

EconS 501 – Micro Theory I

Recitation #3 – Utility and Demand II¹

1. **Cobb-Douglas UMP.** Find the demanded bundle for a consumer whose utility function is $u(x_1, x_2) = x_1^{3/2}x_2$ facing the budget constraint $3x_1 + 4x_2 = 100$.

Answer. We work with the log of u (a strictly increasing transformation, so the argmax is the same) to turn the multiplicative FOCs into additive ones.

Step 1. Log-transform. Let $\tilde{u}(x_1, x_2) = \ln u(x_1, x_2) = \frac{3}{2} \ln x_1 + \ln x_2$. Then the UMP is

$$\max_{x_1, x_2 \geq 0} \frac{3}{2} \ln x_1 + \ln x_2 \quad \text{s.t.} \quad 3x_1 + 4x_2 = 100.$$

Step 2. Lagrangian and FOCs. Form $\mathcal{L} = \frac{3}{2} \ln x_1 + \ln x_2 - \lambda(3x_1 + 4x_2 - 100)$. The FOCs in x_1, x_2, λ read

$$\frac{3}{2x_1} = 3\lambda, \quad \frac{1}{x_2} = 4\lambda, \quad 3x_1 + 4x_2 = 100.$$

Step 3. Eliminate λ and solve. From the first two FOCs, $\lambda = \frac{1}{2x_1} = \frac{1}{4x_2}$, so $4x_2 = 2x_1$, i.e., $x_1 = 2x_2$. Substituting into the budget: $3(2x_2) + 4x_2 = 100$, i.e., $10x_2 = 100$, so

$x_1(3, 4, 100) = 20, \quad x_2(3, 4, 100) = 10.$

Step 4. Value of the Lagrange multiplier. From $\frac{1}{4x_2} = \frac{1}{40}$, the multiplier in the *transformed* problem is $\tilde{\lambda} = 1/40$. This measures the marginal utility of income *measured in “log-utils”*: $\partial \tilde{v} / \partial m = 1/40$, where $\tilde{v}(p, m) = \ln v(p, m)$.

If one instead solves the UMP directly (without the log-transform), the Lagrange multiplier is $\lambda = 20^{3/2}/4$ (a different scale, because $v = e^{\tilde{v}}$).

Intuitively, the log-transform never changes *what* is optimal, but it does change the units in which the Lagrange multiplier “prices” an additional dollar of wealth — something to remember whenever the interpretation of λ^* enters a discussion.

¹Felix Munoz-Garcia, School of Economic Sciences, Washington State University, 103H Hulbert Hall, Pullman, WA. Tel. 509-335-8402. Email: fmunoz@wsu.edu.

2. **The full duality machine.** For $u(x_1, x_2) = x_1^{1/2} x_2^{1/3}$ and the budget $p_1 x_1 + p_2 x_2 = m$, derive (a) the Walrasian demand, (b) the indirect utility function, (c) the Hicksian demand, and (d) the expenditure function.

Where we're going. The four objects are related by two identities:

$$h_\ell(p, u) = \frac{\partial e(p, u)}{\partial p_\ell} \quad (\text{Shephard's lemma}), \quad x(p, m) = h(p, v(p, m)) \quad (\text{the "primal-dual" link}).$$

We will use both. The plan is: solve UMP to get $x(p, m)$; plug into u to get $v(p, m)$; invert to get $e(p, u)$; differentiate to get $h(p, u)$.

Answer. Part (a): Walrasian demand $x(p, m)$.

Step 1. Lagrangian and FOCs. With $\mathcal{L} = x_1^{1/2} x_2^{1/3} - \lambda(p_1 x_1 + p_2 x_2 - m)$,

$$\frac{1}{2} x_1^{-1/2} x_2^{1/3} = \lambda p_1, \quad \frac{1}{3} x_1^{1/2} x_2^{-2/3} = \lambda p_2, \quad p_1 x_1 + p_2 x_2 = m.$$

Step 2. Tangency ratio. Dividing the two FOCs eliminates λ :

$$\frac{\frac{1}{2} x_2 / x_1}{\frac{1}{3} \cdot (\text{same base}) / x_2} = \frac{p_1}{p_2} \iff \frac{3x_2}{2x_1} = \frac{p_1}{p_2} \iff p_2 x_2 = \frac{2}{3} p_1 x_1.$$

Step 3. Use the budget. Substituting into $p_1 x_1 + p_2 x_2 = m$: $p_1 x_1 + \frac{2}{3} p_1 x_1 = m$, so $\frac{5}{3} p_1 x_1 = m$ and hence

$$\boxed{x_1(p, m) = \frac{3}{5} \cdot \frac{m}{p_1}, \quad x_2(p, m) = \frac{2}{5} \cdot \frac{m}{p_2}.$$

The expenditure shares are $3/5$ and $2/5$, matching the “normalized” Cobb-Douglas exponents $1/2$ and $1/3$ (which sum to $5/6$, then divide by $5/6$).

Answer. Part (b): indirect utility $v(p, m)$.

Plug the Walrasian demands into u :

$$v(p, m) = \left(\frac{3}{5} \frac{m}{p_1}\right)^{1/2} \left(\frac{2}{5} \frac{m}{p_2}\right)^{1/3} = \underbrace{\left(\frac{3}{5}\right)^{1/2} \left(\frac{2}{5}\right)^{1/3}}_{\text{constant}} \cdot \frac{m^{5/6}}{p_1^{1/2} p_2^{1/3}}.$$

So v is homogeneous of degree 0 in (p, m) (check: $m^{5/6}/(p_1^{1/2} p_2^{1/3})$ has exponents $5/6 - 1/2 - 1/3 = 0$) and strictly increasing in m , as Proposition 3.D.3 requires.

Answer. Part (c): expenditure function $e(p, u)$.

Step 1. Invert v . At the optimum, $u = v(p, m)$; treating m as the unknown and setting $m = e(p, u)$,

$$u = \left(\frac{3}{5}\right)^{1/2} \left(\frac{2}{5}\right)^{1/3} \frac{e^{5/6}}{p_1^{1/2} p_2^{1/3}}.$$

Step 2. Solve for e . Raise both sides to the $6/5$ power:

$$e(p, u) = \left(\frac{5}{3}\right)^{3/5} \left(\frac{5}{2}\right)^{2/5} p_1^{3/5} p_2^{2/5} u^{6/5}.$$

Homogeneous of degree 1 in p (exponents $3/5 + 2/5 = 1$) and strictly increasing in u , as Proposition 3.E.2 requires.

Answer. Part (d): Hicksian demand $h(p, u)$.

Apply Shephard's lemma, $h_\ell(p, u) = \partial e / \partial p_\ell$:

$h_1(p, u) = \left(\frac{5}{3}\right)^{3/5} \left(\frac{5}{2}\right)^{2/5} \cdot \frac{3}{5} p_1^{-2/5} p_2^{2/5} u^{6/5}, \quad h_2(p, u) = \left(\frac{5}{3}\right)^{3/5} \left(\frac{5}{2}\right)^{2/5} \cdot \frac{2}{5} p_1^{3/5} p_2^{-3/5} u^{6/5}.$

Sanity check. Each Hicksian demand is decreasing in its own price and increasing in the other's — the Cobb-Douglas substitution effect. The next exercise verifies the two round-trip identities that link these four objects.

3. **MWG 3.E.8.** For the Cobb-Douglas utility $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, verify the following two pairs of duality identities:

$$(3.E.1) \quad e(p, v(p, w)) = w \quad \text{and} \quad v(p, e(p, u)) = u;$$

$$(3.E.3) \quad h(p, u) = x(p, e(p, u)) \quad \text{and} \quad x(p, w) = h(p, v(p, w)).$$

The first pair says that expenditure and indirect utility invert each other; the second says that Hicksian and Walrasian demands coincide when wealth equals the minimum expenditure needed to reach utility u .

Building blocks. From Exercise 2 (or, for this exponent choice, directly), the four Cobb-Douglas objects are

$x_1(p, w) = \frac{\alpha w}{p_1},$ $v(p, w) = K \cdot \frac{w}{p_1^\alpha p_2^{1-\alpha}},$ $h_1(p, u) = \frac{\alpha}{K} u p_1^{\alpha-1} p_2^{1-\alpha},$	$x_2(p, w) = \frac{(1-\alpha)w}{p_2},$ $e(p, u) = \frac{u p_1^\alpha p_2^{1-\alpha}}{K},$ $h_2(p, u) = \frac{1-\alpha}{K} u p_1^\alpha p_2^{-\alpha},$
--	--

where $K \equiv \alpha^\alpha(1 - \alpha)^{1-\alpha}$ is a constant that does not depend on (p, w, u) .

Answer. *Identity 1:* $e(p, v(p, w)) = w$.

Step 1. Substitute $v(p, w)$ into e . Take $u = v(p, w) = Kw/(p_1^\alpha p_2^{1-\alpha})$ and plug into the formula for e :

$$e(p, v(p, w)) = \frac{v(p, w) \cdot p_1^\alpha p_2^{1-\alpha}}{K} = \frac{1}{K} \cdot \frac{Kw}{p_1^\alpha p_2^{1-\alpha}} \cdot p_1^\alpha p_2^{1-\alpha} = w.$$

Step 2. Read the identity off. The price factors cancel and the two K 's cancel; only w survives.

Answer. *Identity 2:* $v(p, e(p, u)) = u$.

Step 1. Substitute $e(p, u)$ into v . Take $w = e(p, u) = u p_1^\alpha p_2^{1-\alpha}/K$ and plug into v :

$$v(p, e(p, u)) = K \cdot \frac{e(p, u)}{p_1^\alpha p_2^{1-\alpha}} = K \cdot \frac{1}{p_1^\alpha p_2^{1-\alpha}} \cdot \frac{u p_1^\alpha p_2^{1-\alpha}}{K} = u.$$

Again the price factors and the K 's cancel, leaving u .

Answer. *Identity 3:* $h(p, u) = x(p, e(p, u))$. We check good 1; good 2 is analogous.

Step 1. Substitute $e(p, u)$ into the Walrasian x_1 . Set $w = e(p, u)$ in $x_1(p, w) = \alpha w/p_1$:

$$x_1(p, e(p, u)) = \frac{\alpha e(p, u)}{p_1} = \frac{\alpha}{p_1} \cdot \frac{u p_1^\alpha p_2^{1-\alpha}}{K} = \frac{\alpha}{K} u p_1^{\alpha-1} p_2^{1-\alpha}.$$

Step 2. Compare with $h_1(p, u)$. The formula for h_1 is exactly $\frac{\alpha}{K} u p_1^{\alpha-1} p_2^{1-\alpha}$, so

$$x_1(p, e(p, u)) = h_1(p, u). \quad \checkmark$$

The same calculation with α replaced by $1 - \alpha$ (and p_1, p_2 swapped in the appropriate slots) gives $x_2(p, e(p, u)) = h_2(p, u)$.

Answer. *Identity 4:* $x(p, w) = h(p, v(p, w))$. Again check good 1.

Step 1. Substitute $v(p, w)$ into the Hicksian h_1 . Set $u = v(p, w) = Kw/(p_1^\alpha p_2^{1-\alpha})$ in $h_1(p, u) = (\alpha/K) u p_1^{\alpha-1} p_2^{1-\alpha}$:

$$h_1(p, v(p, w)) = \frac{\alpha}{K} \cdot \frac{Kw}{p_1^\alpha p_2^{1-\alpha}} \cdot p_1^{\alpha-1} p_2^{1-\alpha} = \alpha w \cdot \frac{p_1^{\alpha-1}}{p_1^\alpha} = \frac{\alpha w}{p_1}.$$

Step 2. Compare with $x_1(p, w)$. The right-hand side is precisely $x_1(p, w) = \alpha w/p_1$.

Good 2 works out the same way, giving $h_2(p, v(p, w)) = (1 - \alpha)w/p_2 = x_2(p, w)$.

Intuitively, the identities in (3.E.1) express the fact that “the minimum wealth needed to reach the best utility affordable at w ” is w itself, and vice versa. The identities in (3.E.3) express the parallel fact for demands: if you have exactly the wealth needed to reach utility u , then whether you solve the UMP or the EMP, you end up choosing the same bundle. The two pairs together are what makes the primal–dual approach work: whichever side of the theory is easier to compute, you can always translate to the other.

4. **Slutsky with endowments (two-period consumption).** Dave has utility $u(x_1, x_2)$ over consumption in period 1 (x_1) and period 2 (x_2). He is endowed with $\bar{x} = (\bar{x}_1, \bar{x}_2)$, which he can consume as-is or trade at prices (p_1, p_2) . His budget constraint is

$$p_1 x_1 + p_2 x_2 = p_1 \bar{x}_1 + p_2 \bar{x}_2 \equiv m(p),$$

so *wealth is endogenous* — it depends on prices via the value of the endowment.

- (a) Derive the Slutsky equation in this endowment version of the model.
- (b) Suppose $\bar{x} = (\bar{x}_1, \bar{x}_2)$ and Dave’s optimal choice is such that $x_1 < \bar{x}_1$ (he is a *lender* of the good). If p_1 falls, is he better or worse off? What if p_2 falls?

Answer. *Part (a): the Slutsky equation with endowments.*

Step 1. Starting identity. The primal–dual link gives $h_j(p, u) = x_j(p, e(p, u))$: the Hicksian bundle and the Walrasian bundle coincide when wealth equals the minimum expenditure needed to reach utility u . Differentiate both sides with respect to p_i :

$$\frac{\partial h_j(p, u)}{\partial p_i} = \frac{\partial x_j}{\partial p_i}(p, e(p, u)) + \frac{\partial x_j}{\partial m}(p, e(p, u)) \cdot \frac{\partial e(p, u)}{\partial p_i}.$$

Step 2. Envelope theorem for $\partial e / \partial p_i$ WITH endowments. In the standard model, $\partial e / \partial p_i = h_i(p, u)$ (Shephard’s lemma). With endowments, the EMP is instead

$$e(p, u) = \min_x p \cdot (x - \bar{x}) \quad \text{s.t.} \quad u(x) \geq u,$$

so the objective differs from the standard one by the endowment term $-p \cdot \bar{x}$. Applying the envelope theorem gives

$$\frac{\partial e(p, u)}{\partial p_i} = h_i(p, u) - \bar{x}_i.$$

Interpretation: an increase in p_i raises the cost of what the consumer buys *but also* raises the value of what she owns; the net cost rises by $h_i - \bar{x}_i$ (positive if she’s a net buyer, negative if a net seller).

Step 3. Substitute back. Replace $\partial e/\partial p_i$ in Step 1:

$$\boxed{\frac{\partial x_j(p, m)}{\partial p_i} = \underbrace{\frac{\partial h_j(p, u)}{\partial p_i}}_{\text{substitution effect}} - \underbrace{[x_i(p, m) - \bar{x}_i] \cdot \frac{\partial x_j(p, m)}{\partial m}}_{\text{endowment-corrected wealth effect}}.}$$

Relative to the standard Slutsky equation, only the wealth-effect factor changes: it is scaled by the *net demand* $x_i - \bar{x}_i$ rather than by pure demand x_i . A net seller ($x_i < \bar{x}_i$) gains real wealth when p_i rises, so the sign of the wealth effect flips.

Answer. *Part (b): welfare of a lender when p_1 or p_2 falls.*

Step 1. Identify Dave's position. $x_1 < \bar{x}_1$ means he consumes less of good 1 than he owns, so he sells the excess: he is a *net seller* of good 1 (a lender if x_1 is present consumption) and, by budget balance, a *net buyer* of good 2.

Step 2. Effect of a fall in p_1 (seller's price falls). The endowment is worth less at the new prices: $m(p) = p_1\bar{x}_1 + p_2\bar{x}_2$ falls when p_1 falls. Geometrically, the new budget line still passes through the endowment $\bar{x}$ (Dave can always consume what he owns) but is *flatter* than before; along the x^* side of $\bar{x}$ the new line lies *below* the indifference curve through x^* , so x^* is no longer affordable. He is *worse off*.

Step 3. Effect of a fall in p_2 (buyer's price falls). Again the new budget line pivots about $\bar{x}$ but is now *steeper* than before. Part of it lies *above* the indifference curve through x^* , so bundles that were previously infeasible and yield strictly higher utility become affordable. He is *better off*.

Figure 1 illustrates the two cases with the numerical choice $(p_1^0, p_2^0) = (1, 1)$, $\bar{x} = (5, 2)$, and optimum $x^* = (2, 5)$ on the original budget $B_0 : x_1 + x_2 = 7$; the indifference curve through x^* is drawn from $u(x_1, x_2) = x_1x_2$, i.e., $x_1x_2 = 10$.

5. **A piecewise-linear utility function.** Let $u(x_1, x_2) = \min\{x_2 + 2x_1, x_1 + 2x_2\}$.

- (a) Draw the indifference curve $u = 20$ and shade the upper contour set.
- (b) For which values of p_1/p_2 is the unique optimum at $x_1 = 0$?
- (c) For which values of p_1/p_2 is the unique optimum at $x_2 = 0$?
- (d) If neither x_1 nor x_2 is zero and the optimum is unique, what must be the value of x_1/x_2 ?

Answer. *Part (a): the indifference curve $u = 20$.*

Step 1. Read the two linear pieces. $u(x) = 20$ requires the smaller of the two expressions $x_2 + 2x_1$ and $x_1 + 2x_2$ to equal 20 (and the other to be ≥ 20). Two candidate boundary lines:

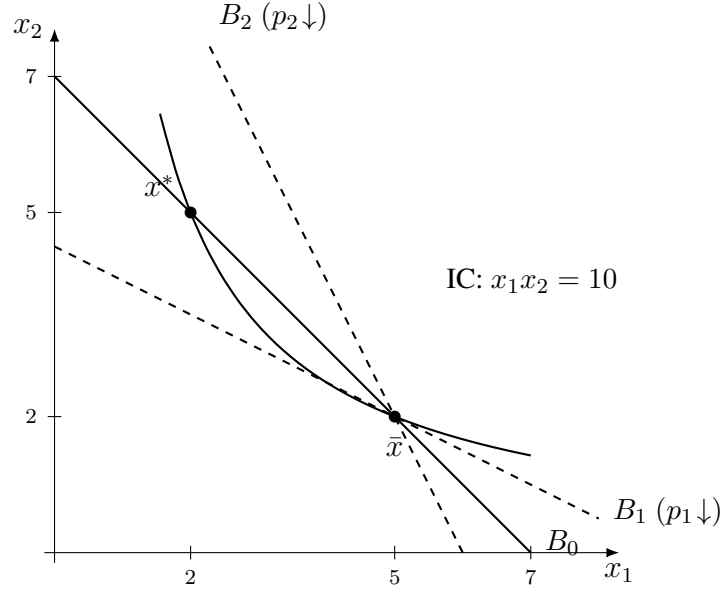


Figure 1: Endowment $\bar{x} = (5, 2)$ and optimal bundle $x^* = (2, 5)$ with $x_1^* < \bar{x}_1$ (Dave is a net seller of good 1). Both dashed lines pivot about $\bar{x}$: B_1 (flatter) is the new budget when p_1 falls, and it lies below the indifference curve through x^* , so Dave is worse off. B_2 (steeper) is the new budget when p_2 falls, and it lies above the IC through x^* , so Dave is better off.

- L_a : $x_2 + 2x_1 = 20$, i.e., $x_2 = 20 - 2x_1$ — y -intercept 20, slope -2 .
- L_b : $x_1 + 2x_2 = 20$, i.e., $x_2 = (20 - x_1)/2$ — y -intercept 10, slope $-1/2$.

Step 2. Where the two lines cross. Solving $20 - 2x_1 = (20 - x_1)/2$ gives $x_1 = 20/3 \approx 6.67$, and then $x_2 = 20/3$. So the two lines meet at the kink $(20/3, 20/3)$, which lies on the 45° line $x_2 = x_1$.

Step 3. Which piece binds where. Along a candidate boundary, the binding constraint is the smaller of the two expressions. For small x_1 (up to $20/3$), the smaller is $x_2 + 2x_1$, so the indifference curve follows L_a . For large x_1 (past $20/3$), the smaller is $x_1 + 2x_2$, so the curve follows L_b . Figure 2 plots the resulting piecewise-linear curve.

Step 4. Upper contour set. The set $\{x : u(x) \geq 20\}$ requires *both* $x_2 + 2x_1 \geq 20$ and $x_1 + 2x_2 \geq 20$, i.e., the region northeast of both lines. This is the shaded region in Figure 2.

Answer. Part (b): corner at $x_1 = 0$.

Step 1. When is the corner optimal? The budget line has slope $-p_1/p_2$. If this is steeper (more negative) than the steepest segment of the indifference curve (slope -2), the budget line rotates past the kink and the tangency point sits on the vertical axis: $x_1 = 0$.

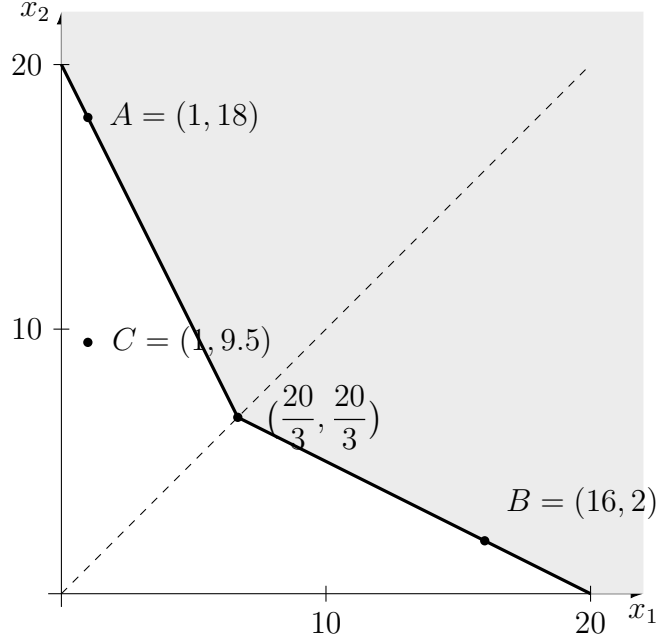


Figure 2: Indifference curve $u = 20$ (thick piecewise-linear curve, kinked at $(\frac{20}{3}, \frac{20}{3})$ on the 45° line). The shaded region is the upper contour set. Bundle $A = (1, 18)$ has $u = \min\{18 + 2, 1 + 36\} = 20$; bundle $B = (16, 2)$ has $u = \min\{2 + 32, 16 + 4\} = 20$; bundle $C = (1, 9.5)$ has $u = \min\{9.5 + 2, 1 + 19\} = 11.5 < 20$, so C is below the indifference curve.

Step 2. Threshold. Steeper than -2 means $-p_1/p_2 < -2$, i.e.,

$$\frac{p_1}{p_2} > 2.$$

The optimum is $x^* = (0, w/p_2)$.

Answer. *Part (c): corner at $x_2 = 0$.*

Symmetrically, if the budget line is flatter than the flatter segment (slope $-1/2$), the tangency sits on the horizontal axis. Flatter than $-1/2$ means $-p_1/p_2 > -1/2$, i.e.,

$$\frac{p_1}{p_2} < \frac{1}{2}.$$

The optimum is $x^* = (w/p_1, 0)$.

Answer. *Part (d): interior optimum at the kink.*

Step 1. The kink is the only interior optimum. A convex piecewise-linear indifference curve like this one has only a single point where it can be tangent to a strictly linear budget line (the kink itself); away from the kink each piece is linear. So an interior optimum must sit at the kink.

Step 2. Location of the kink. The kink is on $x_1 = x_2$, so

$$\frac{x_1}{x_2} = 1.$$

Step 3. Interior Walrasian demand. Setting $x_1 = x_2$ in the budget gives $(p_1 + p_2)x_2 = w$, hence

$$x_1(p, w) = x_2(p, w) = \frac{w}{p_1 + p_2},$$

valid on the range $\frac{1}{2} \leq p_1/p_2 \leq 2$ (so that neither corner from parts (b) or (c) is optimal).

Intuitively, the two linear pieces measure two ways of getting utility — adding a unit of x_1 is worth “2 units” via the first term and “1 unit” via the second (and vice versa for x_2). At the kink the two “ways” agree, and any deviation makes one of them the binding smaller expression.

6. **Savings under a binding cap (I.R.A.).** Under current tax law an individual can save up to \$2,000 per year in an Individual Retirement Account, a savings vehicle with favorable tax treatment. Split the individual’s income Y across consumption C , I.R.A. savings S_1 , and ordinary savings S_2 . Take the reduced-form utility

$$U(C, S_1, S_2) = S_1^\alpha S_2^\beta C^\gamma, \quad \alpha, \beta, \gamma > 0,$$

under the budget $C + S_1 + S_2 = Y$ and the I.R.A. contribution cap $S_1 \leq L$.

- (a) Derive S_1 and S_2 when the cap L is *not* binding.
- (b) Derive S_1 and S_2 when the cap *is* binding.

Answer. *Part (a): cap not binding.*

Step 1. Ignore the inequality and solve as a Cobb-Douglas with three “goods.” Log-transform: $\ln U = \alpha \ln S_1 + \beta \ln S_2 + \gamma \ln C$. The budget is $S_1 + S_2 + C = Y$. This is a standard three-good Cobb-Douglas at prices $(1, 1, 1)$, wealth Y , and normalized exponents summing to $\alpha + \beta + \gamma$.

Step 2. Read off the demands. The Cobb-Douglas rule gives each “good” its share of wealth:

$$S_1 = \frac{\alpha}{\alpha + \beta + \gamma} Y, \quad S_2 = \frac{\beta}{\alpha + \beta + \gamma} Y, \quad C = \frac{\gamma}{\alpha + \beta + \gamma} Y.$$

Step 3. When is this valid? The interior solution above is optimal provided the unconstrained S_1 satisfies $S_1 \leq L$, i.e., provided

$$\frac{\alpha}{\alpha + \beta + \gamma} Y \leq L \iff Y \leq \frac{\alpha + \beta + \gamma}{\alpha} L.$$

If income is modest enough that the “desired” I.R.A. contribution is under the cap, the cap is slack.

Answer. *Part (b): cap binding, $S_1 = L$.*

Step 1. Reduce the problem. When the cap binds, S_1 is pinned at L ; the remaining wealth $Y - L$ must be split between C and S_2 . The subproblem is

$$\max_{S_2, C \geq 0} L^\alpha S_2^\beta C^\gamma \quad \text{s.t.} \quad S_2 + C = Y - L.$$

L^α is a positive constant that drops out of the argmax.

Step 2. Two-good Cobb-Douglas. The remaining problem is a standard Cobb-Douglas in S_2, C with exponents β, γ and wealth $Y - L$. Each takes its share:

$$S_2 = \frac{\beta}{\beta + \gamma} (Y - L), \quad C = \frac{\gamma}{\beta + \gamma} (Y - L), \quad S_1 = L.$$

Step 3. Intuition. When the cap binds, the consumer would like to shovel more into the tax-advantaged account but can’t, so the marginal dollar of savings falls into the ordinary account S_2 instead. Consumption also rises (in absolute terms C falls compared to the unconstrained case because $Y - L < Y$; but the ratio $C/(Y - L)$ is exactly the share $\gamma/(\beta + \gamma)$, higher than the pre-cap share $\gamma/(\alpha + \beta + \gamma)$, because the “missing” α share is re-allocated between C and S_2).

7. **Complement (WARP/SARP from budget data).** A single consumer is observed at three price-wealth pairs (with wealth normalized to $w_t = 1$ throughout), and her chosen bundle exhausts her wealth at each date, $p_t \cdot x_t = 1$. The observations are

$$p_1 = (1, 1), \quad x_1 = \left(\frac{1}{2}, \frac{1}{2}\right); \quad p_2 = (2, 1), \quad x_2 = \left(\frac{1}{4}, \frac{1}{2}\right); \quad p_3 = (2, 2), \quad x_3 = \left(\frac{1}{8}, \frac{3}{8}\right).$$

Do the data satisfy the Weak Axiom (WARP)? Do they satisfy the Strong Axiom (SARP)? Use the matrix M with $M_{ij} = p_i \cdot x_j$ to organize the calculation.

Recall. Given the demand data:

- *WARP is not violated* between t and t' if $p_t \cdot x_{t'} \leq 1$ and $x_t \neq x_{t'}$ imply $p_{t'} \cdot x_t > 1$. Equivalently, if x_t was chosen while $x_{t'}$ was affordable, then x_t must not have been affordable when $x_{t'}$ was chosen.
- *SARP is not violated* if the same conclusion holds along every chain $t_1, t_2, \dots, t_m$ with $p_{t_k} \cdot x_{t_{k+1}} \leq 1$ for $k < m$: at the endpoint, $p_{t_m} \cdot x_{t_1} > 1$.

Answer.

Step 1. Compute the matrix M . With $M_{ij} = p_i \cdot x_j$,

$$M = \begin{bmatrix} p_1 \cdot x_1 & p_1 \cdot x_2 & p_1 \cdot x_3 \\ p_2 \cdot x_1 & p_2 \cdot x_2 & p_2 \cdot x_3 \\ p_3 \cdot x_1 & p_3 \cdot x_2 & p_3 \cdot x_3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} \\ \frac{3}{2} & 1 & \frac{5}{8} \\ 2 & \frac{3}{2} & 1 \end{bmatrix}.$$

The diagonal is all 1's (Walras' law: $p_t \cdot x_t = 1$). An off-diagonal entry $M_{ij} \leq 1$ says "bundle j was affordable when bundle i was chosen," revealing x_i strictly preferred to x_j .

Step 2. Read revealed preferences. The off-diagonal entries ≤ 1 are $M_{12} = 3/4$, $M_{13} = 1/2$, and $M_{23} = 5/8$. So

$$x_1 \succ x_2, \quad x_1 \succ x_3, \quad x_2 \succ x_3.$$

The reverse entries $M_{21} = 3/2$, $M_{31} = 2$, $M_{32} = 3/2$ all strictly exceed 1: none of the "lower-ranked" bundles was affordable when the higher-ranked one was chosen. Good.

Step 3. Check WARP. A WARP violation would require a pair (i, j) with both $M_{ij} \leq 1$ and $M_{ji} \leq 1$ (both directions revealed). Every pair of below-diagonal and above-diagonal entries here has exactly one ≤ 1 and one > 1 ; no reversal. **WARP is not violated.**

Step 4. Check SARP. A SARP violation would require a cycle in the revealed-preference graph. The revealed rankings from Step 2 are all "downward" ($1 \rightarrow 2 \rightarrow 3$) with no arrows going back, so the graph is acyclic. In particular, any chain returning to its starting index would have $p_{t_m} \cdot x_{t_1} > 1$. **SARP is not violated.**

Step 5. Interpret. The three observations are consistent with a single utility-maximizing consumer ranking her bundles $x_1 \succ x_2 \succ x_3$. Whenever a higher-ranked bundle was chosen, the lower-ranked ones were within budget; whenever a lower-ranked bundle was chosen, the higher-ranked ones were out of reach. Figure 3 shows this visually.

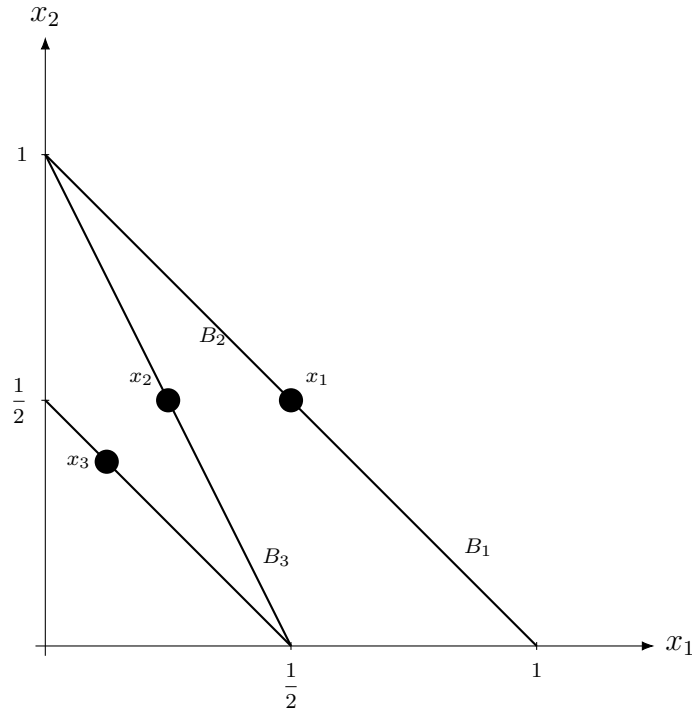


Figure 3: The three budget lines and their chosen bundles. x_2 and x_3 lie *on or below* B_1 (both were affordable when x_1 was chosen: $x_1 \succ x_2, x_3$); x_3 lies below B_2 ($x_2 \succ x_3$); x_1 is strictly outside B_2 and B_3 , and x_2 is outside B_3 , so no lower-ranked bundle could “buy back” a reversal. The revealed-preference arrows all point downward with no cycle: WARP and SARP hold.