EconS 501 – Micro Theory I
Recitation #12 – Externalities[1]

1. **"Pareto-irrelevant" externalities.** Two consumers have preferences

$$U^1 = \left[x_1^1\, x_2^1\right]^{\gamma} \left[x_1^2\, x_2^2\right]^{1-\gamma},$$
$$U^2 = \left[x_1^2\, x_2^2\right]^{\gamma} \left[x_1^1\, x_2^1\right]^{1-\gamma},$$

where superscripts index consumers and subscripts index goods (so x_k^i is consumer i's consumption of good k). Consumer 1's utility depends on consumer 2's bundle, and vice versa — a classic externality. Show that the competitive equilibrium is nonetheless Pareto efficient, and explain why.

Recall. In a two-good economy, efficiency requires each consumer's marginal rate of substitution to equal the price ratio: $\mathrm{MRS}^i_{1,2} = p_1/p_2$ for each i. When both consumers face the same prices, this forces $\mathrm{MRS}^1_{1,2} = \mathrm{MRS}^2_{1,2}$, the standard exchange-efficiency condition.

Answer.

Step 1. Marginal utility of good 1 for consumer 1. Treat consumer 2's bundle as a constant (that's what "taking prices as given" also does under an externality: consumer 1 can't affect the other's consumption). Then

$$\frac{\partial U^1}{\partial x_1^1} = \gamma\left[x_1^1 x_2^1\right]^{\gamma-1}\left(x_2^1\right)\left[x_1^2 x_2^2\right]^{1-\gamma} = \frac{\gamma\, U^1}{x_1^1}.$$

Step 2. Marginal utility of good 2 for consumer 1. Analogously,

$$\frac{\partial U^1}{\partial x_2^1} = \frac{(1-\gamma)\, U^1}{x_2^1}.$$

Step 3. Consumer 1's MRS. Divide:

$$\mathrm{MRS}^1_{1,2} = \frac{\partial U^1/\partial x_1^1}{\partial U^1/\partial x_2^1} = \frac{\gamma}{1-\gamma}\cdot\frac{x_2^1}{x_1^1}.$$

Notice: the externality term $[x_1^2 x_2^2]^{1-\gamma}$ appears in both marginal utilities as a multiplicative constant and *cancels* in the ratio. The MRS of consumer 1 depends only on consumer 1's own bundle.

[1]Felix Munoz-Garcia, School of Economic Sciences, Washington State University, 103H Hulbert Hall, Pullman, WA. Tel. 509-335-8402. Email: fmunoz@wsu.edu.

Step 4. By symmetry. Consumer 2's MRS is $\text{MRS}^2_{1,2} = \dfrac{\gamma}{1-\gamma} \cdot \dfrac{x_2^2}{x_1^2}$, again independent of the externality.

Step 5. Efficiency. Each consumer equates her own MRS to the common price ratio p_1/p_2, so both MRSs are equated. That is the standard exchange-efficiency condition; the competitive equilibrium delivers a Pareto-efficient allocation.

Intuitively, the externality enters each utility as a *multiplicative* factor that depends only on the *other* consumer's bundle — something the price-taking consumer cannot affect. When computing her MRS, this factor cancels, so her optimization is unaffected by the presence of the externality. Externalities that leave every marginal rate of substitution unchanged are "Pareto-irrelevant": they may affect utility *levels*, but not the *trades* the consumers want to make.

2. **Congestion as a negative externality.** A large population of commuters each chooses between the tube (constant 70 minutes) and driving. Driving takes $C(x) = 20 + 60x$ minutes, where $x \in [0, 1]$ is the fraction of the population choosing to drive.

   (a) Plot the two commuting times as functions of x and describe each traveller's decision.

   (b) Find the equilibrium car share x^m when each commuter chooses independently to minimize her own commuting time.

   (c) Find the socially optimal x^o that minimizes total commuting time.

   (d) Compare x^m and x^o; compute the deadweight loss.

   (e) Explain how a per-trip toll could implement the efficient allocation.

Answer. *Part (a): the two commuting-time curves.* Tube time is a horizontal line at 70; car time $C(x) = 20 + 60x$ rises linearly from 20 at $x = 0$ to 80 at $x = 1$. They intersect at $x = 5/6$. A rational commuter chooses car when $C(x) \le 70$ (equivalently, $x \le 5/6$) and tube when $C(x) > 70$. Figure 1 plots both.

Answer. *Part (b): market equilibrium x^m.* In equilibrium the two travel times must be equal (otherwise a commuter would switch to the faster mode):

$$70 = 20 + 60\,x^m \iff x^m = \frac{5}{6} \approx 0.833.$$

Answer. *Part (c): social optimum x^o.*

Step 1. Total commuting time. $T(x) = (20 + 60x)\,x + 70(1 - x) = 60x^2 - 50x + 70.$

Step 2. FOC. $T'(x) = 120x - 50 = 0$, so $x^o = \dfrac{5}{12} \approx 0.417.$

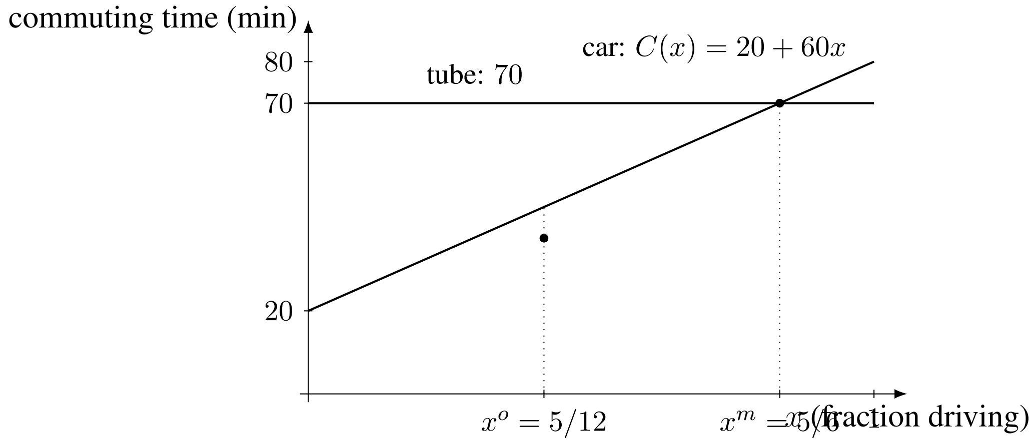

Figure 1: Commuting times as functions of the fraction x driving. Tube time is constant at 70; car time rises linearly. The market equilibrium equates them at $x^m = 5/6$; the social optimum $x^o = 5/12$ is much smaller, reflecting the externality drivers impose on each other via congestion.

Step 3. Confirm minimum. $T''(x) = 120 > 0$: strictly convex, so this is a minimum.

Answer. *Part (d): comparison and deadweight loss.* $x^m = 5/6 > x^o = 5/12$: too many people drive at the market equilibrium. Each individual driver ignores the extra minute that her presence adds to every other driver's commute. Compute the total time under each:

$$T(x^m) = (20 + 60 \cdot \tfrac{5}{6}) \cdot \tfrac{5}{6} + 70 \cdot \tfrac{1}{6} = 70 \cdot \tfrac{5}{6} + 70 \cdot \tfrac{1}{6} = 70.$$
$$T(x^o) = (20 + 60 \cdot \tfrac{5}{12}) \cdot \tfrac{5}{12} + 70 \cdot \tfrac{7}{12} = 45 \cdot \tfrac{5}{12} + 70 \cdot \tfrac{7}{12} = \tfrac{225+490}{12} = \tfrac{715}{12} \approx 59.58.$$

The deadweight loss is $T(x^m) - T(x^o) = 70 - 59.58 \approx 10.42$ minutes per commuter (per trip).

Answer. *Part (e): implementing the optimum via a toll.* Assign each commuter a monetary value ϕ per minute. At the optimum, car takes 45 minutes and tube takes 70, so the "time saving" from driving is 25 minutes, worth 25ϕ. A per-trip toll of exactly 25ϕ makes the marginal commuter indifferent between the two modes at $x = x^o$, so no one prefers to switch. Any smaller toll leaves too many drivers, any larger one drives too many people to the tube. The toll internalizes the congestion externality by charging each driver the marginal external cost her presence imposes.

3. **The tragedy of the commons (Berrytown).** On the outskirts of Berrytown, two berry patches are open to 20 foragers. Each forager chooses a patch and keeps her patch's *average* catch. On patch X the total yield is $F^X = 10\ell_X - \frac{1}{2}\ell_X^2$, where ℓ_X is the number of foragers there. On patch Y the total yield is $F^Y = 5\ell_Y$ (constant returns per forager).

(a) Under free entry, find the total yield.

(b) What limit on ℓ_X maximizes the total yield?

(c) What license fee on patch X implements the optimum?

(d) What does this tell us about property rights and externalities?

Answer. *Part (a): free-entry equilibrium.*

Step 1. Average yields. On patch X, a forager's average yield is $\bar{F}^X = F^X/\ell_X = 10 - \frac{1}{2}\ell_X$; on patch Y, $\bar{F}^Y = F^Y/\ell_Y = 5$.

Step 2. Equalize average yields (free entry). A rational forager switches patches until the average yields equalize:

$$10 - \tfrac{1}{2}\ell_X = 5 \iff \ell_X = 10,\ \ell_Y = 10.$$

Step 3. Total yield. $F^X = 10 \cdot 10 - \frac{1}{2} \cdot 100 = 50$; $F^Y = 5 \cdot 10 = 50$. Total $F^T = 100$.

Answer. *Part (b): socially optimal allocation.*

Step 1. Total yield as a function of ℓ_X. With $\ell_Y = 20 - \ell_X$,

$$F^T(\ell_X) = 10\ell_X - \tfrac{1}{2}\ell_X^2 + 5(20 - \ell_X) = 5\ell_X - \tfrac{1}{2}\ell_X^2 + 100.$$

Step 2. FOC. $\partial F^T/\partial \ell_X = 5 - \ell_X = 0 \Rightarrow \ell_X^o = 5$; $\ell_Y^o = 15$.

Step 3. Total yield at the optimum. $F^T = 5 \cdot 5 - \frac{1}{2} \cdot 25 + 100 = 25 - 12.5 + 100 = 112.5$. The optimum delivers 12.5 more berries than the free-entry equilibrium (100).

Why more people should forage on patch Y? Marginal yield on patch X is $\partial F^X/\partial \ell_X = 10 - \ell_X$; marginal yield on patch Y is a constant 5. At the optimum, marginal yields equate: $10 - \ell_X^o = 5 \Rightarrow \ell_X^o = 5$. But under free entry, individuals compare *average* yields, not marginal ones, and the average is always above the marginal on the diminishing-returns patch X. So free entry over-crowds patch X.

Answer. *Part (c): a license fee implementing the optimum.*

Step 1. Average yield at the optimum on patch X. $\bar{F}^X(\ell_X^o) = F^X(5)/5 = (50 - 12.5)/5 = 7.5$.

Step 2. Set the fee to bring the net average down to patch Y's yield of 5. If a fee of τ berries is charged per forager on patch X, the net average yield there is $7.5 - \tau$. Free-entry equilibrium at $\ell_X = 5$ requires the net average yield on X to equal the average on Y:

$$7.5 - \tau = 5 \iff \tau = 2.5.$$

A license fee of 2.5 berries per forager on patch X prices the externality and reproduces the social optimum.

Answer. *Part (d): property rights.* Suppose patch X were privately owned. The owner would charge each forager for entry and choose ℓ_X to maximize the total yield minus the opportunity cost of each forager (the 5 berries she could pick on patch Y):

$$\max_{\ell_X} \; F^X(\ell_X) - 5\,\ell_X = 10\ell_X - \tfrac{1}{2}\ell_X^2 - 5\ell_X = 5\ell_X - \tfrac{1}{2}\ell_X^2,$$

which has $\ell_X = 5$: the private owner's private problem coincides with the social planner's problem! This is the classical result: *well-defined property rights internalize the commons externality.* The over-crowding of patch X arises because no one owns it and each forager ignores the crowding cost she imposes on the others.

4. **Common oil pool (Utopia).** A perfectly competitive oil industry draws from a single (essentially inexhaustible) pool. Each competitor can sell all the oil it produces at a stable world price $p = \$10$ per barrel; the annual cost of operating a well is \$1,000. Total annual output as a function of the number of wells n is

$$Q = 500\,n - n^2, \qquad q = \frac{Q}{n} = 500 - n \text{ (output per well)}.$$

   (a) Find the competitive equilibrium: number of wells, total output. Is there a divergence between private and social marginal cost?

   (b) If the government nationalizes the pool, what is the optimal number of wells and total output?

   (c) What annual per-well license fee implements the social optimum in the competitive market?

Answer. *Part (a): competitive equilibrium.*

Step 1. Free entry. Every well operates until profit per well is zero: $\pi = p\,q - 1000 = 10(500 - n) - 1000 = 0$.

Step 2. Solve. $5000 - 10n - 1000 = 0 \Rightarrow n^m = 400$. Total output: $Q^m = 500 \cdot 400 - 400^2 = 200{,}000 - 160{,}000 = 40{,}000$ barrels.

Step 3. The externality. Revenue per well is $p \cdot q = 5000 - 10n$, which *declines* in n: drilling another well reduces every other well's output. Private cost of drilling is just \$1,000; social cost is \$1,000 plus the revenue loss imposed on all other wells — a classic negative externality.

Answer. *Part (b): nationalized (social planner's) optimum.*

Step 1. Planner's problem.

$$\max_n \; p\,Q(n) - 1000\,n = 10(500n - n^2) - 1000n = 5000n - 10n^2 - 1000n = 4000n - 10n^2.$$

Step 2. FOC. $4000 - 20n = 0 \Rightarrow n^o = 200$. Total output: $Q^o = 500 \cdot 200 - 200^2 = 100{,}000 - 40{,}000 = 60{,}000$ barrels. Output per well: $q^o = Q^o/n^o = 300$ barrels.

Alternative reading via MVP = MC: total revenue is $5000n - 10n^2$; the marginal value product of a well is $5000 - 20n = 1000 \Rightarrow n = 200$. Same answer, and the interpretation is transparent: the planner sets the marginal value of an extra well equal to its cost.

Answer. *Part (c): license fee.*

Step 1. Break-even at the optimum. At $n = 200$, revenue per well is $p \cdot q^o = 10 \cdot 300 = 3000$. Free-entry break-even under fee τ requires revenue per well − operating cost − fee = 0:

$$3000 - 1000 - \tau = 0 \iff \tau = 2000.$$

Step 2. Interpretation. A \$2,000 annual license fee makes drilling a 201-st well unprofitable at the socially optimal $n = 200$. This is exactly the negative externality that each additional well imposes on the rest of the industry — the fee prices the externality and restores efficiency.

Common thread across Exercises 3 and 4. Whether it's berries in a common patch or oil in a common pool, the tragedy of the commons arises because the private return to entry (average product) exceeds the social return (marginal product). Correcting the wedge requires either (i) internalizing property rights (Exercise 3(d)) or (ii) taxing entry (license fees in both). In each case, the tax equals the externality-induced gap between average and marginal product at the optimum.

5. **Complement (altruistic wealth allocation).** Exercises 1–4 above focused on externalities that individuals impose on *strangers* through their production or consumption choices. This complement studies an *altruistic* externality inside a family: a parent's utility depends directly on how similar her two children's consumption bundles are, and she can influence that similarity by choosing how to split her wealth between them. Each child then optimizes her own bundle independently.

   Nadia has wealth W to split between her two children: Elena receives w_E and Marco receives w_M, with $w_E + w_M = W$. Elena's preferences over peaches (x) and cherries (y) are $u_E(x, y) = \sqrt{xy}$; Marco's are $u_M(x, y) = x^\alpha y^{1-\alpha}$. Prices are p for peaches and $q = 1$ for cherries; each child spends her transfer at these prices. Nadia's own preferences are

$$V_N = -\big(x_E - x_M\big)^2 - \big(y_E - y_M\big)^2$$

   — she dislikes *inequality* in her children's consumption bundles.

(a) Derive each child's demand functions.

(b) Express V_N in terms of (w_E, W, p, α).

(c) Find Nadia's optimal split w_E^*; when is an equal split $w_E^* = W/2$ optimal?

(d) How does w_E^* depend on total wealth W?

(e) How does w_E^* depend on the peach price p? Interpret.

Answer. *Part (a): children's demands.*

Elena has Cobb-Douglas preferences with equal weights: $x_E = w_E/(2p)$ and $y_E = w_E/2$. Marco has Cobb-Douglas with weight α: $x_M = \alpha\, w_M/p$ and $y_M = (1-\alpha)\, w_M$.

Answer. *Part (b): V_N as a function of (w_E, W, p, α).*

Substituting $w_M = W - w_E$ and the child demands into V_N:

$$V_N = -\Big(\frac{w_E - 2\alpha(W - w_E)}{2p}\Big)^2 - \Big(\frac{w_E}{2} - (1-\alpha)(W - w_E)\Big)^2.$$

Answer. *Part (c): optimal split.*

Step 1. FOC. Set $\partial V_N/\partial w_E = 0$. After expansion and simplification, the linear-in-W solution is

$$w_E^* = W \cdot \frac{2\alpha(1+2\alpha) + 2p^2(3-2\alpha)(1-\alpha)}{(1+2\alpha)^2 + p^2(3-2\alpha)^2}.$$

Step 2. When is an equal split optimal? Setting $w_E^* = W/2$ and simplifying,

$$(2\alpha - 1)\big[(1+2\alpha) - p^2(3-2\alpha)\big] = 0.$$

So $w_E^* = W/2$ iff either

- $\alpha = \frac{1}{2}$ (the children have identical Cobb-Douglas weights), or
- $p = \sqrt{(1+2\alpha)/(3-2\alpha)}$ (a knife-edge peach price).

For generic (α, p) neither condition holds, and the equal split is *not* optimal: Nadia deliberately transfers more to whichever child would otherwise consume too differently from the other.

Answer. *Part (d): dependence on W.* Since w_E^* is proportional to W, the *share* of wealth going to each child is constant — the marginal split equals the average split. Whether Elena's share is above or below $1/2$ depends on the sign of $(2\alpha-1)\big[(1+2\alpha) - p^2(3-2\alpha)\big]$, exactly the factor that had to vanish in part (c).

Answer. *Part (e): dependence on p.* Differentiating w_E^* with respect to p (or checking limiting behavior),

$$w_E^* \to \frac{2\alpha}{1+2\alpha} W \text{ as } p \to 0, \qquad w_E^* \to \frac{2(1-\alpha)}{3-2\alpha} W \text{ as } p \to \infty.$$

The direction of the move is determined by the sign of $(1-2\alpha)$: if Marco is *less* peach-oriented than Elena ($\alpha < \frac{1}{2}$), higher peach prices push Nadia to give more to Elena (whose peach share is $\frac{1}{2}$, the larger of the two); if Marco is *more* peach-oriented ($\alpha > \frac{1}{2}$), higher peach prices push wealth toward Marco.

Intuitively, the peach price changes how each child's peach purchase reacts to a wealth transfer. As the price shifts, the two children's optimally chosen peach quantities diverge, and Nadia reshuffles wealth to keep their consumption bundles as similar as possible. This is a positive-externality mirror of the negative externalities in Exercises 1–4: Nadia's utility depends on both children's consumption, so her choice of split reflects the "sum of individual utilities" logic that internalizes the family-level inequality externality, in the same way a Pigouvian tax internalizes a firm's environmental externality.