EconS 501 – Micro Theory I
Recitation #2 – Utility and Demand I[1]

1. **MWG 3.D.1.** Verify that the Walrasian demand function generated by the Cobb-Douglas utility function satisfies the following three properties:

   (i) *Homogeneity of degree zero in* (p, w)*:* $x(\alpha p, \alpha w) = x(p, w)$ for every (p, w) and every scalar $\alpha > 0$.

   (ii) *Walras' law:* $p \cdot x = w$ for every $x \in x(p, w)$.

   (iii) *Convexity/uniqueness:* If preferences are convex (so u is quasiconcave), $x(p, w)$ is a convex set; if preferences are strictly convex (so u is strictly quasiconcave), $x(p, w)$ is a single point.

   Setup. Take $u(x_1, x_2) = A\, x_1^{\alpha} x_2^{1-\alpha}$ with $A > 0$ and $\alpha \in (0, 1)$.[2] The Utility Maximization Problem (UMP) reads

$$\max_{x_1, x_2 \geq 0} \; A\, x_1^{\alpha} x_2^{1-\alpha} \qquad \text{subject to} \qquad p_1 x_1 + p_2 x_2 = w.$$

   Answer. *Deriving the Walrasian demand.*

   Step 1. First-order conditions. Using the Lagrangian $\mathcal{L} = A\, x_1^{\alpha} x_2^{1-\alpha} - \lambda\big(p_1 x_1 + p_2 x_2 - w\big)$, the FOCs are

$$\alpha A\, x_1^{\alpha-1} x_2^{1-\alpha} = \lambda\, p_1, \qquad (1-\alpha) A\, x_1^{\alpha} x_2^{-\alpha} = \lambda\, p_2.$$

   Step 2. Tangency condition. Dividing the two FOCs,

$$\frac{\alpha}{1-\alpha} \cdot \frac{x_2}{x_1} = \frac{p_1}{p_2} \qquad \Longleftrightarrow \qquad p_2 x_2 = \frac{1-\alpha}{\alpha}\, p_1 x_1.$$

   Step 3. Substitute into the budget. Plugging $p_2 x_2 = \frac{1-\alpha}{\alpha} p_1 x_1$ into $p_1 x_1 + p_2 x_2 = w$ gives $\frac{1}{\alpha} p_1 x_1 = w$, so

$$\boxed{x_1(p, w) = \frac{\alpha\, w}{p_1}, \qquad x_2(p, w) = \frac{(1-\alpha)\, w}{p_2}.}$$

   Each Walrasian demand is a fixed share of wealth divided by its own price — the well-known constant-expenditure-share property of Cobb-Douglas.

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[2] The results all extend to the more general Cobb-Douglas $u(x) = A\, x_1^{\alpha} x_2^{\beta}$; see MWG 3.D.6.

Answer. *Property (i): homogeneity of degree zero in (p,w).* Scale prices and wealth by any $\lambda > 0$:

$$x_1(\lambda p, \lambda w) = \frac{\alpha\,\lambda w}{\lambda p_1} = \frac{\alpha w}{p_1} = x_1(p,w), \qquad x_2(\lambda p, \lambda w) = \frac{(1-\alpha)\,\lambda w}{\lambda p_2} = x_2(p,w).$$

The λ's cancel. *Intuitively,* a proportional rescaling of every price and of wealth changes only the units of account and leaves the real budget set — and therefore the optimal choice — untouched.

Answer. *Property (ii): Walras' law.* Plug the demands into $p_1x_1 + p_2x_2$:

$$p_1\,\frac{\alpha w}{p_1} + p_2\,\frac{(1-\alpha)w}{p_2} = \alpha w + (1-\alpha)w = w.$$

The consumer spends her entire wealth. *Intuitively,* Cobb-Douglas preferences are strictly monotone, so leaving any income unspent is never optimal.

Answer. *Property (iii): convexity and uniqueness.*

Step 1. Convexity of preferences via the slope of the indifference curve. Along an indifference curve $u(x_1,x_2) = \bar{u}$, the slope is

$$\left.\frac{dx_2}{dx_1}\right|_{u=\bar{u}} = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{\alpha A x_1^{\alpha-1}x_2^{1-\alpha}}{(1-\alpha)A x_1^{\alpha}x_2^{-\alpha}} = -\frac{\alpha}{1-\alpha}\cdot\frac{x_2}{x_1}.$$

Preferences are (strictly) convex if this slope *increases* (becomes less negative) as x_1 rises along the curve.

Step 2. Check the sign of the derivative. Along the indifference curve, an increase in x_1 is offset by a decrease in x_2; because the slope depends on x_2/x_1, an increase in x_1 shrinks the ratio x_2/x_1 and hence shrinks $|dx_2/dx_1|$. Formally,

$$\frac{\partial}{\partial x_1}\left[-\frac{\alpha}{1-\alpha}\cdot\frac{x_2}{x_1}\right] = \frac{\alpha}{1-\alpha}\cdot\frac{x_2}{x_1^2} > 0.$$

The slope of the indifference curve is strictly increasing in x_1, so indifference curves are bowed toward the origin. Preferences are *strictly* convex.

Step 3. Uniqueness. From the closed-form expressions in the boxed formula, at any (p,w) the pair $\big(x_1(p,w), x_2(p,w)\big)$ is uniquely pinned down — the Walrasian demand is a single point, not a set. This is no accident: a strictly convex indifference curve is tangent to the (linear) budget line at exactly one point.

2. **MWG 3.D.2.** Verify that the indirect utility function

$$v(p,w) = \big[\alpha\ln\alpha + (1-\alpha)\ln(1-\alpha)\big] + \ln w - \alpha\ln p_1 - (1-\alpha)\ln p_2$$

satisfies:

(i) Homogeneous of degree zero in (p, w);

(ii) Strictly increasing in w and non-increasing in p_ℓ for every ℓ;

(iii) Quasiconvex in (p, w); that is, the set $\{(p, w) : v(p, w) \le \bar{v}\}$ is convex for every $\bar{v}$;

(iv) Continuous in (p, w).

Where does v come from? It is the indirect utility of a Cobb-Douglas consumer with $u = x_1^\alpha x_2^{1-\alpha}$. Plugging the Walrasian demands from Exercise 1 into the log-transformed utility $\ln u = \alpha \ln x_1 + (1-\alpha)\ln x_2$ delivers exactly the formula above.

Handy shorthand. Let $K \equiv \alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)$ (a constant that does not depend on p or w), so

$$v(p, w) = K + \ln w - \alpha \ln p_1 - (1-\alpha)\ln p_2.$$

Answer. *Property (i): homogeneity of degree zero.* For $\lambda > 0$,

$$\begin{aligned} v(\lambda p, \lambda w) &= K + \ln(\lambda w) - \alpha \ln(\lambda p_1) - (1-\alpha)\ln(\lambda p_2) \\ &= K + \big[\ln \lambda + \ln w\big] - \alpha\big[\ln \lambda + \ln p_1\big] - (1-\alpha)\big[\ln \lambda + \ln p_2\big] \\ &= K + \ln w - \alpha \ln p_1 - (1-\alpha)\ln p_2 + \underbrace{\big[1 - \alpha - (1-\alpha)\big]}_{=0} \ln \lambda \\ &= v(p, w). \end{aligned}$$

The $\ln \lambda$ terms cancel because the coefficients $\{1, -\alpha, -(1-\alpha)\}$ sum to zero.

Answer. *Property (ii): monotonicity.* Compute the partial derivatives:

$$\frac{\partial v}{\partial w} = \frac{1}{w} > 0, \qquad \frac{\partial v}{\partial p_1} = -\frac{\alpha}{p_1} < 0, \qquad \frac{\partial v}{\partial p_2} = -\frac{1-\alpha}{p_2} < 0.$$

Hence v is strictly increasing in w and strictly decreasing in each p_ℓ (so a fortiori non-increasing). *Intuitively,* more wealth relaxes the budget, and higher prices tighten it.

Answer. *Property (iii): quasiconvexity in (p, w).*

Step 1. What we need to show. Fix any $\bar{v}$. The lower contour set

$$\mathcal{L}(\bar{v}) = \big\{(p, w) : v(p, w) \le \bar{v}\big\}$$

must be convex.

Step 2. Rewrite the level condition in a friendlier form. Because exp is strictly increasing, $v(p, w) \le \bar{v}$ is equivalent to $\exp\big(v(p, w) - K\big) \le \exp(\bar{v} - K)$, i.e.,

$$\frac{w}{p_1^\alpha \, p_2^{1-\alpha}} \le C, \qquad \text{where } C \equiv \exp(\bar{v} - K).$$

Equivalently,

$$w \;\le\; C\cdot \underbrace{p_1^{\alpha}\,p_2^{1-\alpha}}_{\equiv\,\phi(p)}.$$

Step 3. The price aggregator $\phi(p)$ is concave in p. With $\alpha+(1-\alpha)=1$, $\phi(p)=p_1^{\alpha}p_2^{1-\alpha}$ is a Cobb-Douglas function of p_1, p_2. Such a function is concave on the positive orthant (this is a standard fact: its log is $\alpha\ln p_1+(1-\alpha)\ln p_2$, a concave function; and any function whose log is concave and that is nonnegative is itself concave when its exponents sum to at most one). Hence

$$\phi\big(\lambda p+(1-\lambda)p'\big) \;\ge\; \lambda\,\phi(p)+(1-\lambda)\,\phi(p').$$

Step 4. Show $\mathcal{L}(\bar v)$ is convex. Take any $(p,w),(p',w')\in\mathcal{L}(\bar v)$: by Step 2, $w\le C\,\phi(p)$ and $w'\le C\,\phi(p')$. For any $\lambda\in[0,1]$, form $(p'',w'')=\lambda(p,w)+(1-\lambda)(p',w')$. Then

$$w''=\lambda w+(1-\lambda)w' \;\le\; C\big[\lambda\,\phi(p)+(1-\lambda)\,\phi(p')\big] \;\le\; C\,\phi(p''),$$

where the second inequality uses concavity of ϕ from Step 3. Hence $(p'',w'')\in\mathcal{L}(\bar v)$, so the set is convex.

Intuitively, the level set of v is the set of (p,w) pairs delivering utility at most $\bar v$. The Cobb-Douglas indirect utility rises linearly in wealth and is a concave (Cobb-Douglas) aggregate of prices; so "below-$\bar v$" is bounded above by the concave curve $w=C\,\phi(p)$, and a set below a concave curve is convex.

Answer. *Property (iv): continuity.* v is a sum of the continuous function $\ln w$ and two continuous functions $-\alpha\ln p_1$, $-(1-\alpha)\ln p_2$ on the open orthant $\{(p,w): p_1,p_2,w>0\}$, so v itself is continuous there.

3. **MWG 3.D.6.** Consider a three-good consumer with utility

$$u(x)=(x_1-b_1)^{\alpha}(x_2-b_2)^{\beta}(x_3-b_3)^{\gamma},\qquad \alpha,\beta,\gamma>0,$$

defined on the region $x_\ell>b_\ell$ for $\ell=1,2,3$. Here b_ℓ is a "subsistence" quantity of good ℓ.

(a) Why can you assume without loss of generality that $\alpha+\beta+\gamma=1$? Do so for the rest of the problem.

(b) Write the FOCs for the UMP and derive the Walrasian demand and indirect utility. (This is the Linear Expenditure System, due to Stone, 1954.)

(c) Verify that these functions satisfy Propositions 3.D.2 and 3.D.3.

Answer. *Part (a): normalizing $\alpha+\beta+\gamma=1$.*

Step 1. Consider a monotone transformation of u. Let

$$\widetilde{u}(x) = u(x)^{1/(\alpha+\beta+\gamma)} = (x_1-b_1)^{\tilde{\alpha}}(x_2-b_2)^{\tilde{\beta}}(x_3-b_3)^{\tilde{\gamma}}, \qquad \tilde{\alpha} = \frac{\alpha}{\alpha+\beta+\gamma},\ \tilde{\beta} = \frac{\beta}{\alpha+\beta+\gamma},\ \tilde{\gamma} = \frac{\gamma}{\alpha+\beta+\gamma}$$

By construction, $\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma}=1$.

Step 2. $\widetilde{u}$ represents the same preferences. Since $u \geq 0$ on the region of interest and $t \mapsto t^{1/(\alpha+\beta+\gamma)}$ is strictly increasing on $[0,\infty)$, $\widetilde{u}$ is a strictly monotone transformation of u, so both represent the same preference relation.

Hence we lose no generality by working with the exponents $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ summing to 1; rename them α,β,γ from now on.

Answer. *Part (b): FOCs and closed-form demand.*

Step 1. Log-transform. Working with the monotone transformation

$$\ln u(x) = \alpha\ln(x_1-b_1) + \beta\ln(x_2-b_2) + \gamma\ln(x_3-b_3)$$

simplifies the FOCs without changing the argmax.

Step 2. Lagrangian and FOCs. With Lagrangian $\mathcal{L} = \ln u(x) - \mu\big(\sum_\ell p_\ell x_\ell - w\big)$, the FOCs are

$$\frac{\alpha}{x_1-b_1} = \mu p_1, \qquad \frac{\beta}{x_2-b_2} = \mu p_2, \qquad \frac{\gamma}{x_3-b_3} = \mu p_3,$$

together with the budget $p_1x_1+p_2x_2+p_3x_3=w$.

Step 3. Solve for expenditure on each good above subsistence. Rearranging the FOCs,

$$p_1(x_1-b_1) = \frac{\alpha}{\mu}, \qquad p_2(x_2-b_2) = \frac{\beta}{\mu}, \qquad p_3(x_3-b_3) = \frac{\gamma}{\mu}.$$

Sum all three, apply $\alpha+\beta+\gamma=1$, and use the budget:

$$\underbrace{\sum_\ell p_\ell x_\ell}_{=w} - \underbrace{\sum_\ell p_\ell b_\ell}_{\equiv p\cdot b} = \frac{\alpha+\beta+\gamma}{\mu} = \frac{1}{\mu} \quad\Longleftrightarrow\quad \frac{1}{\mu} = w - p\cdot b.$$

Step 4. Substitute back to obtain the Walrasian demand. For $\ell=1,2,3$,

$$\boxed{x_\ell(p,w) = b_\ell + \frac{s_\ell}{p_\ell}\big(w-p\cdot b\big), \qquad s_1=\alpha,\ s_2=\beta,\ s_3=\gamma.}$$

The consumer first buys the "subsistence" bundle b, then splits the leftover "discretionary income" $w-p\cdot b$ into the fixed shares α,β,γ across the three goods.

Step 5. *Indirect utility.* Plug into the objective $u(x) = \prod_\ell (x_\ell - b_\ell)^{s_\ell}$:

$$v(p,w) = \prod_{\ell=1}^{3} \Big(\frac{s_\ell(w - p\cdot b)}{p_\ell}\Big)^{s_\ell} = \big(w - p\cdot b\big)^{\alpha+\beta+\gamma} \prod_{\ell=1}^{3} \Big(\frac{s_\ell}{p_\ell}\Big)^{s_\ell} = \big(w - p\cdot b\big)\cdot \frac{\alpha^\alpha \beta^\beta \gamma^\gamma}{p_1^\alpha p_2^\beta p_3^\gamma}.$$

Answer. *Part (c): checking Propositions 3.D.2 and 3.D.3.*

Step 1. *Homogeneity of degree zero of demand.* Scale by $\lambda > 0$: $p \to \lambda p$, $w \to \lambda w$. Then $\lambda w - (\lambda p)\cdot b = \lambda(w - p\cdot b)$, so

$$x_\ell(\lambda p, \lambda w) = b_\ell + \frac{s_\ell}{\lambda p_\ell}\cdot \lambda(w - p\cdot b) = b_\ell + \frac{s_\ell(w - p\cdot b)}{p_\ell} = x_\ell(p,w).$$

Step 2. *Walras' law.*

$$\sum_\ell p_\ell x_\ell(p,w) = \sum_\ell p_\ell b_\ell + \underbrace{\Big(\sum_\ell s_\ell\Big)}_{=1}\big(w - p\cdot b\big) = p\cdot b + (w - p\cdot b) = w.$$

Step 3. *Uniqueness.* The formula in Step 4 above gives a single number for each x_ℓ; the demand is a singleton for every (p,w) in the interior of the domain.

Step 4. *Homogeneity of degree zero of v.* $v(\lambda p, \lambda w) = \lambda(w - p\cdot b)\cdot(\alpha^\alpha\beta^\beta\gamma^\gamma)/\big[\lambda^{\alpha+\beta+\gamma}\prod p_\ell^{s_\ell}\big] = \lambda^{1-1}v(p,w) = v(p,w)$, since $\alpha+\beta+\gamma = 1$.

Step 5. *Monotonicity of v.* $\partial v/\partial w > 0$ (income helps), and $\partial v/\partial p_\ell < 0$ for every ℓ (a rise in any price makes the consumer worse off), because both $(w - p\cdot b)$ decreases in p_ℓ and $p_\ell^{-s_\ell}$ decreases in p_ℓ.

Step 6. *Quasiconvexity of v.* Take logs:

$$\ln v(p,w) = \ln(w - p\cdot b) + \sum_\ell s_\ell \ln s_\ell - \sum_\ell s_\ell \ln p_\ell.$$

The set $\{(p,w) : \ln v \le \bar{V}\}$ is a level set of a concave function of $(w - p\cdot b)$ minus a sum of convex functions of p_ℓ; when the level sets of the corresponding transformed function are checked in the appropriate $(\ln p_\ell, \ln w)$ chart, they are convex. Hence so is the level set of v itself.

Step 7. *Continuity.* v is a product of continuous functions on the open domain $\{w > p\cdot b\}$, so v is continuous there.

4. **MWG 3.D.7.** We observe the same consumer at two price-wealth pairs. At (p^0, w^0) she chooses $x^0 = (4, 4)$; at (p^1, w^1) her choice x^1 satisfies $p^1 \cdot x^1 = w^1$. Throughout this exercise use the concrete example $(p^0, w^0) = \big((1,1), 8\big)$ and $(p^1, w^1) = \big((1,2), 14\big)$, so the two budget lines cross at $(2, 6)$ and x^0 is strictly inside the new budget set $\big(p^1 \cdot x^0 = 12 < 14\big)$. (These specific numbers are the ones behind Figures 1–4.)

   Determine the region of permissible choices for x^1 under each of the following assumptions.

   (a) x^0 and x^1 are consistent with utility maximization (WARP alone).

   (b) Preferences are quasilinear in the first good.

   (c) Preferences are quasilinear in the second good.

   (d) Both goods are normal.

   (e) Preferences are homothetic.

   For parts (b)–(e) assume, in addition, that u is differentiable.

   Answer. *Part (a): WARP.*

   Step 1. Restate WARP for demand data. If two bundles x^0 and x^1 are both affordable at (p^1, w^1) (so $p^1 \cdot x^0 \leq w^1$ and $p^1 \cdot x^1 \leq w^1$) and the consumer chooses x^1, then x^0 must not be affordable at (p^0, w^0) when x^1 is; equivalently, if $x^0 \neq x^1$ and x^0 was affordable at (p^1, w^1), then x^1 must not have been affordable at (p^0, w^0): $p^0 \cdot x^1 > w^0$.

   Step 2. Check the premise here. Given the numbers, $p^1 \cdot x^0 = 1 \cdot 4 + 2 \cdot 4 = 12 < 14 = w^1$, so x^0 was strictly inside the new budget set. If $x^1 \neq x^0$, then WARP forces $p^0 \cdot x^1 > w^0$, i.e.,

   $$x_1^1 + x_2^1 > 8.$$

   Step 3. Combine with Walras' law on the new budget. We also have $p^1 \cdot x^1 = w^1$, i.e.,

   $$x_1^1 + 2x_2^1 = 14.$$

   Solving the second for $x_2^1 = (14 - x_1^1)/2$ and plugging into the WARP inequality: $x_1^1 + (14 - x_1^1)/2 > 8$, i.e., $x_1^1/2 > 1$, i.e., $x_1^1 > 2$.

   Step 4. Bold region. The permissible set for x^1 is the portion of the new budget line with $x_1^1 > 2$: the segment from just past $(2, 6)$ down to $(14, 0)$.

   Answer. *Part (b): quasilinear in the first good.*

   Step 1. Utility form. Quasilinear in good 1 means $u(x) = x_1 + g(x_2)$ with g increasing and concave (so $\partial u / \partial x_1 = 1$ everywhere; see MWG Exercise 3.C.5(b)).

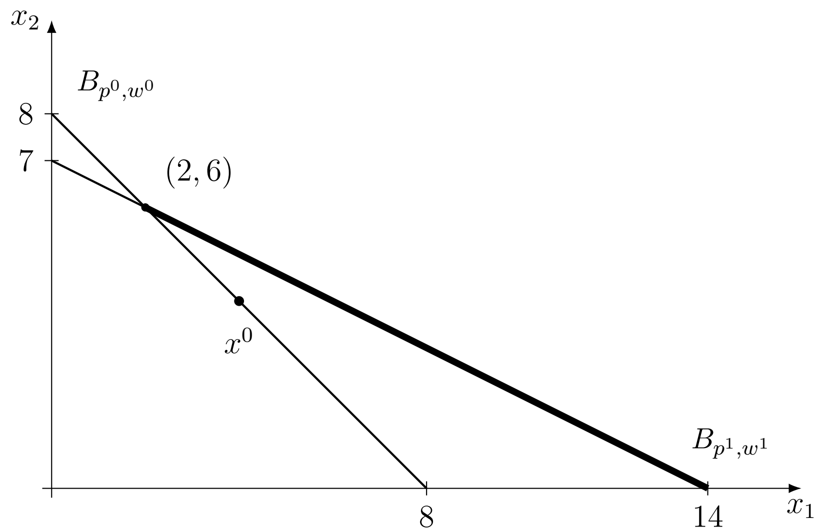

Figure 1: Part (a) — WARP. The permissible region for x^1 is the thick portion of the new budget line, from $(2, 6)$ to $(14, 0)$.

Step 2. FOC. At an interior optimum, tangency requires

$$\frac{\partial u/\partial x_2}{\partial u/\partial x_1} = g'(x_2) = \frac{p_2}{p_1}.$$

Step 3. Apply at $t = 0$ and $t = 1$. With $x_2^0 = 4$ and $p_2^0/p_1^0 = 1$: $g'(4) = 1$. With $p_2^1/p_1^1 = 2$: $g'(x_2^1) = 2$. Because g is concave, g' is decreasing; the larger value $g' = 2$ is attained at a smaller argument, so

$$x_2^1 < x_2^0 = 4.$$

Step 4. Translate into x_1^1. On the new budget line $x_1^1 + 2x_2^1 = 14$, the constraint $x_2^1 < 4$ is equivalent to $x_1^1 > 6$. Combined with the WARP restriction (Part a) $x_1^1 > 2$, the binding condition here is $x_1^1 > 6$.

Step 5. Bold region. The permissible set is the segment of the new budget line from just past $(6, 4)$ down to $(14, 0)$.

Answer. *Part (c): quasilinear in the second good.*

Step 1. Utility form and FOC. Now $u(x) = f(x_1) + x_2$ with f increasing and concave, so $\partial u/\partial x_2 = 1$ and

$$f'(x_1) = \frac{p_1}{p_2}.$$

Step 2. Apply at $t = 0$ and $t = 1$. With $x_1^0 = 4$ and $p_1^0/p_2^0 = 1$: $f'(4) = 1$. With $p_1^1/p_2^1 = 1/2$: $f'(x_1^1) = 1/2$. Because f' is decreasing, the smaller value $f' = 1/2$ is

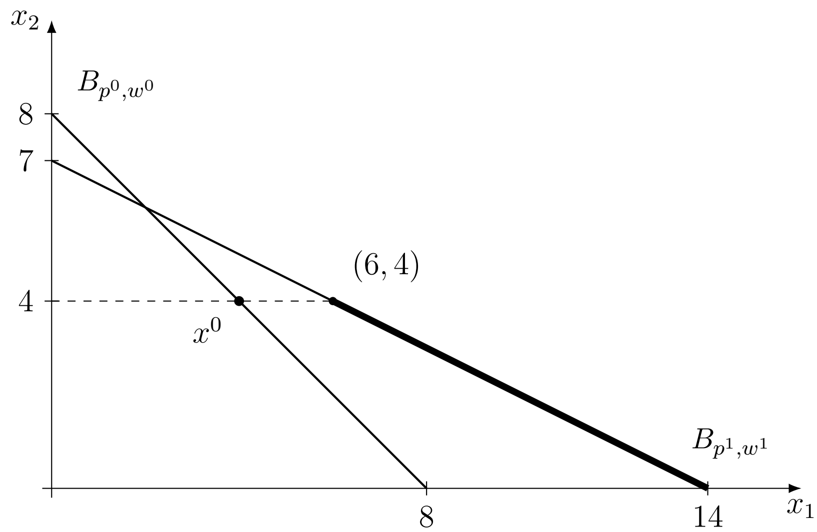

Figure 2: Part (b) — quasilinear in good 1. The FOC forces $x_2^1 < 4$, so x^1 lies on the new budget line with $x_1^1 > 6$.

attained at a larger argument, so

$$x_1^1 > x_1^0 = 4.$$

Step 3. Translate. On the new budget, $x_1^1 > 4$ is equivalent to $x_2^1 < 5$ (since $x_2^1 = (14 - x_1^1)/2 < 5$).

Step 4. Bold region. The permissible set is the segment of the new budget line from just past $(4, 5)$ down to $(14, 0)$.

Answer. *Part (d): both goods are normal.*

Step 1. Wealth vs. substitution effect. Because $p^1 \cdot x^0 < w^1$, at prices p^1 the consumer has strictly more purchasing power than she needs to buy x^0: the price change from (p^0, w^0) to (p^1, w^1) increases her real wealth. Decompose the total change in demand into a Slutsky substitution effect (holding real wealth fixed) and a wealth effect.

Step 2. Effect on x_1. The relative price of good 1 fell (p_1/p_2 went from 1 to 1/2). The substitution effect on x_1 is therefore *positive*. The wealth effect on x_1 is also *positive* because good 1 is normal. Total: $x_1^1 > x_1^0 = 4$.

Step 3. Effect on x_2. The substitution effect on x_2 is *negative* (good 2 became relatively more expensive), while the wealth effect on x_2 is *positive* (good 2 is normal). The two effects push in opposite directions, so we cannot sign the change in x_2^1 without more information.

Step 4. Bold region. On the new budget line, $x_1^1 > 4$ gives the same segment as in Part (c): from just past $(4, 5)$ down to $(14, 0)$. See Figure 3.

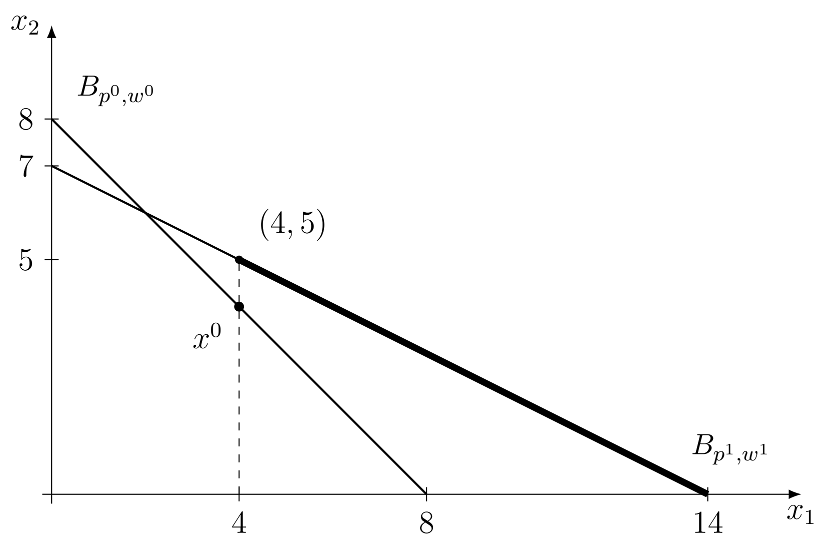

Figure 3: Part (c) — quasilinear in good 2. The FOC forces $x_1^1 > 4$, so x^1 lies on the new budget line with $x_2^1 < 5$.

Answer. *Part (e): homothetic preferences.*

Step 1. Key property of homothetic preferences. With homothetic $\succsim$, the MRS depends only on the ratio x_2/x_1. Along any ray from the origin, the MRS is constant; along a *flatter* ray (smaller x_2/x_1, i.e., more of good 1 per unit of good 2), the indifference curves are *less steep.*

Step 2. Ray through x^0. The ray from the origin through $x^0 = (4,4)$ is the 45° line $x_2 = x_1$; along it the MRS equals its value at x^0, which the FOC pins down as $p_1^0/p_2^0 = 1$.

Step 3. At the new prices. At any optimum for $t = 1$, the FOC gives MRS $= p_1^1/p_2^1 = 1/2 < 1$. So the indifference curve at x^1 is *flatter* than at x^0, which by the homotheticity property means x^1 lies on a *flatter* ray than x^0's, i.e., with $x_2^1/x_1^1 < 1$, equivalently

$$x_1^1 > x_2^1.$$

Step 4. Translate. On the new budget line, $x_1^1 > x_2^1$ means $x_1^1 > 14/3 \approx 4.67$; the ray $x_2 = x_1$ hits the new budget line at $\big(14/3, 14/3\big)$.

Step 5. Bold region. The permissible set is the segment of the new budget line from just past $\big(14/3, 14/3\big)$ down to $(14, 0)$.

5. **Complement (continuity and utility representation).** Let $\succsim$ be a preference relation on a convex set $X \subseteq \mathbb{R}^L$ (or, in part (b), on $X = [0,1]$).

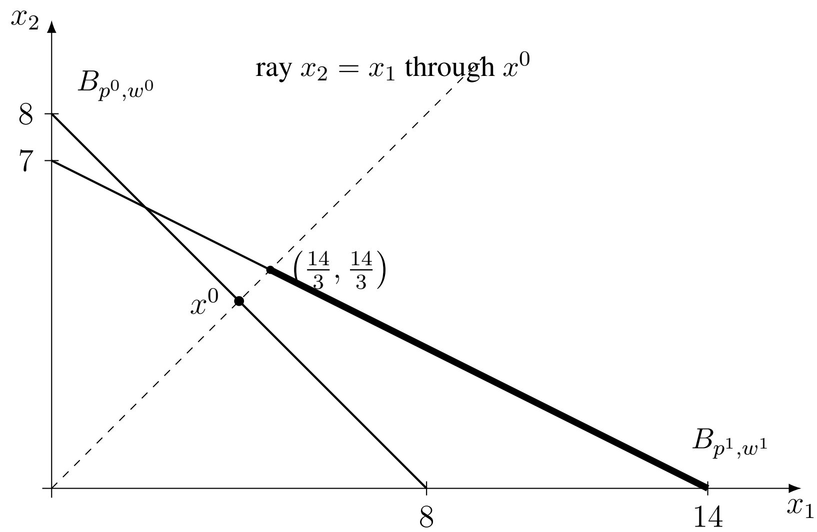

Figure 4: Part (e) — homothetic preferences. The dashed line is the 45° ray through x^0; homotheticity forces x^1 onto a flatter ray, i.e., with $x_1^1 > x_2^1$.

(a) Assume $\succsim$ is rational and continuous. Show that for any bundles $x, y, z \in X$ with $x \succsim y \succsim z$, there exists a weight $\lambda \in [0,1]$ such that the mixture $\lambda x + (1-\lambda)z$ is indifferent to y.

(b) Does there exist a rational preference on $[0,1]$ that admits a real-valued utility representation but is *not* continuous? Either provide an example or explain why no such example can be constructed.

Why this pairs with today's problems. Every exercise above assumed a smooth utility function and computed with it. Part (a) explains why we can even work along the "connecting segment" between two bundles, and part (b) reminds us that having a utility index is genuinely weaker than having a *continuous* preference.

Answer. *Part (a): an intermediate-value argument for indifference.*

Step 1. Parameterize the segment. For $\alpha \in [0,1]$ define $w(\alpha) = \alpha x + (1-\alpha)z$. Since $x, z \in X$ and X is convex, $w(\alpha) \in X$ for every $\alpha \in [0,1]$. Note $w(1) = x$ and $w(0) = z$.

Step 2. Two contour sets. Consider

$$A^+ = \{\alpha \in [0,1] : w(\alpha) \succsim y\}, \qquad A^- = \{\alpha \in [0,1] : y \succsim w(\alpha)\}.$$

Both are nonempty: $1 \in A^+$ (because $w(1) = x \succsim y$) and $0 \in A^-$ (because $y \succsim z = w(0)$).

Step 3. Both are closed. The map $\alpha \mapsto w(\alpha)$ is continuous. Continuity of $\succsim$ means the upper contour set $\{v : v \succsim y\}$ and the lower contour set $\{v : y \succsim v\}$ are closed in

X. The preimages of closed sets under a continuous map are closed, so A^+ and A^- are closed subsets of $[0,1]$.

Step 4. Their union is everything. Completeness of $\succsim$ implies that for every α, either $w(\alpha) \succsim y$ or $y \succsim w(\alpha)$ (or both). Hence $A^+ \cup A^- = [0,1]$.

Step 5. Connectedness forces a common point. $[0,1]$ is connected, and a connected set cannot be written as the union of two disjoint nonempty closed sets. Since A^+ and A^- are nonempty and closed and cover $[0,1]$, they must intersect: there is $\lambda \in A^+ \cap A^-$.

Step 6. Conclude. Any λ in the intersection satisfies both $w(\lambda) \succsim y$ and $y \succsim w(\lambda)$, i.e., $w(\lambda) \sim y$. That is,

$$\lambda\, x + (1-\lambda)\, z \;\sim\; y.$$

Intuitively, sliding the weight from z (weakly worse than y) to x (weakly better) moves the bundle continuously through preference space, so it must cross an indifference class of y at some intermediate point.

Answer. *Part (b): yes; a jump at an endpoint works.*

Step 1. Construction. On $X = [0,1]$ define

$$u(x) = \begin{cases} x, & 0 \le x < 1, \\ -1, & x = 1, \end{cases}$$

and let $\succsim$ be the preference relation induced by u (that is, $x \succsim y \iff u(x) \ge u(y)$).

Step 2. $\succsim$ is rational. Any preference induced by a real-valued function is complete (real numbers are totally ordered by $\ge$) and transitive (transitivity of $\ge$). So $\succsim$ is rational.

Step 3. $\succsim$ is representable. Trivially: by construction, u is a utility representation.

Step 4. $\succsim$ is not continuous. Consider the sequence $x_n = 1 - 1/n \to 1$. For large n, $u(x_n) = 1 - 1/n > \frac{1}{2}$, so $x_n \succ \frac{1}{2}$. Passing to the limit, $u(1) = -1 < \frac{1}{2}$, so $\frac{1}{2} \succ 1$. Thus the upper contour set $\{x : x \succsim \frac{1}{2}\}$ contains every x_n but not the limit $x = 1$; this set is not closed, so $\succsim$ fails continuity at $x = \frac{1}{2}$.

Intuitively, the utility index takes a downward jump at $x = 1$: alternatives just below 1 are highly preferred, but $x = 1$ itself is worse than $\frac{1}{2}$. Utility ranks the alternatives just fine, but the ordering has a discontinuity that continuity of $\succsim$ would forbid. The moral: "$\succsim$ admits a utility representation" is strictly weaker than "$\succsim$ is continuous."