EconS 501 – Micro Theory I
Recitation #11 – Monopoly[1]

1. **Monopoly PMP with three cost structures.** A monopolist faces demand $Q = 70 - p$ (equivalently $p = 70 - Q$).

   (a) Constant costs: $AC = MC = 6$. Find (Q^*, p^*, π^*).

   (b) Quadratic costs: $C(Q) = 0.25Q^2 - 5Q + 300$. Repeat.

   (c) Cubic costs: $C(Q) = 0.0133Q^3 - 5Q + 250$. Repeat.

   (d) Sketch demand, MR, and the three MC curves; identify the monopolist's optimum on each.

   Recall. Marginal revenue for a linear demand $p = 70 - Q$ is $MR = 70 - 2Q$: same intercept, twice the slope. The monopolist's PMP rule is $MR = MC$.

   Answer. *Part (a): constant* $MC = 6$.

   Step 1. Set $MR = MC$. $70 - 2Q = 6 \Rightarrow Q^* = 32$.

   Step 2. Recover price and profit. $p^* = 70 - 32 = 38$; $\pi^* = (p^* - MC)Q^* = (38 - 6) \cdot 32 = 1024$.

   Answer. *Part (b): quadratic costs.*

   Step 1. Compute MC. $MC(Q) = C'(Q) = 0.5Q - 5$.

   Step 2. Set $MR = MC$. $70 - 2Q = 0.5Q - 5 \Rightarrow 2.5Q = 75 \Rightarrow Q^* = 30$.

   Step 3. Recover price and profit. $p^* = 70 - 30 = 40$; $C(30) = 0.25(900) - 5(30) + 300 = 225 - 150 + 300 = 375$; $\pi^* = 40 \cdot 30 - 375 = 1200 - 375 = 825$.

   Answer. *Part (c): cubic costs.*

   Step 1. Compute MC. $MC(Q) = C'(Q) = 0.0399Q^2 - 5$.

   Step 2. Set $MR = MC$. $70 - 2Q = 0.0399Q^2 - 5$, i.e., $0.0399Q^2 + 2Q - 75 = 0$. Quadratic formula: $Q = \left(-2 + \sqrt{4 + 4 \cdot 0.0399 \cdot 75}\right)/(2 \cdot 0.0399)$. The positive root is $Q^* = 25$.

   Step 3. Recover price and profit. $p^* = 70 - 25 = 45$; $C(25) = 0.0133(15\,625) - 5(25) + 250 = 207.8 - 125 + 250 = 332.8$; $\pi^* = 45 \cdot 25 - 332.8 = 1125 - 332.8 = 792.2$.

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Answer. *Part (d): visual summary.* Figure 1 overlays demand, MR, and the three MC curves. Each optimum sits at the intersection of MR with the corresponding MC; reading the price off the demand curve above that Q gives (p^*, Q^*).

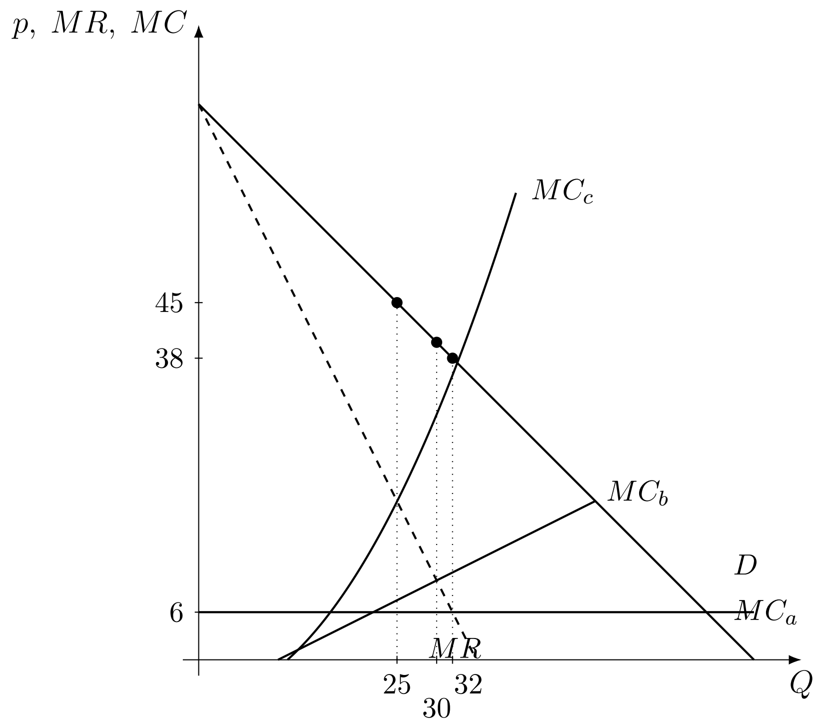

Figure 1: Demand, MR, and the three MC curves from parts (a)–(c). The three monopoly optima (Q^*, p^*) are $(32, 38)$, $(30, 40)$, and $(25, 45)$; each is the intersection of MR with the corresponding MC, with the price then read off the demand curve above that quantity. Axes are scaled so one axis unit represents 10 units of Q (horizontal) and \$10 of price (vertical).

Intuitively, moving from constant to quadratic to cubic marginal costs makes the marginal cost curve steeper (more sharply increasing in output). Each step shrinks the profit-maximizing output, raises the price, and reduces total profit. The monopolist's profit-making ability is constrained by both the demand curve (which limits the maximum price at each Q) and the cost structure (which sets the marginal cost of expanding output).

2. **Alternative PMP and the inverse elasticity pricing rule.** The monopolist can equivalently be modeled as choosing the price p (with the market determining $q(p)$). Write the corresponding PMP and verify that the IEPR

$$p\left(1 + \frac{1}{\varepsilon}\right) = c'(q)$$

still holds.

Answer.

Step 1. Set up. The price-choice PMP is $\max_p \; p \cdot q(p) - c(q(p))$.

Step 2. FOC. Differentiate in p using the product and chain rules:

$$q(p) + p \cdot q'(p) - c'(q) \cdot q'(p) = 0.$$

Step 3. Rearrange to expose the elasticity. Factor out $q'(p)$ on the terms that carry it:

$$p + \frac{q(p)}{q'(p)} - c'(q) = 0 \iff p\Big(1 + \frac{q(p)}{p\, q'(p)}\Big) = c'(q).$$

The elasticity of demand is $\varepsilon = p\, q'(p)/q(p)$, so $q(p)/(p\, q'(p)) = 1/\varepsilon$. Substituting,

$$p\Big(1 + \frac{1}{\varepsilon}\Big) = c'(q).$$

The same IEPR emerges whether the monopolist chooses the quantity (the standard $MR = MC$ derivation) or the price. Choosing p and letting the market determine q is equivalent to choosing q and letting the market determine p: both problems have the same FOC and hence the same solution.

3. **Monopoly subsidies.** A government wants to correct the allocative distortion of a monopoly through a subsidy.
   (a) Why won't a lump-sum subsidy work?
   (b) Show graphically (or algebraically) how a per-unit subsidy t can move the monopolist toward the competitive quantity.
   (c) Show that the socially optimal per-unit subsidy satisfies $t/p = -1/\varepsilon$, and interpret.

Answer. *Part (a): lump-sum subsidy doesn't work.*

A lump-sum subsidy T enters the monopolist's profit as an additive constant: $\pi = p \cdot Q - C(Q) + T$. The FOC $MR = MC$ is unchanged by an additive constant, so Q^* (and hence p^*) are unchanged. The subsidy transfers surplus from the government to the monopoly but does not close the allocative wedge.

Answer. *Part (b): per-unit subsidy shifts the effective MC.*

With a per-unit subsidy t, profit becomes $\pi = p \cdot Q - [C(Q) - tQ]$; the effective marginal cost is $MC(Q) - t$. Setting $MR = MC - t$ replaces the standard optimum with one at a higher Q and lower p. Increasing t continuously moves the monopolist along the demand curve toward the competitive quantity where $p = MC$.

Answer. *Part (c): socially optimal per-unit subsidy is $t/p = -1/\varepsilon$.*

Step 1. What social optimality requires. Social efficiency demands $p = MC$ at the optimum.

Step 2. What monopoly PMP delivers with subsidy. From part (b), the monopoly sets $MR = MC - t$, i.e., $MR + t = MC$.

Step 3. Combine. At the socially optimal outcome, $p = MC = MR + t$, so $t = p - MR$.

Step 4. Substitute MR from the IEPR. $MR = p(1 + 1/\varepsilon)$, so

$$t = p - p\Big(1 + \frac{1}{\varepsilon}\Big) = -\frac{p}{\varepsilon} \iff \frac{t}{p} = -\frac{1}{\varepsilon}.$$

Since $\varepsilon < 0$, $-1/\varepsilon > 0$, so the optimal subsidy is positive.

Intuitively, the monopoly creates a price–marginal-cost gap of $-p/\varepsilon$ (the Lerner index times p). The optimal subsidy just closes that gap: pay the monopolist the exact wedge on every unit, and it faces the same effective incentives as a competitive firm.

4. **Taxing a monopoly.** An ad valorem tax at rate t cuts the monopoly's net price from p to $p(1-t)$.

   (a) Linear demand, constant MC: show $p_A - p_B < t\,p_A$ (price rises by less than the tax).

   (b) Constant-elasticity demand: show $p_A - p_B = t\,p_A$ (full pass-through).

   (c) Describe a case where $p_A - p_B > t\,p_A$ (over-shifting).

   (d) A specific (per-unit) tax τ raising the same revenue reduces output more (and raises price more) than the ad valorem tax. Show this.

Recall. Rewriting the IEPR under an ad valorem tax: $p(1-t)(1+1/\varepsilon) = MC$, so

$$p = \frac{MC}{(1-t)(1+1/\varepsilon)} = \frac{MC}{1-t} \cdot \frac{1}{1+1/\varepsilon}.$$

The price depends on t both through the wedge $1/(1-t)$ and through the elasticity ε (which may itself move with the equilibrium price).

Answer. *Part (a): linear demand and constant MC $\Rightarrow$ under-shifting.*

Step 1. Elasticity along linear demand. On a linear demand $Q = A - Bp$, elasticity $|\varepsilon| = Bp/Q$ rises with p: higher price $\Rightarrow$ more elastic.

Step 2. Effect of the tax. The tax pushes p up (from p_B pre-tax to p_A post-tax), which increases $|\varepsilon|$, which *shrinks* the factor $1/(1+1/\varepsilon)$ (since $-1 < 1/\varepsilon < 0$ becomes less negative, $1 + 1/\varepsilon$ rises, so $1/(1+1/\varepsilon)$ falls).

Step 3. Compare pre- and post-tax. Formally,

$$p_A = \frac{MC}{1-t} \cdot \frac{1}{1+1/\varepsilon_A} < \frac{MC}{1-t} \cdot \frac{1}{1+1/\varepsilon_B} = \frac{p_B}{1-t}.$$

So $p_A(1-t) < p_B$, i.e., $p_A - t\,p_A < p_B$, i.e., $p_A - p_B < t\,p_A$. Under-shifting.

Answer. *Part (b): constant-elasticity demand ⇒ full pass-through.*

With $q(p) = Ap^{\varepsilon}$ constant-elasticity, $\varepsilon_A = \varepsilon_B = \varepsilon$, so the factor $1/(1+1/\varepsilon)$ is the same pre- and post-tax. The comparison in Step 3 becomes an equality:

$$p_A = \frac{p_B}{1-t} \iff p_A(1-t) = p_B \iff p_A - p_B = t\,p_A.$$

Why the contrast with (a)? Under linear demand, higher prices push the monopoly into a *more elastic* region, where its markup shrinks; that partly absorbs the tax. Under constant-elasticity demand, the markup is invariant in p, so nothing absorbs the tax and it is fully passed through.

Answer. *Part (c): over-shifting.* Combine (b)'s constant elasticity with an *upward-sloping* MC. Then MC itself rises when output falls after the tax, adding to the pre-tax price on top of the $1/(1-t)$ inflation. Formally, $p_A = [MC_A/(1-t)] \cdot [1/(1+1/\varepsilon)]$ with $MC_A > MC_B$, so $p_A > p_B/(1-t)$, i.e., $p_A - p_B > t\,p_A$.

Answer. *Part (d): specific tax reduces output more than an equal-revenue ad valorem tax.*

Step 1. Effective marginal revenue under each tax. Under an ad valorem tax at rate t, the monopoly's net revenue on the marginal unit is $MR_a = (1-t)\cdot MR = MR - t\cdot MR$. Under a specific tax τ, it is $MR_s = MR - \tau$.

Step 2. Match the revenue. At a common output Q, revenues match when $\tau\cdot Q = t\cdot p\cdot Q$, i.e., $\tau = t\cdot p$. Then $\tau = t\cdot p > t\cdot MR$ because $p > MR$ (demand lies above MR whenever ε is finite and negative). So $MR - \tau < MR - t \cdot MR$, i.e., $MR_s < MR_a$.

Step 3. Compare with the same MC. The monopoly's optimum satisfies $MR_{\text{net}} = MC$. Under the specific tax the net MR curve lies lower than under the ad valorem tax (Step 2), so the intersection with a (constant or upward-sloping) MC curve occurs at a smaller Q:

$$Q_s < Q_a \implies p_s > p_a.$$

The specific tax reduces output more (and raises price more) than an ad valorem tax that collects the same revenue on the pre-tax output. Under perfect competition the two would be equivalent, but the monopoly's markup makes them differ.

5. **Pricing with discontinuous demand (G-Jeans).** A monopoly holds the design patent for G-Jeans. There are $n_H = 200$ high-income consumers with reservation price $V^H = 20$ and $n_L = 300$ low-income consumers with reservation price $V^L = 10$. Each buys at most one pair. The unit cost is $c = 5$.

   (a) Draw the aggregate demand curve.

   (b) Uniform pricing (no price discrimination): find the profit-maximizing price and profit.

   (c) Perfect price discrimination between the two groups: compute profit and show it strictly exceeds uniform-pricing profit.

Answer. *Part (a): step demand.* The aggregate demand $Q(p)$ is

$$Q(p) = \begin{cases} 0 & \text{if } p > 20, \\ 200 & \text{if } 10 < p \leq 20, \\ 500 & \text{if } p \leq 10. \end{cases}$$

Figure 2 shows the step-shaped demand.

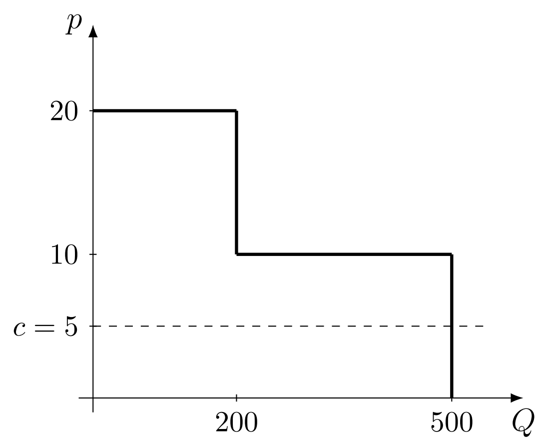

Figure 2: Aggregate demand for G-Jeans. Two "steps" at prices \$20 and \$10, corresponding to the willingness-to-pay of the high- and low-income groups. The dashed line marks the unit cost $c = \$5$.

Answer. *Part (b): uniform pricing.*

Only two candidate prices give positive sales: $p = 20$ (serve only H) or $p = 10$ (serve both). Compute profit under each:

$$\pi(p = 20) = (20 - 5) \cdot 200 = 15 \cdot 200 = 3\,000,$$
$$\pi(p = 10) = (10 - 5) \cdot 500 = 5 \cdot 500 = 2\,500.$$

The high price wins: $p^* = 20$ and $\pi^* = 3\,000$. The low-income group is priced out.

Answer. *Part (c): perfect price discrimination.*

With perfect discrimination the monopoly charges each group its own willingness-to-pay:

$$\pi_{PD} = (20 - 5) \cdot 200 + (10 - 5) \cdot 300 = 3\,000 + 1\,500 = 4\,500.$$

This exceeds the uniform-pricing profit of 3 000 by 1 500 — the extra surplus captured from the low-income group.

Price discrimination cannot reduce profit: the monopoly can always mimic uniform pricing by charging the same price to both groups. The fact that the profit-maximizing prices differ across groups (and profit strictly rises) shows that the extra flexibility of discrimination strictly helps.

6. **Third-degree price discrimination (concert tickets).** A monopoly orchestra sells tickets to nonstudents (N) and students (S). The two demands are

$$q_N = \frac{240}{p_N^2}, \qquad q_S = \frac{540}{p_S^3},$$

with total cost $C(Q) = 2Q$ where $Q = q_N + q_S$. Assume the orchestra can price-discriminate between the two groups and compute prices and quantities.

Answer.

Step 1. Read off the demand elasticities. $q_N = 240\, p_N^{-2}$ has constant elasticity $\varepsilon_N = -2$; $q_S = 540\, p_S^{-3}$ has constant elasticity $\varepsilon_S = -3$.

Step 2. Apply IEPR to each market separately. With $MC = 2$ constant,

$$p_N\left(1 + \frac{1}{-2}\right) = 2 \iff p_N = 4,$$

and

$$p_S\left(1 + \frac{1}{-3}\right) = 2 \iff p_S = 3.$$

Students face the more elastic demand and pay the lower price — the standard third-degree discrimination result.

Step 3. Recover quantities. $q_N = 240/16 = 15$; $q_S = 540/27 = 20$.

Intuitively, the profit-maximizing markup in each market is $1/|\varepsilon|$ (the Lerner index equals the inverse elasticity). Students have the more elastic demand, so their optimal markup is smaller, so their price is closer to marginal cost. Discrimination lets the monopolist charge each group exactly the markup its own elasticity supports.

7. **Comparative statics: demand-shift parameter.** A monopoly faces inverse demand $p(q,t)$, where t shifts the demand curve (e.g., a fashion parameter). Assume constant marginal cost c.

   (a) Derive dq/dt in general form.

   (b) Specialize to additively separable inverse demand $p(q,t) = a(q) + b(t)$.

Answer. *Part (a): general implicit-function-theorem expression.*

Step 1. Set up the PMP. $\max_q \; p(q,t)\, q - c\, q$; FOC

$$\Phi(q,t) \equiv p(q,t) + p_q(q,t)\, q - c = 0.$$

Step 2. Compute the partials.

$$\frac{\partial \Phi}{\partial t} = p_t(q,t) + p_{qt}(q,t)\, q,$$
$$\frac{\partial \Phi}{\partial q} = 2\, p_q(q,t) + p_{qq}(q,t)\, q \qquad \text{(the SOC of the PMP; negative at an interior optimum).}$$

Step 3. Implicit function theorem.

$$\frac{dq}{dt} = -\frac{\partial \Phi/\partial t}{\partial \Phi/\partial q} = -\frac{p_t + p_{qt}\, q}{2\, p_q + p_{qq}\, q}.$$

Answer. *Part (b): additive separability* $p(q,t) = a(q) + b(t)$.

Then $p_t = b'(t)$ and $p_{qt} = 0$, so the numerator of the general formula becomes just $b'(t)$. Also $p_q = a'(q)$ and $p_{qq} = a''(q)$, so the denominator is $2a'(q) + a''(q)\, q$. Hence

$$\frac{dq}{dt} = -\frac{b'(t)}{2\, a'(q) + a''(q)\, q}.$$

The sign is unambiguous when $b'(t) > 0$ (positive demand shift) and a is well-behaved so the denominator is negative (interior maximum): then $dq/dt > 0$. The monopoly's output rises with the demand shifter, but the magnitude depends on how curved the inverse-demand function is.

Intuitively, additive separability means the demand shift $b(t)$ affects the intercept but not the slope of inverse demand. The monopoly's optimal response depends only on how much extra revenue an additional unit brings (numerator) versus how much marginal revenue falls with quantity (denominator).

8. **Complement (a monopolist choosing how many machines to run).** Exercises 1–7 above examined the monopolist's price-quantity choice; this complement adds a discrete margin, the number of production units the monopolist operates. A patent-holding monopolist produces with K identical machines, each with strictly convex per-machine cost $c(q)$ for output q. An extra machine costs $F > 0$ to install. Inverse demand is $P(q)$, differentiable and strictly decreasing.

   (a) For a given K (with each machine producing a positive amount), characterize the per-machine output and the firm's profit.

   (b) Derive the condition that pins down the optimal number of machines K^*.

   (c) Take the specific example $P(q) = A - q$ and $c(q) = q^2$. Find total output, profit as a function of K, and the optimal K^*.

Answer. *Part (a): output per machine and profit at fixed K.*

Step 1. Set up. With total output $Q = \sum_{k=1}^{K} q_k$, profit is

$$\pi = P(Q)\,Q - \sum_{k=1}^{K} c(q_k) - K\,F.$$

Step 2. FOC in q_k. For each active machine,

$$\frac{\partial \pi}{\partial q_k} = P(Q) + P'(Q)\,Q - c'(q_k) = MR(Q) - c'(q_k) = 0,$$

so $c'(q_k) = MR(Q)$ for every k. The common $MR(Q)$ on the right forces every q_k to satisfy the same equation; strict convexity of c (so c' is strictly increasing) makes the solution unique. Hence all machines produce the same amount, $q_k = Q/K$.

Step 3. Read off profit. With $q_k = Q/K$ for every k, profit collapses to

$$\pi(Q, K) = P(Q)\,Q - K\,c(Q/K) - K\,F.$$

Answer. *Part (b): optimal number of machines.*

Step 1. FOC in K (treating K as continuous and using the envelope in Q). The optimal Q satisfies the FOC from part (a), so its derivative w.r.t. K affects profit only through the direct dependence on K:

$$\frac{d\pi}{dK} = -\frac{d}{dK}\big[K\,c(Q/K)\big] - F = 0.$$

Step 2. Differentiate $K\,c(Q/K)$. Using the product rule,

$$\frac{d}{dK}\big[K\,c(Q/K)\big] = c(Q/K) + K\,c'(Q/K)\cdot(-Q/K^2) = c(Q/K) - \frac{Q}{K}\,c'(Q/K).$$

Step 3. Combine and rearrange. Setting the total derivative to zero and multiplying by -1,

$$\boxed{\frac{Q}{K}\, c'\Big(\frac{Q}{K}\Big) - c\Big(\frac{Q}{K}\Big) = F.}$$

The left-hand side is the "per-machine operating surplus" generated by strict convexity of c: revenue at marginal cost minus average cost, times per-machine output. Adding a machine is worth it up to the point where that surplus equals the fixed cost F.

Answer. *Part (c): the specific example $P(q) = A - q$, $c(q) = q^2$.*

Step 1. Per-machine FOC. $c'(q) = 2q$; $MR(Q) = A - 2Q$. Setting $c'(Q/K) = MR(Q)$: $2(Q/K) = A - 2Q$, i.e.,

$$Q\Big(\frac{2}{K} + 2\Big) = A \iff Q = \frac{A\,K}{2(K+1)}.$$

Step 2. Profit as a function of K.

$$\pi(K) = P(Q)\,Q - K\,(Q/K)^2 - K\,F = (A - Q)Q - \frac{Q^2}{K} - K\,F.$$

Substituting $Q = AK/[2(K+1)]$ and simplifying (using $A - Q = A(K+2)/[2(K+1)]$),

$$\pi(K) = \frac{A^2\,K}{4(K+1)} - K\,F.$$

Step 3. Optimal K. Differentiate in K: $\pi'(K) = \dfrac{A^2\,[4(K+1) - 4K]}{16(K+1)^2} - F = \dfrac{A^2}{4(K+1)^2} - F$. Setting equal to zero,

$$\frac{A^2}{4(K+1)^2} = F \iff K^* = \frac{A}{2\sqrt{F}} - 1,$$

rounded to the nearest feasible integer.

Intuitively, the per-machine "operating surplus" at the specific-example optimum is q_k^2, and the optimal-K condition sets this equal to F. The firm runs more machines when demand is strong (A large) or when installing another machine is cheap (F small). Convexity of the per-machine cost is what makes running multiple machines valuable in the first place; without it (constant returns per machine), the firm would concentrate all production in a single machine.