EconS 501 – Micro Theory I
Recitation #1 – Preferences and Choice[1]

1. **MWG 1.B.2.** Prove property (ii) of Proposition 1.B.1: if a preference relation $\succsim$ is rational (i.e., complete and transitive), then

   (a) $\sim$ is reflexive ($x \sim x$ for all x), transitive (if $x \sim y$ and $y \sim z$, then $x \sim z$), and symmetric (if $x \sim y$, then $y \sim x$); and

   (b) if $x \succ y \succsim z$, then $x \succ z$.

   Recall the definitions of the two derived relations. For any two bundles x, y,

   - $x \sim y$ ("x is indifferent to y") means both $x \succsim y$ and $y \succsim x$; and
   - $x \succ y$ ("x is strictly preferred to y") means $x \succsim y$ and *not* $y \succsim x$.

   Both properties below reduce to using this definition and the completeness/transitivity of $\succsim$.

   Answer. *Part (a).*

   (i) Reflexivity of $\sim$. *Step 1.* Completeness of $\succsim$ requires that for every pair of bundles, one is at least as good as the other. Applying completeness to the pair (x, x) gives $x \succsim x$.

   Step 2. Since $x \succsim x$ and $x \succsim x$ both hold, the definition of $\sim$ gives $x \sim x$.

   Hence $x \sim x$ for every $x \in X$, so $\sim$ is reflexive.

   (ii) Transitivity of $\sim$. Suppose $x \sim y$ and $y \sim z$. We must show $x \sim z$, i.e., $x \succsim z$ *and* $z \succsim x$.

   Step 1. First direction: $x \succsim z$. From $x \sim y$, the definition of $\sim$ gives $x \succsim y$. From $y \sim z$, it gives $y \succsim z$. Transitivity of $\succsim$ then combines $x \succsim y$ with $y \succsim z$ to yield $x \succsim z$.

   Step 2. Second direction: $z \succsim x$. From $x \sim y$, the definition also gives $y \succsim x$. From $y \sim z$, it gives $z \succsim y$. Transitivity of $\succsim$ combines $z \succsim y$ with $y \succsim x$ to yield $z \succsim x$.

   Step 3. Both $x \succsim z$ and $z \succsim x$ hold, so by definition $x \sim z$.

   Hence $\sim$ is transitive.

   (iii) Symmetry of $\sim$. Suppose $x \sim y$.

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Step 1. By definition of $\sim$, we have both $x \succsim y$ and $y \succsim x$.

Step 2. These are exactly the two conditions for $y \sim x$ (just written in the reverse order).

Hence $y \sim x$, and $\sim$ is symmetric.

Answer. *Part (b): if $x \succ y \succsim z$, then $x \succ z$.* We argue by contradiction.

Step 1. Set up the contradiction hypothesis. Suppose the premise $x \succ y \succsim z$ holds but the conclusion $x \succ z$ fails. Since $\succsim$ is complete (so between x and z at least one of $x \succsim z$ or $z \succsim x$ must hold), failing $x \succ z$ means $z \succsim x$ must hold.

Step 2. Derive a consequence via transitivity. Combining $y \succsim z$ (part of the premise) with $z \succsim x$ (from Step 1), transitivity of $\succsim$ yields

$$y \succsim x.$$

Step 3. Identify the contradiction. The premise says $x \succ y$, which by definition means $x \succsim y$ and *not* $y \succsim x$. But Step 2 just showed $y \succsim x$. These two statements cannot both hold.

Step 4. Conclude. The contradiction rules out $z \succsim x$, so the only remaining possibility is $x \succ z$.

2. **MWG 1.C.1.** Consider the choice structure $(\mathcal{B}, C(\cdot))$ with

$$\mathcal{B} = \Big\{\{x,y\},\ \{x,y,z\}\Big\} \qquad \text{and} \qquad C(\{x,y\}) = \{x\}.$$

Show that if $(\mathcal{B}, C(\cdot))$ satisfies the Weak Axiom (WARP), then $C(\{x,y,z\})$ must be one of $\{x\}$, $\{z\}$, or $\{x,z\}$.

Recall the Weak Axiom of Revealed Preference (WARP). If in some menu B containing x and y the element x is chosen, then in every other menu B' that also contains both x and y, whenever y is chosen we must also have x chosen.

In plain English: if x is ever chosen when y is on offer, then no menu containing both x and y can pick y but leave x out.

Answer. We proceed by ruling out the possibilities for $C(\{x,y,z\})$ one by one, focusing on the alternative y.

Step 1. y cannot be in $C(\{x,y,z\})$. Suppose, seeking a contradiction, that $y \in C(\{x,y,z\})$.

- Set $B = \{x,y\}$ and $B' = \{x,y,z\}$. Both menus contain x and y, and $x \in C(B)$ (given).

- WARP applied to this pair of menus says: since x is chosen from B and y is chosen from B' (our supposition), we must have x chosen from B as well — but this is already the case, so we need to look at the symmetric implication. Restated concretely: WARP requires that whenever $y \in C(B')$ (and $x \in B'$), the choice from $B = \{x, y\}$ must include y. That is, $y \in C(\{x, y\})$.
- But $C(\{x, y\}) = \{x\}$, so $y \notin C(\{x, y\})$. Contradiction.

Hence $y \notin C(\{x, y, z\})$.

Step 2. $C(\{x, y, z\})$ is nonempty and a subset of $\{x, z\}$. A choice correspondence returns a nonempty subset of the menu, and Step 1 rules y out. The only nonempty subsets of $\{x, z\}$ are $\{x\}$, $\{z\}$, and $\{x, z\}$.

Hence

$$C(\{x, y, z\}) = \{x\}, \qquad C(\{x, y, z\}) = \{z\}, \qquad \text{or} \qquad C(\{x, y, z\}) = \{x, z\}.$$

Intuitively, y was already "beaten" by x in the two-element menu; enlarging the menu to include z cannot revive y without contradicting the revealed ranking. The alternative z, on the other hand, has never been compared to anything, so it is free to be chosen (alone, together with x, or not at all).

3. **MWG 2.D.3(b).** Consider an extension of the Walrasian budget set to an arbitrary consumption set $X \subseteq \mathbb{R}^L$,

$$B_{p,w} = \{x \in X : p \cdot x \leq w\}.$$

Show that if X is convex, then $B_{p,w}$ is convex as well.

Answer. To show $B_{p,w}$ is convex we must show that whenever two bundles lie in $B_{p,w}$, so does every convex combination of them.

Step 1. Take two arbitrary elements of $B_{p,w}$. Pick $x, x' \in B_{p,w}$ and a weight $\lambda \in [0, 1]$, and form the convex combination

$$x'' = \lambda x + (1 - \lambda)x'.$$

Step 2. Check $x'' \in X$. Because $x, x' \in X$ and X is convex, $x'' \in X$ by definition of convexity.

Step 3. Check the budget constraint $p \cdot x'' \leq w$. Compute

$$p \cdot x'' = p \cdot \left[\lambda x + (1 - \lambda)x'\right] = \lambda\,(p \cdot x) + (1 - \lambda)(p \cdot x').$$

Since $x \in B_{p,w}$ gives $p \cdot x \leq w$ and $x' \in B_{p,w}$ gives $p \cdot x' \leq w$,

$$\lambda\,(p \cdot x) + (1 - \lambda)(p \cdot x') \leq \lambda\,w + (1 - \lambda)\,w = w.$$

Step 4. Conclude. $x'' \in X$ and $p \cdot x'' \le w$, so $x'' \in B_{p,w}$. Hence $B_{p,w}$ is convex.

Intuitively, the budget frontier is a hyperplane (a flat surface) in X; a convex intersection of a convex set X with the "below" half-space $\{x : p \cdot x \le w\}$ (itself convex) stays convex.

4. **MWG 2.D.4.** Show that the budget set in Figure 2.D.4 is not convex.

Setup. A worker has 24 hours to allocate between leisure and labor. For each of the first 8 hours of labor the wage is s; for each additional hour beyond 8, the wage is $s' > s$ (an overtime premium). The budget frontier in the (leisure, consumption) plane is therefore piecewise linear:

- From $A = (24, 0)$ up to $K = (16,\, 8s)$ the slope is $-s$ (each hour of labor gives up one hour of leisure and buys s units of consumption); the kink K is the point of exactly 8 hours of work.
- From K up to $C = (0,\, 8s+16s')$ the slope is $-s'$; since $s' > s$, this second segment is *steeper.*

Figure 1 draws the frontier, the shaded budget set below it, and the ingredients of the proof.

Answer. To show non-convexity, it is enough to exhibit *one* pair of feasible points whose convex combination leaves the budget set. We take A and any point B on the upper (steeper) segment, then show the chord from A to B passes above the kink K.

Step 1. Pick a target point B on the overtime segment. Fix any $M > 8s$ and let $B = (\ell_B, M)$ be the point of the frontier at consumption M. To reach consumption M (which exceeds $8s$) the worker must work 8 regular-wage hours and $(M - 8s)/s'$ overtime hours, spending $8 + (M - 8s)/s'$ hours on labor in total. Hence

$$\ell_B = 24 - \left[8 + \frac{M - 8s}{s'}\right] = 16 - \frac{M - 8s}{s'}.$$

Step 2. Form a convex combination whose leisure coordinate is exactly 16. Consider $\lambda A + (1 - \lambda)B$ for a weight $\lambda \in (0, 1)$ to be chosen. The horizontal coordinate is $\lambda \cdot 24 + (1 - \lambda) \cdot \ell_B$; setting this equal to 16 gives one linear equation for λ:

$$\lambda \cdot 24 + (1 - \lambda)\left(16 - \frac{M - 8s}{s'}\right) = 16.$$

Solving,

$$\lambda = \frac{\frac{M-8s}{s'}}{8 + \frac{M-8s}{s'}} \in (0, 1).$$

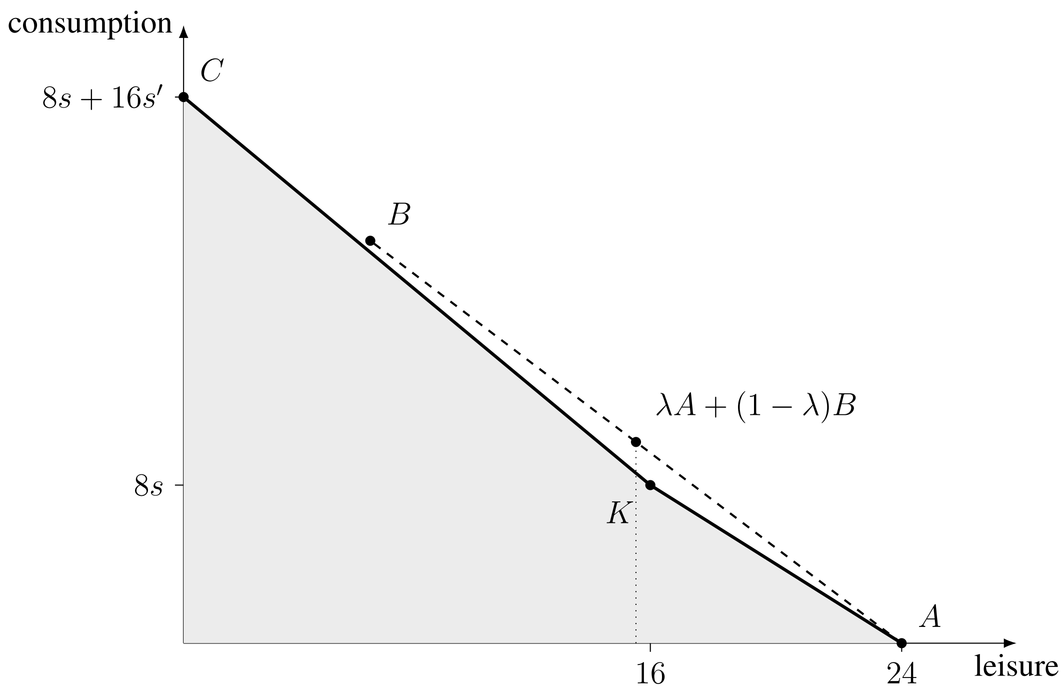

Figure 1: The budget set (shaded) is not convex. Endpoints: $A = (24, 0)$ and $C = (0,\, 8s + 16s')$. The kink at 8 hours of work is $K = (16,\, 8s)$, and $B = \big(16 - \frac{M-8s}{s'},\, M\big)$ is any point on the overtime (upper) segment. The dashed chord from A to B leaves the budget set at leisure $= 16$, where it lies strictly above K.

Why do we choose leisure $= 16$? Because that is the leisure coordinate of the kink K, so it is the point of the frontier the chord must beat if we want the chord to lie outside the set.

Step 3. Compute the consumption coordinate of this convex combination. With the λ above,

$$\lambda \cdot 0 + (1-\lambda) \cdot M = M\,\frac{8}{8 + \frac{M-8s}{s'}}.$$

(The first term vanishes because $A = (24, 0)$ has consumption 0.)

Step 4. Rewrite the frontier value at leisure $= 16$ in a comparable form. At leisure $= 16$ the frontier is exactly K, i.e., consumption $= 8s$. We manipulate $8s$ to look like the expression above. Multiply numerator and denominator by M/s:

$$8s = 8s \cdot \frac{M/s}{M/s} = M\,\frac{8}{M/s}.$$

Next, rewrite M/s by adding and subtracting $8s$ inside: $M/s = (M - 8s + 8s)/s = 8 + (M-8s)/s$. Hence

$$8s = M\,\frac{8}{8 + \frac{M-8s}{s}}.$$

Step 5. Compare the two heights. The two expressions differ only in whether the denominator uses s' or s:

$$\underbrace{M\,\frac{8}{8+\frac{M-8s}{s'}}}_{\text{chord at leisure }=16} \qquad \text{vs.} \qquad \underbrace{M\,\frac{8}{8+\frac{M-8s}{s}}}_{\text{frontier at leisure }=16=8s}.$$

Since $s' > s$, the fraction $\dfrac{M-8s}{s'}$ is *smaller* than $\dfrac{M-8s}{s}$, so the left denominator is smaller and the left fraction is *larger*:

$$M\,\frac{8}{8+\frac{M-8s}{s'}} \;>\; M\,\frac{8}{8+\frac{M-8s}{s}} = 8s.$$

Step 6. Conclude. At leisure $=16$ the chord from A to B sits strictly above the frontier value $8s$. So the chord contains a point that is outside the budget set, and convexity fails.

Intuitively, the overtime wage $s' > s$ makes the frontier steeper past K. Chords that span the kink bulge outward past the frontier, so the set lying below the frontier has a "notch" at K that keeps it from being convex.

5. **MWG 3.B.1.** Show the following:

   (a) If $\succsim$ is strongly monotone, then it is monotone.

   (b) If $\succsim$ is monotone, then it is locally nonsatiated.

Recall the three definitions (all for bundles in $X \subseteq \mathbb{R}^L$).

- **Monotonicity.** For all $x, y \in X$,

  (m1) $x_k \geq y_k$ for every k implies $x \succsim y$; and

  (m2) $x_k > y_k$ for every k — written $x \gg y$ — implies $x \succ y$.

  (More of every good makes the bundle at least as good; strictly more of every good makes it strictly better.)

- **Strong monotonicity.** For all $x, y \in X$, $x_k \geq y_k$ for every k *and* $x \neq y$ implies $x \succ y$. (Adding even a single unit of a single good makes the bundle strictly better.)

- **Local nonsatiation (LNS).** For every $x \in X$ and every $\varepsilon > 0$, there exists $y \in X$ with $\|y - x\| \leq \varepsilon$ and $y \succ x$. (No bundle is a local best: there is always a strictly preferred bundle arbitrarily close.)

Note the logical ordering the exercise is pinning down:

$$\text{strong monotonicity} \implies \text{monotonicity} \implies \text{LNS}.$$

Answer. *Part (a): strong monotonicity implies monotonicity.* Assume $\succsim$ is strongly monotone. We verify each part of the monotonicity definition.

Step 1. Verify (m1): if $x_k \geq y_k$ for every k, then $x \succsim y$. Split into two sub-cases.

- If $x = y$, then $x \succsim y$ trivially (by reflexivity, which follows from completeness).
- If $x \neq y$, then $x_k \geq y_k$ for every k *and* $x \neq y$ hold, so strong monotonicity gives $x \succ y$, and in particular $x \succsim y$.

Either way $x \succsim y$.

Step 2. Verify (m2): if $x \gg y$, then $x \succ y$. From $x \gg y$ we get $x_k > y_k \geq y_k$ for every k (so $x_k \geq y_k$ for every k) and $x \neq y$ (since at least one strict inequality holds). Strong monotonicity gives $x \succ y$ directly.

Both parts of the monotonicity definition hold, so $\succsim$ is monotone.

Answer. *Part (b): monotonicity implies LNS.* Assume $\succsim$ is monotone. Fix any $x \in X \subseteq \mathbb{R}^L$ and any $\varepsilon > 0$. We construct a bundle y that is (i) close to x (within distance ε) and (ii) strictly preferred to x.

Step 1. Construct y. Let $e = (1, 1, \ldots, 1) \in \mathbb{R}^L$ (the vector of ones) and define

$$y = x + \frac{\varepsilon}{\sqrt{L}}\, e,$$

so y is x shifted by $\varepsilon/\sqrt{L}$ in every coordinate. Then $y_k - x_k = \varepsilon/\sqrt{L} > 0$ for every k, i.e., $y \gg x$.

Step 2. Verify closeness: $\|y - x\| \leq \varepsilon$. With the Euclidean norm,

$$\|y - x\| = \sqrt{\sum_{k=1}^{L} (y_k - x_k)^2} = \sqrt{\sum_{k=1}^{L} \left(\frac{\varepsilon}{\sqrt{L}}\right)^2} = \sqrt{L \cdot \frac{\varepsilon^2}{L}} = \varepsilon.$$

The factor $1/\sqrt{L}$ was chosen exactly so the L equal shifts add up (in Euclidean distance) to ε.

Step 3. Verify strict preference: $y \succ x$. Since $y \gg x$ and $\succsim$ is monotone, part (m2) of the definition gives $y \succ x$.

Because $x \in X$ and $\varepsilon > 0$ were arbitrary, LNS holds at every point, so $\succsim$ is locally nonsatiated.

6. **MWG 3.B.3.** Draw a convex preference relation that is locally nonsatiated but is not monotone.

Why this exercise is subtle. Part (b) of the previous exercise showed monotonicity implies LNS. The converse can *fail*: LNS is a weaker property, and there are preferences that satisfy LNS and convexity without being monotone. We need to construct one.

Answer. A clean construction on $\mathbb{R}^2_+$: interpret good 1 as a *desirable* good and good 2 as a *bad* — a nuisance the consumer would pay to reduce (commuting time, ambient noise, pollution exposure, effort). Preferences are then *increasing* in x_1 and *decreasing* in x_2. A convenient utility representation is

$$u(x_1, x_2) = x_1 - x_2,$$

though many other choices work. Figure 2 draws three indifference curves in this family; they are upward-sloping (extra x_1 compensates the consumer for extra x_2), convex, and utility rises as one moves down and to the right.

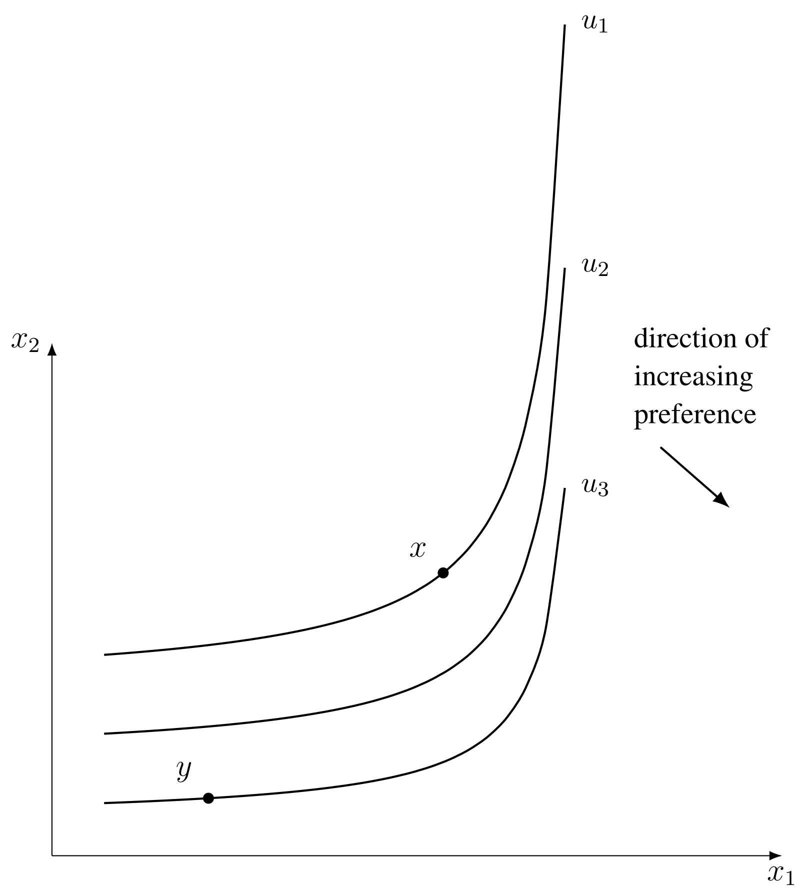

Figure 2: Convex, locally nonsatiated, but not monotone preferences on $\mathbb{R}^2_+$. Good 1 is desirable, good 2 is a bad; utility rises to the lower right, so $u_1 < u_2 < u_3$. The bundle x has more of both coordinates than y (so $x \gg y$), yet y lies on the higher indifference curve u_3, so $y \succ x$.

We verify the three properties in turn.

Step 1. Preferences are not monotone. Read the bundles from the figure: $x = (4.5, 3.25)$ lies on u_1 and $y = (1.8, 0.66)$ lies on u_3, with $u_3 > u_1$. So

$$x \gg y \quad \text{yet} \quad y \succ x,$$

which flatly contradicts monotonicity (which would require $y \succsim x$ to *fail*, not to hold strictly). The reason is that moving from y to x raises the bad (x_2) by more than the gain from raising the good (x_1).

Step 2. Preferences are convex. The upper contour set $\{v : v \succsim x\}$ is the region *below and to the right* of the indifference curve through x (that is where utility is at least $u(x)$). Each such region is a convex subset of $\mathbb{R}^2_+$, so convexity holds.

Step 3. Preferences are locally nonsatiated. At any bundle $x = (x_1, x_2)$, moving a small amount in the direction of *more* x_1 and *less* x_2 raises utility. Concretely, for any $\varepsilon > 0$ the bundle

$$y_\varepsilon = \left(x_1 + \tfrac{\varepsilon}{\sqrt{2}},\ x_2 - \tfrac{\varepsilon}{\sqrt{2}}\right)$$

satisfies $\|y_\varepsilon - x\| = \varepsilon$ and $u(y_\varepsilon) = u(x) + \varepsilon\sqrt{2} > u(x)$, so $y_\varepsilon \succ x$. Because we can always move in this direction (there is no "bliss point" at which utility is maximized locally), LNS holds *everywhere*.

Aside on other examples. A textbook alternative is a preference relation with a bliss point (a^*, b^*), drawn as concentric ovals with utility rising toward the center; that picture is convex and non-monotone as well, but LNS fails *at the bliss point itself* (there is no strictly preferred nearby bundle). The good-plus-bad example above avoids that wrinkle and gives LNS on the whole domain.

7. **Complement (WARP in a real-world choice).** Two friends, Maria and Diego, sit down at a café whose posted menu offers *espresso*, *cappuccino*, and *latte*. Looking at the menu, Maria says "I'll have the latte," and Diego says "the cappuccino for me." At that point the barista adds, "Today's off-menu special is a mocha." Maria immediately switches: "In that case I'll have the mocha." Diego reconsiders too: "If mocha is on the table, then I'll go with the espresso." Do Maria's and Diego's choices satisfy the Weak Axiom (WARP)? Be precise.

   Recall (WARP, restated for observed choices). If in some menu B containing both a and b the alternative a is chosen (revealing $a \succsim b$), then in any other menu B' that also contains both a and b, the choice cannot pick b while leaving a out; that is, revealed rankings between the same pair are not allowed to reverse across menus.

   Answer. Write the alternatives as e (espresso), c (cappuccino), l (latte), and m (mocha). The two observed menus are $B_1 = \{e, c, l\}$ and $B_2 = \{e, c, l, m\}$, with choices

$$\text{Maria:} \quad C_M(B_1) = \{l\}, \qquad C_M(B_2) = \{m\};$$
$$\text{Diego:} \quad C_D(B_1) = \{c\}, \qquad C_D(B_2) = \{e\}.$$

Step 1. Maria satisfies WARP. From B_1 Maria's choice reveals $l \succ_M e$ and $l \succ_M c$. From B_2 her choice reveals $m \succ_M e$, c, l. To check WARP we must look for a pair on which the revealed rankings clash *across* the two menus.

- The pair $\{l, m\}$ appears only in B_2, so no cross-menu comparison arises.
- The pairs $\{e, l\}$, $\{c, l\}$: the second menu ranks m above both e and c, but this says nothing about how l compares to e or c inside B_2. Nothing chosen in B_2 overturns $l \succ_M e$ or $l \succ_M c$.
- The pairs $\{e, c\}$: not ranked from either menu (neither e nor c is chosen).

No revealed ranking is reversed, so Maria satisfies WARP. Intuitively, mocha simply becomes her new most-preferred item; the ranking of the other three is not disturbed.

Step 2. Diego violates WARP. Look at the pair $\{c, e\}$, which appears in *both* menus.

- In $B_1 = \{e, c, l\}$ Diego chose c while e was available, which reveals $c \succsim_D e$ (in fact strictly).
- In $B_2 = \{e, c, l, m\}$ Diego chose e while c was available, which reveals $e \succsim_D c$ (again strictly).

The same pair $\{c, e\}$ is ranked in opposite directions across the two menus — a direct WARP violation.

Intuitively, WARP requires the ranking of options already present not to be overturned by the mere availability of a new option. Maria respects this (she promotes the newcomer to the top and leaves the old ranking alone); Diego does not (the mere availability of mocha, which he did not even choose, flipped his ranking between cappuccino and espresso).