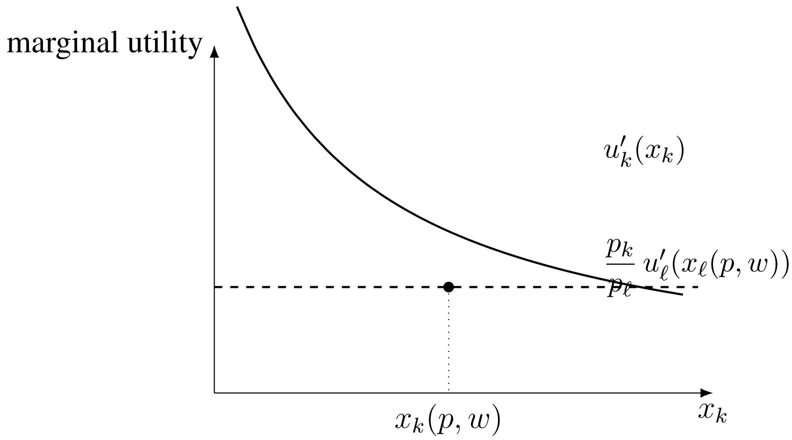

Figure 1: The tangency condition (†) determines $x_k(p,w)$ as the intersection of the decreasing curve $u_k'(x_k)$ with the horizontal line $(p_k/p_\ell)\, u_\ell'(x_\ell(p,w))$.

Intuitively, additive separability + concavity forces demand to spread any wealth increase across *all* goods rather than pulling it out of some good. There is no "income-inferior good" hiding inside a separable Cobb-Douglas, quasilinear-in-x_1, or CES with common exponent.

4. **Labor supply when leisure is inferior.** If leisure is an inferior good, what is the slope of the labor supply curve?

   Setup. Denote leisure by L, the wage by w, and non-labor income by m. Labor hours are $\bar{L} - L$ where $\bar{L}$ is the time endowment; so the labor supply curve has slope $\partial(\bar{L} - L)/\partial w = -\partial L/\partial w$. Whether the labor supply is upward-sloping (typical) or downward-sloping depends on the sign of $\partial L/\partial w$.

   Answer.

   Step 1. Slutsky decomposition for leisure demand. With wealth m and wage w (the price of leisure), the standard Slutsky equation with endowments (Recitation #3, Exercise 3) gives

$$\frac{\partial L(w,m)}{\partial w} = \underbrace{\frac{\partial L^S(w,u)}{\partial w}}_{\text{substitution effect (SE)}} - \underbrace{\left[L(w,m) - \bar{L}\right] \cdot \frac{\partial L(w,m)}{\partial m}}_{\text{endowment-adjusted income effect}}.$$

   Because the consumer sells labor from her endowment $\bar{L}$, the net demand for leisure is $L - \bar{L} < 0$, so $L - \bar{L} = -(\bar{L} - L)$. Substituting,

$$\frac{\partial L}{\partial w} = \frac{\partial L^S}{\partial w} + (\bar{L} - L) \cdot \frac{\partial L}{\partial m} = \underbrace{\frac{\partial L^S}{\partial w}}_{\leq 0} + \underbrace{(\bar{L} - L)}_{\text{hours worked} > 0} \cdot \frac{\partial L}{\partial m}. \tag{‡}$$

Own-price substitution effects are always non-positive, so $\partial L^S/\partial w \leq 0$.

Step 2. Case A: leisure normal ($\partial L/\partial m > 0$). Both terms in (‡) can pull in opposite directions: SE ≤ 0 (fewer leisure hours when w rises), income effect ≥ 0 (a higher wage effectively raises income, boosting leisure). The sign of $\partial L/\partial w$ is *ambiguous* — exactly the backward-bending labor supply story.

Step 3. Case B: leisure inferior ($\partial L/\partial m < 0$). Now the endowment-adjusted income term is $(\bar{L} - L) \cdot (\partial L/\partial m) < 0$. Combined with $\partial L^S/\partial w \leq 0$:

$$\frac{\partial L}{\partial w} \;=\; \underbrace{\frac{\partial L^S}{\partial w}}_{\leq 0} \;+\; \underbrace{(\bar{L} - L) \cdot \frac{\partial L}{\partial m}}_{<0} \;<\; 0.$$

Both effects push leisure down when the wage rises. The sign of $\partial L/\partial w$ is unambiguously negative.

Step 4. Read the labor supply slope. Since labor supply is $\bar{L} - L$,

$$\frac{\partial(\bar{L} - L)}{\partial w} = -\frac{\partial L}{\partial w} > 0.$$

When leisure is inferior, labor supply is *unambiguously upward-sloping* in the wage — no backward bend can occur.

Intuitively, the backward-bending case relies on the income effect (higher wage $\to$ richer $\to$ more leisure) overpowering the substitution effect. If leisure is inferior, higher income actually pulls leisure *down*, so both channels reinforce the substitution effect: more work at higher wages, without exception.

5. **Complement (recovering u from v).** Exercises 1–2 above went in the direction $u \to v, e, h$. This complement goes in the *reverse* direction: given the indirect utility $v(p, w)$, reconstruct the primitive utility $u(x)$. The key inversion formula is

$$u(x) = \inf_{p \gg 0} \; v\big(p,\, p \cdot x\big),$$

which says: at bundle x, look at every price vector under which x is just affordable ($w = p \cdot x$), read off the corresponding indirect utility, and take the infimum. For a quasiconcave u this returns $u(x)$ itself.

   (a) Take the Cobb-Douglas $u(x_1, x_2) = \sqrt{x_1 x_2}$. Compute $v(p, w)$; then verify that $\inf_{p \gg 0} v(p,\, p \cdot x) = \sqrt{x_1 x_2}$.

   (b) Repeat with the Leontief $u(x_1, x_2) = \min\{x_1, x_2\}$.

Answer. *Part (a):* $u(x_1, x_2) = \sqrt{x_1 x_2}$.

Step 1. Compute $v(p,w)$. u is monotone in x_1x_2, so the UMP is equivalent to $\max x_1x_2$ subject to $p_1x_1 + p_2x_2 \leq w$, a Cobb-Douglas with equal exponents. The Walrasian demand is $x_\ell(p,w) = w/(2p_\ell)$, and

$$v(p,w) = \sqrt{\frac{w}{2p_1} \cdot \frac{w}{2p_2}} = \frac{w}{2\sqrt{p_1p_2}}.$$

Step 2. Plug $w = p \cdot x$ into v. At the bundle $x = (x_1, x_2)$,

$$v(p,\, p \cdot x) = \frac{p_1x_1 + p_2x_2}{2\sqrt{p_1p_2}}.$$

The expression depends on p_1 and p_2 only through their ratio; let $t = p_1/p_2$, so $\sqrt{p_1p_2} = p_2\sqrt{t}$ and $p_1x_1 + p_2x_2 = p_2(tx_1 + x_2)$. Substituting,

$$v(p, p \cdot x) = \frac{tx_1 + x_2}{2\sqrt{t}} = \frac{1}{2}\Big(x_1\sqrt{t} + \frac{x_2}{\sqrt{t}}\Big).$$

Step 3. Minimize over $t > 0$. Differentiate with respect to $s = \sqrt{t}$: the interior FOC is $x_1 - x_2/s^2 = 0$, i.e., $s^2 = x_2/x_1$, so $t = x_2/x_1$. At this t,

$$v(p, p \cdot x) = \frac{1}{2}\Big(x_1\sqrt{x_2/x_1} + \frac{x_2}{\sqrt{x_2/x_1}}\Big) = \frac{1}{2}\Big(\sqrt{x_1x_2} + \sqrt{x_1x_2}\Big) = \sqrt{x_1x_2}.$$

The infimum is attained and equals $\sqrt{x_1x_2} = u(x_1, x_2)$. Recovered.

Answer. *Part (b):* $u(x_1, x_2) = \min\{x_1, x_2\}$.

Step 1. Compute $v(p,w)$. Leontief demand pins the consumer at $x_1 = x_2$. Plug into the budget: $(p_1 + p_2)x_1 = w$, so $x_\ell = w/(p_1 + p_2)$ and

$$v(p,w) = \frac{w}{p_1 + p_2}.$$

Step 2. Plug $w = p \cdot x$ into v.

$$v(p,\, p \cdot x) = \frac{p_1x_1 + p_2x_2}{p_1 + p_2} = \frac{p_1}{p_1 + p_2}\, x_1 + \frac{p_2}{p_1 + p_2}\, x_2.$$

This is a *convex combination* of x_1 and x_2, with weights $\lambda = p_1/(p_1 + p_2) \in (0,1)$ and $1 - \lambda$.

Step 3. Minimize over $p \gg 0$. A convex combination of two numbers is bounded below by their minimum, and the infimum $\min\{x_1, x_2\}$ is approached by putting essentially all weight on the smaller coordinate (let $p_k/(p_1+p_2) \to 1$ for the good with the smaller x_k). Hence

$$\inf_{p \gg 0} v(p,\, p \cdot x) = \min\{x_1, x_2\} = u(x_1, x_2).$$

Intuitively, the inversion formula reflects the same duality as Roy and Shephard: v carries all the information in u, and the operation "for a given x, take the worst indirect utility across price systems that make x just affordable" picks out the utility of the bundle itself. When u is Cobb-Douglas the minimizing price ratio is the one that makes x optimal at those prices ($t = x_2/x_1$); when u is Leontief the minimizing prices push essentially all wealth against the smaller coordinate. Either way, the inversion returns $u(x)$.