

EconS 501 – Micro Theory I

Recitation #2 – Utility and Demand I¹

1. **MWG 3.D.1.** Verify that the Walrasian demand function generated by the Cobb-Douglas utility function satisfies the following three properties:

- (i) *Homogeneity of degree zero in (p, w) :* $x(\alpha p, \alpha w) = x(p, w)$ for every (p, w) and every scalar $\alpha > 0$.
- (ii) *Walras' law:* $p \cdot x = w$ for every $x \in x(p, w)$.
- (iii) *Convexity/uniqueness:* If preferences are convex (so u is quasiconcave), $x(p, w)$ is a convex set; if preferences are strictly convex (so u is strictly quasiconcave), $x(p, w)$ is a single point.

Setup. Take $u(x_1, x_2) = A x_1^\alpha x_2^{1-\alpha}$ with $A > 0$ and $\alpha \in (0, 1)$.² The Utility Maximization Problem (UMP) reads

$$\max_{x_1, x_2 \geq 0} A x_1^\alpha x_2^{1-\alpha} \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 = w.$$

Answer. *Deriving the Walrasian demand.*

Step 1. First-order conditions. Using the Lagrangian $\mathcal{L} = A x_1^\alpha x_2^{1-\alpha} - \lambda(p_1 x_1 + p_2 x_2 - w)$, the FOCs are

$$\alpha A x_1^{\alpha-1} x_2^{1-\alpha} = \lambda p_1, \quad (1 - \alpha) A x_1^\alpha x_2^{-\alpha} = \lambda p_2.$$

Step 2. Tangency condition. Dividing the two FOCs,

$$\frac{\alpha}{1 - \alpha} \cdot \frac{x_2}{x_1} = \frac{p_1}{p_2} \quad \Longleftrightarrow \quad p_2 x_2 = \frac{1 - \alpha}{\alpha} p_1 x_1.$$

Step 3. Substitute into the budget. Plugging $p_2 x_2 = \frac{1 - \alpha}{\alpha} p_1 x_1$ into $p_1 x_1 + p_2 x_2 = w$ gives $\frac{1}{\alpha} p_1 x_1 = w$, so

$$x_1(p, w) = \frac{\alpha w}{p_1}, \quad x_2(p, w) = \frac{(1 - \alpha) w}{p_2}.$$

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²The results all extend to the more general Cobb-Douglas $u(x) = A x_1^\alpha x_2^\beta$; see MWG 3.D.6.

Each Walrasian demand is a fixed share of wealth divided by its own price — the well-known constant-expenditure-share property of Cobb-Douglas.

Answer. *Property (i): homogeneity of degree zero in (p, w) .* Scale prices and wealth by any $\lambda > 0$:

$$x_1(\lambda p, \lambda w) = \frac{\alpha \lambda w}{\lambda p_1} = \frac{\alpha w}{p_1} = x_1(p, w), \quad x_2(\lambda p, \lambda w) = \frac{(1 - \alpha) \lambda w}{\lambda p_2} = x_2(p, w).$$

The λ 's cancel. *Intuitively*, a proportional rescaling of every price and of wealth changes only the units of account and leaves the real budget set — and therefore the optimal choice — untouched.

Answer. *Property (ii): Walras' law.* Plug the demands into $p_1 x_1 + p_2 x_2$:

$$p_1 \frac{\alpha w}{p_1} + p_2 \frac{(1 - \alpha) w}{p_2} = \alpha w + (1 - \alpha) w = w.$$

The consumer spends her entire wealth. *Intuitively*, Cobb-Douglas preferences are strictly monotone, so leaving any income unspent is never optimal.

Answer. *Property (iii): convexity and uniqueness.*

Step 1. Convexity of preferences via the slope of the indifference curve. Along an indifference curve $u(x_1, x_2) = \bar{u}$, the slope is

$$\left. \frac{dx_2}{dx_1} \right|_{u=\bar{u}} = -\frac{\partial u / \partial x_1}{\partial u / \partial x_2} = -\frac{\alpha A x_1^{\alpha-1} x_2^{1-\alpha}}{(1 - \alpha) A x_1^\alpha x_2^{-\alpha}} = -\frac{\alpha}{1 - \alpha} \cdot \frac{x_2}{x_1}.$$

Preferences are (strictly) convex if this slope *increases* (becomes less negative) as x_1 rises along the curve.

Step 2. Check the sign of the derivative. Along the indifference curve, an increase in x_1 is offset by a decrease in x_2 ; because the slope depends on x_2/x_1 , an increase in x_1 shrinks the ratio x_2/x_1 and hence shrinks $|dx_2/dx_1|$. Formally,

$$\frac{\partial}{\partial x_1} \left[-\frac{\alpha}{1 - \alpha} \cdot \frac{x_2}{x_1} \right] = \frac{\alpha}{1 - \alpha} \cdot \frac{x_2}{x_1^2} > 0.$$

The slope of the indifference curve is strictly increasing in x_1 , so indifference curves are bowed toward the origin. Preferences are *strictly* convex.

Step 3. Uniqueness. From the closed-form expressions in the boxed formula, at any (p, w) the pair $(x_1(p, w), x_2(p, w))$ is uniquely pinned down — the Walrasian demand is a single point, not a set. This is no accident: a strictly convex indifference curve is tangent to the (linear) budget line at exactly one point.

2. **MWG 3.D.2.** Verify that the indirect utility function

$$v(p, w) = [\alpha \ln \alpha + (1 - \alpha) \ln(1 - \alpha)] + \ln w - \alpha \ln p_1 - (1 - \alpha) \ln p_2$$

satisfies:

- (i) Homogeneous of degree zero in (p, w) ;
- (ii) Strictly increasing in w and non-increasing in p_ℓ for every ℓ ;
- (iii) Quasiconvex in (p, w) ; that is, the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for every $\bar{v}$;
- (iv) Continuous in (p, w) .

Where does v come from? It is the indirect utility of a Cobb-Douglas consumer with $u = x_1^\alpha x_2^{1-\alpha}$. Plugging the Walrasian demands from Exercise 1 into the log-transformed utility $\ln u = \alpha \ln x_1 + (1 - \alpha) \ln x_2$ delivers exactly the formula above.

Handy shorthand. Let $K \equiv \alpha \ln \alpha + (1 - \alpha) \ln(1 - \alpha)$ (a constant that does not depend on p or w), so

$$v(p, w) = K + \ln w - \alpha \ln p_1 - (1 - \alpha) \ln p_2.$$

Answer. *Property (i): homogeneity of degree zero.* For $\lambda > 0$,

$$\begin{aligned} v(\lambda p, \lambda w) &= K + \ln(\lambda w) - \alpha \ln(\lambda p_1) - (1 - \alpha) \ln(\lambda p_2) \\ &= K + [\ln \lambda + \ln w] - \alpha [\ln \lambda + \ln p_1] - (1 - \alpha) [\ln \lambda + \ln p_2] \\ &= K + \ln w - \alpha \ln p_1 - (1 - \alpha) \ln p_2 + \underbrace{[1 - \alpha - (1 - \alpha)] \ln \lambda}_{=0} \\ &= v(p, w). \end{aligned}$$

The $\ln \lambda$ terms cancel because the coefficients $\{1, -\alpha, -(1 - \alpha)\}$ sum to zero.

Answer. *Property (ii): monotonicity.* Compute the partial derivatives:

$$\frac{\partial v}{\partial w} = \frac{1}{w} > 0, \quad \frac{\partial v}{\partial p_1} = -\frac{\alpha}{p_1} < 0, \quad \frac{\partial v}{\partial p_2} = -\frac{1 - \alpha}{p_2} < 0.$$

Hence v is strictly increasing in w and strictly decreasing in each p_ℓ (so a fortiori non-increasing). *Intuitively*, more wealth relaxes the budget, and higher prices tighten it.

Answer. *Property (iii): quasiconvexity in (p, w) .*

Step 1. What we need to show. Fix any $\bar{v}$. The lower contour set

$$\mathcal{L}(\bar{v}) = \{(p, w) : v(p, w) \leq \bar{v}\}$$

must be convex.

Step 2. Rewrite the level condition in a friendlier form. Because $\exp$ is strictly increasing, $v(p, w) \leq \bar{v}$ is equivalent to $\exp(v(p, w) - K) \leq \exp(\bar{v} - K)$, i.e.,

$$\frac{w}{p_1^\alpha p_2^{1-\alpha}} \leq C, \quad \text{where } C \equiv \exp(\bar{v} - K).$$

Equivalently,

$$w \leq C \cdot \underbrace{p_1^\alpha p_2^{1-\alpha}}_{\equiv \phi(p)}.$$

Step 3. The price aggregator $\phi(p)$ is concave in p . With $\alpha + (1 - \alpha) = 1$, $\phi(p) = p_1^\alpha p_2^{1-\alpha}$ is a Cobb-Douglas function of p_1, p_2 . Such a function is concave on the positive orthant (this is a standard fact: its log is $\alpha \ln p_1 + (1 - \alpha) \ln p_2$, a concave function; and any function whose log is concave and that is nonnegative is itself concave when its exponents sum to at most one). Hence

$$\phi(\lambda p + (1 - \lambda)p') \geq \lambda \phi(p) + (1 - \lambda) \phi(p').$$

Step 4. Show $\mathcal{L}(\bar{v})$ is convex. Take any $(p, w), (p', w') \in \mathcal{L}(\bar{v})$: by Step 2, $w \leq C \phi(p)$ and $w' \leq C \phi(p')$. For any $\lambda \in [0, 1]$, form $(p'', w'') = \lambda(p, w) + (1 - \lambda)(p', w')$. Then

$$w'' = \lambda w + (1 - \lambda)w' \leq C[\lambda \phi(p) + (1 - \lambda) \phi(p')] \leq C \phi(p''),$$

where the second inequality uses concavity of ϕ from Step 3. Hence $(p'', w'') \in \mathcal{L}(\bar{v})$, so the set is convex.

Intuitively, the level set of v is the set of (p, w) pairs delivering utility at most $\bar{v}$. The Cobb-Douglas indirect utility rises linearly in wealth and is a concave (Cobb-Douglas) aggregate of prices; so “below- $\bar{v}$ ” is bounded above by the concave curve $w = C \phi(p)$, and a set below a concave curve is convex.

Answer. *Property (iv): continuity.* v is a sum of the continuous function $\ln w$ and two continuous functions $-\alpha \ln p_1$, $-(1 - \alpha) \ln p_2$ on the open orthant $\{(p, w) : p_1, p_2, w > 0\}$, so v itself is continuous there.

3. MWG 3.D.6. Consider a three-good consumer with utility

$$u(x) = (x_1 - b_1)^\alpha (x_2 - b_2)^\beta (x_3 - b_3)^\gamma, \quad \alpha, \beta, \gamma > 0,$$

defined on the region $x_\ell > b_\ell$ for $\ell = 1, 2, 3$. Here b_ℓ is a “subsistence” quantity of good ℓ .

- (a) Why can you assume without loss of generality that $\alpha + \beta + \gamma = 1$? Do so for the rest of the problem.

- (b) Write the FOCs for the UMP and derive the Walrasian demand and indirect utility. (This is the Linear Expenditure System, due to Stone, 1954.)
- (c) Verify that these functions satisfy Propositions 3.D.2 and 3.D.3.

Answer. Part (a): normalizing $\alpha + \beta + \gamma = 1$.

Step 1. Consider a monotone transformation of u . Let

$$\tilde{u}(x) = u(x)^{1/(\alpha+\beta+\gamma)} = (x_1 - b_1)^{\tilde{\alpha}}(x_2 - b_2)^{\tilde{\beta}}(x_3 - b_3)^{\tilde{\gamma}}, \quad \tilde{\alpha} = \frac{\alpha}{\alpha + \beta + \gamma}, \tilde{\beta} = \frac{\beta}{\alpha + \beta + \gamma}, \tilde{\gamma} = \frac{\gamma}{\alpha + \beta + \gamma}.$$

By construction, $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = 1$.

Step 2. $\tilde{u}$ represents the same preferences. Since $u \geq 0$ on the region of interest and $t \mapsto t^{1/(\alpha+\beta+\gamma)}$ is strictly increasing on $[0, \infty)$, $\tilde{u}$ is a strictly monotone transformation of u , so both represent the same preference relation.

Hence we lose no generality by working with the exponents $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ summing to 1; rename them α, β, γ from now on.

Answer. Part (b): FOCs and closed-form demand.

Step 1. Log-transform. Working with the monotone transformation

$$\ln u(x) = \alpha \ln(x_1 - b_1) + \beta \ln(x_2 - b_2) + \gamma \ln(x_3 - b_3)$$

simplifies the FOCs without changing the argmax.

Step 2. Lagrangian and FOCs. With Lagrangian $\mathcal{L} = \ln u(x) - \mu(\sum_{\ell} p_{\ell} x_{\ell} - w)$, the FOCs are

$$\frac{\alpha}{x_1 - b_1} = \mu p_1, \quad \frac{\beta}{x_2 - b_2} = \mu p_2, \quad \frac{\gamma}{x_3 - b_3} = \mu p_3,$$

together with the budget $p_1 x_1 + p_2 x_2 + p_3 x_3 = w$.

Step 3. Solve for expenditure on each good above subsistence. Rearranging the FOCs,

$$p_1(x_1 - b_1) = \frac{\alpha}{\mu}, \quad p_2(x_2 - b_2) = \frac{\beta}{\mu}, \quad p_3(x_3 - b_3) = \frac{\gamma}{\mu}.$$

Sum all three, apply $\alpha + \beta + \gamma = 1$, and use the budget:

$$\underbrace{\sum_{\ell} p_{\ell} x_{\ell}}_{=w} - \underbrace{\sum_{\ell} p_{\ell} b_{\ell}}_{\equiv p \cdot b} = \frac{\alpha + \beta + \gamma}{\mu} = \frac{1}{\mu} \iff \frac{1}{\mu} = w - p \cdot b.$$

Step 4. Substitute back to obtain the Walrasian demand. For $\ell = 1, 2, 3$,

$$x_{\ell}(p, w) = b_{\ell} + \frac{s_{\ell}}{p_{\ell}}(w - p \cdot b), \quad s_1 = \alpha, \quad s_2 = \beta, \quad s_3 = \gamma.$$

The consumer first buys the “subsistence” bundle b , then splits the leftover “discretionary income” $w - p \cdot b$ into the fixed shares α, β, γ across the three goods.

Step 5. Indirect utility. Plug into the objective $u(x) = \prod_{\ell} (x_{\ell} - b_{\ell})^{s_{\ell}}$:

$$v(p, w) = \prod_{\ell=1}^3 \left(\frac{s_{\ell}(w - p \cdot b)}{p_{\ell}} \right)^{s_{\ell}} = (w - p \cdot b)^{\alpha+\beta+\gamma} \prod_{\ell=1}^3 \left(\frac{s_{\ell}}{p_{\ell}} \right)^{s_{\ell}} = (w - p \cdot b) \cdot \frac{\alpha^{\alpha} \beta^{\beta} \gamma^{\gamma}}{p_1^{\alpha} p_2^{\beta} p_3^{\gamma}}.$$

Answer. *Part (c): checking Propositions 3.D.2 and 3.D.3.*

Step 1. Homogeneity of degree zero of demand. Scale by $\lambda > 0$: $p \rightarrow \lambda p$, $w \rightarrow \lambda w$. Then $\lambda w - (\lambda p) \cdot b = \lambda(w - p \cdot b)$, so

$$x_{\ell}(\lambda p, \lambda w) = b_{\ell} + \frac{s_{\ell}}{\lambda p_{\ell}} \cdot \lambda(w - p \cdot b) = b_{\ell} + \frac{s_{\ell}(w - p \cdot b)}{p_{\ell}} = x_{\ell}(p, w).$$

Step 2. Walras' law.

$$\sum_{\ell} p_{\ell} x_{\ell}(p, w) = \sum_{\ell} p_{\ell} b_{\ell} + \underbrace{\left(\sum_{\ell} s_{\ell} \right)}_{=1} (w - p \cdot b) = p \cdot b + (w - p \cdot b) = w.$$

Step 3. Uniqueness. The formula in Step 4 above gives a single number for each x_{ℓ} ; the demand is a singleton for every (p, w) in the interior of the domain.

Step 4. Homogeneity of degree zero of v . $v(\lambda p, \lambda w) = \lambda(w - p \cdot b) \cdot (\alpha^{\alpha} \beta^{\beta} \gamma^{\gamma}) / [\lambda^{\alpha+\beta+\gamma} \prod p_{\ell}^{s_{\ell}}] = \lambda^{1-1} v(p, w) = v(p, w)$, since $\alpha + \beta + \gamma = 1$.

Step 5. Monotonicity of v . $\partial v / \partial w > 0$ (income helps), and $\partial v / \partial p_{\ell} < 0$ for every ℓ (a rise in any price makes the consumer worse off), because both $(w - p \cdot b)$ decreases in p_{ℓ} and $p_{\ell}^{-s_{\ell}}$ decreases in p_{ℓ} .

Step 6. Quasiconvexity of v . Take logs:

$$\ln v(p, w) = \ln(w - p \cdot b) + \sum_{\ell} s_{\ell} \ln s_{\ell} - \sum_{\ell} s_{\ell} \ln p_{\ell}.$$

The set $\{(p, w) : \ln v \leq \bar{V}\}$ is a level set of a concave function of $(w - p \cdot b)$ minus a sum of convex functions of p_{ℓ} ; when the level sets of the corresponding transformed function are checked in the appropriate $(\ln p_{\ell}, \ln w)$ chart, they are convex. Hence so is the level set of v itself.

Step 7. Continuity. v is a product of continuous functions on the open domain $\{w > p \cdot b\}$, so v is continuous there.

4. **MWG 3.D.7.** We observe the same consumer at two price-wealth pairs. At (p^0, w^0) she chooses $x^0 = (4, 4)$; at (p^1, w^1) her choice x^1 satisfies $p^1 \cdot x^1 = w^1$. Throughout this exercise use the concrete example $(p^0, w^0) = ((1, 1), 8)$ and $(p^1, w^1) = ((1, 2), 14)$, so the two budget lines cross at $(2, 6)$ and x^0 is strictly inside the new budget set ($p^1 \cdot x^0 = 12 < 14$). (These specific numbers are the ones behind Figures 1–4.)

Determine the region of permissible choices for x^1 under each of the following assumptions.

- (a) x^0 and x^1 are consistent with utility maximization (WARP alone).
- (b) Preferences are quasilinear in the first good.
- (c) Preferences are quasilinear in the second good.
- (d) Both goods are normal.
- (e) Preferences are homothetic.

For parts (b)–(e) assume, in addition, that u is differentiable.

Answer. *Part (a): WARP.*

Step 1. Restate WARP for demand data. If two bundles x^0 and x^1 are both affordable at (p^1, w^1) (so $p^1 \cdot x^0 \leq w^1$ and $p^1 \cdot x^1 \leq w^1$) and the consumer chooses x^1 , then x^0 must not be affordable at (p^0, w^0) when x^1 is; equivalently, if $x^0 \neq x^1$ and x^0 was affordable at (p^1, w^1) , then x^1 must not have been affordable at (p^0, w^0) : $p^0 \cdot x^1 > w^0$.

Step 2. Check the premise here. Given the numbers, $p^1 \cdot x^0 = 1 \cdot 4 + 2 \cdot 4 = 12 < 14 = w^1$, so x^0 was strictly inside the new budget set. If $x^1 \neq x^0$, then WARP forces $p^0 \cdot x^1 > w^0$, i.e.,

$$x_1^1 + x_2^1 > 8.$$

Step 3. Combine with Walras' law on the new budget. We also have $p^1 \cdot x^1 = w^1$, i.e.,

$$x_1^1 + 2x_2^1 = 14.$$

Solving the second for $x_2^1 = (14 - x_1^1)/2$ and plugging into the WARP inequality: $x_1^1 + (14 - x_1^1)/2 > 8$, i.e., $x_1^1/2 > 1$, i.e., $x_1^1 > 2$.

Step 4. Bold region. The permissible set for x^1 is the portion of the new budget line with $x_1^1 > 2$: the segment from just past $(2, 6)$ down to $(14, 0)$.

Answer. *Part (b): quasilinear in the first good.*

Step 1. Utility form. Quasilinear in good 1 means $u(x) = x_1 + g(x_2)$ with g increasing and concave (so $\partial u / \partial x_1 = 1$ everywhere; see MWG Exercise 3.C.5(b)).

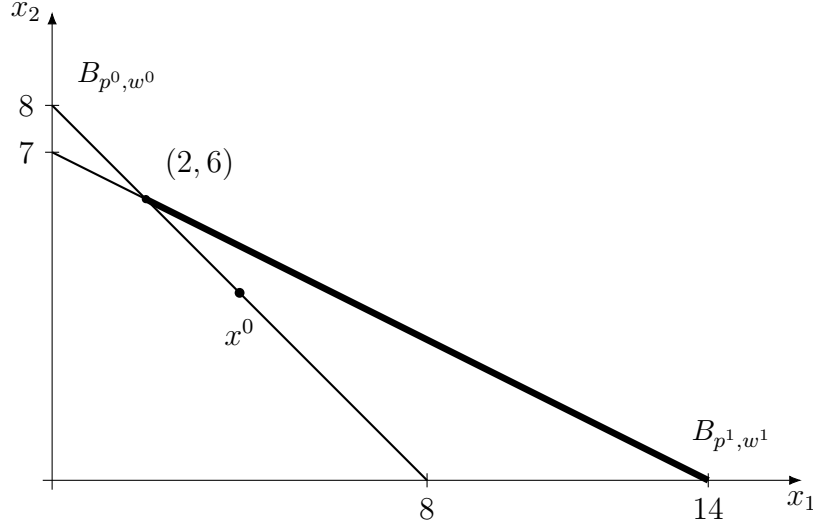


Figure 1: Part (a) — WARP. The permissible region for x^1 is the thick portion of the new budget line, from $(2, 6)$ to $(14, 0)$.

Step 2. FOC. At an interior optimum, tangency requires

$$\frac{\partial u / \partial x_2}{\partial u / \partial x_1} = g'(x_2) = \frac{p_2}{p_1}.$$

Step 3. Apply at $t = 0$ and $t = 1$. With $x_2^0 = 4$ and $p_2^0/p_1^0 = 1$: $g'(4) = 1$. With $p_2^1/p_1^1 = 2$: $g'(x_2^1) = 2$. Because g is concave, g' is decreasing; the larger value $g' = 2$ is attained at a smaller argument, so

$$x_2^1 < x_2^0 = 4.$$

Step 4. Translate into x_1^1 . On the new budget line $x_1^1 + 2x_2^1 = 14$, the constraint $x_2^1 < 4$ is equivalent to $x_1^1 > 6$. Combined with the WARP restriction (Part a) $x_1^1 > 2$, the binding condition here is $x_1^1 > 6$.

Step 5. Bold region. The permissible set is the segment of the new budget line from just past $(6, 4)$ down to $(14, 0)$.

Answer. Part (c): *quasilinear in the second good.*

Step 1. Utility form and FOC. Now $u(x) = f(x_1) + x_2$ with f increasing and concave, so $\partial u / \partial x_2 = 1$ and

$$f'(x_1) = \frac{p_1}{p_2}.$$

Step 2. Apply at $t = 0$ and $t = 1$. With $x_1^0 = 4$ and $p_1^0/p_2^0 = 1$: $f'(4) = 1$. With $p_1^1/p_2^1 = 1/2$: $f'(x_1^1) = 1/2$. Because f' is decreasing, the smaller value $f' = 1/2$ is

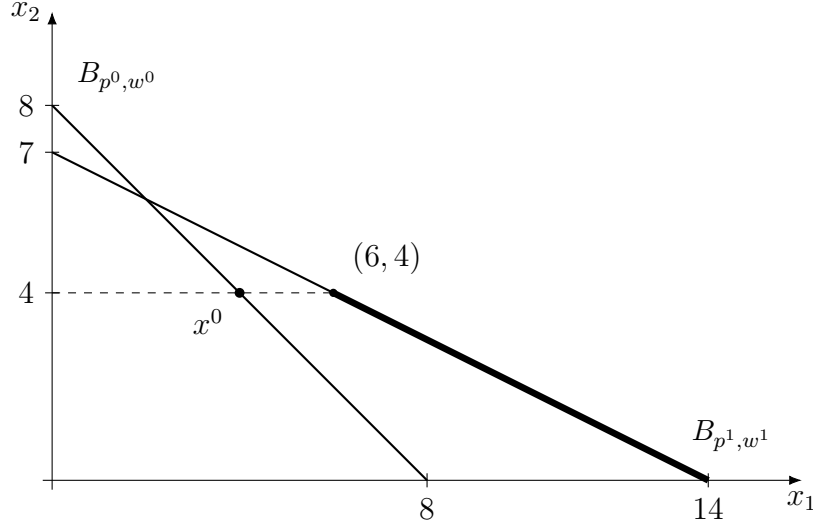


Figure 2: Part (b) — quasilinear in good 1. The FOC forces $x_2^1 < 4$, so x^1 lies on the new budget line with $x_1^1 > 6$.

attained at a larger argument, so

$$x_1^1 > x_1^0 = 4.$$

Step 3. Translate. On the new budget, $x_1^1 > 4$ is equivalent to $x_2^1 < 5$ (since $x_2^1 = (14 - x_1^1)/2 < 5$).

Step 4. Bold region. The permissible set is the segment of the new budget line from just past $(4, 5)$ down to $(14, 0)$.

Answer. *Part (d): both goods are normal.*

Step 1. Wealth vs. substitution effect. Because $p^1 \cdot x^0 < w^1$, at prices p^1 the consumer has strictly more purchasing power than she needs to buy x^0 : the price change from (p^0, w^0) to (p^1, w^1) increases her real wealth. Decompose the total change in demand into a Slutsky substitution effect (holding real wealth fixed) and a wealth effect.

Step 2. Effect on x_1 . The relative price of good 1 fell (p_1/p_2 went from 1 to $1/2$). The substitution effect on x_1 is therefore *positive*. The wealth effect on x_1 is also *positive* because good 1 is normal. Total: $x_1^1 > x_1^0 = 4$.

Step 3. Effect on x_2 . The substitution effect on x_2 is *negative* (good 2 became relatively more expensive), while the wealth effect on x_2 is *positive* (good 2 is normal). The two effects push in opposite directions, so we cannot sign the change in x_2^1 without more information.

Step 4. Bold region. On the new budget line, $x_1^1 > 4$ gives the same segment as in Part (c): from just past $(4, 5)$ down to $(14, 0)$. See Figure 3.

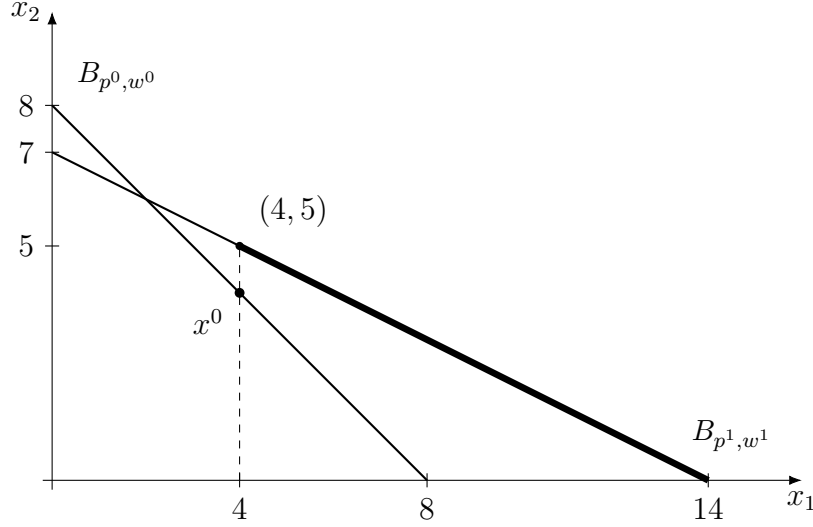


Figure 3: Part (c) — quasilinear in good 2. The FOC forces $x_1^1 > 4$, so x^1 lies on the new budget line with $x_2^1 < 5$.

Answer. *Part (e): homothetic preferences.*

Step 1. Key property of homothetic preferences. With homothetic $\succsim$, the MRS depends only on the ratio x_2/x_1 . Along any ray from the origin, the MRS is constant; along a *flatter* ray (smaller x_2/x_1 , i.e., more of good 1 per unit of good 2), the indifference curves are *less steep*.

Step 2. Ray through x^0 . The ray from the origin through $x^0 = (4, 4)$ is the 45° line $x_2 = x_1$; along it the MRS equals its value at x^0 , which the FOC pins down as $p_1^0/p_2^0 = 1$.

Step 3. At the new prices. At any optimum for $t = 1$, the FOC gives $\text{MRS} = p_1^1/p_2^1 = 1/2 < 1$. So the indifference curve at x^1 is *flatter* than at x^0 , which by the homotheticity property means x^1 lies on a *flatter* ray than x^0 's, i.e., with $x_2^1/x_1^1 < 1$, equivalently

$$x_1^1 > x_2^1.$$

Step 4. Translate. On the new budget line, $x_1^1 > x_2^1$ means $x_1^1 > 14/3 \approx 4.67$; the ray $x_2 = x_1$ hits the new budget line at $(14/3, 14/3)$.

Step 5. Bold region. The permissible set is the segment of the new budget line from just past $(14/3, 14/3)$ down to $(14, 0)$.

5. **Complement (continuity and utility representation).** Let $\succsim$ be a preference relation on a convex set $X \subseteq \mathbb{R}^L$ (or, in part (b), on $X = [0, 1]$).

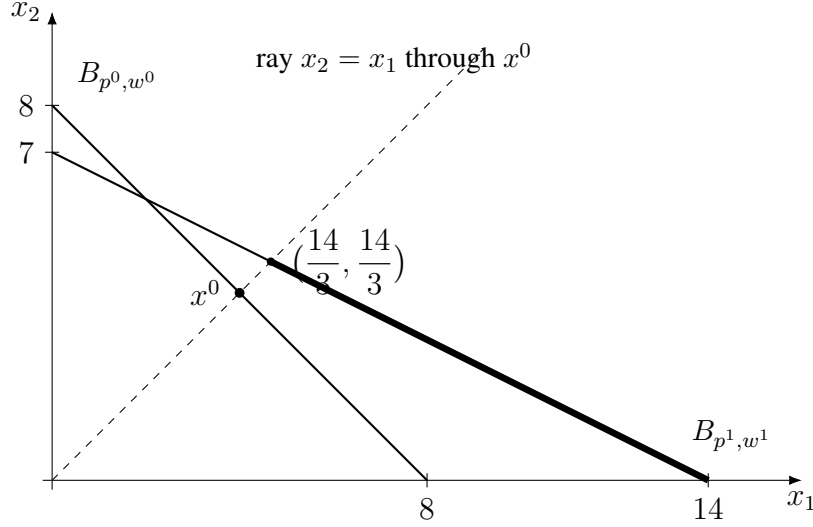


Figure 4: Part (e) — homothetic preferences. The dashed line is the 45° ray through x^0 ; homotheticity forces x^1 onto a flatter ray, i.e., with $x_1^1 > x_2^1$.

- (a) Assume $\succsim$ is rational and continuous. Show that for any bundles $x, y, z \in X$ with $x \succsim y \succsim z$, there exists a weight $\lambda \in [0, 1]$ such that the mixture $\lambda x + (1 - \lambda)z$ is indifferent to y .
- (b) Does there exist a rational preference on $[0, 1]$ that admits a real-valued utility representation but is *not* continuous? Either provide an example or explain why no such example can be constructed.

Why this pairs with today's problems. Every exercise above assumed a smooth utility function and computed with it. Part (a) explains why we can even work along the “connecting segment” between two bundles, and part (b) reminds us that having a utility index is genuinely weaker than having a *continuous* preference.

Answer. *Part (a): an intermediate-value argument for indifference.*

Step 1. Parameterize the segment. For $\alpha \in [0, 1]$ define $w(\alpha) = \alpha x + (1 - \alpha)z$. Since $x, z \in X$ and X is convex, $w(\alpha) \in X$ for every $\alpha \in [0, 1]$. Note $w(1) = x$ and $w(0) = z$.

Step 2. Two contour sets. Consider

$$A^+ = \{\alpha \in [0, 1] : w(\alpha) \succsim y\}, \quad A^- = \{\alpha \in [0, 1] : y \succsim w(\alpha)\}.$$

Both are nonempty: $1 \in A^+$ (because $w(1) = x \succsim y$) and $0 \in A^-$ (because $y \succsim z = w(0)$).

Step 3. Both are closed. The map $\alpha \mapsto w(\alpha)$ is continuous. Continuity of $\succsim$ means the upper contour set $\{v : v \succsim y\}$ and the lower contour set $\{v : y \succsim v\}$ are closed in

X . The preimages of closed sets under a continuous map are closed, so A^+ and A^- are closed subsets of $[0, 1]$.

Step 4. Their union is everything. Completeness of $\succsim$ implies that for every α , either $w(\alpha) \succsim y$ or $y \succsim w(\alpha)$ (or both). Hence $A^+ \cup A^- = [0, 1]$.

Step 5. Connectedness forces a common point. $[0, 1]$ is connected, and a connected set cannot be written as the union of two disjoint nonempty closed sets. Since A^+ and A^- are nonempty and closed and cover $[0, 1]$, they must intersect: there is $\lambda \in A^+ \cap A^-$.

Step 6. Conclude. Any λ in the intersection satisfies both $w(\lambda) \succsim y$ and $y \succsim w(\lambda)$, i.e., $w(\lambda) \sim y$. That is,

$$\lambda x + (1 - \lambda) z \sim y.$$

Intuitively, sliding the weight from z (weakly worse than y) to x (weakly better) moves the bundle continuously through preference space, so it must cross an indifference class of y at some intermediate point.

Answer. *Part (b): yes; a jump at an endpoint works.*

Step 1. Construction. On $X = [0, 1]$ define

$$u(x) = \begin{cases} x, & 0 \leq x < 1, \\ -1, & x = 1, \end{cases}$$

and let $\succsim$ be the preference relation induced by u (that is, $x \succsim y \iff u(x) \geq u(y)$).

Step 2. $\succsim$ is rational. Any preference induced by a real-valued function is complete (real numbers are totally ordered by $\geq$) and transitive (transitivity of $\geq$). So $\succsim$ is rational.

Step 3. $\succsim$ is representable. Trivially: by construction, u is a utility representation.

Step 4. $\succsim$ is not continuous. Consider the sequence $x_n = 1 - 1/n \rightarrow 1$. For large n , $u(x_n) = 1 - 1/n > \frac{1}{2}$, so $x_n \succ \frac{1}{2}$. Passing to the limit, $u(1) = -1 < \frac{1}{2}$, so $\frac{1}{2} \succ 1$. Thus the upper contour set $\{x : x \succsim \frac{1}{2}\}$ contains every x_n but not the limit $x = 1$; this set is not closed, so $\succsim$ fails continuity at $x = \frac{1}{2}$.

Intuitively, the utility index takes a downward jump at $x = 1$: alternatives just below 1 are highly preferred, but $x = 1$ itself is worse than $\frac{1}{2}$. Utility ranks the alternatives just fine, but the ordering has a discontinuity that continuity of $\succsim$ would forbid. The moral: “ $\succsim$ admits a utility representation” is strictly weaker than “ $\succsim$ is continuous.”