EconS 503 - Advanced Microeconomics - II
Midterm Exam #2 - Answer key

1. **Public good game under incomplete information.** Suppose that there are $N \geq 2$ players and that a public good is supplied if only if at least one player contributes. If the public good is supplied, every player enjoys a benefit of 1. The cost of contributing is denoted as θ_i for player i, and is independently drawn from the cumulative distribution function $F(\cdot)$ on $[\underline{\theta}, \overline{\theta}]$ where the lower and upper bounds satisfy $\underline{\theta} < 1 < \overline{\theta}$. Every player i privately observes his contribution cost, θ_i, but does not observe his rivals' costs, θ_{-i}, although he knows that they are drawing their contribution costs from $F(\cdot)$.

   (a) Find the BNE of the game in this setting.

   - We seek to find a BNE where every player i contributes if and only if his contribution cost, θ_i, satisfies $\theta_i \leq \theta_i^*$. Therefore, when $\theta_i = \theta_i^*$ player i is indifferent between contributing and not contributing. If player i contributes, C, his expected utility is

$$\begin{aligned} EU_i(C|\theta_i^*) &= \underbrace{[\Pr(\text{Some other player contributes})1 - \theta_i^*]}_{\text{Utility if } i \text{ and } j \neq i \text{ contribute}} \\ &\quad + \underbrace{[\Pr(\text{No other player contributes})1 - \theta_i^*]}_{\text{Utility if only } i \text{ contributes}} \end{aligned}$$

     Since player i contributes, his benefit from the public good is 1, which occurs both when some other players $j \neq i$ contribute and when he does not. Similarly, if player i doesn't contribute, NC, his expected utility is

$$\begin{aligned} EU_i(NC|\theta_i^*) &= \underbrace{[\Pr(\text{Some other player contributes})1 - 0]}_{\text{Utility if } j \neq i \text{ contributes}} \\ &\quad + \underbrace{[\Pr(\text{No other player contributes})0 - 0]}_{\text{Utility if no player contributes}} \end{aligned}$$

     because the public good is only supplied if some other players $j \neq i$ contribute, and player i's cost from not contributing is zero.

   - Therefore, player i is indifferent between contributing and not contributing when $EU_i(C|\theta_i^*) = EU_i(NC|\theta_i^*)$, which entails

$$\begin{aligned} &\Pr(\text{Some other player contributes})1 + \Pr(\text{No other player contributes})1 - \theta_i^* \\ = \; &\Pr(\text{Some other player contributes})1 + \Pr(\text{No other player contributes})0, \end{aligned}$$

     which simplifies to

$$\theta_i^* = \Pr(\text{No other player contributes}).$$

     We now specify the probability that no other player contributes. Recall that each rival does not contribute when his cost is sufficiently high, $\theta_i > \theta_i^*$, which occurs with probability $1 - F(\theta_i^*)$, implying that $N-1$ players do not contribute to the public good with probability $[1 - F(\theta_i^*)]^{N-1}$, that is,

$$\theta_i^* = [1 - F(\theta_i^*)]^{N-1}.$$

Evaluating this indifference condition at the lower bound, $\underline{\theta}$, where $F(\underline{\theta}) = 0$, we obtain that

$$\underline{\theta} = 1.$$

And evaluating the indifference condition at $\theta_i = 1$, we find that

$$1 = [1 - F(1)]^{N-1},$$

which simplifies to $F(1) = 0$. Therefore, there is at least one BNE where the indifferent player θ_i^* lies in $(\underline{\theta}, 1)$.

(b) *Uniformly distributed costs.* Assume that θ_i is uniformly distributed on $[\underline{\theta}, \overline{\theta}]$, where $\underline{\theta} = 0$ and $\overline{\theta} = 2$, so that $F(\theta_i) = \frac{\theta_i}{2}$, and that $N = 2$ play the game. Use your results from part (a) to find a BNE where only one player contributes to the public good.

- Using the indifference condition found in part (a), we have that

$$\theta_i^* = \left(1 - \frac{\theta_i^*}{2}\right)^{2-1}.$$

Rearranging, yields

$$\theta_i^* = 1 - \frac{\theta_i^*}{2}.$$

Solving for θ_i^*, we obtain that

$$\theta_i^* = \frac{2}{3},$$

so every player i contributes to the public good if his cost satisfies $\theta_i < \frac{2}{3}$, but does not contribute otherwise.

(c) Still in the context of part (b), find the BNE when $N = 3$, $N = 4$, and $N = 10$. How are the equilibrium results affected? Interpret.

- When the number of players grows from $N = 2$ to $N = 3$, the above expression becomes

$$\theta_i^* = \left(1 - \frac{\theta_i^*}{2}\right)^{3-1}$$

which yields

$$(\theta_i^*)^{\frac{1}{2}} = 1 - \frac{\theta_i^*}{2}$$

which further simplifies to $\theta_i^* + 2(\theta_i^*)^{\frac{1}{2}} - 2 = 0$. Solving for θ_i^*, we obtain

$$\theta_i^* = 4 - 2\sqrt{3} \simeq 0.54.$$

- Similarly, when N grows to $N = 4$, the cutoff keeps decreasing to $\theta_i^* = 0.46$; and, finally, when $N = 10$, the cutoff further decreases to $\theta_i^* = 0.27$. Therefore, as the number of players increases, every player becomes less willing to contribute (i.e., condition $\theta_i \leq \theta_i^*$ holds under more restrictive conditions). Intuitively, this occurs because every player anticipates that the probability of some rival's cost satisfying condition $\theta_i \leq \theta_i^*$ increases, thus making the supply of the public good more likely.

2. **First-price auction with asymmetrically distributed valuations.** Most applications generally assume that all bidders independently draw their valuation from a *common* distribution, $F(v_i)$. In this exercise, we analyze how our equilibrium results are affected by relaxing this assumption. Consider a first-price auction with two risk-neutral bidders, i and j, independently drawing their valuations for the object from the following cumulative distribution functions $F_i(v_i) = v_i^{\alpha}$ and $F_j(v_j) = v_j^{\gamma}$, respectively, where $\alpha \neq \gamma > 0$. For simplicity, assume that $v_i, v_j \in [0, 1]$.

   (a) Find the equilibrium bidding function for bidder i and j.

   - *Writing expected utility.* We can write bidder i's expected utility maximization problem (UMP) as follows:

$$\max_{b_i \geq 0} \ \Pr(win) \times (v_i - b_i),$$

   which denotes the probability of winning the object times bidder i's net payoff from winning, $v_i - b_i$, because he values the object at v_i and pays his bid b_i for it.

   - *Finding the probability of winning.* At this point, we write the probability of winning as follows

$$\begin{aligned} \Pr(win) &= \Pr\{b_i > b_j\} \\ &= \Pr\{b_i > b_j(v_j)\} \end{aligned}$$

   and inverting by $b_j^{-1}(\cdot)$, yields

$$\Pr\{b_j^{-1}(b_i) > b_j^{-1}(b_j(v_j))\} = \Pr\{b_j^{-1}(b_i) > v_j\} = \left(b_j^{-1}(b_i)\right)^{\gamma}$$

   where the first equality uses $b_j^{-1}(b_j(v_j)) = v_j$, while the second equality considers that bidder j's valuation is distributed according to $F_j(v_j) = v_j^{\gamma}$. Therefore, the above expected utility maximization problem can be rewritten as

$$\max_{b_i \geq 0} \ \left(b_j^{-1}(b_i)\right)^{\gamma} \times (v_i - b_i).$$

   - *First order condition.* Differentiating with respect to b_i, yields

$$-\left(b_j^{-1}(b_i)\right)^{\gamma} + \gamma(v_i - b_i)\left(b_j^{-1}(b_i)\right)^{\gamma-1} \frac{\partial b_j^{-1}(b_i)}{\partial b_i} = 0.$$

Because $b_j(v_i) = b_i$, we can write that $b_j^{-1}(b_i) = v_i$. Therefore, $\frac{\partial b_j^{-1}(b_i)}{\partial b_i} = \frac{1}{b'\left(b_j^{-1}(b_i)\right)} = \frac{1}{b'(v_i)}$, implying that the above first-order condition simplifies to

$$-v_i^\gamma + \gamma(v_i - b_i)v_i^{\gamma-1}\frac{1}{b'(v_i)} = 0$$

or

$$\gamma v_i^\gamma = \gamma v_i^{\gamma-1} b_i + v_i^\gamma b'(v_i)$$

The right side is $\frac{\partial\left[v_i^\gamma b_i(v_i)\right]}{\partial v_i}$, which helps us rewrite this expression as

$$\gamma v_i^\gamma = \frac{\partial\left[v_i^\gamma b_i(v_i)\right]}{\partial v_i}.$$

Integrating both sides, yields

$$\int_0^{v_i} \gamma x^\gamma dx = v_i^\gamma b_i(v_i)$$

and solving for $b_i(v_i)$, we obtain the equilibrium bidding function, as follows

$$b_i(v_i) = \frac{1}{v_i^\gamma}\int_0^{v_i} \gamma x^\gamma dx.$$

Solving the integral, we can find a more precise expression for this bidding function, that is,

$$\begin{aligned} b_i(v_i) &= \frac{1}{v_i^\gamma}\int_0^{v_i} \gamma x^\gamma dx \\ &= \frac{1}{v_i^\gamma}\left[\frac{\gamma}{1+\gamma}x^{\gamma+1}\right]_0^{v_i} \\ &= \frac{1}{v_i^\gamma}\frac{\gamma}{1+\gamma}v_i^{\gamma+1} \\ &= \frac{\gamma}{1+\gamma}v_i \end{aligned}$$

where, intuitively, the term $0 < \frac{\gamma}{1+\gamma} < 1$ captures the extent of bid shading. Operating similarly for bidder j, we obtain that his equilibrium bidding function is

$$b_j(v_j) = \frac{\alpha}{1+\alpha}v_j.$$

- Finally, comparing $b_i(v_i)$ and $b_j(v_j)$, we claim that bidder i wins the auction if $\frac{\gamma}{1+\gamma}v_i > \frac{\alpha}{1+\alpha}v_j$, which holds if $v_i > \frac{\alpha(1+\gamma)}{\gamma(1+\alpha)}v_j$. Intuitively, this occurs if bidder i has a sufficiently higher valuation than bidder j. If both bidders have the same valuation $v_i = v_j$ but $\gamma > \alpha$, we have that $F_j(v_j) = v_j^\gamma$ first-order stochastically dominates $F_i(v_i) = v_i^\alpha$, thus bidder j assigning a larger probability weight on high valuations than bidder i.

(b) *Symmetrically distribution values.* Assume now that $\alpha = \gamma > 0$. How are your above results affected? How are equilibrium bids affected by a marginal increase in α? Interpret.

- When $\alpha = \gamma$, bidder i's bidding function becomes $b_i(v_i) = \frac{\alpha}{1+\alpha} v_i$, and that of bidder j is symmetric, that is, $b_j(v_j) = \frac{\alpha}{1+\alpha} v_j$. In that context, bidder i wins the auction if his valuation is higher than that of bidder j, $v_i > v_j$, and a marginal increase in α induces every bidder to submit (weakly) more aggressive bids because

$$\frac{\partial b_i(v_i)}{\partial \alpha} = \frac{1}{(1+\alpha)^2} v_i \geq 0.$$

  Intuitively, an increase in α entails that the common cumulative distribution function assigns a larger probability weight on high valuations. As a consequence, for any given value v_i that bidder i privately observes, he knows that the probability that his rival draws a high valuation is, essentially, increasing in α, and thus responds submitting a higher bid b_i.
- For illustration purposes, figure 1 depicts the cumulative distribution function and figure 2 plots the corresponding equilibrium bidding function.

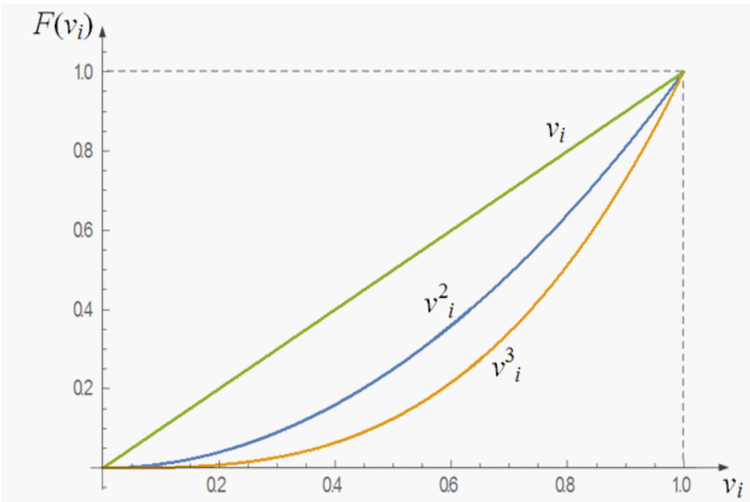


Figure 1. $F(v_i)$ evaluated at $\alpha = 2$ and at $\alpha = 3$.

If $\alpha = \gamma = 2$, equilibrium bidding functions become $b_i(v_i) = \frac{2}{3} v_i$ for every bidder i, which does not coincide with that in the standard first-price auction with two bidders independently drawing their valuation from a common, uniform, distribution. A similar argument applies if $\alpha = \beta = 3$, where the equilibrium bidding function becomes $b_i(v_i) = \frac{3}{4} v_i$, thus reducing bid shading. In the limit where $\alpha \to \infty$, bidders do not shade their bids since

$$\lim_{\alpha \to \infty} b_i(v_i) = \lim_{\alpha \to \infty} \left(1 - \frac{1}{1+\alpha}\right) v_i = v_i$$

where the common cumulative distribution function $F_i(v_i)$ assigns all probability weight to $v_i = 1$. In this context, every bidder, knowing for sure that the other bidder has a valuation of \$1 with certainty, can only have positive possibility to win the object if he does not shade his bid and submits a bid equal to his valuation at \$1, thus forming a tie with the other bidder.

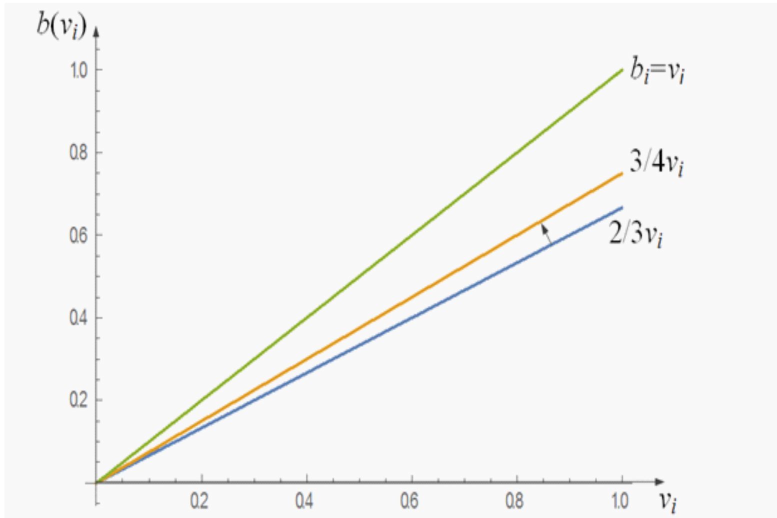


Figure 2. Equilibrium bidding function evaluated at $\alpha = 2$ and at $\alpha = 3$.

(c) *Uniformly distributed valuations.* How are the equilibrium results affected if $\alpha = \gamma = 1$?

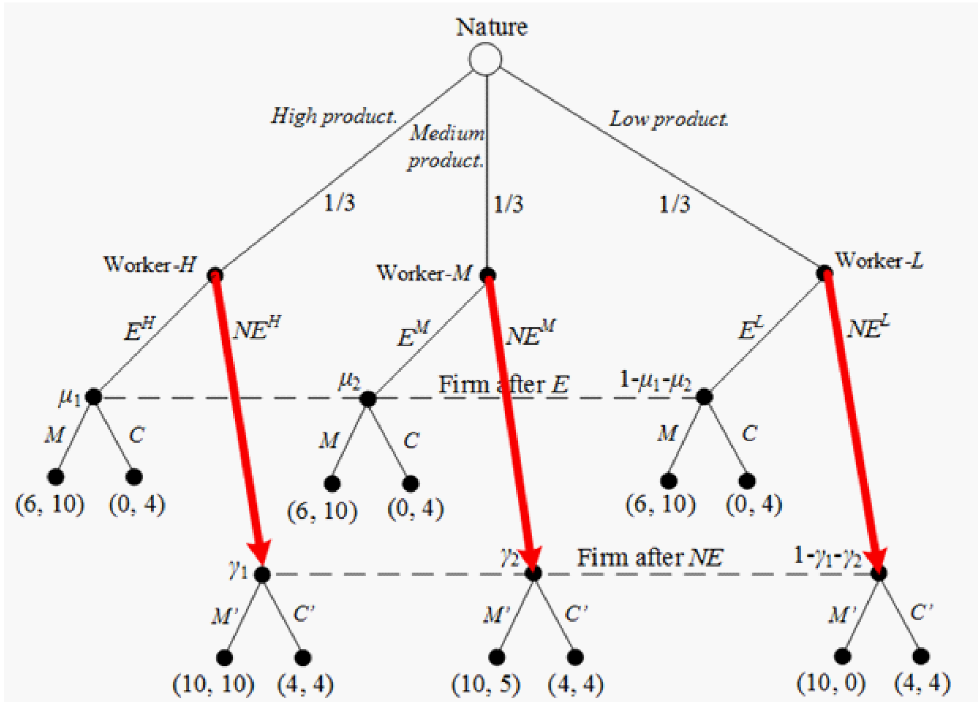
- If $\alpha = \gamma = 1$, both bidder's valuations are distributed according to a uniform distribution, that is, $F_i(v_i) = v_i$, for every bidder i. In this context, equilibrium bidding functions simplify to

$$b_i(v_i) = \frac{1}{2}v_i,$$

as in the first-price auction with two bidders independently drawing their valuation from a common uniform distribution, as in previous exercises discussed in class.

3. **Applying the Intuitive Criterion to the labor market signaling game with three worker types.** Consider an extension of the labor market signaling game that we analyzed in class, where we now allow for the worker to have three types (high, medium, or low productivity, each of them equality likely). The worker privately observes her own type, then chooses whether or not to acquire education, and the firm responds hiring him as a manager or cashier.

(a) The figure below depicts a pooling strategy profile where no worker type acquires education, (NE^L, NE^M, NE^H). Show that this strategy profile can be sustained as a PBE, and explain if you need additional restrictions on the off-the-equilibrium beliefs (μ_1, μ_2, and $1 - \mu_1 - \mu_2$).



- *Firm's updated beliefs.*
  - After observing no education, the firm updates its belief to $\gamma_1 = \gamma_2 = \frac{1}{3}$ that coincide with its prior.
  - After observing education, the firm leaves its off-the-equilibrium belief unrestricted, that is, $\mu_1, \mu_2 \in [0, 1]$.
- *Firm's responses.*
  - After observing no education, the firm responds hiring the worker as a manager since

$$\begin{aligned} EU_F(M') &= \frac{1}{3}10 + \frac{1}{3}5 + \frac{1}{3}0 = 5 \\ &> \frac{1}{3}4 + \frac{1}{3}4 + \frac{1}{3}4 = 4 = EU_F(C') \end{aligned}$$

  - After observing education, the firm hires the worker as a manager when

$$\begin{aligned} EU_F(M) &= 10\mu_1 + 10\mu_2 + 10(1 - \mu_1 - \mu_2) = 10 \\ &\geq 4 = 4\mu_1 + 4\mu_2 + 4(1 - \mu_1 - \mu_2) = EU_F(C) \end{aligned}$$

    Therefore, the firm hires the worker as a manager, both after observing education and no education.
- *Worker's choices.* Since the firm responds hiring the worker as a manager regardless of whether she acquires education or not, the worker prefers to acquire no education. We can see that in the figure: every worker type earns 10 from being hired as a manager without acquiring education, but only 6 otherwise.

(b) Does the PBE found in part (a) survive the Intuitive Criterion?

- *Step 1.* We consider the pooling PBE (NE^H, NE^M, NE^L).
- *Step 2.* We identify an off-the-equilibrium message, Education in this case, since no worker type chooses Education in this PBE.
- *Step 3.* We now find which worker types (sender) can benefit when deviating to E.
  - The high-productivity worker earns 10 in equilibrium, and the highest payoff he can earn by deviating to E^H is 6. Therefore, he cannot benefit by deviating from NE^H to E^H.
  - The medium-productivity worker earns 10 in equilibrium, and the highest payoff he can earn by deviating to E^M is 6. Therefore, he cannot benefit by deviating from NE^M to E^M.
  - The low-productivity worker earns 10 in equilibrium, and the highest payoff he can earn by deviating to E^L is 6. Hence, he cannot benefit by deviating from NE^L to E^L.
- *Step 4.* Given our results in Step 3, the firm cannot restrict its off-the-equilibrium belief upon observing education, μ_1 and μ_2, since no worker types can profitably benefit from such a deviation.
- Therefore, we can conclude that the pooling PBE (NE^H, NE^M, NE^L) survives the Intuitive Criterion.