EconS 503 - Advanced Microeconomics - II
Final Exam - Answer key

1. **Contracting with input providers.** Consider a firm contracting inputs from a company that produces two types of inputs, A with probability α and B with probability $1-\alpha$. Intuitively, one can interpret these two types as if the firm purchases inputs from a company without observing the input's quality, but knowing that the frequency of A-type is α. The firm's profits from a known quality $i = \{A, B\}$ are $10q - p$, where q denotes the input units purchased and p represents the (lump-sum) price that the firm pays for the inputs.

   A company producing input i has total costs $c_i(q) = \gamma^i q^2$ from producing q units, which is increasing and convex in output, and where $\gamma^B < \gamma^A$. Therefore, input provider i earns profits $p - \gamma^i q^2$. For simplicity, assume that its reservation profit from rejecting a contract is zero.

   (a) *Symmetric information.* As a benchmark, let us first solve the principal's problem (firm buying inputs) when it can perfectly observe the input type i. Find the contract $\left(q^{SI}, p^{SI}\right)$, specifying the number of units ordered from input provider i and the lump-sum price that the principal pays to the input provider.

   - The principal observes the type of input provider it faces, and solves the following profit maximization problem

     $$\max_{q,p\geq 0} \; 10q - p$$

     subject to

     $$p - \gamma^i q^2 \geq 0 \tag{PC}$$

     The participation constraint, PC, must hold with equality, $p = \gamma^i q^2$. Inserting this result into the principal's objective function, we obtain

     $$\max_{q\geq 0} \; 10q - \underbrace{\gamma^i q^2}_{p}.$$

     Differentiating with respect to q, yields $10 - 2\gamma^i q = 0$ and, rearranging, we find that the principals orders

     $$q^{SI} = \frac{5}{\gamma^i} \text{ units}$$

     from input provider i. Since $\gamma^B < \gamma^A$ by assumption, the principal orders fewer units from the high-cost input provider (company A) than from the low-cost one (company B).

   - *Price.* Therefore, equilibrium price is $p^{SI} = \gamma^i \left(\frac{5}{\gamma^i}\right)^2 = \frac{25}{\gamma^i}$. Since $\gamma^B < \gamma^A$ by assumption, the principal pays a lower lump-sum price, p^{SI}, when ordering units from the high-cost input provider (company A) than from the low-cost one (company B).

- *Profits.* In this symmetric information context, the input provider makes no profit, but the principal (firm ordering inputs) earns

$$\pi^{SI} = 10q^{SI} - p^{SI} = 10\frac{5}{\gamma^i} - \frac{25}{\gamma^i} = \frac{25}{\gamma^i}.$$

As above, since $\gamma^B < \gamma^A$, the principal earns a lower profit when facing a high-cost input provider (company A) than when facing a low-cost one (company B).

(b) *Asymmetric information.* Assume now that the principal cannot observe the input provider's type. Find the optimal contract pair, $\left(q^A, p^A\right)$ and $\left(q^B, p^B\right)$, in this context.

- The principal solves the following expected profit maximization problem

$$\max_{q^A, p^A, q^B, p^B} \quad \alpha\left(10q^A - p^A\right) + (1-\alpha)(10q^B - p^B)$$

subject to

$$\begin{aligned} p^A - \gamma^A (q^A)^2 &\geq 0 & (\text{PC}_A) \\ p^B - \gamma^B (q^B)^2 &\geq 0 & (\text{PC}_B) \\ p^A - \gamma^A (q^A)^2 &\geq p^B - \gamma^A (q^B)^2 & (\text{IC}_A) \\ p^B - \gamma^B (q^B)^2 &\geq p^A - \gamma^B (q^A)^2 & (\text{IC}_B) \end{aligned}$$

The PC constraints, as usual, indicate that the input provider prefers to accept the contract offered by the principal than rejecting it (and earn a zero profit), which holds for every input provider i. The incentive compatibility conditions, IC, mean that each input provider i prefers its contract, (q^i, p^i), than that meant for the other type of input provider, (q^j, p^j), where $j \neq i$. As a remark, note that the input provider's type is fixed on both sides of every IC condition, i.e., γ^A in IC_A and γ^B in IC_B.

- If IC_B and PC_A hold, we have that

$$p^B - \gamma^B (q^B)^2 \underbrace{\geq}_{\text{From IC}_B} p^A - \gamma^B (q^A)^2 \underbrace{>}_{\text{From } \gamma^B < \gamma^A} p^A - \gamma^A (q^A)^2 \underbrace{\geq}_{\text{From PC}_A} 0$$

which, taking the first and last term, implies $p^B - \gamma^B (q^B)^2 > 0$. Therefore, PC_B must also hold, meaning that we can ignore this constraint, which is slack, in our subsequent analysis. The Lagrangian of the above problem is

$$\begin{aligned} L \;=\; & \alpha\left(10q^A - p^A\right) + (1-\alpha)(10q^B - p^B) \\ & + \lambda\left[p^A - \gamma^A (q^A)^2\right] \\ & + \mu_A\left[p^A - \gamma^A (q^A)^2 - p^B + \gamma^A (q^B)^2\right] \\ & + \mu_B\left[p^B - \gamma^B (q^B)^2 - p^A + \gamma^B (q^A)^2\right]. \end{aligned}$$

Differentiating with respect to p_A and p_B, yields

$$\begin{aligned} \frac{\partial L}{\partial p_A} &= -\alpha + \lambda + \mu_A - \mu_B = 0, \text{ and} \\ \frac{\partial L}{\partial p_B} &= -(1-\alpha) - \mu_A + \mu_B = 0. \end{aligned}$$

Summing these two first-order conditions, we obtain

$$[-\alpha+\lambda+\mu_A-\mu_B]+[-(1-\alpha)-\mu_A+\mu_B]=\lambda-1=0$$

or $\lambda = 1$, which implies that PC_A must hold with strict equality, $p^A - \gamma^A(q^A)^2 = 0$.

- Differentiating with respect to q_A and q_B, we obtain

$$\begin{aligned}\frac{\partial L}{\partial q_A} &= 10\alpha - 2q^A\left(\lambda\gamma^A+\gamma^A\mu_A-\gamma^B\mu_B\right)=0, \text{ and}\\ \frac{\partial L}{\partial q_B} &= 10(1-\alpha)+2q^B\left(\gamma^A\mu_A-\gamma^B\mu_B\right)=0.\end{aligned}$$

At this point, we recall that $\lambda = 1$, inserted in the above first-order conditions. In addition, we can consider that IC_A holds with strict inequality, implying that its Lagrange multiplier is $\mu_A = 0$. (We confirm this property below.) Inserting $\mu_A = 0$ in the above first-order conditions, yields

$$\begin{aligned}\frac{\partial L}{\partial p_A} &= 1-\alpha-\mu_B=0,\\ \frac{\partial L}{\partial p_B} &= -(1-\alpha)+\mu_B=0,\\ \frac{\partial L}{\partial q_A} &= 10\alpha-2q^A\left(\gamma^A-\gamma^B\mu_B\right)=0, \text{ and}\\ \frac{\partial L}{\partial q_B} &= 10(1-\alpha)-2q^B\gamma^B\mu_B=0.\end{aligned}$$

From $\frac{\partial L}{\partial p_A}$, we find that $\mu_B = 1-\alpha$. Inserting this result in $\frac{\partial L}{\partial q_A}$, yields $\frac{\partial L}{\partial q_A} = 10\alpha - 2q^A\left[\gamma^A - \gamma^B(1-\alpha)\right] = 0$ that, after solving for q^A, entails

$$q^A = \frac{5\alpha}{\gamma^A-\gamma^B(1-\alpha)}.$$

Inserting $\mu_B = 1-\alpha$ into $\frac{\partial L}{\partial q_B}$, we find that $\frac{\partial L}{\partial q_B} = 10(1-\alpha)-2q^B\gamma^B(1-\alpha) = 0$ that simplifies to $10-2q^B\gamma^B = 0$ and, solving for q^B, yields

$$q^B = \frac{5}{\gamma^B}.$$

Finally, we know that PC_A holds with strict equality, $p^A - \gamma^A(q^A)^2 = 0$, or $p^A = \gamma^A(q^A)^2$, which helps us identify optimal price p^A, that is,

$$p^A = \gamma^A\left(\frac{5\alpha}{\gamma^A-\gamma^B(1-\alpha)}\right)^2.$$

In addition, since IC_B holds with strict equality, $p^B - \gamma^B(q^B)^2 = p^A - \gamma^B(q^A)^2$, which entails

$$\begin{aligned} p^B &= \gamma^B(q^B)^2 + p^A - \gamma^B(q^A)^2 \\ &= \gamma^B(q^B)^2 + \underbrace{\gamma^A(q^A)^2}_{p^A} - \gamma^B(q^A)^2 \\ &= \gamma^B(q^B)^2 + (\gamma^A - \gamma^B)(q^A)^2 \\ &= \gamma^B\left(\frac{5}{\gamma^B}\right)^2 + \underbrace{(\gamma^A - \gamma^B)\left(\frac{5\alpha}{\gamma^A - \gamma^B(1-\alpha)}\right)^2}_{+} \end{aligned}$$

meaning that price is higher than that under symmetric information, with the "price premium" being captured by the last term $(\gamma^A - \gamma^B)\left(\frac{5\alpha}{\gamma^A - \gamma^B(1-\alpha)}\right)^2$.

- *Constraint IC_A holds with strict inequality.* We must now check that $p^A - \gamma^A(q^A)^2 > p^B - \gamma^A(q^B)^2$. Inserting our equilibrium results, we obtain that

$$\underbrace{\gamma^A(q^A)^2}_{p^A} - \gamma^A(q^A)^2 > \underbrace{\left[\gamma^B(q^B)^2 - (\gamma^A - \gamma^B)(q^A)^2\right]}_{p^B} - \gamma^A(q^B)^2$$

which simplifies to

$$\begin{aligned} 0 &> \left(\gamma^B - \gamma^A\right)\left[(q^B)^2 + (q^A)^2\right] \\ &= \left(\gamma^B - \gamma^A\right)\left[\left(\frac{5}{\gamma^B}\right)^2 + \left(\frac{5\alpha}{\gamma^A - \gamma^B(1-\alpha)}\right)^2\right] \end{aligned}$$

which holds, since $\gamma^B < \gamma^A$ by assumption, and the term in brackets is positive. Therefore, IC_A holds with strict inequality.

(c) Compare your equilibrium results under symmetric and asymmetric information.

- *Equilibrium output.* Recall from part (a) that under symmetric information the principal offers, to each type i, $q^{SI} = \frac{5}{\gamma^i}$. The equilibrium quantity under asymmetric information, as found in part (b), coincides with that under symmetric information for company B), $q^B = \frac{5}{\gamma^B}$, implying that there is no distortion for the low-cost type.
For company A, we find that

$$q^A = \frac{5\alpha}{\gamma^A - \gamma^B(1-\alpha)} < \frac{5}{\gamma^A} = q^{SI}$$

since this inequality simpflies to $\alpha(\gamma^A - \gamma^B) < (\gamma^A - \gamma^B)$, or $\alpha < 1$. Therefore, the principal distorts downward the quantity assigned to the high-cost type A, in order to reduce the information rent that must be left to type B.

- *Equilibrium price.* Under symmetric information, equilibrium prices are $p^{SI} = \gamma^i(q^{SI})^2$ for every input provider i. Starting with company A, we find that

$$p^A = \gamma^A(q^A)^2 < \gamma^A(q^{SI})^2 = p^{SI}$$

since $q^A < q^{SI}$, as shown above. Hence, the high-cost firm receives a lower transfer under asymmetric than symmetric information.

For company B, we found that $p^B = \gamma^B(q^B)^2 = \frac{25}{\gamma^B}$ under symmetric information, and

$$\begin{aligned} p^B &= \gamma^B(q^B)^2 + (\gamma^A - \gamma^B)(q^A)^2 \\ &= \frac{25}{\gamma^B} + (\gamma^A - \gamma^B)(q^A)^2 \end{aligned}$$

under asymmetric information in part (b). Comparing them, we find that

$$p^B = \frac{25}{\gamma^B} + (\gamma^A - \gamma^B)(q^A)^2 > \frac{25}{\gamma^B} = p^{SI}$$

since this inequality simplifies to $(\gamma^A - \gamma^B)(q^A)^2 > 0$, which holds given that $\gamma^A > \gamma^B$ by definition. Therefore, even though the quantity of type B is unaffected, this firm receives a higher lump-sum transfer under asymmetric than symmetric information, giving rise to an information rent, required to satisfy IC_B.

2. **When goods are bads.** An exchange economy consists of two consumers (A and B) with utility function

$$u^i(x_1^i, x_2^i) = x_1^i(4 - x_2^i) \ \text{ for consumer } i = \{A, B\}$$

So, the first commodity is a "good" for each consumer, whereas the second commodity is a "bad" for each consumer. Their initial endowments are $\omega^A = (4, 3)$ and $\omega^B = (1, 0)$.

(a) Find the consumers' Walrasian demand functions.

- *Consumer A.* Starting with consumer A, his maximization problem is

$$\max_{x_1^A, x_2^A \geq 0} \quad x_1^A(4 - x_2^A)$$
$$\text{subject to } \ p_1 x_1^A + p_2 x_2^A = 4p_1 + 3p_2$$

Because his utility is increasing in x_1^A whenever $x_2^A < 4$, the consumer will always choose the largest feasible amount of x_1^A for a given x_2^A. Hence, we can solve for x_1^A in the budget line to obtain

$$x_1^A = \frac{4p_1 + 3p_2 - p_2 x_2^A}{p_1}.$$

Inserting it into the utility function, yields

$$u^A = \frac{4p_1 + 3p_2 - p_2 x_2^A}{p_1}(4 - x_2^A)$$

  - If $p_2 \geq 0$, consuming the bad (good 2) reduces A's utility and uses up budget. Therefore, the consumers sets $x_2^A = 0$, implying that all income is spent on good 1, that is, $x_1^A = \frac{4p_1 + 3p_2}{p_1}$.

– If $p_2 < 0$, taking some units of good 2 increases A's purchasing power, but still decreases his utility through $(4 - x_2^A)$. His optimal choice balances these two effects. The interior FOC yields Walrasian demand

$$x_2^A = \frac{4p_1 + 7p_2}{2p_2}$$

which is positive if $4p_1 + 7p_2 > 0$, i.e., when $p_1 > \frac{7}{4}p_2$. In this context, his demand for good 1 is

$$x_1^A = \frac{4p_1 + 3p_2 - p_2 x_2^A}{p_1} = \frac{4p_1 + 3p_2 - p_2 \frac{4p_1+7p_2}{2p_2}}{p_1} = \frac{4p_1 + p_2}{2p_1}.$$

- *Consumer B*. His income is $1p_1 + 0p_2 = p_1$ since he only has one unit of good 1 in his endowment, and no units of good 2. Following a similar argument as for consumer A, we find that

$$x_1^B = \frac{p_1 - p_2 x_2^B}{p_1}.$$

Inserting it into the utility function, yields

$$u^B = \frac{p_1 - p_2 x_2^B}{p_1}(4 - x_2^B)$$

– If $p_2 \geq 0$, consuming the bad (good 2) reduces B's utility and uses up budget. Therefore, the consumers sets $x_2^B = 0$, implying that all income is spent on good 1, that is, $x_1^B = \frac{p_1}{p_1} = 1$ unit. As a consequence, consumer B is happy with his original endowment and does not want to trade with consumer A.

– If $p_2 < 0$, following a similar argument as for consumer A, we obtain an interior candidate of

$$x_1^A = \frac{p_1 + 4p_2}{2p_2}$$

which is positive if and only if $p_1 < -4p_2$. In this case, his demand for good 1 is $x_1^B = \frac{p_1 - 2p_2}{p_1}$.

(b) Show that an allocation is Pareto optimal if and only if $x_1^A + x_2^A = 4$.

- Any Pareto optimal allocation must be feasible. At any feasible allocation with $x_1^i > 0$ and $x_2^i < 4$ units, the marginal utilities become $MU_1^i = 4 - x_2^i$ and $MU_2^i = -x_1^i$, implying that the $MRS = -\frac{4 - x_2^i}{x_1^i}$ for every consumer i. Using the feasibility condition, $x_1^B = 5 - x_1^A$ for good 1 and, similarly, $x_2^B = 3 - x_2^A$ for good 2. Therefore, the condition given in the exercise, $x_1^A + x_2^A = 4$, can be rewritten as $x_1^A = 4 - x_2^A$, and the above MRS becomes

$$MRS^A = -\frac{4 - x_2^A}{x_1^A} = -\frac{4 - x_2^A}{4 - x_2^A} = -1.$$

A similar argument applies for consumer B, where $4 - x_2^B = 4 - (3 - x_2^A) = 1 + x_2^A$ and $x_1^B = 5 - x_1^A = 1 + x_2^A$ since $x_1^A = 4 - x_2^A$. Then, his MRS becomes

$$MRS^B = -\frac{4 - x_2^B}{x_1^B} = -\frac{1 + x_2^A}{1 + x_2^A} = -1.$$

Therefore, along the line $x_1^A + x_2^A = 4$, we have shown that $MRS_A = MRS_B$, implying that both consumer's indifference curves are tangent to each other.

(c) Draw the Edgeworth Box.

- Since good 2 is a bad, we find that consumer A's utility is increasing as he moves towards the lower right-hand corner of the Edgeworth box, whereas consumer B's utility increases as he moves to the upper left-hand corner of the Edgeworth box, as shown in figure 6.12.

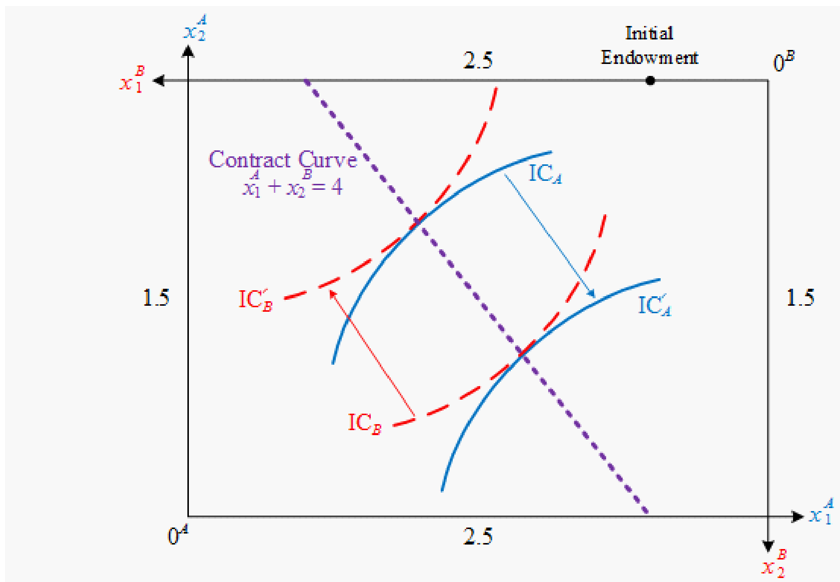

Figure 6.12. Edgeworth box and the contract curve.

(d) Find the competitive equilibria in this economy (remembering that good 2 is a bad.)

- Because good 2 is a bad for both consumers, a WEA cannot have $p_2 \geq 0$. If this was the case, both consumers would optimally choose zero amounts of good 2, $x_2^A = x_2^B = 0$, so the market for good 2 could not clear since its total endowment is 3 units. Hence, in any WEA, we must have that $p_2 < 0$. Using the Walrasian demand in the case that $p_2 < 0$ found in part (a), and inserting them into the feasibility constraint for good 1, $x_1^A + x_1^B = 5$, yields

$$x_1^A + x_1^B = \frac{4p_1 - p_2}{2p_1} + \frac{p_1 - 4p_2}{2p_1} = 5$$

which simplifies to $\frac{p_2}{p_1} = -1$. Therefore, the WEA price ratio is $\frac{p_2}{p_1} = -1$. Inserting this price ratio, $\frac{p_2}{p_1} = -1$ or $p_2 = -p_1$, into the Walrasian demands found in part (a), we obtain that

$$x_1^A = \frac{4p_1 - (-p_1)}{2p_1} = \frac{5}{2} \quad \text{and} \quad x_2^A = \frac{4p_1 + 7(-p_1)}{2(-p_1)} = \frac{3}{2}$$

for consumer A, and

$$x_1^B = \frac{p_1 - 4(-p_1)}{2p_1} = \frac{5}{2} \text{ and } x_2^B = \frac{p_1 + 4(-p_1)}{2(-p_1)} = \frac{3}{2}$$

for consumer B. Hence, every consumer purchases $\frac{5}{2}$ units of good 1 and $\frac{3}{2}$ of good 2.

(e) What happens to the set of competitive equilibria in the economy if consumer A is given the right to dump her endowment of the second good on consumer B without compensating consumer B?

- *PEA.* If consumer A can dump her endowment of the second good on consumer B without compensating him, the new endowments become $\omega^A = (4, 0)$ and $\omega^B = (1, 3)$, while the aggregate endowment remains $(5, 3)$. This endowment satisfies the $x_1^A + x_2^A = 4$ Pareto efficiency requirement, and thus neither consumer can do any better through trading without hurting the other consumer.
- *WEA.* We seek to show that the allocation with no trade (new endowments) is a WEA. To show this, notice that consumer A's MRS at $(x_1^A, x_2^A) = (4, 0)$ is

$$MRS^A = \frac{MU_1^A}{MU_2^A} = -\frac{4-0}{4} = -1.$$

which is equal to the WEA price ratio found in part (d), $\frac{p_2}{p_1} = -1$. Similarly, consumer B's MRS at $(x_1^B, x_2^B) = (1, 3)$ is

$$MRS^B = \frac{MU_1^B}{MU_2^B} = -\frac{4-3}{1} = -1.$$

which is also equal to the WEA price ratio found in part (d), $\frac{p_2}{p_1} = -1$. Finally, we verify the market clearing conditions, $x_1^A + x_1^B = 4 + 1 = 5$ for good 1, and $x_2^A + x_2^B = 0 + 3 = 3$ for good 2.
- This WEA without trade can be seen in figure 6.13.

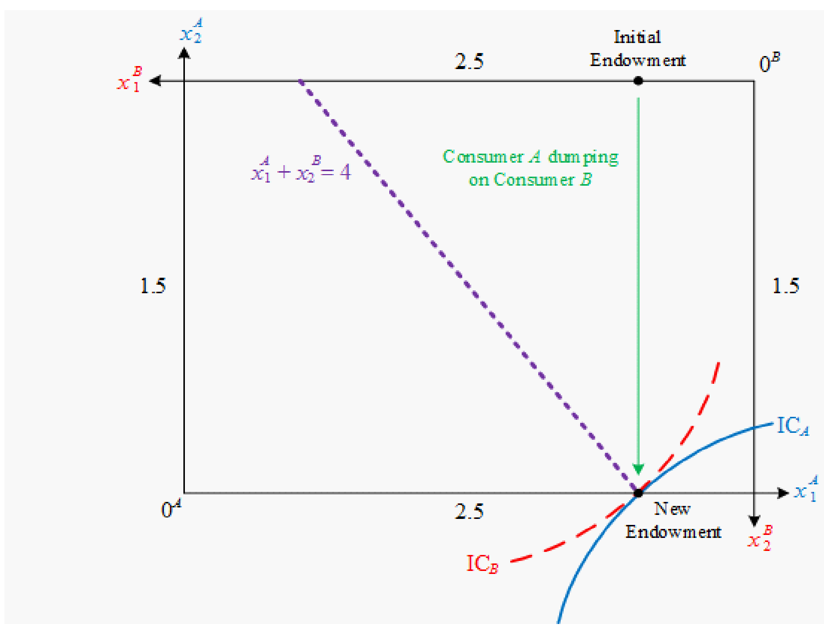

Figure 6.13. WEA when we allow for dumping from consumer A to B.

3. **Optimal taxation using mechanism design.** Consider a government needing to raise a fixed sum of money, \$$S$, through income taxes. There are two types of workers, high productivity (H) and low productivity (L), and the output (gross income) produced by each is given by

$$q^k = \theta^k e^k, \quad \text{where} \quad k = H, L$$

where e^k is the amount of effort exerted by a worker of type k and the productivity parameter satisfies $\theta^H > \theta^L$. Hence, for a given effort level, the high-productivity worker generates a larger amount of output than the low-productivity worker. The utility function of a worker with type k is

$$v^k = q^k - t^k - g(e^k)$$

where t^k is the tax on a worker of type k, and $g(\cdot)$ is a strictly increasing and convex function in effort, i.e., $g' > 0$ and $g'' > 0$. The government has no interest in the inequality of utility outcomes and so just seeks to maximize the expected social welfare

$$W = pv^H + (1 - p)v^L$$

where p is the proportion of H-type workers.

(a) What is the government's budget constraint?

- If S is the amount to be raised per person in the economy, the government's budget constraint is

$$S \leq pt^H + (1 - p)t^L$$

which must hold with equality at the optimum. Indeed, if the sum of taxes exceeded S the government could increase individual utility and therefore social welfare by cutting taxes. Hence, we can write the constraint as

$$S = pt^H + (1 - p)t^L.$$

(b) *Complete information.* If the government was perfectly informed about the worker's type, find the socially optimal taxes and the associated output levels.

- If the government was perfectly informed about the worker's type, the government maximizes each type's utility subject to its budget constraint and participation constraints of both types of worker. That is, since $q^k = \theta^k e^k$, then $e^k = \frac{q^k}{\theta^k}$, and the planner's problem when observing a type-θ^k worker is

$$\max_{q^k \geq 0} \; q^k - t^k - g\left(\frac{q^k}{\theta^k}\right)$$

$$\text{subject to} \quad S = t^k$$

or, inserting the constraint into the objective function,

$$\max_{q^k \geq 0} \; q^k - S - g\left(\frac{q^k}{\theta^k}\right)$$

Taking FOCs with respect to q^k for type k's utility function, we obtain $1 - \frac{1}{\theta^k} g'\left(\frac{q^k}{\theta^k}\right) = 0$ or, more compactly,

$$\theta^k = g'\left(\frac{q^k}{\theta^k}\right)$$

(c) *Parametric example (Complete information).* Assume that the cost of effort function is $g(e^k) = (e^k)^2$, so its derivatives are $g' = 2e^k \geq 0$ and $g'' = 2 > 0$; as required. Evaluate the FOCs found in part (b) for the complete information context assuming that productivity parameters are $\theta^H = 1$ and $\theta^L = \frac{1}{2}$. Find the optimal values of q^H and q^L. [*Hint*: After finding the equilibrium output and effort levels for each worker type, you can define the difference between a worker's output and his tax payment as $y^k \equiv q^k - t^k$ or, alternatively, his taxes as $t^k \equiv q^k - y^k$.]

- *Output.* In the case of a high-productivity worker, the FOC found in part (b), $\theta^H = g'\left(\frac{q^H}{\theta^H}\right) = 0$, yields $1 = 2\frac{q^H}{1}$, that is, $q^H = \frac{1}{2}$. And in the case of a low-productivity worker, we obtain $\frac{1}{2} = 2\frac{q^L}{\frac{1}{2}}$, implying that $q^L = \frac{1}{8}$.
- *Effort.* Hence, optimal effort levels, $e^k = \frac{q^k}{\theta^k}$, are $e^H = \frac{1/2}{1} = \frac{1}{2}$ and $e^L = \frac{1/8}{1/2} = \frac{1}{4}$
- *Taxes.* Substituting the optimal output levels into each type's participation constraints, we obtain

$$y^H = g(e^H) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$y^L = g(e^L) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

where we define $y^k \equiv q^k - t^k$ to yield taxes $t^k \equiv q^k - y^k$ that help us find the associated optimal taxes under complete information

$$\begin{aligned} t^H &= q^H - y^H = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}, \text{ and} \\ t^L &= q^L - y^L = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}. \end{aligned}$$

Intuitively, the output, effort, and tax on the high-productivity worker is higher than on the low-type of worker.

(d) *Incomplete information.* Assuming that the government cannot observe the worker's type, write the government's objective function in terms of q^H, q^L, p, and S.

- By definition, the objective function is $pv^H + [1 - p]v^L$. Substituting the utility function of each type of worker, yields

$$p[q^H - t^H - g(e^H)] + (1-p)\left[q^L - t^L - g(e^L)\right]$$

which can be rewritten as

$$p[q^H - g(e^H)] + (1-p)\left[q^L - g(e^L)\right] - \underbrace{\left[pt^H + (1-p)\, t^L\right]}_{S}$$

Substituting the government's budget constraint in the last term, we obtain

$$p[q^H - g(e^H)] + (1-p)\,[q^L - g(e^L)] - S.$$

Finally, since $e^k = \frac{q^k}{\theta^k}$, we can express the government's objective function function of output and taxes alone (without including effort levels explicitly).

$$p\left[q^H - g\left(\frac{q^H}{\theta^H}\right)\right] + (1-p)\left[q^L - g\left(\frac{q^L}{\theta^L}\right)\right] - S$$

(e) Using the government's objective function you identified in part (d), write down the government's optimization problem. [*Hint*: You can use the same identity as in part (c), $y^k \equiv q^k - t^k$, to simplify your calculations. After using y^k, recall that the set of choice variables for the social planner changes from $\left(q^H, q^L, t^H, t^L\right)$ to $\left(q^H, q^L, y^H, y^L\right)$.]

- The government's problem is to maximize the expected welfare found in part (c), by choice of taxes t^H and t^L. In addition, since $q^k = \theta^k e^k$ we can solve for e^k obtaining $e^k = \frac{q^k}{\theta^k}$, which helps us express the government's problem as a function of output and taxes alone (without including effort levels explicitly).

$$\max_{q^H,q^L,t^H,t^L} \quad p\left[q^H - g\left(\frac{q^H}{\theta^H}\right)\right] + (1-p)\left[q^L - g\left(\frac{q^L}{\theta^L}\right)\right] - S$$

subject to

$$q^L - t^L - g\left(\frac{q^L}{\theta^L}\right) \geq q^H - t^H - g\left(\frac{q^H}{\theta^L}\right) \tag{IC$_L$}$$

$$q^H - t^H - g\left(\frac{q^H}{\theta^H}\right) \geq q^L - t^L - g\left(\frac{q^L}{\theta^H}\right) \tag{IC$_H$}$$

$$q^L - t^L - g\left(\frac{q^L}{\theta^L}\right) \geq 0 \tag{PC$_L$}$$

$$q^H - t^H - g\left(\frac{q^H}{\theta^H}\right) \geq 0 \tag{PC$_H$}$$

For compactness, we denote $y^k \equiv q^k - t^k$ in the IC and PC constraints, which helps us simplify the problem as follows (note that the choice variables now changed, as taxes are implicitly embedded in y^k).

$$\max_{q^H,q^L,y^H,y^L} \quad p\left[q^H - g\left(\frac{q^H}{\theta^H}\right)\right] + (1-p)\left[q^L - g\left(\frac{q^L}{\theta^L}\right)\right] - S$$

subject to

$$y^L - g\left(\frac{q^L}{\theta^L}\right) \geq y^H - g\left(\frac{q^H}{\theta^L}\right) \tag{IC$_L$}$$

$$y^H - g\left(\frac{q^H}{\theta^H}\right) \geq y^L - g\left(\frac{q^L}{\theta^H}\right) \tag{IC$_H$}$$

$$y^L - g\left(\frac{q^L}{\theta^L}\right) \geq 0 \tag{PC$_L$}$$

$$y^H - g\left(\frac{q^H}{\theta^H}\right) \geq 0 \tag{PC$_H$}$$

- As in similar screening models, note that PC_L must hold with equality (no information rents for the low-type worker), entailing that

$$y^L = g\left(\frac{q^L}{\theta^L}\right)$$

and, similarly, IC_H holds with equality, implying that

$$y^H - g\left(\frac{q^H}{\theta^H}\right) = \underbrace{g\left(\frac{q^L}{\theta^L}\right)}_{y^L} - g\left(\frac{q^L}{\theta^H}\right).$$

Rearranging this expression, we obtain

$$y^H = g\left(\frac{q^H}{\theta^H}\right) + g\left(\frac{q^L}{\theta^L}\right) - g\left(\frac{q^L}{\theta^H}\right).$$

Inserting the expressions of y^L and y^H that we found above into the maximization problem, we obtain the following program (where we already removed PC_L and IC_H, leaving us with only two constraints: PC_H and IC_L):

$$\max_{q^H, q^L} \quad p\left[q^H - g\left(\frac{q^H}{\theta^H}\right)\right] + (1-p)\left[q^L - g\left(\frac{q^L}{\theta^L}\right)\right] - S$$

subject to

$$\underbrace{g\left(\frac{q^L}{\theta^L}\right)}_{y^L} - g\left(\frac{q^L}{\theta^L}\right) \geq \underbrace{g\left(\frac{q^H}{\theta^H}\right) + g\left(\frac{q^L}{\theta^L}\right) - g\left(\frac{q^L}{\theta^H}\right)}_{y^H} - g\left(\frac{q^H}{\theta^L}\right) \tag{IC$_L$}$$

$$\underbrace{g\left(\frac{q^H}{\theta^H}\right) + g\left(\frac{q^L}{\theta^L}\right) - g\left(\frac{q^L}{\theta^H}\right)}_{y^H} - g\left(\frac{q^H}{\theta^H}\right) \geq 0 \tag{PC$_H$}$$

Letting λ and μ be the Lagrange multipliers for constraints IC_L and PC_H, respectively, we obtain that

$$\begin{aligned}
\mathcal{L} \;=\; & p\left[q^H - g\left(\frac{q^H}{\theta^H}\right)\right] + (1-p)\left[q^L - g\left(\frac{q^L}{\theta^L}\right)\right] - S \\
& +\lambda\left[g\left(\frac{q^L}{\theta^H}\right) + g\left(\frac{q^H}{\theta^L}\right) - g\left(\frac{q^H}{\theta^H}\right) - g\left(\frac{q^L}{\theta^L}\right)\right] \\
& +\mu\left[g\left(\frac{q^L}{\theta^L}\right) - g\left(\frac{q^L}{\theta^H}\right)\right]
\end{aligned}$$

(f) Find the solution to the government's problem in part (d). Compare your answer to the complete information solution found in part (b).

- Taking the FOC with respect to q^H, we find

$$p\left[1-\frac{1}{\theta^H}g'\left(\frac{q^H}{\theta^H}\right)\right]+\lambda\left[\frac{1}{\theta^L}g'\left(\frac{q^H}{\theta^L}\right)-\frac{1}{\theta^H}g'\left(\frac{q^H}{\theta^H}\right)\right]=0$$

and taking the FOC with respect to q^L, we have

$$(1-p)\left[1-\frac{1}{\theta^L}g'\left(\frac{q^L}{\theta^L}\right)\right]+\lambda\left[\frac{1}{\theta^H}g'\left(\frac{q^L}{\theta^H}\right)-\frac{1}{\theta^L}g'\left(\frac{q^L}{\theta^L}\right)\right]$$
$$+\mu\left[\frac{1}{\theta^L}g'\left(\frac{q^L}{\theta^L}\right)-\frac{1}{\theta^H}g'\left(\frac{q^L}{\theta^H}\right)\right]=0$$

- The PC_H must hold with strict inequality and, similarly, IC_L must hold with strict inequality, meaning that $\lambda=\mu=0$, which simplifies the above FOCs to

$$p\left[1-\frac{1}{\theta^H}g'\left(\frac{q^H}{\theta^H}\right)\right]=0$$
$$(1-p)\left[1-\frac{1}{\theta^L}g'\left(\frac{q^L}{\theta^L}\right)\right]=0$$

In other words, we get the complete-information solution

$$\theta^H=g'\left(\frac{q^H}{\theta^H}\right)\quad\text{and}\quad\theta^L=g'\left(\frac{q^L}{\theta^L}\right)$$

(g) *Parametric example (Incomplete information).* Assume the same cost of effort function as in the parametric example developed in part (c), $g(e^k)=(e^k)^2$, and the same set of productivity parameters $\theta^H=1$ and $\theta^L=\frac{1}{2}$. In addition, consider that both types of workers are equally likely, i.e., $p=\frac{1}{2}$. Find the optimal values of q^H and q^L in the incomplete information setting. Then, find the optimal y^H and y^L, where $y^k\equiv q^k-t^k$.

- Since in both information contexts optimal outputs are given by the same FOC, we have that

$$q^H=\frac{1}{2}\quad\text{and}\quad q^L=\frac{1}{8}$$

(as shown in the parametric example of part (c)). However, taxes under complete and incomplete information do not coincide. In order to show this, let us first recall that, under complete information, we found that

$$t^H_{CI}=\frac{1}{4},\quad\text{and}\quad t^L_{CI}=\frac{1}{16}$$

Let us now find optimal taxes under incomplete information. Using PC_L, which also holds with equality, we find that $y^L_{II}=g(e^L)=\left(\frac{1}{4}\right)^2=\frac{1}{16}$

- Similarly, using condition IC_H, which holds with equality, we obtain

$$\begin{aligned} y_{II}^H - g\left(\frac{q^H}{\theta^H}\right) &= y_{II}^L + g\left(\frac{q^L}{\theta^H}\right) \\ &\iff y_{II}^H - \frac{1}{4} = \frac{1}{16} + \left(\frac{1}{8}\right)^2 \\ &\iff y_{II}^H = \frac{21}{64} > \frac{1}{4} \end{aligned}$$

for the high-type.

- Regarding taxes, we can next operate as in the case of complete information in part (c), that is, using $t^k \equiv q^k - y^k$ to find the optimal taxes under incomplete information, as follows

$$\begin{aligned} t_{II}^H &= q^H - y_{II}^H = \frac{1}{2} - \frac{21}{64} = \frac{11}{64}, \text{ and} \\ t_{II}^L &= q^L - y_{II}^L = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}. \end{aligned}$$

Hence, under incomplete information, the high-type worker pays lower taxes and obtains a higher utility, ultimately generating an information rent of

$$\begin{aligned} \Delta v^H &= v_{II}^H - v_{CI}^H \\ &= \left(\frac{1}{2} - \frac{11}{64}\right) - \left(\frac{1}{4} - 0\right) \\ &= \frac{5}{64} > 0. \end{aligned}$$

However, the low-type worker's information rent is zero. That is, he produces the same first-best output and tax under both information contexts. In summary, while output levels suffer no information distortion (i.e., they coincide under both information contexts), taxes differ, thus giving rise to a lower tax for the high-productivity worker and the same tax for the low-productivity worker.