EconS 501 – Micro Theory I
Recitation #3 – Utility and Demand II[1]

1. **Cobb-Douglas UMP.** Find the demanded bundle for a consumer whose utility function is $u(x_1, x_2) = x_1^{3/2} x_2$ facing the budget constraint $3x_1 + 4x_2 = 100$.

   Answer. We work with the log of u (a strictly increasing transformation, so the argmax is the same) to turn the multiplicative FOCs into additive ones.

   Step 1. Log-transform. Let $\widetilde{u}(x_1, x_2) = \ln u(x_1, x_2) = \frac{3}{2} \ln x_1 + \ln x_2$. Then the UMP is

   $$\max_{x_1, x_2 \geq 0} \frac{3}{2} \ln x_1 + \ln x_2 \qquad \text{s.t.} \qquad 3x_1 + 4x_2 = 100.$$

   Step 2. Lagrangian and FOCs. Form $\mathcal{L} = \frac{3}{2} \ln x_1 + \ln x_2 - \lambda(3x_1 + 4x_2 - 100)$. The FOCs in x_1, x_2, λ read

   $$\frac{3}{2x_1} = 3\lambda, \qquad \frac{1}{x_2} = 4\lambda, \qquad 3x_1 + 4x_2 = 100.$$

   Step 3. Eliminate λ and solve. From the first two FOCs, $\lambda = \frac{1}{2x_1} = \frac{1}{4x_2}$, so $4x_2 = 2x_1$, i.e., $x_1 = 2x_2$. Substituting into the budget: $3(2x_2) + 4x_2 = 100$, i.e., $10x_2 = 100$, so

   $$\boxed{x_1(3, 4, 100) = 20, \qquad x_2(3, 4, 100) = 10.}$$

   Step 4. Value of the Lagrange multiplier. From $\frac{1}{4x_2} = \frac{1}{40}$, the multiplier in the *transformed* problem is $\widetilde{\lambda} = 1/40$. This measures the marginal utility of income *measured in "log-utils"*: $\partial \widetilde{v} / \partial m = 1/40$, where $\widetilde{v}(p, m) = \ln v(p, m)$.

   If one instead solves the UMP directly (without the log-transform), the Lagrange multiplier is $\lambda = 20^{3/2}/4$ (a different scale, because $v = e^{\widetilde{v}}$).

   Intuitively, the log-transform never changes *what* is optimal, but it does change the units in which the Lagrange multiplier "prices" an additional dollar of wealth — something to remember whenever the interpretation of λ^* enters a discussion.

2. **The full duality machine.** For $u(x_1, x_2) = x_1^{1/2} x_2^{1/3}$ and the budget $p_1 x_1 + p_2 x_2 = m$, derive (a) the Walrasian demand, (b) the indirect utility function, (c) the Hicksian demand, and (d) the expenditure function.

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Where we're going. The four objects are related by two identities:

$$h_\ell(p,u) = \frac{\partial e(p,u)}{\partial p_\ell} \quad \text{(Shephard's lemma)}, \qquad x(p,m) = h\big(p,\, v(p,m)\big) \quad \text{(the "primal–dual" link)}.$$

We will use both. The plan is: solve UMP to get $x(p,m)$; plug into u to get $v(p,m)$; invert to get $e(p,u)$; differentiate to get $h(p,u)$.

Answer. *Part (a): Walrasian demand $x(p,m)$.*

Step 1. Lagrangian and FOCs. With $\mathcal{L} = x_1^{1/2}x_2^{1/3} - \lambda(p_1x_1 + p_2x_2 - m)$,

$$\tfrac{1}{2}x_1^{-1/2}x_2^{1/3} = \lambda p_1, \qquad \tfrac{1}{3}x_1^{1/2}x_2^{-2/3} = \lambda p_2, \qquad p_1x_1 + p_2x_2 = m.$$

Step 2. Tangency ratio. Dividing the two FOCs eliminates λ:

$$\frac{\tfrac{1}{2}x_2/x_1}{\tfrac{1}{3}\cdot(\text{same base})/x_2} = \frac{p_1}{p_2} \iff \frac{3x_2}{2x_1} = \frac{p_1}{p_2} \iff p_2x_2 = \tfrac{2}{3}\,p_1x_1.$$

Step 3. Use the budget. Substituting into $p_1x_1 + p_2x_2 = m$: $p_1x_1 + \frac{2}{3}p_1x_1 = m$, so $\frac{5}{3}p_1x_1 = m$ and hence

$$\boxed{x_1(p,m) = \frac{3}{5}\cdot\frac{m}{p_1}, \qquad x_2(p,m) = \frac{2}{5}\cdot\frac{m}{p_2}.}$$

The expenditure shares are $3/5$ and $2/5$, matching the "normalized" Cobb-Douglas exponents $1/2$ and $1/3$ (which sum to $5/6$, then divide by $5/6$).

Answer. *Part (b): indirect utility $v(p,m)$.*

Plug the Walrasian demands into u:

$$v(p,m) = \left(\tfrac{3}{5}\tfrac{m}{p_1}\right)^{1/2}\left(\tfrac{2}{5}\tfrac{m}{p_2}\right)^{1/3} = \underbrace{\left(\tfrac{3}{5}\right)^{1/2}\left(\tfrac{2}{5}\right)^{1/3}}_{\text{constant}}\cdot\frac{m^{5/6}}{p_1^{1/2}\,p_2^{1/3}}.$$

So v is homogeneous of degree 0 in (p,m) (check: $m^{5/6}/(p_1^{1/2}p_2^{1/3})$ has exponents $5/6 - 1/2 - 1/3 = 0$) and strictly increasing in m, as Proposition 3.D.3 requires.

Answer. *Part (c): expenditure function $e(p,u)$.*

Step 1. Invert v. At the optimum, $u = v(p,m)$; treating m as the unknown and setting $m = e(p,u)$,

$$u = \left(\tfrac{3}{5}\right)^{1/2}\left(\tfrac{2}{5}\right)^{1/3}\frac{e^{5/6}}{p_1^{1/2}p_2^{1/3}}.$$

Step 2. Solve for e. Raise both sides to the 6/5 power:

$$e(p,u) = \left(\tfrac{5}{3}\right)^{3/5}\left(\tfrac{5}{2}\right)^{2/5} p_1^{3/5}\, p_2^{2/5}\, u^{6/5}.$$

Homogeneous of degree 1 in p (exponents $3/5 + 2/5 = 1$) and strictly increasing in u, as Proposition 3.E.2 requires.

Answer. *Part (d): Hicksian demand $h(p,u)$.*

Apply Shephard's lemma, $h_\ell(p,u) = \partial e/\partial p_\ell$:

$$\boxed{h_1(p,u) = \left(\tfrac{5}{3}\right)^{3/5}\left(\tfrac{5}{2}\right)^{2/5} \cdot \tfrac{3}{5}\, p_1^{-2/5}\, p_2^{2/5}\, u^{6/5}, \qquad h_2(p,u) = \left(\tfrac{5}{3}\right)^{3/5}\left(\tfrac{5}{2}\right)^{2/5} \cdot \tfrac{2}{5}\, p_1^{3/5}\, p_2^{-3/5}\, u^{6/5}.}$$

Sanity check. Each Hicksian demand is decreasing in its own price and increasing in the other's — the Cobb-Douglas substitution effect. The next exercise verifies the two round-trip identities that link these four objects.

3. **MWG 3.E.8.** For the Cobb-Douglas utility $u(x_1,x_2) = x_1^{\alpha}x_2^{1-\alpha}$, verify the following two pairs of duality identities:

$$\begin{array}{llcl} \text{(3.E.1)} & e\big(p,\, v(p,w)\big) = w & \text{and} & v\big(p,\, e(p,u)\big) = u; \\ \text{(3.E.3)} & h(p,u) = x\big(p,\, e(p,u)\big) & \text{and} & x(p,w) = h\big(p,\, v(p,w)\big). \end{array}$$

The first pair says that expenditure and indirect utility invert each other; the second says that Hicksian and Walrasian demands coincide when wealth equals the minimum expenditure needed to reach utility u.

Building blocks. From Exercise 2 (or, for this exponent choice, directly), the four Cobb-Douglas objects are

$$\begin{aligned} x_1(p,w) &= \frac{\alpha w}{p_1}, & x_2(p,w) &= \frac{(1-\alpha)w}{p_2}, \\ v(p,w) &= K \cdot \frac{w}{p_1^{\alpha}\, p_2^{1-\alpha}}, & e(p,u) &= \frac{u\, p_1^{\alpha}\, p_2^{1-\alpha}}{K}, \\ h_1(p,u) &= \frac{\alpha}{K}\, u\, p_1^{\alpha-1} p_2^{1-\alpha}, & h_2(p,u) &= \frac{1-\alpha}{K}\, u\, p_1^{\alpha} p_2^{-\alpha}, \end{aligned}$$

where $K \equiv \alpha^{\alpha}(1-\alpha)^{1-\alpha}$ is a constant that does not depend on (p,w,u).

Answer. *Identity 1: $e(p,\, v(p,w)) = w$.*

Step 1. Substitute $v(p,w)$ into e. Take $u = v(p,w) = Kw/(p_1^{\alpha}p_2^{1-\alpha})$ and plug into the formula for e:

$$e\big(p,\, v(p,w)\big) = \frac{v(p,w)\cdot p_1^{\alpha}p_2^{1-\alpha}}{K} = \frac{1}{K}\cdot \frac{Kw}{p_1^{\alpha}p_2^{1-\alpha}} \cdot p_1^{\alpha}p_2^{1-\alpha} = w.$$

Step 2. *Read the identity off.* The price factors cancel and the two K's cancel; only w survives.

Answer. *Identity 2:* $v(p,\, e(p,u)) = u$.

Step 1. *Substitute* $e(p,u)$ *into* v. Take $w = e(p,u) = u\, p_1^{\alpha} p_2^{1-\alpha}/K$ and plug into v:

$$v\big(p,\, e(p,u)\big) = K \cdot \frac{e(p,u)}{p_1^{\alpha} p_2^{1-\alpha}} = K \cdot \frac{1}{p_1^{\alpha} p_2^{1-\alpha}} \cdot \frac{u\, p_1^{\alpha} p_2^{1-\alpha}}{K} = u.$$

Again the price factors and the K's cancel, leaving u.

Answer. *Identity 3:* $h(p,u) = x(p,\, e(p,u))$. We check good 1; good 2 is analogous.

Step 1. *Substitute* $e(p,u)$ *into the Walrasian* x_1. Set $w = e(p,u)$ in $x_1(p,w) = \alpha w/p_1$:

$$x_1\big(p,\, e(p,u)\big) = \frac{\alpha\, e(p,u)}{p_1} = \frac{\alpha}{p_1} \cdot \frac{u\, p_1^{\alpha}\, p_2^{1-\alpha}}{K} = \frac{\alpha}{K}\, u\, p_1^{\alpha-1}\, p_2^{1-\alpha}.$$

Step 2. *Compare with* $h_1(p,u)$. The formula for h_1 is exactly $\dfrac{\alpha}{K}\, u\, p_1^{\alpha-1}\, p_2^{1-\alpha}$, so

$$x_1\big(p,\, e(p,u)\big) = h_1(p,u). \checkmark$$

The same calculation with α replaced by $1-\alpha$ (and p_1, p_2 swapped in the appropriate slots) gives $x_2(p, e(p,u)) = h_2(p,u)$.

Answer. *Identity 4:* $x(p,w) = h(p,\, v(p,w))$. Again check good 1.

Step 1. *Substitute* $v(p,w)$ *into the Hicksian* h_1. Set $u = v(p,w) = Kw/(p_1^{\alpha} p_2^{1-\alpha})$ in $h_1(p,u) = (\alpha/K)\, u\, p_1^{\alpha-1} p_2^{1-\alpha}$:

$$h_1\big(p,\, v(p,w)\big) = \frac{\alpha}{K} \cdot \frac{Kw}{p_1^{\alpha} p_2^{1-\alpha}} \cdot p_1^{\alpha-1} p_2^{1-\alpha} = \alpha w \cdot \frac{p_1^{\alpha-1}}{p_1^{\alpha}} = \frac{\alpha w}{p_1}.$$

Step 2. *Compare with* $x_1(p,w)$. The right-hand side is precisely $x_1(p,w) = \alpha w/p_1$.

Good 2 works out the same way, giving $h_2(p, v(p,w)) = (1-\alpha)w/p_2 = x_2(p,w)$.

Intuitively, the identities in (3.E.1) express the fact that "the minimum wealth needed to reach the best utility affordable at w" is w itself, and vice versa. The identities in (3.E.3) express the parallel fact for demands: if you have exactly the wealth needed to reach utility u, then whether you solve the UMP or the EMP, you end up choosing the same bundle. The two pairs together are what makes the primal–dual approach work: whichever side of the theory is easier to compute, you can always translate to the other.

4. **Slutsky with endowments (two-period consumption).** Dave has utility $u(x_1, x_2)$ over consumption in period 1 (x_1) and period 2 (x_2). He is endowed with $\bar{x} = (\bar{x}_1, \bar{x}_2)$, which he can consume as-is or trade at prices (p_1, p_2). His budget constraint is

$$p_1 x_1 + p_2 x_2 \;=\; p_1 \bar{x}_1 + p_2 \bar{x}_2 \;\equiv\; m(p),$$

so *wealth is endogenous* — it depends on prices via the value of the endowment.

   (a) Derive the Slutsky equation in this endowment version of the model.

   (b) Suppose $\bar{x} = (\bar{x}_1, \bar{x}_2)$ and Dave's optimal choice is such that $x_1 < \bar{x}_1$ (he is a *lender* of the good). If p_1 falls, is he better or worse off? What if p_2 falls?

Answer. *Part (a): the Slutsky equation with endowments.*

Step 1. Starting identity. The primal-dual link gives $h_j(p,u) = x_j\big(p,\, e(p,u)\big)$: the Hicksian bundle and the Walrasian bundle coincide when wealth equals the minimum expenditure needed to reach utility u. Differentiate both sides with respect to p_i:

$$\frac{\partial h_j(p,u)}{\partial p_i} = \frac{\partial x_j}{\partial p_i}\Big(p,\, e(p,u)\Big) + \frac{\partial x_j}{\partial m}\Big(p,\, e(p,u)\Big) \cdot \frac{\partial e(p,u)}{\partial p_i}.$$

Step 2. Envelope theorem for $\partial e/\partial p_i$ WITH endowments. In the standard model, $\partial e/\partial p_i = h_i(p,u)$ (Shephard's lemma). With endowments, the EMP is instead

$$e(p,u) = \min_x \; p \cdot (x - \bar{x}) \quad \text{s.t.} \quad u(x) \geq u,$$

so the objective differs from the standard one by the endowment term $-p \cdot \bar{x}$. Applying the envelope theorem gives

$$\frac{\partial e(p,u)}{\partial p_i} = h_i(p,u) - \bar{x}_i.$$

Interpretation: an increase in p_i raises the cost of what the consumer buys *but also* raises the value of what she owns; the net cost rises by $h_i - \bar{x}_i$ (positive if she's a net buyer, negative if a net seller).

Step 3. Substitute back. Replace $\partial e/\partial p_i$ in Step 1:

$$\boxed{\frac{\partial x_j(p,m)}{\partial p_i} = \underbrace{\frac{\partial h_j(p,u)}{\partial p_i}}_{\text{substitution effect}} - \underbrace{\big[x_i(p,m) - \bar{x}_i\big] \cdot \frac{\partial x_j(p,m)}{\partial m}}_{\text{endowment-corrected wealth effect}}.}$$

Relative to the standard Slutsky equation, only the wealth-effect factor changes: it is scaled by the *net demand* $x_i - \bar{x}_i$ rather than by pure demand x_i. A net seller ($x_i < \bar{x}_i$) gains real wealth when p_i rises, so the sign of the wealth effect flips.

Answer. *Part (b): welfare of a lender when p_1 or p_2 falls.*

Step 1. Identify Dave's position. $x_1 < \bar{x}_1$ means he consumes less of good 1 than he owns, so he sells the excess: he is a *net seller* of good 1 (a lender if x_1 is present consumption) and, by budget balance, a *net buyer* of good 2.

Step 2. Effect of a fall in p_1 (seller's price falls). The endowment is worth less at the new prices: $m(p) = p_1\bar{x}_1 + p_2\bar{x}_2$ falls when p_1 falls. Geometrically, the new budget line still passes through the endowment $\bar{x}$ (Dave can always consume what he owns) but is *flatter* than before; along the x^* side of $\bar{x}$ the new line lies *below* the indifference curve through x^*, so x^* is no longer affordable. He is *worse off.*

Step 3. Effect of a fall in p_2 (buyer's price falls). Again the new budget line pivots about $\bar{x}$ but is now *steeper* than before. Part of it lies *above* the indifference curve through x^*, so bundles that were previously infeasible and yield strictly higher utility become affordable. He is *better off.*

Figure 1 illustrates the two cases with the numerical choice $(p_1^0, p_2^0) = (1,1)$, $\bar{x} = (5,2)$, and optimum $x^* = (2,5)$ on the original budget $B_0 : x_1 + x_2 = 7$; the indifference curve through x^* is drawn from $u(x_1, x_2) = x_1x_2$, i.e., $x_1x_2 = 10$.

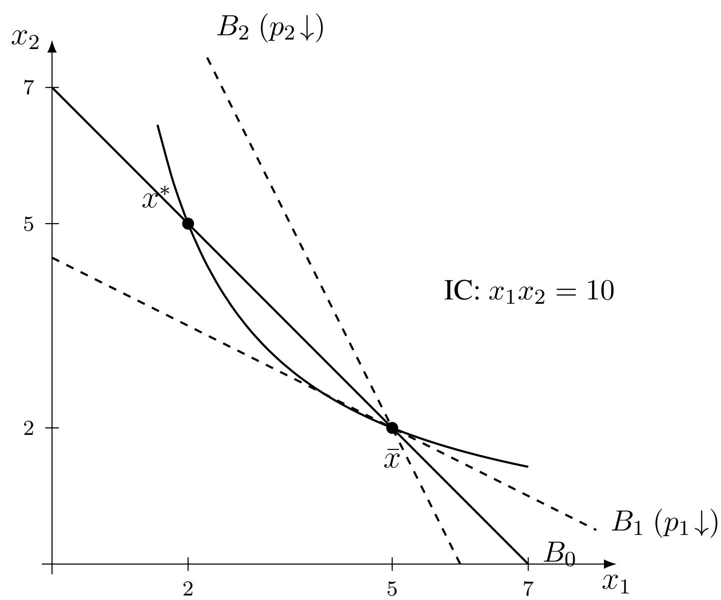

Figure 1: Endowment $\bar{x} = (5,2)$ and optimal bundle $x^* = (2,5)$ with $x_1^* < \bar{x}_1$ (Dave is a net seller of good 1). Both dashed lines pivot about $\bar{x}$: B_1 (flatter) is the new budget when p_1 falls, and it lies below the indifference curve through x^*, so Dave is worse off. B_2 (steeper) is the new budget when p_2 falls, and it lies above the IC through x^*, so Dave is better off.

5. **A piecewise-linear utility function.** Let $u(x_1, x_2) = \min\{x_2 + 2x_1,\ x_1 + 2x_2\}$.

   (a) Draw the indifference curve $u = 20$ and shade the upper contour set.

(b) For which values of p_1/p_2 is the unique optimum at $x_1 = 0$?

(c) For which values of p_1/p_2 is the unique optimum at $x_2 = 0$?

(d) If neither x_1 nor x_2 is zero and the optimum is unique, what must be the value of x_1/x_2?

Answer. *Part (a): the indifference curve $u = 20$.*

Step 1. Read the two linear pieces. $u(x) = 20$ requires the smaller of the two expressions $x_2 + 2x_1$ and $x_1 + 2x_2$ to equal 20 (and the other to be ≥ 20). Two candidate boundary lines:

- L_a: $x_2 + 2x_1 = 20$, i.e., $x_2 = 20 - 2x_1$ — y-intercept 20, slope -2.
- L_b: $x_1 + 2x_2 = 20$, i.e., $x_2 = (20 - x_1)/2$ — y-intercept 10, slope $-1/2$.

Step 2. Where the two lines cross. Solving $20 - 2x_1 = (20 - x_1)/2$ gives $x_1 = 20/3 \approx 6.67$, and then $x_2 = 20/3$. So the two lines meet at the kink $\big(20/3,\, 20/3\big)$, which lies on the 45° line $x_2 = x_1$.

Step 3. Which piece binds where. Along a candidate boundary, the binding constraint is the smaller of the two expressions. For small x_1 (up to 20/3), the smaller is $x_2 + 2x_1$, so the indifference curve follows L_a. For large x_1 (past 20/3), the smaller is $x_1 + 2x_2$, so the curve follows L_b. Figure 2 plots the resulting piecewise-linear curve.

Step 4. Upper contour set. The set $\{x : u(x) \geq 20\}$ requires *both* $x_2 + 2x_1 \geq 20$ and $x_1 + 2x_2 \geq 20$, i.e., the region northeast of both lines. This is the shaded region in Figure 2.

Answer. *Part (b): corner at $x_1 = 0$.*

Step 1. When is the corner optimal? The budget line has slope $-p_1/p_2$. If this is steeper (more negative) than the steepest segment of the indifference curve (slope -2), the budget line rotates past the kink and the tangency point sits on the vertical axis: $x_1 = 0$.

Step 2. Threshold. Steeper than -2 means $-p_1/p_2 < -2$, i.e.,

$$\frac{p_1}{p_2} > 2.$$

The optimum is $x^* = \big(0,\, w/p_2\big)$.

Answer. *Part (c): corner at $x_2 = 0$.*

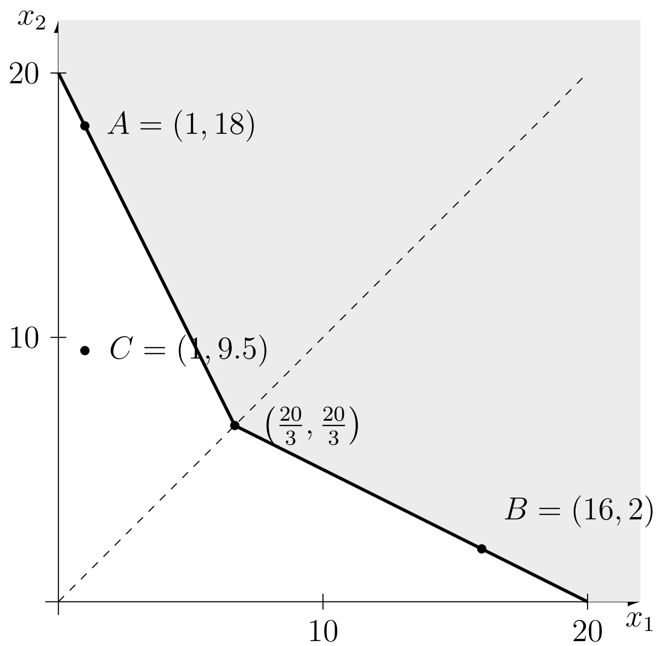

Figure 2: Indifference curve $u = 20$ (thick piecewise-linear curve, kinked at $\left(\frac{20}{3}, \frac{20}{3}\right)$ on the 45° line). The shaded region is the upper contour set. Bundle $A = (1, 18)$ has $u = \min\{18 + 2, 1 + 36\} = 20$; bundle $B = (16, 2)$ has $u = \min\{2 + 32, 16 + 4\} = 20$; bundle $C = (1, 9.5)$ has $u = \min\{9.5 + 2, 1 + 19\} = 11.5 < 20$, so C is below the indifference curve.

Symmetrically, if the budget line is flatter than the flatter segment (slope $-1/2$), the tangency sits on the horizontal axis. Flatter than $-1/2$ means $-p_1/p_2 > -1/2$, i.e.,

$$\frac{p_1}{p_2} < \frac{1}{2}.$$

The optimum is $x^* = \big(w/p_1,\, 0\big)$.

Answer. *Part (d): interior optimum at the kink.*

Step 1. The kink is the only interior optimum. A convex piecewise-linear indifference curve like this one has only a single point where it can be tangent to a strictly linear budget line (the kink itself); away from the kink each piece is linear. So an interior optimum must sit at the kink.

Step 2. Location of the kink. The kink is on $x_1 = x_2$, so

$$\frac{x_1}{x_2} = 1.$$

Step 3. Interior Walrasian demand. Setting $x_1 = x_2$ in the budget gives $(p_1 + p_2)x_2 = w$, hence

$$x_1(p, w) = x_2(p, w) = \frac{w}{p_1 + p_2},$$

valid on the range $\frac{1}{2} \le p_1/p_2 \le 2$ (so that neither corner from parts (b) or (c) is optimal).

Intuitively, the two linear pieces measure two ways of getting utility — adding a unit of x_1 is worth "2 units" via the first term and "1 unit" via the second (and vice versa for x_2). At the kink the two "ways" agree, and any deviation makes one of them the binding smaller expression.

6. **Savings under a binding cap (I.R.A.).** Under current tax law an individual can save up to \$2,000 per year in an Individual Retirement Account, a savings vehicle with favorable tax treatment. Split the individual's income Y across consumption C, I.R.A. savings S_1, and ordinary savings S_2. Take the reduced-form utility

$$U(C, S_1, S_2) = S_1^{\alpha}\, S_2^{\beta}\, C^{\gamma}, \qquad \alpha, \beta, \gamma > 0,$$

under the budget $C + S_1 + S_2 = Y$ and the I.R.A. contribution cap $S_1 \le L$.

(a) Derive S_1 and S_2 when the cap L is *not* binding.

(b) Derive S_1 and S_2 when the cap *is* binding.

Answer. *Part (a): cap not binding.*

Step 1. Ignore the inequality and solve as a Cobb-Douglas with three "goods." Log-transform: $\ln U = \alpha \ln S_1 + \beta \ln S_2 + \gamma \ln C$. The budget is $S_1 + S_2 + C = Y$. This is a standard three-good Cobb-Douglas at prices $(1, 1, 1)$, wealth Y, and normalized exponents summing to $\alpha + \beta + \gamma$.

Step 2. Read off the demands. The Cobb-Douglas rule gives each "good" its share of wealth:

$$S_1 = \frac{\alpha}{\alpha + \beta + \gamma} Y, \qquad S_2 = \frac{\beta}{\alpha + \beta + \gamma} Y, \qquad C = \frac{\gamma}{\alpha + \beta + \gamma} Y.$$

Step 3. When is this valid? The interior solution above is optimal provided the unconstrained S_1 satisfies $S_1 \le L$, i.e., provided

$$\frac{\alpha}{\alpha + \beta + \gamma} Y \;\le\; L \quad \Longleftrightarrow \quad Y \;\le\; \frac{\alpha + \beta + \gamma}{\alpha} L.$$

If income is modest enough that the "desired" I.R.A. contribution is under the cap, the cap is slack.

Answer. *Part (b): cap binding, $S_1 = L$.*

Step 1. Reduce the problem. When the cap binds, S_1 is pinned at L; the remaining wealth $Y - L$ must be split between C and S_2. The subproblem is

$$\max_{S_2, C \geq 0} \; L^{\alpha} S_2^{\beta} C^{\gamma} \qquad \text{s.t.} \qquad S_2 + C = Y - L.$$

L^{α} is a positive constant that drops out of the argmax.

Step 2. Two-good Cobb-Douglas. The remaining problem is a standard Cobb-Douglas in S_2, C with exponents β, γ and wealth $Y - L$. Each takes its share:

$$S_2 = \frac{\beta}{\beta + \gamma}\,(Y - L), \qquad C = \frac{\gamma}{\beta + \gamma}\,(Y - L), \qquad S_1 = L.$$

Step 3. Intuition. When the cap binds, the consumer would like to shovel more into the tax-advantaged account but can't, so the marginal dollar of savings falls into the ordinary account S_2 instead. Consumption also rises (in absolute terms C falls compared to the unconstrained case because $Y - L < Y$; but the ratio $C/(Y - L)$ is exactly the share $\gamma/(\beta + \gamma)$, higher than the pre-cap share $\gamma/(\alpha + \beta + \gamma)$, because the "missing" α share is re-allocated between C and S_2).

7. **Complement (WARP/SARP from budget data).** A single consumer is observed at three price-wealth pairs (with wealth normalized to $w_t = 1$ throughout), and her chosen bundle exhausts her wealth at each date, $p_t \cdot x_t = 1$. The observations are

$$p_1 = (1, 1),\; x_1 = \left(\tfrac{1}{2}, \tfrac{1}{2}\right); \quad p_2 = (2, 1),\; x_2 = \left(\tfrac{1}{4}, \tfrac{1}{2}\right); \quad p_3 = (2, 2),\; x_3 = \left(\tfrac{1}{8}, \tfrac{3}{8}\right).$$

Do the data satisfy the Weak Axiom (WARP)? Do they satisfy the Strong Axiom (SARP)? Use the matrix M with $M_{ij} = p_i \cdot x_j$ to organize the calculation.

Recall. Given the demand data:

- *WARP is not violated* between t and t' if $p_t \cdot x_{t'} \leq 1$ and $x_t \neq x_{t'}$ imply $p_{t'} \cdot x_t > 1$. Equivalently, if x_t was chosen while $x_{t'}$ was affordable, then x_t must not have been affordable when $x_{t'}$ was chosen.
- *SARP is not violated* if the same conclusion holds along every chain $t_1, t_2, \ldots, t_m$ with $p_{t_k} \cdot x_{t_{k+1}} \leq 1$ for $k < m$: at the endpoint, $p_{t_m} \cdot x_{t_1} > 1$.

Answer.

Step 1. Compute the matrix M. With $M_{ij} = p_i \cdot x_j$,

$$M = \begin{bmatrix} p_1 \cdot x_1 & p_1 \cdot x_2 & p_1 \cdot x_3 \\ p_2 \cdot x_1 & p_2 \cdot x_2 & p_2 \cdot x_3 \\ p_3 \cdot x_1 & p_3 \cdot x_2 & p_3 \cdot x_3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} \\ \frac{3}{2} & 1 & \frac{5}{8} \\ 2 & \frac{3}{2} & 1 \end{bmatrix}.$$

The diagonal is all 1's (Walras' law: $p_t \cdot x_t = 1$). An off-diagonal entry $M_{ij} \le 1$ says "bundle j was affordable when bundle i was chosen," revealing x_i strictly preferred to x_j.

Step 2. Read revealed preferences. The off-diagonal entries ≤ 1 are $M_{12} = 3/4$, $M_{13} = 1/2$, and $M_{23} = 5/8$. So

$$x_1 \succ x_2, \qquad x_1 \succ x_3, \qquad x_2 \succ x_3.$$

The reverse entries $M_{21} = 3/2$, $M_{31} = 2$, $M_{32} = 3/2$ all strictly exceed 1: none of the "lower-ranked" bundles was affordable when the higher-ranked one was chosen. Good.

Step 3. Check WARP. A WARP violation would require a pair (i, j) with both $M_{ij} \le 1$ and $M_{ji} \le 1$ (both directions revealed). Every pair of below-diagonal and above-diagonal entries here has exactly one ≤ 1 and one > 1; no reversal. **WARP is not violated.**

Step 4. Check SARP. A SARP violation would require a cycle in the revealed-preference graph. The revealed rankings from Step 2 are all "downward" ($1 \to 2 \to 3$) with no arrows going back, so the graph is acyclic. In particular, any chain returning to its starting index would have $p_{t_m} \cdot x_{t_1} > 1$. **SARP is not violated.**

Step 5. Interpret. The three observations are consistent with a single utility-maximizing consumer ranking her bundles $x_1 \succ x_2 \succ x_3$. Whenever a higher-ranked bundle was chosen, the lower-ranked ones were within budget; whenever a lower-ranked bundle was chosen, the higher-ranked ones were out of reach. Figure 3 shows this visually.

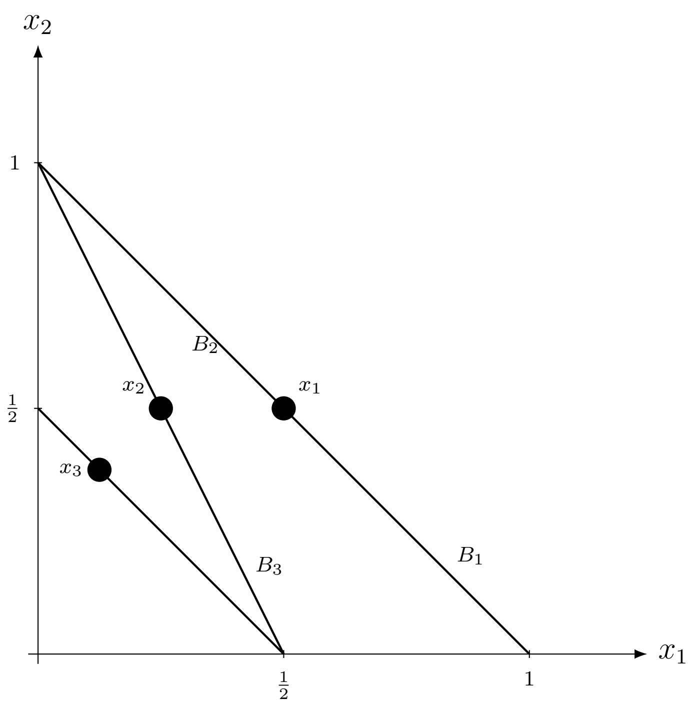

Figure 3: The three budget lines and their chosen bundles. x_2 and x_3 lie *on or below* B_1 (both were affordable when x_1 was chosen: $x_1 \succ x_2, x_3$); x_3 lies below B_2 ($x_2 \succ x_3$); x_1 is strictly outside B_2 and B_3, and x_2 is outside B_3, so no lower-ranked bundle could "buy back" a reversal. The revealed-preference arrows all point downward with no cycle: WARP and SARP hold.