EconS 424 - Strategy and Game Theory
Final Exam - Answer key

1. **Ultimatum bargaining game with altruistic players.** Consider the ultimatum bargaining game we discussed in class. If an agreement is reached, the responder earns the division that the proposer offered, d, while the proposer earns the remaining surplus, $1-d$. Assume, however, that the proposer's utility from this payoff distribution is

$$(1-d)+\alpha_P d,$$

where $\alpha_P \in [0,1]$ represents his degree of altruism. Similarly, the responder's utility from this payoff distribution is

$$d+\alpha_R(1-d),$$

where $\alpha_R \in [0,1]$ denotes his degree of altruism. Intuitively, every player cares about his own payoff (first component of every utility function) but also enjoys seeing the other player receive a higher payoff (second component of the utility function). When $\alpha_i = 0$, player $i = \{P, R\}$ does not exhibit altruistic preferences; when $\alpha_i \in (0,1)$ he cares more about his own payoff than his rival's; and when $\alpha_i = 1$, he assigns the same weight to his payoff and those of his rival. For generality, we allow for altruistic preferences to satisfy $\alpha_P > \alpha_R$, $\alpha_P < \alpha_R$, or $\alpha_P = \alpha_R$.

(a) Find the responder's best response. How is it affected by his altruism parameter, α_R?

- Using backward induction, we first focus on the last mover (the responder). In particular, the responder accepts any offer d from the proposer such that:

$$d+\alpha_R(1-d) \geq 0$$

since the payoff he obtains from rejecting the offer is zero. Solving for d, we obtain

$$d \geq -\frac{\alpha_R}{1-\alpha_R}$$

However, since $\alpha_R \in [0,1]$, we find that $-\frac{\alpha_R}{1-\alpha_R} \leq 0$, implying that the responder accepts any offer $d \geq 0$. Intuitively, the altruism parameter would make the responder accept offers from the proposer even if they were negative! Since offers cannot be negative, his acceptance rule, $d \geq 0$, coincides with that in the standard setting where the responder exhibits no altruism, $\alpha_R = 0$.

(b) Find the proposer's offer in equilibrium. How is it affected by his altruism parameter, α_P? How is it affected by the responder's altruism parameter, α_R? Interpret.

- Anticipating such a response from the responder, the proposer chooses the level of d that maximizes his utility. That is,

$$\max_{d \geq 0} \quad (1-d)+\alpha_P d$$

Differentiating with respect to d, we obtain $-1 + \alpha_P$. Since α_P satisfies $\alpha_P \in [0, 1]$ by definition, this derivative is negative, $-1 + \alpha_P \leq 0$, for all values of the altruism parameter α_P. As a result, the optimal division that the proposer offers is $d^* = 0$, which is unaffected by his altruism parameter, α_P.

Therefore, the proposer's utility in equilibrium is $(1 - d) + \alpha_P d = 1$ and the responder's utility is $d + \alpha_R(1 - d) = \alpha_R$.

(c) How are your results affected if the proposer exhibits stronger altruistic preferences than the responder, $\alpha_P > \alpha_R$? What if, instead, the responder has stronger altruistic preferences?

- The equilibrium offer from the proposer, $d^* = 0$, and the acceptance rule that the responder uses, $d \geq 0$, are unaffected by the altruism parameter of either player. Therefore, our equilibrium results are unchanged when altruism parameters satisfy $\alpha_P > \alpha_R$, $\alpha_P = \alpha_R$, or $\alpha_P < \alpha_R$.

(d) Evaluate your equilibrium results at $\alpha_P = \alpha_R = 0$. Interpret.

- When $\alpha_P = \alpha_R = 0$, the equilibrium offer is still $d^* = 0$, entailing that the proposer keeps the whole surplus, $1 - d^* = 1$, as in the standard ultimatum bargaining game where players do not exhibit altruistic preferences.

(e) Repeat parts (a)-(c) assuming that altruism parameters satisfy $\alpha_i \geq 0$ for every player i, thus not being restricted to $\alpha_i \in [0, 1]$.

- *Responder.* The responder accepts any offer d from the proposer such that $d + \alpha_R(1 - d) \geq 0$ which, solving for d, yields

$$d \geq -\frac{\alpha_R}{1 - \alpha_R}$$

so that the responder accepts any offer $d \geq 0$.

- *Proposer.* Anticipating such a response from the responder, the proposer chooses the level of d that maximizes his utility. That is,

$$\max_{d \geq 0} \quad (1 - d) + \alpha_P d$$

Differentiating with respect to d, we obtain $-1 + \alpha_P$. Since $\alpha_P \geq 0$ in this setting, we find that: (1) when $\alpha_P \leq 1$, this derivative is negative, $-1 + \alpha_P < 0$, and the proposer offers $d^* = 0$; but (2) when $\alpha_P > 1$, the derivative becomes positive, $-1 + \alpha_P > 0$, implying that the proposer offers $d^* = 1$ (the highest division of the surplus). Intuitively, when the weight he assigns to the responder's payoff is weakly lower than to his own payoff, $\alpha_P \leq 1$, he offers the lowest division that guarantees acceptance; whereas when the weight that he assigns to the responder's payoff is larger than to his own, $\alpha_P > 1$, he offers the highest division of the surplus.

- Our equilibrium results are, again, unaffected by whether the proposer exhibits stronger altruistic preferences than the responder, $\alpha_P > \alpha_R$, the same, $\alpha_P = \alpha_R$, or lower than the responder, $\alpha_P < \alpha_R$.

2. **Partial collusion in quantity competition.** Consider an industry with two firms competing in quantities, facing inverse demand function $p(Q) = 1 - Q$, where $Q = q_1 + q_2$ denotes aggregate output; and having the same marginal cost of production, c, which satisfies $1 > c \geq 0$.

   In this context, consider the GTS with partial collusion, as follows:

   - In period $t = 1$, every firm i chooses output $q_i = \alpha q_i^C + (1-\alpha) q_i^{NE}$, where $q_i^C = \frac{1-c}{4}$ denotes firm i's collusive output and $q_i^{NE} = \frac{1-c}{3}$ represents the NE output.
   - In all subsequent periods, $t > 1$, every firm i chooses output q_i if both firms produced q_i in every previous period. Otherwise, every firm i reverts to the NE of the stage game, choosing $q_i = \frac{1-c}{3}$ thereafter.

   Answer the following questions:

   (a) Find the minimal discount factor sustaining the above GTS (partial collusion) as a function of parameter α, $\underline{\delta}(\alpha)$.

   - **Partial collusion:** When every firm i chooses partial collusion output

$$\begin{aligned} q_i^P &= \alpha q_i^C + (1-\alpha) q_i^{NE} \\ &= \alpha \frac{1-c}{4} + (1-\alpha)\frac{1-c}{3} \\ &= \frac{(4-\alpha)(1-c)}{12}, \end{aligned}$$

   and its profits become

$$\begin{aligned} \pi_i^P &= (1 - q_i^P - q_j^P - c) q_i^P \\ &= \left(1 - \frac{2(4-\alpha)(1-c)}{12} - c\right) \frac{(4-\alpha)(1-c)}{12} \\ &= \frac{(4-\alpha)(2+\alpha)(1-c)^2}{72}. \end{aligned}$$

   Therefore, at any period t, when firms partially collude, every firm $i = \{1, 2\}$ earns a payoff steam of

$$\pi_i^P + \pi_i^P \delta + \pi_i^P \delta^2 + \ldots = \frac{\pi_i^P}{1-\delta}$$

   where $\delta \in [0, 1]$ denotes the discount factor.

   - **Deviation:** We need to find the optimal deviation that, conditional on firm j choosing the collusion output, maximizes firm i's profit. That is, firm j sticks to cooperation (choosing q_j^P), which means that firm i's profits from deviating to q_i^D are

$$\begin{aligned} \pi_i^D &= (1 - q_i^D - q_j^P - c) q_i^D \\ &= \left(1 - q_i^D - \frac{(4-\alpha)(1-c)}{12} - c\right) q_i^D \end{aligned}$$

Differentiating with respect to q_i^D, we obtain

$$1 - 2q_i^D - \frac{(4-\alpha)(1-c)}{12} - c = 0$$

and, after simplifying, $q_i^D = \frac{(8+\alpha)(1-c)}{24}$. In this context, firm i's deviation profits are

$$\begin{aligned}
\pi_i^D &= (1 - q_i^D - q_j^P - c)q_i^D \\
&= \left(1 - \frac{(8+\alpha)(1-c)}{24} - \frac{(4-\alpha)(1-c)}{12} - c\right)\frac{(8+\alpha)(1-c)}{24} \\
&= \left(\frac{(8+\alpha)(1-c)}{24}\right)^2
\end{aligned}$$

However, in all subsequent periods, every firm reverts to the NE of the stage game, choosing $q_i^{NE} = \frac{1-c}{3}$ thereafter, yielding profits of

$$\begin{aligned}
\pi_i^{NE} &= (1 - 2q_i^{NE} - c)q_i^{NE} \\
&= \left(1 - \frac{2(1-c)}{3} - c\right)\frac{1-c}{3} \\
&= \frac{(1-c)^2}{9}.
\end{aligned}$$

Hence, the payoff stream that the deviating firm obtains is:

$$\pi_i^D + \pi_i^{NE}\delta + \pi_i^{NE}\delta^2 + \ldots = \pi_i^D + \frac{\delta}{1-\delta}\pi_i^{NE}.$$

Therefore, partial collusion can be sustained in equilibrium if

$$\frac{\pi_i^P}{1-\delta} \geq \pi_i^D + \frac{\delta}{1-\delta}\pi_i^{NE}$$

or

$$\frac{(4-\alpha)(2+\alpha)(1-c)^2}{72}\frac{1}{1-\delta} \geq \left(\frac{(8+\alpha)(1-c)}{24}\right)^2 + \frac{\delta}{1-\delta}\frac{(1-c)^2}{9}$$

which simplifies to

$$(16\alpha + \alpha^2)\delta \geq 9\alpha^2$$

Rearranging and solving for δ, we find

$$\delta \geq \frac{9\alpha}{16+\alpha} \equiv \underline{\delta}(\alpha).$$

Thus, the minimal discount factor sustaining partial collusion is $\underline{\delta}(\alpha) \equiv \frac{9\alpha}{16+\alpha}$.

(b) How is $\underline{\delta}(\alpha)$ affected by an increase in α? Interpret.

- Differentiating the minimum discount factor, $\underline{\delta}(\alpha)$, with respect α, we have

$$\frac{\partial \underline{\delta}(\alpha)}{\partial \alpha} = \frac{144}{(16+\alpha)^2} \geq 0$$

Therefore, as collusion becomes "purer " (closer to the monopoly output, when $\alpha = 1$), the minimal discount factor sustaining partial collusion increases, entailing that it becomes more difficult for firms to reach and sustain collusion.

(c) Evaluate $\underline{\delta}(\alpha)$ at $\alpha = 0$ and at $\alpha = 1$. Interpret.

- At $\alpha = 0$, the minimal discount factor sustaining collusion is $\underline{\delta}(0) = \frac{9\times 0}{16+0} = 0$, where both firms play a Cournot game choosing the NE output $q_i^{NE} = \frac{1-c}{3}$. In this context, firms have no incentives to deviate from the NE of the stage game, and the result in part (a) holds for all values of $\delta \in [0,1]$.
- At $\alpha = 1$, the minimal discount factor sustaining collusion is $\underline{\delta}(1) = \frac{9\times 1}{16+1} = \frac{9}{17}$, where both firms seek to form a cartel choosing half of the monopoly output as their collusive output, $q_i^C = \frac{1-c}{4}$. In this context, firms need to assign a sufficiently high importance to future payoffs $\delta \in \left[\frac{9}{17}, 1\right]$ for the cartel agreement to be sustained.

3. **Labor market signaling when education is productivity enhancing - Separating equilibria.** Consider the labor market signaling game shown in figure 1. A worker privately observes whether he has High productivity or Low productivity with equal probability, and then decides whether to acquire some education that he will be able to use as a signal about his productivity level. The firm that is thinking of hiring him as a manager or a cashier without observing his productivity (only whether the worker acquired a college education or not). Additionally, there is no innative productivity differential between a low and a high type worker when they acquire no education but increases by α for the high type worker and by β for the low type worker.

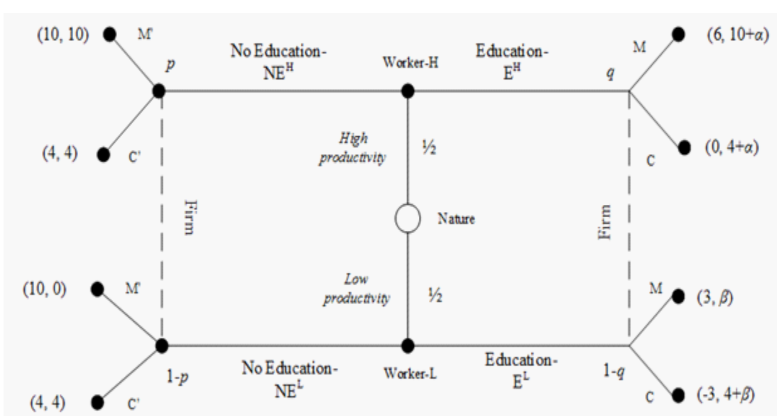


Figure 1 - Labor market signaling when education is productivity enhancing.

(a) *Separating* $NE^H E^L$. Can you sustain a separating equilibrium where only the low productivity worker acquires education?

- Figure 2 depicts the separating strategy profile in which only the low productivity worker chooses to acquire education.

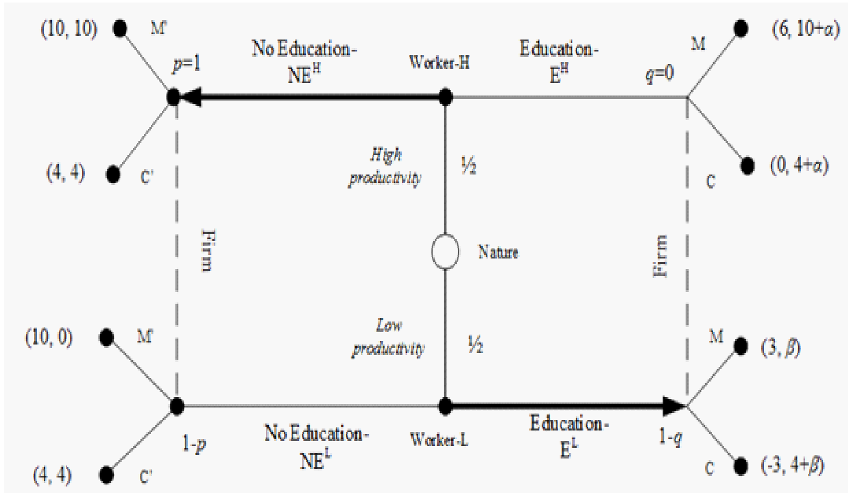


Figure 2

- *Step 1. Responder's beliefs.*
  Firm's updated beliefs about the worker's type are $p = 1$ after observing No education, since such a message must only originate from a high-productivity worker in this strategy profile; and $q = 0$ after observing Education, given that such a message is only sent by a low-productivity worker in this strategy profile. Graphically, $p = 1$ implies that the firm is convinced to be in the upper node after observing no education (in the left-hand side of the tree); whereas $q = 0$ entails that the firm believes to be in the lower node upon observing an educated worker (in the right-hand side of the tree).
- *Step 2. Firm's optimal response given its beliefs.*
  - After observing "No Education" the firm responds with M' since, conditional on being in the upper left-hand node of Figure 3, the firm's profit from M', 10, exceeds that from C', 4.
  - After observing "Education" the firm chooses C, receiving $4+\beta$, instead of M, which yields a payoff of β. Figure 3 summarizes the optimal responses

of the firm, M' and C.

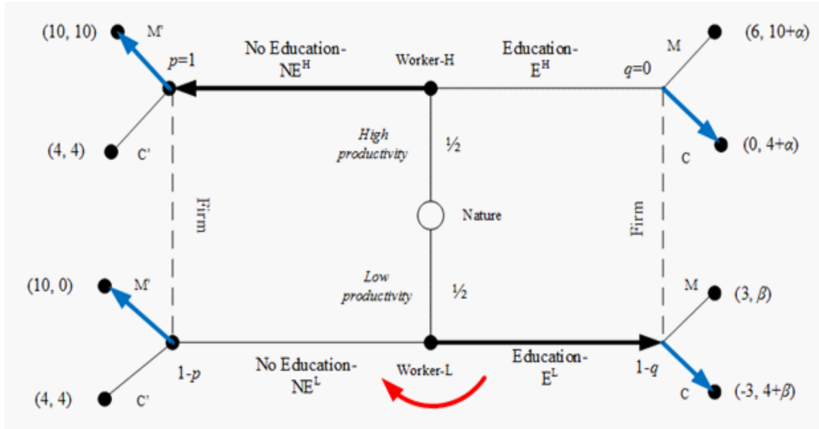


Figure 3

- *Step 3. Optimal messages from the worker.* Given steps 1 and 2, the worker's optimal actions are:

  – When the worker is a high-productivity type, he does not deviate from "No Education" (behaving as prescribed) since his payoff from no education and be recognized as a high productivity worker by the firm (thus being hired as a manager), 10, exceeds that of acquiring education and be identified as a low-productivity worker (and thus be hired as a cashier), 0; as indicated in the upper part of Fig. 3.
  – When the worker is of low type, he deviates from "education", since his payoff from acquiring education and be identified as a high-productivity worker, 10, exceeds that from not acquiring education, -3, such that $NE^H E^L$ cannot be sustained as a separating PBE of this game.

(b) *Separating $E^H NE^L$.* Can you sustain a separating equilibrium where only the high productivity worker acquires education?

- Figure 4 depicts this separating strategy profile, whereby only the high productivity worker acquires education, as indicated in the thick arrows E^H and NE^L.

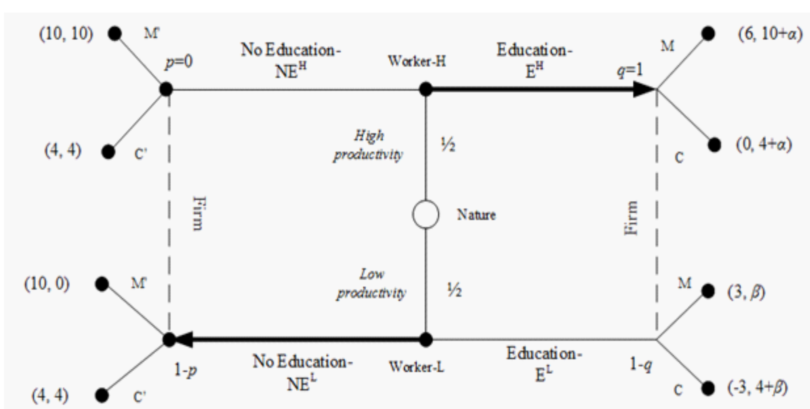


Figure 4

- *Step 1. Responder's beliefs.*
  Firm's beliefs about the worker's type can be updated using Bayes' rule, as follows:

$$q = p(H|E) = \frac{p(H) \times p(E|H)}{p(E)} = \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + (1 - \frac{1}{2}) \times 0} = 1$$

  Intuitively, upon observing that the worker acquires education, the firm infers that the worker must be of high productivity, i.e., $q = 1$, since only this type of worker acquires education in this separating strategy profile; while No education conveys the opposite information, i.e., $p = 0$, thus implying that the worker is not of high productivity but instead of low productivity. Graphically, the firm restricts its attention to the upper right-hand corner, i.e., $q = 1$, and to the lower left-hand corner, i.e., $p = 0$.

- *Step 2. Firm's optimal response given its beliefs.*
  - After observing "Education" the firm responds with M. Graphically, the firm is convinced to be located in the upper right-hand corner of the game tree since $q = 1$. In this corner, the best response of the firm is M, which provides a payoff of $10 + \alpha$, rather than C, which only yields a payoff of $4 + \alpha$.
  - After observing "No Education" the firm responds with C'. In this case the firm is convinced to be located in the lower left-hand corner of the game tree given that $p = 0$. In such a corner, the firm's best response is C', providing a payoff of 4, rather than M', which yields a zero payoff. Figure 5 illustrates these optimal responses for the firm.

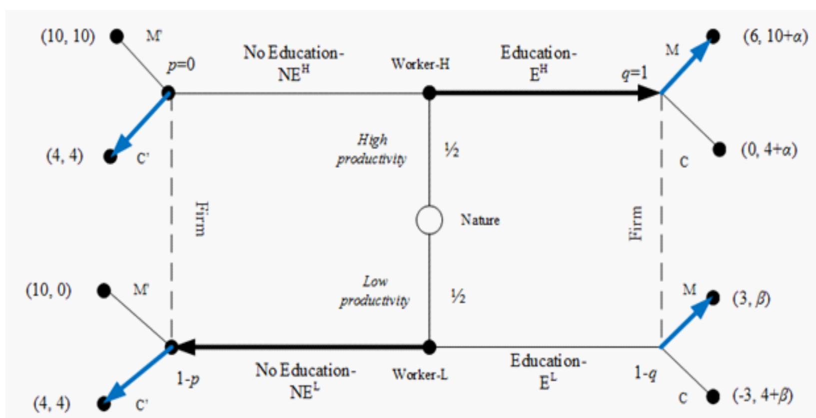


Figure 5

- *Step 3. Optimal messages from the worker.* Given the previous steps 1 and 2, let us now find the worker's optimal actions:
  - When the worker is of high-productivity type, he acquires education since deviating to no-education implies a lower payoff .
  - When he is a low-productivity worker, he does not deviate from "No education", since $4 > 3$, as indicated in the lower part of the game tree. Therefore, $E^H NE^L$ can be supported as a PBE.

4. **Labor market signaling when education is productivity enhancing - Pooling equilibria.** Consider the setting in exercise 1.

(a) *Pooling $E^H E^H$.* Can you sustain a pooling equilibrium where both worker types acquire education?

- Figure 6 describes the pooling strategy profile in which all types of workers acquire education by graphically shading branches E^H and E^L of the tree.

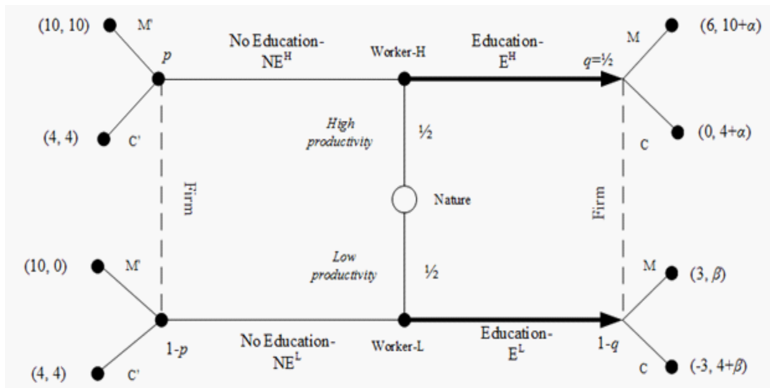


Figure 6

- *Step 1. Responder's beliefs.*
  - Upon observing the equilibrium message of Education, the firm cannot further update its beliefs about the worker's type. That is, its beliefs are

$$q = p(H|E) = \frac{p(H) \times p(E|H)}{p(E)} = \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + (1 - \frac{1}{2}) \times 1} = \frac{1}{2}$$

    which coincide with the prior probability of the worker being of high productivity, i.e., $q = \frac{1}{2}$. Intuitively, since both types of workers acquire education, observing a worker with education does not help the firm to further restrict its beliefs.
  - After observing the off-the-equilibrium message of No Education, the firm's beliefs are

$$p = p(H|NE) = \frac{p(H) \times p(NE|H)}{p(NE)} = \frac{\frac{1}{2} \times 0}{\frac{1}{2} \times 0 + (1 - \frac{1}{2}) \times 0} = \frac{0}{0}$$

    and must then be left unrestricted, i.e., $p \in [0, 1]$.
- *Step 2. Firm's optimal response given its beliefs.*
  - Given the previous beliefs, after observing "Education" (in equilibrium): if the firm responds hiring the worker as a manager (M), it obtains an expected payoff of

$$EU_F(M) = \frac{1}{2}(10 + \alpha) + \frac{1}{2}\beta = \frac{10 + \alpha + \beta}{2}$$

If, instead, the firm hires him as a cashier (C), its expected payoff is only

$$EU_F(C) = \frac{1}{2}(4+\alpha) + \frac{1}{2}(4+\beta) = \frac{8+\alpha+\beta}{2}$$

Thus inducing the firm to hire the worker as a Manager (M).

- After observing "No Education" (off-the-equilibrium): the firm obtains a expected payoffs of

$$EU_F(M') = p \times 10 + (1-p) \times 0 = 10p$$

when it hires the worker as a manager (M'), and

$$EU_F(C') = p \times 4 + (1-p) \times 4 = 4$$

when it hires him as a cashier (C').[1] Hence, the firm prefers to hire him as a manager (M') after observing no education if and only if $10p > 4$, or $p > \frac{2}{5}$: Otherwise, the firm hires the worker as a cashier (C').

- *Step 3. Optimal messages from the worker.*
Given the previous steps 1 and 2, the worker's optimal actions must be divided into two cases (one where the firm's off-the-equilibrium beliefs satisfy $p > \frac{2}{5}$, thus implying that the firm responds hiring the worker as a manager when he does not acquire education, and another case in which $p \leq \frac{2}{5}$ entailing that the firm hires the worker as a cashier upon observing that he acquires education):
  - *Case 1*: When $p > \frac{2}{5}$, the firm responds hiring him as a manager when he does not acquire education (M'), as a depicted in Fig.7 (see left-hand side of the figure).
  In this setting, if the worker is a high-productivity type, he deviates from "Education," where he only obtains a payoff of 6, to "No Education," where his payoff increases to 10, as indicated in the upper part of the game tree. As a consequence, the pooling strategy profile in which both workers acquire education ($E^H E^L$) cannot be sustained as a PBE when

[1] Since firm's beliefs upon observing the off-the-equilibrium message of No Education, p, had to be left unrestricted, we must express the above expected utilities as a function of p.

off-the-equilibrium beliefs satisfy $p > \frac{2}{5}$.

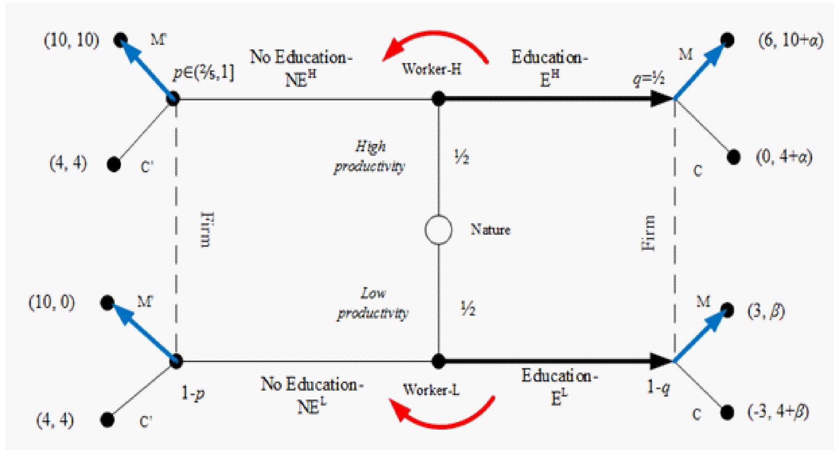


Figure 7

- *Case 2*: When $p \leq \frac{2}{5}$, the firm responds hiring the worker as a cashier (C') after observing the off-the-equilibrium message of No education, as a depicted in the shaded branches on the left-hand side of Fig. 8.
  In this context, if the worker is a low-productivity type, he deviates from "Education" where his payoff is 3, to "No education", where his payoff is 4. For the high type worker, he chooses "No Education," which yields a payoff of 4, which lies below his payoff from "Education", where his payoff is only 6. Therefore, the pooling strategy profile in which both types of workers acquire education ($E^H E^L$) cannot be sustained when the off-the equilibrium beliefs satisfy $p \leq \frac{2}{5}$.

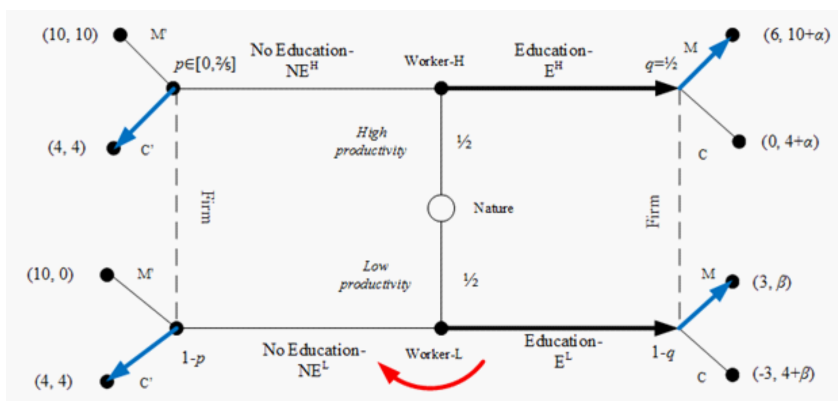


Figure 8

(b) *Pooling* $NE^H NE^L$. Can you sustain a pooling equilibrium where both worker types do not acquire education?

- Figure 9 depicts the pooling strategy profile in which both types of worker choose No Education by shading branches NE^H and NE^L (see thick arrows).

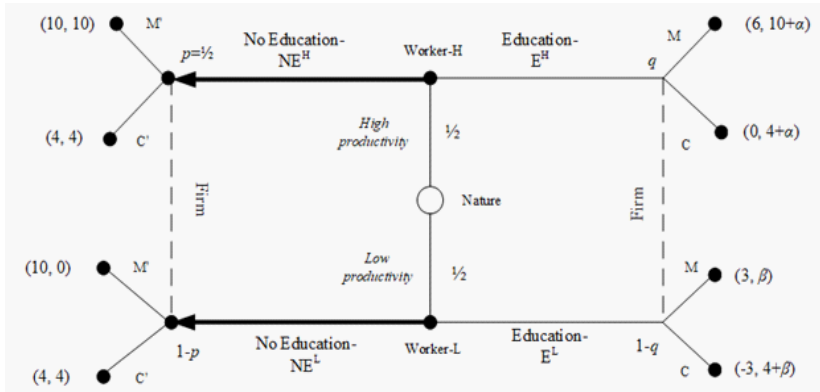


Figure 9

- *Step 1. Responder's beliefs.*
  Analogous to the previous pooling strategy profile, the firm's equilibrium beliefs (after observing No Education) coincide with the prior probability of a high type, $p = \frac{1}{2}$; while its off-the-equilibrium beliefs (after observing Education) are left unrestricted, i.e., $q \in [0, 1]$.
- *Step 2. Firm's optimal response given its beliefs.*
  - Given the previous beliefs, after observing "No Education" (in equilibrium): if the firm hires the worker as a manager (M'), it obtains an expected payoff of

    $$EU_F(M') = \frac{1}{2}(10 + \alpha) + \frac{1}{2}\beta = \frac{10 + \alpha + \beta}{2}$$

    and if it hires him as a cashier (C') its expected payoff is only

    $$EU_F(C') = \frac{1}{2}(4 + \alpha) + \frac{1}{2}(4 + \beta) = \frac{8 + \alpha + \beta}{2}$$

    leading the firm to hire the worker as a manager (M') after observing the equilibrium message of No Education.
  - After observing "Education" (off-the-equilibrium): the expected payoff the firm obtains from hiring the worker as a manager (M) or cashier (C) are, respectively,

    $$EU_F(M) = q \times 10 + (1 - q) \times 0 = 10q$$

    and

    $$EU_F(C) = q \times 4 + (1 - q) \times 4 = 4.$$

  Hence, the firm responds hiring the worker as a manager (M) after observing the off-the-equilibrium message of Education if and only if $10q > 4$, or $q > \frac{2}{5}$. Otherwise, the firm hires the worker as a cashier.

- *Step 3. Optimal messages from the worker.* Given the previous steps 1 and 2, let us find the worker's optimal actions. In this case, we will also need to split our analysis into two cases (one in which $q > \frac{2}{5}$ and thus the firm hires the worker as a manager upon observing the off-the-equilibrium message of Education, and the case in which $q \leq \frac{2}{5}$, in which the firm responds hiring him as a cashier):
  - *Case 1*: When $q > \frac{2}{5}$, the firm responds hiring the worker as a manager (M) when he acquires education, as depicted in Fig.10 (see thick shaded arrows on the right-hand side of the figure).
    If the worker is a high-productivity type, he plays "No Education" since $10 > 6$; as indicated in the shaded branches of upper part of the game tree. If he is a low-productivity type he plays "No Education" since $10 > 3$ Hence, the pooling strategy profile in which no worker type acquires education, $NE^H NE^L$, can be supported as a PBE when off-the-equilibrium beliefs satisfy $q > \frac{2}{5}$.

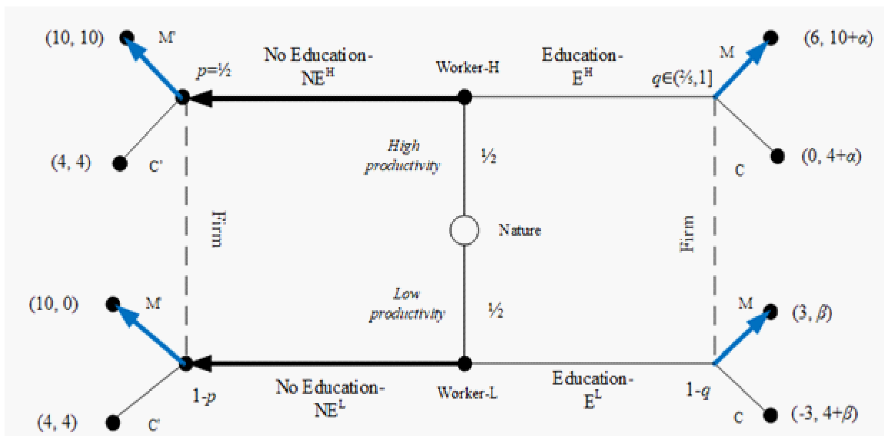


Figure 10

  - *Case 2*: When $q \leq \frac{2}{5}$, the firm responds hiring the worker as a cashier (C) upon observing the off-the-equilibrium message of Education, as depicted in the thick shaded branches at the right-hand side of Fig. 11.
    If the worker is a high-productivity type, he does not deviate from No Education since his payoff from NE^H, 10, exceeds that he would obtain by deviating to E^H of 0; as indicated in the shaded branches at the upper

part of the game tree.

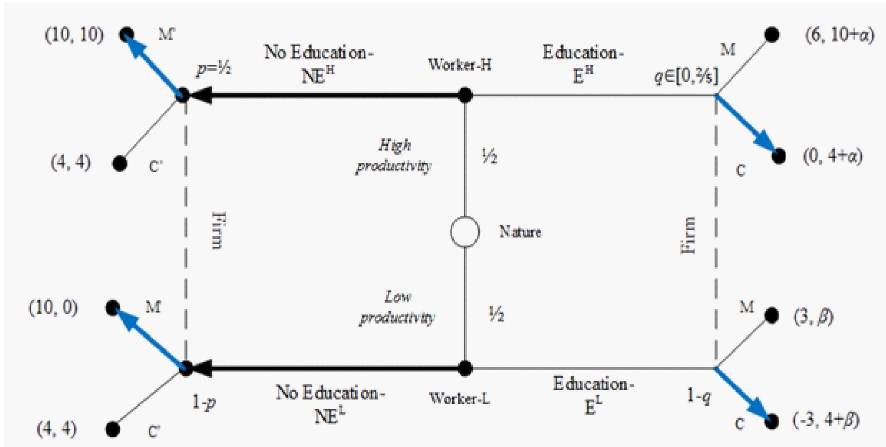


Figure 11

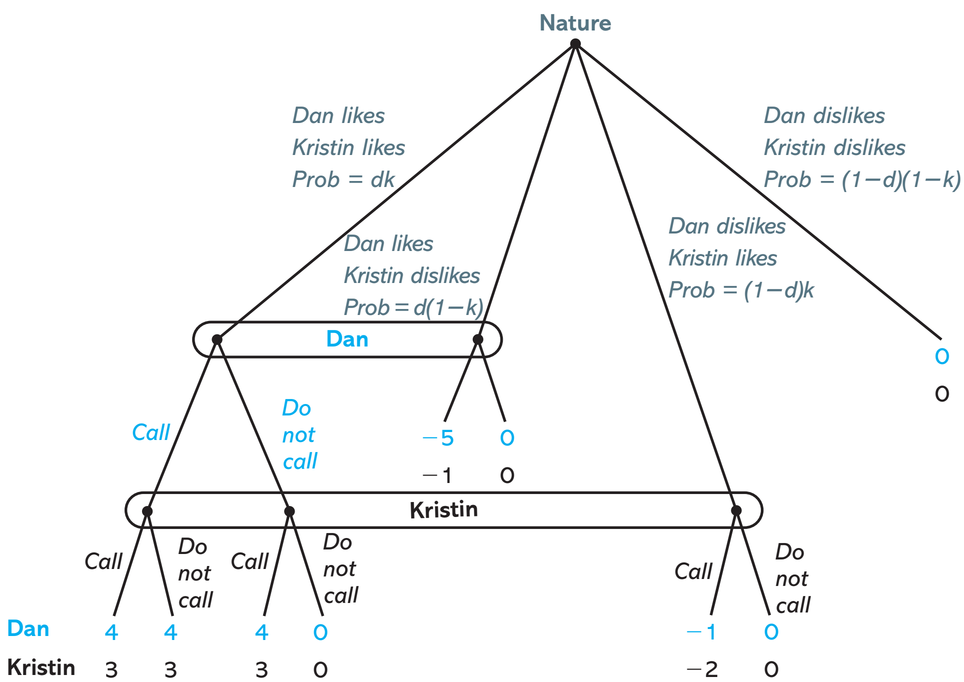
– If he is a low type worker, he does deviate from No education since his payoff 10 is above the payoff he would obtain by deviating to E^L, -3; as indicated in the lower part of Fig 11. Therefore, the pooling strategy profile in with no type of worker acquires education, $NE^H NE^L$, can be supported when off-the-equilibrium beliefs satisfy $q \leq \frac{2}{5}$.

3. Exercises from Harrington (both editions should work)**:**

- Chapter 10: Exercise 19.
- Chapter 12: Exercise 6.
- See answer key to these exercises in the scanned pages below.

So player 2 is indifferent. If she chooses d (and player 3 chooses y), then player 1's optimal strategy is a/a, and we don't have a BNE. If instead she chooses c (and player 3 chooses x), then player 1's optimal strategy is b/a, and we still don't have a BNE. Hence, there is no BNE that has player 1 choose strategy b/b.

19. Dan and Kristin had their first date last night and each is deciding whether to call the other. Each wants to call only if they both are interested. Dan is then of two types: he likes Kristin (which Nature chooses with probability d) or he does not (with probability $1 - d$). Similarly, Kristin is of two types: she likes Dan (with probability k) or does not (probability $1 - k$). Assume $0 < k < 1$ and $0 < d < 1$. To simplify matters (without any loss of generality), assume that Dan (Kristin) has a choice of whether or not to call only when he (she) likes Kristin (Dan). If a person does not like the other then he or she has no decision. The Bayesian game is shown below. (As a side note, which you can ignore if you don't understand, d is the probability that Dan likes Kristin conditional on Kristin liking Dan, and k is the probability that Kristin likes Dan conditional on Dan liking Kristin. Got it? If not, press the IGNORE button.)



a. When is it a BNE for Dan to call (when he likes Kristin) but Kristin not to call (when she likes Dan)?

ANSWER: Given that Dan will call if he is interested, Kristin's strategy is optimal because she only wants to speak with Dan when they both like each other, in which case Dan will call anyway and she gets a payoff of 3 regardless of whether she calls or not. If you don't feel comfortable with this reasoning, let's do the math. When Kristin likes Dan and given that Dan will call if he likes Kristin, Kristin's expected payoff from calling is $d \times 3 + (1 - d)(-2)$ while the expected payoff from not calling is $d \times 3 + (1 - d) \times 0$. Hence, not calling is optimal for her.

On the other hand, given that Kristin will not call, the expected payoff Dan gets from calling is $k \times 4 + (1 - k) \times (-5)$, and the payoff from not calling is 0. His strategy is optimal if and only if

$$k \times 4 + (1 - k) \times (-5) \geq 0 \Rightarrow 4k - 5 + 5k \geq 0 \Rightarrow k \geq \frac{5}{9}$$

b. When is it a BNE for Kristin to call but not Dan?

ANSWER: Similar as in (a), given that Kristin will call if she is interested, Dan's strategy is optimal because he only wants to speak with her if both are interested. On the other hand, given that Dan will not call, the expected payoff to Kristin if she calls is $d \times 3 + (1 - d) \times (-2)$, and the payoff of not calling is 0. So her strategy is optimal if and only if

$$d \times 3 + (1 - d) \times (-2) \geq 0 \Rightarrow 3d - 2 + 2d \geq 0 \Rightarrow d \geq \frac{2}{5}$$

c. When is it a BNE for both to call?

ANSWER: Never because if the other person is expected to call when interested then one is better off not calling and avoiding the unpleasant outcome of calling when the other person is not interested. For the math, suppose that Kristin will call if she likes Dan. If Dan likes Kristin then his expected payoff from calling is $k \times 4 + (1 - k) \times (-5)$ and the expected payoff from not calling is $k \times 4 + (1 - k) \times 0$. Hence, it is optimal for him not to call. Conversely, it is also the case that not calling is optimal for Kristin given that Dan is going to call. There is then no BNE in which both call.

d. When is it a BNE for neither to call?

ANSWER: By the analysis in (a), when $k > \frac{5}{9}$, if Kristin chooses not to call, then it is optimal for Dan to call. When $k \leq \frac{5}{9}$, if Kristin chooses not to call, then it is optimal for Dan to not call as well. Similarly by the analysis in (b), when $d > \frac{2}{5}$, if Dan chooses not to call then it is optimal for Kristin to call. When $d \leq \frac{2}{5}$, if Dan chooses not to call then it is optimal for Kristin to not call as well. Hence, $k \leq \frac{5}{9}$ and $d \leq \frac{2}{5}$ are necessary and sufficient for there to exist a BNE in which neither person calls.

e. When is it a BNE for them to randomize? Find the mixed-strategy BNE.

ANSWER: Let x denote the probability that Dan calls and y denote the probability that Kristin calls. Dan is indifferent between calling and not calling if

$$\text{Call: } k \times 4 + (1 - k) \times (-5) = k \times y \times 4 \text{: Do not call}$$

$$y = \frac{9k - 5}{4k}$$

Since we are looking for a BNE in which they randomize, we want $0 < y < 1$. $\frac{9k-5}{4k} < 1$ when $k < 1$ and $\frac{9k-5}{4k} > 0$ when $k > \frac{5}{9}$. Kristin is indifferent between calling and not calling if

$$\text{Call: } d \times 3 + (1 - d) \times (-2) = d \times x \times 3 \text{: Do not call}$$

$$x = \frac{5d - 2}{3d}$$

We want $0 < x < 1$. $\frac{5d-2}{3d} < 1$ when $d < 1$ and $\frac{5d-2}{3d} > 0$ when $d > \frac{2}{5}$. In sum, if $k > \frac{5}{9}$ and $d > \frac{2}{5}$ then it is a BNE for Dan to call with probability $\frac{5d-2}{3d}$ when he is interested and for Kristin to call with probability $\frac{9k-5}{4k}$ when she is interested.

20. Consider a market with two firms that are competing in quantities (as modeled in the Appendix to Chapter 6). Inverse market demand is $P = 1 - q_1 - q_2$ where q_i is the quantity of firm i. Each firm has a constant marginal cost of producing. Firm 1's marginal cost is zero so its profit is $(1 - q_1 - q_2)q_1$. Firm 2's marginal cost is c so its profit is $(1 - q_1 - q_2)q_2 - cq_2$. c is private information to firm 2 and equals 0 with probability 1/2 and equals 1/4 with probability $\frac{1}{2}$. Firm 2 is then fully informed, while firm 1 does not know firm 2's marginal cost. Nature chooses the value for c and then firms simultaneously choose quantities. Find the BNE.

Receiver's strategy:

If the message is m_1, then choose a.

If the message is m_2, then choose a.

Receiver's beliefs:

If the message is m_1, then assign probability .6 to the sender's being type t_1 and probability .4 to the sender's being type t_2.

If the message is m_2, then assign probability .6 to the sender's being type t_1 and probability .4 to the sender's being type t_2.

There are other babbling equilibria that differ in terms of the message sent by the sender and the receiver's beliefs in response to a message that the sender never sends (according to his strategy). For any babbling equilibrium, it must be the case that the receiver ends up choosing action a because, given posterior beliefs are the same as prior beliefs, action a maximizes the receiver's expected payoff with a payoff of 3.4.

c. Suppose the probability that the sender is type t_1 is p and the probability that the sender is type t_2 is $1 - p$. Find the values for p such that there is a pooling perfect Bayes–Nash equilibrium in which the receiver chooses action b.

ANSWER: For any pooling equilibrium, the sender's strategy has him choose the same message—let it be m_1—for any type and, in response to observing that message, the receiver's beliefs are her prior beliefs. Thus, in response to observing m_1, the receiver's expected payoffs from her various actions are

$$\text{Action } a: p \times 3 + (1 - p) \times 4 = 4 - p$$

$$\text{Action } b: p \times 4 + (1 - p) \times 1 = 1 + 3p$$

$$\text{Action } c: p \times 0 + (1 - p) \times 5 = 5 - 5p.$$

For it to be optimal to choose action b, it must be the case that

$$1 + 3p \geq 4 - p \text{ and } 1 + 3p \geq 5 - 5p \Rightarrow p \geq \frac{3}{4} \text{ and } p \geq \frac{1}{2}.$$

Thus, if $p < \frac{3}{4}$, then the receiver does not choose action b at a pooling equilibrium as she would prefer action a. If $p \geq \frac{3}{4}$, then there is a pooling equilibrium in which the receiver chooses action b, and it is

Sender's strategy:

If my type is t_1, then choose m_1.

If my type is t_2, then choose m_1.

Receiver's strategy:

If the message is m_1, then choose b.

If the message is m_2, then choose b.

Receiver's beliefs:

If the message is m_1, then assign probability p to the sender's being type t_1 and probability $1 - p$ to the sender's being type t_2.

If the message is m_2, then assign probability p to the sender's being type t_1 and probability $1 - p$ to the sender's being type t_2.

6. Suppose Grace and Lisa are to go to dinner. Lisa is visiting Grace from out of town, and they are to meet at a local restaurant. When Lisa lived in town, they had two favorite restaurants: Bel Loc Diner and the Corner Stable. Of course, Lisa's information is out of date, but Grace knows which is better these days. Assume that the probability that the Bel Loc Diner is better is $p > \frac{1}{2}$ and the probability that the Corner Stable is better is $1 - p$. Nature determines which restaurant Grace thinks is better. Grace then sends a message to Lisa, either "Let's go to the Bel Loc Diner," "Let's go to the Corner Stable," or "I don't know [which is better]." Lisa receives the message, and then Grace and Lisa simultaneously decide which restaurant to go to. Payoffs are such that Grace and Lisa want to go to the same restaurant, but they prefer it to be the one that Grace thinks is better. More specifically, if, in fact, the Bel Loc Diner is better, then the payoffs from their actions are as shown in the first figure on page 483. If, instead, the Corner Stable is better, then the second figure on page 483 describes the payoffs.

Payoffs When the Bel Loc Diner Is Better

Grace \ Lisa	*Bel Loc Diner*	*Corner Stable*
Bel Loc Diner	2,2	0,0
Corner Stable	0,0	1,1

Payoffs When the Corner Stable Is Better

Grace \ Lisa	*Bel Loc Diner*	*Corner Stable*
Bel Loc Diner	1,1	0,0
Corner Stable	0,0	2,2

a. Find a perfect Bayes–Nash equilibrium in which Grace and Lisa always go to the better restaurant.

ANSWER: Consider the following separating strategy profile:

Grace's strategy:

If the Bel Loc Diner is better, then say "Let's go to the Bel Loc Diner" and go to the Bel Loc Diner.

If the Corner Stable is better, then say "Let's go to the Corner Stable" and go to the Corner Stable.

If Grace tells Lisa "I don't know," then go to the Bel Loc Diner regardless which restaurant is better.

Lisa's strategy:

If Grace says "Let's go to the Bel Loc Diner," then go to the Bel Loc Diner.

If Grace says "Let's go to the Corner Stable," then go to the Corner Stable.

If Grace says "I don't know," then go to the Bel Loc Diner.

Lisa's beliefs:

If Grace says "Let's go to the Bel Loc Diner," then assign probability 1 to the Bel Loc Diner's being better (and Grace going to the Bel Loc Diner).

If Grace says "Let's go to the Corner Stable," then assign probability 1 to the Corner Stable's being better (and Grace going to the Corner Stable).

If Grace says "I don't know," then assign probability p to the Bel Loc Diner's being better (and Grace going to the Bel Loc Diner).

To show this is a perfect Bayes-Nash equilibrium, first consider Grace's strategy. Given Lisa's strategy, she'll go to whatever restaurant Grace suggests. If the Bel Loc Diner is better, Grace gets a payoff of 2 from saying "Let's go to the Bel Loc Diner" and a payoff of 1 from saying "Let's go to the Corner Stable," so she prefers the former. If the Corner Stable is better, Grace gets a payoff of 2 from saying "Let's go to the Corner Stable" and a payoff of 1 from saying "Let's go to the Bel Loc Diner," so she prefers the former. If she says "I don't know," then she should go to the Bel Loc Diner, as that yields a payoff of either 1 or 2, while the payoff from going to the Corner Stable is 0. Thus, Grace's strategy is optimal.

Turning to Lisa, first note that her beliefs are consistent. Given Grace's strategy, the specified beliefs for the messages "Let's go to the Bel Loc Diner" and "Let's go to the Corner Stable" are correct. Since Grace's strategy never has her say "I don't know," it doesn't provide any guidance as to what Lisa should infer. Thus, any beliefs could be assigned here, including the prior set of beliefs. As to her strategy, since, when Grace says "Let's go to the Bel Loc Diner," then Grace intends to go to the Bel Loc Diner, Lisa wants to do so as well since she'll get a payoff of 2 rather than 0. When Grace says "Let's go to the Corner Stable," she intends to go to the Corner Stable and Lisa wants to do so as well since she'll get a payoff of 2 rather than 0.

b. Find a pooling perfect Bayes–Nash equilibrium.

ANSWER: Consider the following pooling strategy profile:

Grace's strategy: Regardless of which restaurant is better, say "I don't know" and go to the Bel Loc Diner.

Lisa's strategy: Regardless of what Grace says, go to the Bel Loc Diner.

Lisa's beliefs: Regardless of what Grace says, assign probability p to the Bel Loc Diner's being better (and expect Grace to go to the Bel Loc Diner).

Given Lisa intends to go to the Bel Loc Diner regardless what Grace says, it is clearly optimal for Grace to go there as well. Lisa's beliefs are consistent and, given those beliefs, it is optimal for her to go the Bel Loc Diner since she expects Grace to go there.

7. Let's reexamine the Courtship game from Section 11.3, but suppose there is no gift. The extensive form of this game is shown below. The private information in this setting is whether Jack cares deeply about Rose and thus would like to marry her and, similarly, whether Rose cares deeply about Jack and would like to marry him. Only each person knows whether he or she truly loves the other. Assume that each person loves the other with probability p, where $0 < p < 1$. Thus, the probability that they are "meant for each other"—that is, the probability that Jack loves Rose and Rose loves Jack— is p^2.

After learning their types, Jack and Rose face the following sequence of decisions: Jack starts by deciding whether to suggest to Rose that they have premarital sex. If he does make such a suggestion, then Rose either accepts or declines. If she accepts, then they have sex. After this round of decisions and actions, either they marry (if they love each other) or they don't (if one or both does not love the other). In particular, we'll assume that the marriage decision—which will not be explicitly modeled, but rather will be implicit in the payoffs—is independent of whether or not they have sex. Jack's payoff depends on whether they have sex and whether they love each other (and thus marry). Jack desires sex from Rose regardless of whether he loves her, and the gain in his payoff from it is $s > 0$. If he and Rose prove to be in love and thus marry, Jack assigns a value of $m > 0$ to marriage.

Thus, if Jack has sex with Rose and they marry (because it turns out that they love each other), then his payoff is the sum of those two terms: $m + s$. If he has sex, but marriage does not ensue, then his payoff is only s. Finally, if he neither has sex nor marries, then his payoff is zero. Like Jack, Rose values their being in love and marrying by an amount m. As for sex, her biggest concern is to not have it with someone for whom marriage is not in her future. Rose's payoff from having sex with Jack and then marrying him is $m + s$, just like Jack's payoff. However, her payoff from having sex and then not marrying Jack (which occurs if she doesn't love Jack and/or he doesn't love her) is $u < 0$. Finally, her payoff is zero from neither sex nor marriage. Show that there is no perfect Bayes–Nash equilibrium in which premarital sex occurs.