EconS 424 – Strategy and Game Theory
Midterm #2 – Answer Key

Exercise #1
Solving the game by backward induction, we first analyze the output decision of the last mover (firm 3).

Firm 3 (last stage). This firm maximizes the following profits:

$$\pi_3 = (18 - q_1 - q_2 - q_3)q_3 - (5 + 2q_3)$$

Differentiating with respect to q_3, we obtain

$$18 - q_1 - q_2 - 2q_3 - 2 = 0$$

So, firm 3's BRF is

$$q_3(q_1, q_2) = 8 - \frac{q_1 + q_2}{2}$$

Firm 2. Firm 2 anticipates firm 3's BRF. Inserting firm 3's BRF into firm 2's maximization problem yields,

$$\max_{q_2 \geq 0} \left[18 - q_1 - q_2 - \left(8 - \frac{q_1 + q_2}{2}\right)\right] q_2 - (5 + 2q_2)$$

Differentiating with respect to q_2, we obtain

$$18 - q_1 - 2q_2 - 8 + \frac{1}{2}(q_1 + 2q_2) - 2 = 0$$

Therefore, solving for q_2, we find that firm 2's BRF is

$$q_2(q_1) = 8 - \frac{q_1}{2}$$

Firm 1 (first mover). Inserting firm 3's and firm 2'as BRFs into firm 1's maximization problem yields,

$$\max_{q_1 \geq 0} \left[18 - q_1 - \left(8 - \frac{q_1}{2}\right) - \left(8 - \frac{q_1 + \left(8 - \frac{q_1}{2}\right)}{2}\right)\right] q_1 - (5 + 2q_1)$$

Differentiating with respect to q_1, we obtain

$$18 - 2q_1 - 8 + \frac{1}{2}2q_1 - 8 + \frac{1}{2}\left(2q_1 + 8 - \frac{1}{2}2q_1\right) - 2 = 0$$

Rearranging and solving for q1, we find firm 1's equilibrium output $q_1 = 8$ units. Therefore, in equilibrium, firm 2 produces $q_2(8) = 8 - \frac{8}{2} = 4$ units, and firm 3 produces $q_3(8,4) = 8 - \frac{8+4}{2} = 8 - 6 = 2$ units.

Exercise #2 – Cournot competition under incomplete information about market demand

a) Firm 1 (informed firm):

If *high* demand:

$$\pi_1^H = (20 - q_1^H - q_2) * q_1^H - 2q_1^H$$

Taking FOCs with respect to q_1^H,

$$20 - 2q_1^H - q_2 - 2 = 0 \ \rightarrow 18 - q_2 = 2q_1^H$$

Hence, solving for q_1^H, we obtain firm 1's best response function when facing a high demand

$$BRF_1^H \rightarrow q_1^H(q_2) = 9 - \frac{1}{2}q_2$$

If *low* demand:

$$\pi_1^L = (8 - q_1^L - q_2)q_1^L - 2q_1^L$$

Taking FOCs with respect to q_1^L,

$$8 - 2q_1^L - q_2 - 2 = 0 \rightarrow 6 - q_2 = 2q_1^L$$

Hence, solving for q_1^L, we obtain firm 1's best response function when facing a low demand

$$BRF_1^L \rightarrow q_1^L(q_2) = 3 - \frac{1}{2}q_2$$

b) Firm 2 (uninformed firm): Expected profit for Firm 2

$$\mathrm{E}\pi_2 = \frac{2}{3}\Big((20 - q_1^H - q_2)q_2 - 2q_2\Big) + \frac{1}{3}\Big((8 - q_1^L - q_2)q_2 - 2q_2\Big)$$

$$= \frac{2}{3}(20q_2 - q_1^H q_2 - q_2^2) + \frac{1}{3}(8q_2 - q_1^L q_2 - q_2^2) - 2q_2$$

Taking FOCs with respect to q_2,

$$\frac{40}{3} - \frac{2}{3}q_1^H - \frac{4}{3}q_2 + \frac{8}{3} - \frac{1}{3}q_1^L - \frac{2}{3}q_2 - 2 = 0$$

And solving for q_2, we obtain firm 2's best response function.

$$BRF_2 \rightarrow q_2(q_1^H, q_1^L) = 7 - \frac{2}{6}q_1^H - \frac{1}{6}q_1^L$$

c) Plugging q_1^H and q_1^L into firm 2's best response function, we find

$$q_2 = 7 - \frac{1}{3}\left[9 - \frac{1}{2}q_2\right] - \frac{1}{6}\left[3 - \frac{1}{2}q_2\right]$$

and solving for q_2, we obtain firm 2's equilibrium output level, $q_2 = 4.66$.

We can now find firm 1's equilibrium output level. First, plugging q_2=4.66 into $q_1^H = 9 - \frac{1}{2}q_2$, we obtain:

$$q_1^H = 9 - \frac{1}{2}(4.66) = 6.7$$

Similarly, plugging q_2=4.66 into $q_1^L = 3 - \frac{1}{2}q_2$,we obtain.

$$q_1^L = 3 - \frac{1}{2}(4.66) = 0.66$$

Summarizing, the BNE of this incomplete information Cournot game is:

$$(q_1^H, q_1^L, q_2) = (6.7, 0.66, 4.66)$$

Exercise #3

a) First, we list all the strategies for both the worker and firm. For the worker, his strategy set is

$$\{(NE^H, NE^L), (NE^H, E^L), (E^H, NE^L), (E^H, E^L)\}$$

where the first element of each strategy regards to that when his type is *High productivity*, and the second element regards to *Low Productivity*.

The strategy set for the firm is

$$\{(M', M), (M', C), (C', M), (C', C)\}$$

where the first element in each strategy regards to *No Education* is observed, and the second regards to *Education.*

b) Then we draw and fill in the following payoff matrix with players' expected payoffs: Each cell in the payoff matrix is calculated by taking the expected payoff of each strategy combination. For example, in the cell where the worker chooses strategy (E^H, NE^L) and the firm chooses strategy (M', C), we can calculate the payoff for the worker by taking the linear combination of M' in the lower left-hand corner and C in the upper-right hand corner of the figure (Since they correspond with (E^H, NE^L) and (M', C)). The calculation for the worker is

$$EU_W = \frac{2}{3}(10) + \frac{1}{3}(0) = \frac{20}{3}$$

and that for the firm as

$$EU_F = \frac{2}{3}(0) + \frac{1}{3}(4) = \frac{4}{3}$$

		Firm			
		M’, M	M’, C	C’, M	C’, C
Worker	NE^H, NE^L	$\underline{10}$, 10/3	$\underline{10}$, 10/3	4, $\underline{4}$	$\underline{4}$, $\underline{4}$
	NE^H, E^L	16/3, 10/3	4/3, $\underline{6}$	10/3, 4/3	-2/3, 4
	E^H, NE^L	26/3, 10/3	20/3, 4/3	$\underline{14/3}$, $\underline{6}$	8/3, 4
	E^H, E^L	4, 10/3	-2, $\underline{4}$	4, 10/3	-2, $\underline{4}$

c) As we can see from the above payoff matrix, there are two BNEs in this game:

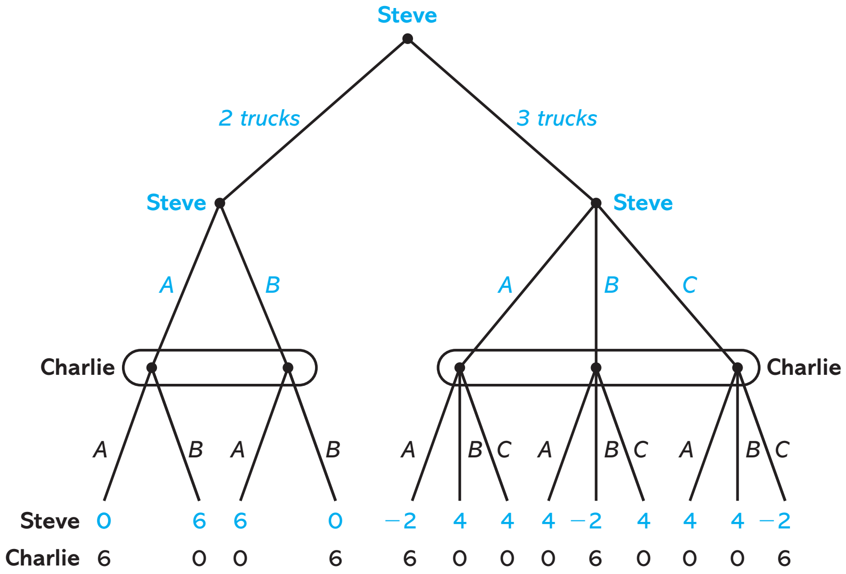
- (*No Education if High Productivity, No Education if Low Productivity*), (*Cashier if No Education, Cashier if Education*)
- (*Education if High Productivity, No Education if Low Productivity*), (*Cashier if No Education, Manager if Education*)

d) Both of these BNEs are also the two PBEs we found in class. The first one corresponds to the pooling strategy profile (NE^H, NE^L) where it is PBE if off-the-equilibrium beliefs satisfy $\mu > \frac{2}{5}$. The second BNE corresponds to the separating strategy profile (E^H, NE^L). Therefore, in this case the BNE and PBE solution concepts yield the exact same equilibrium predictions.

13. In the film *The Italian Job*, Steve is trying to move his stash of gold without his nemesis Charlie knowing where the gold is going. Steve knows that Charlie is watching Steve's house, where the gold is stored. Steve's plan is to have multiple armored trucks pull into his multi-car garage and to put the gold in one of the trucks. Charlie must then figure out which truck has the gold. Before all that happens, Steve decides whether to rent two or three trucks, where the cost of a third truck is 2 units of payoff. Charlie observes how many trucks arrive, so he knows how many trucks were rented. Steve decides whether to put the gold in truck A or B (when he rents two trucks) or in truck A, B, or C (when he rents three trucks). After Steve has loaded the gold onto one of the trucks, Charlie decides which truck to follow (A or B when two trucks are used; A, B, or C when three trucks are used) without knowing which truck has the gold. If Charlie ends up following the truck with the gold, his payoff is 6 and Steve's payoff is 0. If Charlie follows a truck without the gold, his payoff is 0 and Steve's payoff is 6. Recall that if Steve chose three trucks, then 2 must be subtracted from his payoff. (*Note:* In the film, Charlie had his computer geek Lyle figure out which truck had the gold by measuring how much the tires were compressed and thus which truck was carrying a heavier load. We're not allowing for that here.)

a. Write down the extensive form of the game.

ANSWER:



b. For the subgame in which Steve chose two trucks, find the Nash equilibrium in mixed strategies.

ANSWER: Steve chooses A with probability $\frac{1}{2}$, and Charlie chooses A with probability $\frac{1}{2}$. The strategic form game is

		Charlie	
		A	B
Steve	A	0,6	6,0
	B	6,0	0,6

It is easy to verify there is no Nash equilibrium in pure strategies. To find a Nash equilibrium in mixed strategies, suppose that Steve chooses A with probability x. For Charlie to be willing to mix, he must be indifferent between A and B, i.e. $6(1 - x) = 6x$ which means $x = 0.5$. The same reasoning applies to Charlie and he will choose A with 0.5 probability as well.

c. For the subgame in which Steve chose three trucks, find the Nash equilibrium in mixed strategies.

ANSWER: Steve chooses A with probability $\frac{1}{3}$, B with probability $\frac{1}{3}$, and C with probability $\frac{1}{3}$; Charlie chooses A with probability $\frac{1}{3}$, B with probability $\frac{1}{3}$, and C with probability $\frac{1}{3}$. Steve's expected payoff is 4 less the cost of a third truck which is a payoff of 2. The strategic form is

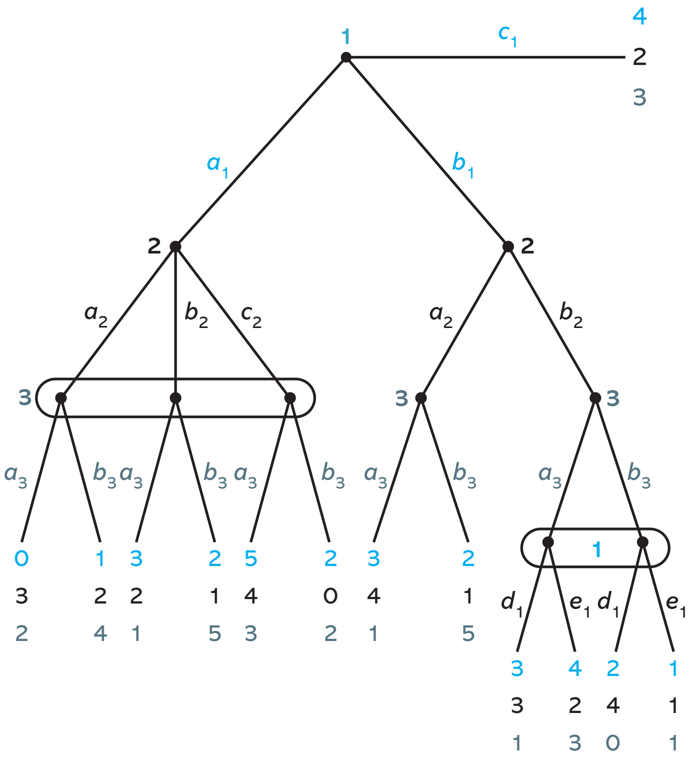
		Charlie		
		A	***B***	***C***
Steve	***A***	−2,6	4,0	4,0
	B	4,0	−2,6	4,0
	C	4,0	4,0	−2,6

It is easy to verify there is no Nash equilibrium in pure strategies. To find a Nash equilibrium in mixed strategies, suppose that Steve chooses A with probability x, and B with probability y. For Charlie to be willing to randomize over A, B, and C, he must be indifferent among all three options, i.e. $6x = 6y = 6(1 - x - y)$, which implies that $x = y = \frac{1}{3}$. Suppose that Charlie chooses A with probability a, and B with probability b. For Steve willing to be content to randomize over A, B, and C, it must be that $-2a + 4(1 - a) = -2b + 4(1 - b) = -2(1 - a - b) + 4(a + b)$, which implies that $a = b = \frac{1}{3}$.

d. For the game, find an SPNE in mixed strategies.

ANSWER: If Steve chooses 2 trucks then, given the Nash equilibrium for the two truck subgame, his expected payoff is 3. If Steve chooses 3 trucks then, given the Nash equilibrium for the three truck subgame, his expected payoff is $(\frac{2}{3}) \times 6 + (\frac{1}{3}) \times 0 - 2 = 2$. Thus, Steve will choose 2 trucks. The SPNE then is: Steve chooses two trucks, randomizes equally over trucks A and B when he chooses 2 trucks, and randomizes equally over trucks A, B, and C when he chooses 3 trucks; and Charlie randomizes equally over trucks A and B when Steve chooses 2 trucks, and randomizes equally over trucks A, B, and C when Steve chooses 3 trucks.

14. Consider this extensive form game on the top of page 354. The top number is player 1's payoff, the middle number is player 2's payoff, and the bottom number is player 3's payoff.



unique Nash equilibrium of d_1. Thus, if (a_1, a_2) is chosen then the payoffs are (4,2). If (a_1, b_2) is chosen then the payoffs are (1,2). If (b_1, a_2) is chosen then the payoffs are (2,1). And if (b_1, b_2) is chosen then the payoffs are (3,2). With those as the resulting payoffs, the game starting from the initial decision nodes:

Player 2

Player 1		a_2	b_2
	a_1	4,2	1,2
	b_1	2,1	3,2

It has two Nash equilibria: (a_1, a_2) and (b_1, b_2). There are then two SPNE: ($a_1/d_1/c_1/d_1$, a_2/d_2) and ($b_1/d_1/c_1/d_1$, b_2/d_2).

17. Player 1 is seeking to get a project approved by the local government, which will require bribing a few government officials. Three officials are relevant to the approval process: players 2, 3, and 4. Player 1 has access to player 2, while player 2 has access to players 3 and 4. In stage 1, player 1 offers a bribe to player 2, denoted b_2, which is any integer from {0,1, . . . , 100}. The bribe is of the form: "If the project is approved, then I'll pay you b_2." In stage 2, player 2 accepts or rejects the bribe. If he rejects it, then the project is not approved and the game is over. If player 2 accepts the bribe (which, recall, is paid only if the project is eventually approved), then the game moves to stage 3, which has player 2 simultaneously offer bribes to player 3 and 4, which are denoted b_3, and b_4, respectively. b_3, and b_4 are any integers from {0, 1, . . . , b_2,} for which their sum does not exceed b_2; thus, player 2 can offer bribes financed by the bribe given to him by player 1. In stage 4, players 3 and 4 each decide whether to accept or reject the bribe made to him by player 2. If both players 3 and 4 reject the bribes, then the project is not approved, no bribes are paid to any players, and the game is over. If, out of players 3 and 4, at least one player accepts a bribe, then the project is approved and all bribes are paid to those players who accepted bribes. Player 1's payoff is 0 when the project is not approved and is 100 minus the bribe paid to player 2 when the project is approved. For player 2, if the project is not approved, then his payoff is 4.5; if the project is approved, then it equals the bribe he received from player 1 less the bribes he paid to players 3 and 4. For player 3 (4), his payoff equals 9.5 (14.5) when the project is not approved and/or he rejected the bribe, and equals the bribe he was offered when the project is approved and he accepted the bribe. Using SPNE, is the project approved, and, if so, what bribes are given out?

ANSWER: Consider the subgame in which players 3 and 4 decide whether or not to accept the bribes made to them. These bribes were simultaneously made by player 2 so player 3 does not know the bribe made to player 4 and player 4 does not know the bribe made to player 3. However, this lack of information will prove unimportant for determining optimal behavior. That the game has gotten to this stage means that player 2 accepted his bribe in which case all that is needed for the project to be approved (and for bribes to be paid) is for player 3 and/or 4 to accept their bribes. Recall that if the project is not approved and/or a player rejects a bribe then player 3's payoff is 9.5, and player 4's payoff is 14.5. Hence, it is optimal for player 3 to accept the bribe if and only if $b_3 \geq 10$ (recall that bribes are in integers) and it is optimal for player 4 to accept his bribe if and only if $b_4 \geq 15$. Thus, if the subgame has $b_3 \geq 10$ and $b_4 \geq 15$ then the unique Nash equilibrium is that both accept; if $b_3 \geq 10$ and $b_4 < 15$ then the unique Nash equilibrium is that player 3 accepts and player 4 rejects; if $b_3 < 10$ and $b_4 \geq 15$ then the unique Nash equilibrium is that player 3 rejects and player 4 accepts; and if $b_3 < 10$ and $b_4 < 15$ then the unique Nash equilibrium is that both reject.

Consider the subgame in which player 2 accepted the bribe and is to decide how much to bribe players 3 and 4. Recall that if the project is not approved then player 2's payoff is 4.5. First note that a necessary condition for the project to be approved is $b_3 \geq 10$ and/or $b_4 \geq 15$ so that at least one of those two players will accept the bribe. Thus, the smallest amount of total bribes sufficient to get project approval is 10 which involves offering a bribe of 10 to player 3 (who'll accept) and 0 to player 4 (who'll reject).

Let us show that if $b_2 \leq 14$ then Nash equilibrium has player 2 proposing sufficiently small amounts to players 3 and 4 that the bribes are rejected. In order to get the project approved, player 2 must offer player 3 a bribe of 10 which then leaves player 2 with a payoff of $b_2 - 10$ which is less than or equal to 4 when $b_2 \leq 14$. Given that player 2's payoff when the project is not approved is 4.5, he prefers that the project is not approved. Nash equilibrium is then any values for b_3 and b_4 such that $b_3 < 10$ and $b_4 < 15$ so that the bribes are rejected and the project is not approved.

If instead $b_2 \geq 15$ then player 2 can offer a bribe of 10 to player 3 which he'll accept, the project will be approved, and player 2's payoff is $b_2 - 10 \geq 5$ and thus exceeds his payoff of 4.5 from the project's disapproval. Thus, if $b_2 \geq 15$ then player 2 prefers to offer bribes that will result in the project's approval. Conditional on the project being approved, player 2's payoff is decreasing in the bribes he gives players 3 and 4. He then wants to offer the minimum sum of bribes to players 3 and 4 that will result in at least one of them accepting the bribe, which means offering 10 to player 3 and 0 to player 4.

Consider the subgame in which player 2 has been offered a bribe. If $b_2 \leq 14$ then, even if he accepts it, the project will not be approved because he will subsequently offer sufficiently small bribes to players 3 and 4 that neither of them will accept. Hence, there are two Nash equilibria: player 2 accepts and player 2 rejects. If $b_2 \geq 15$ then player 2's payoff from rejecting is 4.5 and from accepting is $b_2 - 10$ which is strictly greater than 4.5. Hence, there is a unique Nash equilibrium: player 2 accepts.

Consider the the subgame in which player 1 decides on the bribe to offer player 2. If $b_2 \leq 14$ then, as just argued, ensuing equilibrium play results in the project's disapproval so player 1's payoff is 0. If $b_2 \geq 15$ then, as just argued, ensuing equilibrium play results in the project's approval so player 1's payoff is $100 - b_2$. Thus, player 1 wants to give player 2 the minimum bribe necessary to induce player 2 both to accept it and to make bribes to players 3 and 4 so that at least one of them will accept. This implies $b_2 = 15$.

The project is approved with player 2 receiving a bribe of 15 and player 3 receiving a bribe of 10.

18. The owner of an item has decided to sell it using a first-price sealed bid auction. Three bidders are participating. Bidder 1 assigns a value of 10.7 to the item, bidder 2's value is 15.3, and bidder 3's value is 19.4. The game has two stages. In stage 1, the seller selects a reserve price r, which is the lowest price for which she'll sell it at auction. The reserve price is allowed to be any nonnegative integer. In stage 2, the three bidders simultaneously submit bids, after having learned the reserve price. A bid is any nonnegative integer that does not exceed a bidder's value. If the highest bid is greater than or equal to r then the item is awarded to the highest bidder who then pays a price equal to her bid. In that case, the winning bidder's payoff is her value minus the price paid for the item. If a bidder does not win the item, then her payoff is 0. If two bidders submitted the highest bid (and it is at least as great as the reserve price), then each has probability 1/2 of being declared the winner (and getting the item at a price equal to her bid) and probability 1/2 of not being the winner (with payoff of 0). If all three bidders submitted the highest bid (and it is at least as great as the reserve price) then each has probability 1/3 of being declared the winner (and getting the item at a price equal to her bid) and probability 2/3 of not being the winner (with payoff of 0). If the item is sold at auction, then the seller's payoff equals the price for which it is sold, and her payoff is 0 if it is not sold.

a. Describe the general form of a strategy for each player.

ANSWER: A strategy for the seller is a value for r. A strategy for a bidder is a bid for each possible value of the reserve price.

b. Using SPNE, what is the reserve price, who wins the item, and what is the winning bid?

ANSWER: Consider the stage 2 auction. If the reserve price exceeds all bidders' valuations then, given that bidders are constrained not to bid above their valuations, all bidders must submit bids below the reserve price in which case no one wins and each has a payoff of zero. Thus, if $r \geq 20$ then any strategy profile is a Nash equilibrium and the item is not sold.

Cooperation and Reputation:
Applications of Repeated Interaction with Infinitely Lived Players

14

1. Consider the infinitely repeated game in which the stage game is shown here. Each player's payoff is the present value of her payoff stream, where the discount factor is δ.

Player 2

		w	*x*	*y*	*z*
Player 1	*a*	2,2	3,1	2,0	4,−4
	b	1,3	4,4	3,1	2,3
	c	0,2	1,3	7,7	3,9
	d	−4,4	3,2	9,3	0,0

a. Define a grim-trigger strategy that results in player 1's choosing *c* and player 2's choosing *y*, and state conditions for that strategy's resulting in an SPNE.

ANSWER: This game has two Nash equilibria, (a, w) and (b, x). Thus, consider a strategy for player 1 in which, in period 1, he chooses *c*; in a future period, chooses *c* if the outcome was (c, y) in all past periods; and, in a future period, chooses *b* if the outcome was not (c, y) in some past period. For player 2, in period 1, she chooses *y*; in a future period, chooses *y* if the outcome was (c, y) in all past periods; and, in a future period, chooses *x* if the outcome was not (c, y) in some past period. The equilibrium condition is

$$\frac{7}{1-\delta} \geq 9 + \delta\left(\frac{4}{1-\delta}\right) \Rightarrow \delta \geq \frac{2}{5}.$$

b. Consider the following strategy profile: In period 1, player 1 chooses *c*. In any other period, player 1 chooses *c* if, in the previous period, the outcome was either (*c*, *y*) or (*d*, *z*); otherwise he chooses *d*. In period 1, player 2 chooses *y*. In any other period, player 2 chooses *y* if, in the previous period, the outcome was either (*c*, *y*) or (*d*, *z*); otherwise she chooses *z*. Derive conditions for this profile to result in an SPNE.

ANSWER: Suppose it is either period 1 or it is some future period in which the outcome was either (c, y) or (d, z) in the previous period. The prescribed action of *c* for player 1 is optimal if

$$7 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 9 + \delta \times 0 + \delta^2 \times 7 + \cdots \Rightarrow 7 + 7\delta \geq 9 \Rightarrow \delta \geq \frac{2}{7}.$$

Now consider a history in which in the previous period the outcome was neither (c, y) nor (d, z). The prescribed action of *d* for player 1 is optimal if

$$0 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 4 + \delta \times 0 + \delta^2 \times 7 + \cdots \Rightarrow 7\delta \geq 4 \Rightarrow \delta \geq \frac{4}{7}.$$

The conditions are the same for player 2. It is then a subgame perfect Nash equilibrium if

$$\delta \geq \frac{2}{7} \text{ and } \delta \geq \frac{4}{7}; \text{ or } \delta \geq \frac{4}{7}.$$

c. Consider the following strategy profile: In period 1, player 1 chooses *c*. In any other period, player 1 chooses *c* if, in the previous period, the outcome was either (*c, y*), (*d, w*), or (*a, z*); he chooses *d* if, in the previous period, player 2 chose *y* and player 1 did not choose *c;* he chooses *a* if, in the previous period, player 1 chose *c* and player 2 did not choose *y;* otherwise he chooses *b*. In period 1, player 2 chooses *y*. In any other period, player 2 chooses *y* if, in the previous period, the outcome was either (*c, y*), (*d, w*), or (*a, z*); she chooses *w* if, in the previous period, player 2 chose *y* and player 1 did not choose *c;* she chooses *z* if, in the previous period, player 1 chose *c* and player 2 did not choose *y;* otherwise she chooses *x*. Derive conditions for this profile to yield an SPNE.

ANSWER: Consider period 1 or a future period in which the previous period's outcome was either (c, y), (d, w), or (a, z). Player 1's prescribed action of c is optimal if

$$7 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 9 + \delta \times (-4) + \delta^2 \times 7 + \cdots$$

$$\Rightarrow 7 + 7\delta \geq 9 - 4\delta \Rightarrow \delta \geq \frac{2}{11}.$$

Next, consider a history in which, in the previous period, player 1 did not choose c and player 2 chose y. Player 1's prescribed action of d is optimal if

$$-4 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 2 + \delta \times 4 + \delta^2 \times 4 + \cdots$$

$$\Rightarrow -4 + 7\frac{\delta}{1-\delta} \geq 2 + 4\frac{\delta}{1-\delta} \Rightarrow \delta \geq \frac{2}{3}.$$

Next, consider a history in which, in the previous period, player 1 chose c and player 2 did not choose y. Player 1's prescribed action of a is optimal since any other action lowers the current period payoff and reduces the future payoff stream from

$$\delta \times 7 + \delta^2 \times 7 + \cdots$$

to

$$\delta \times 4 + \delta^2 \times 4 + \cdots.$$

Finally, for any other history, player 1 is to choose b. Note that b maximizes his current payoff given player 2 is to choose x. Furthermore, the future payoff is the same since, come next period, the previous period's history will involve player 2 choosing x and thus the outcome will be (b, x). This applies as well to all ensuing periods. The analysis is analogous for player 2. This strategy profile is then a subgame perfect Nash equilibrium if

$$\delta \geq \frac{2}{11} \text{ and } \delta \geq \frac{2}{3} \Rightarrow \delta \geq \frac{2}{3}.$$

2. As early as the 1820s, the custom of "pairing off" had developed in Congress. If a member of Congress was to miss a formal vote, he would arrange beforehand with a member on the opposing side of the issue for the two not to vote. In modeling this situation, consider two members of Congress—Representatives Smith and Jones—who, on a regular basis, would like to miss a House vote in order to take care of other business. Representative Smith would prefer to be away every three periods, starting with period 1 (hence periods 1, 4, 7, . . .), and receives a value of 3 from being away. Representative Jones would prefer to be away every three periods, starting with period 2 (hence, periods 2, 5, 8, . . .), and also receives value of 3. Call these periods the representatives' "traveling periods." In each such period, there is a House vote, and Smith receives a value of 5 from being in attendance and voting and a value of −5 if Jones is in attendance and votes. Analogously, Jones earns 5 from being in attendance and voting and −5 if Smith is in attendance and votes. Thus, if both are in attendance and vote, then Smith and Jones each have a payoff of

10. In each period, two players simultaneously choose between two actions: *up* and *down*. If both choose *up*, then each receives a payoff of 10 with probability .8 and a payoff of 3 with probability .2. If both choose *down*, then they both receive a payoff of 5 for sure. If one chooses *down* and the other chooses *up*, then the former receives a payoff of 10 for sure and the latter receives a payoff of 10 with probability .6 and a payoff of 3 with probability .4. When choosing an action, each player knows his past choices and both players' past payoffs, but neither knows what the other player actually chose in the past. This stage game is infinitely repeated, where each player's discount factor is δ. Consider the following symmetric strategy pair: In period 1, both players choose *up*. In any other period, choose *up* if both players received the same payoff (either 3, 5, or 10) last period; otherwise choose *down*. Derive conditions whereby this strategy pair is an SPNE.

ANSWER: To begin, derive a player's payoff when it is period 1 or if in the previous period they both had the same payoff, given that they use this strategy (which implies that they choose *up* in the current period). Denoting that payoff as V, it is defined as

$$V = .8 \times 10 + .2 \times 3 + \delta V \Leftrightarrow V = \frac{8.6}{1-\delta}.$$

If it is period 1 or if in the previous period they both had the same payoff, then a player finds it optimal to choose *up* rather than *down* when

$$8.6 + \delta \times 8.6 + \delta^2 \times 8.6 + \cdots \geq 10 + .6 \times [\delta \times 8.6 + \delta^2 \times 8.6 + \cdots] \\ + .4 \times [\delta \times 5 + \delta^2 \times 8.6 + \cdots].$$

If she chooses *down*, then her current payoff is 10. With probability .6, the other player's payoff is 10, in which case the future payoff is V as both will choose *up* next period. With probability .4, the other player's payoff is 3, in which case the payoff next period is 5—as both choose *down*—but after that they return to choosing *up*. Cancelling common terms,

$$8.6 + 8.6\delta \geq 10 + 5.16\delta + 2\delta \Rightarrow \delta \geq \frac{1.4}{1.44} \simeq .97.$$

Now suppose, in the previous period, the players' payoffs were not the same. A player's strategy has him choose *down*, and that is preferable to *up* when

$$5 + \delta \times 8.6 + \delta^2 \times 8.6 + \cdots \geq .6 \times 10 + .4 \times 3 + [\delta \times 5 + \delta^2 \times 8.6 + \cdots].$$

By choosing *down*, a player gets 5 today, and then both players return to choosing *up*. If she chooses *up* today, then her expected current payoff is .6 × 10 + .4 × 3. As players' payoffs will differ, then both players will choose *down* tomorrow—yielding a payoff of 5—and *up* thereafter. Cancelling common terms,

$$5 + 8.6\delta \geq 7.2 + 5\delta \Rightarrow \delta \geq \frac{2.2}{3.6} \simeq .61.$$

In sum, this symmetric strategy pair is a subgame perfect Nash equilibrium if

$$\delta \geq .97 \text{ and } \delta \geq .61 \Rightarrow \delta \geq .97.$$

11. In a village, there are $n \geq 2$ households. In each period, each household works and earns money. After paying for necessities, the household is left with $x > 0$ dollars. Each household only considers using this residual money to buy appliances. There are an infinite number of different types of appliances that each household would like to buy: a washer, a dryer, a television set, and so forth. For simplicity, each appliance has the same price of $p > 0$ and the same lifetime utility, which is denoted $Z > 0$ (where Z is measured in dollars). Assume $Z > p$ so that the value of an appliance to a household exceeds its price.

At the start of the game, each household has zero savings. Assume that no interest is paid on savings. A household's total payoff is calculated as shown here and is based only on buying appliances. Imagine that dollars are worthless except when used to buy appliances. If a household buys appliances in periods $t_1, t_2, \ldots,$ then its payoff is

$$\delta^{t_1-1}Z + \delta^{t_2-1}Z + \cdots \text{ or } \sum_{i=1}^{\infty} \delta^{t_i-1}Z$$

where δ is the discount factor and $0 < \delta < 1$. One final assumption is that $p = nx$; that is, the price of an appliance happens to be an integer multiple of what a household can save in each period, and this multiple happens to equal the number of households. In each period, a household has $n + 1$ choices: save x, spend its savings to buy an appliance (assuming that its savings is at least p), or give x to another household (of which there are $n - 1$ households). Let us provide two descriptions of behavior. The *self-sufficiency rule* has a household save x dollars each period until savings is nx, at which time it buys an appliance. Thus, this rule results in a household buying an appliance in periods $n, 2n, 3n, \ldots$. Alternatively, the *rotational credit rule* has all households giving x dollars to household i in period t when $t = i, n + i, 2n + i, \ldots$ and household i buys an appliance. Thus, when used by all households, the rotational credit rule has household 1 buying an appliance in period 1, household 2 buying an appliance in period 2, . . . , household n buying an appliance in period n, household 1 buying an appliance in period $n + 1$, and so forth.

a. Derive the payoff to a household from using the self-sufficiency rule.

ANSWER: With the self-sufficiency rule, a household purchases an appliance every n periods starting with period n. Its payoff is then:

$$\delta^{n-1}Z + \delta^{2n-1}Z + \cdots = \sum\nolimits_{r=1}^{\infty}\delta^{rn-1}Z = \frac{\delta^{n-1}z}{1-\delta^{n}}.$$

The last equality is derived as follows. Define S as

$$S = \delta^{n-1} + \delta^{2n-1} + \delta^{3n-1} + \cdots$$

Now multiply each side by δ^n,

$$\delta^n S = \delta^{2n-1} + \delta^{3n-1} + \delta^{4n-1} + \cdots$$

Subtracting the second line from the first line yields

$$S - \delta^n S = \delta^{n-1} \Rightarrow S = \frac{\delta^{n-1}}{1-\delta^{n}}.$$

b. Consider the following strategy profile. In period 1, each player uses the rotational credit rule. In period $t \geq 2$: 1) if all players have acted according to the rotational credit rule in all past periods, then use the rotational credit rule; and 2) if one or more players did not act according to the rotational credit rule in some past period, then use the self-sufficiency rule. Derive the conditions for this strategy profile to be an SPNE.

ANSWER: The space of histories can be partitioned into $n + 1$ sets. There are n sets of the following form: 1') period 1 and period t where $t = 1, n + 1, 2n + 1, \ldots$ and in all past periods all players acted according to the rotational credit rule; 2') period t where $t = 2, n + 2, 2n + 2, \ldots$ and in all past periods all players acted according to the rotational credit rule; . . . ; and n') period t where $t = n, 2n, 3n, \ldots$ and in all past periods all players acted according to the rotational credit rule. Note that a history in set h' results in household h buying an appliance. Finally, there is the last set of histories: $n + 1$')$t \geq 2$ and, in some past period, one or more players did not act according to the rotational credit rule. For all histories within each of these $n + 1$ sets, the prescribed behavior is the same.

Consider household 1 and a history in set 1′. Its strategy has it buy an appliance which results in a total payoff of

$$Z + \delta^n Z + \delta^{2n} Z + \cdots = \frac{Z}{1-\delta^n}.$$

Implicit in this expression is that the other households act according to their strategies in the current period which means giving their savings to household 1 and all players acting according to their strategies in future periods. Alternatively, household 1 can choose not to buy an appliance. It then has savings of $(n+1)x$ in period 2 so, according to the self-sufficiency rule, it buys an appliance. This leaves savings of x going into period 3. Come period $n+1$, it has savings of nx and thus buys an appliance and subsequently buys one every n periods thereafter. Its total payoff in that case is

$$\delta Z + \delta^n Z + \delta^{2n} Z + \cdots$$

which is strictly lower than that from buying in period 1 (note that the first term is smaller and the other terms are equal). It is easy to see that the remaining option—giving x to another household—is inferior as it yields a total payoff of

$$\delta Z + \delta^{n+1} Z + \delta^{2n+1} Z + \cdots = \frac{\delta Z}{1-\delta^n}.$$

Thus, household 1's strategy is clearly optimal for 1′ histories.

Next consider household 1 and a history in set 2′. Its strategy has it give x dollars to household 2 and this yields a total payoff of

$$\delta^{n-1} Z + \delta^{2n-1} Z + \cdots = \frac{\delta^{n-1} Z}{1-\delta^n},$$

as it doesn't get to buy an appliance for another $n-1$ periods. The other possible actions are to give it to another household (which is clearly nonoptimal) or to retain its savings in which case households move to the self-sufficiency rule. With the latter case, household 1's total payoff is then

$$\delta^{n-1} Z + \delta^{2n-1} Z + \cdots = \frac{\delta^{n-1} Z}{1-\delta^n}.$$

as it has enough savings by period $n+1$ to buy an appliance (and will buy an appliance every n periods thereafter). As the payoffs from the two actions are the same, household 1 is content to act according to its strategy.

Next consider household 1 and a history in set m' where $m \in \{3, 4, \ldots, n\}$. The total payoff from its strategy is:

$$\delta^{n-m+1} Z + \delta^{2n-m+1} Z + \cdots = \frac{\delta^{n-m+1} Z}{1-\delta^n},$$

while its total payoff from not giving its savings to household m is:

$$\delta^{n-1} Z + \delta^{2n-1} Z + \cdots = \frac{\delta^{n-1} Z}{1-\delta^n},$$

since household 1 won't have enough saved up for another $n-1$ periods. The strategy is optimal because

$$\frac{\delta^{n-m+1} Z}{1-\delta^n} > \frac{\delta^{n-1} Z}{1-\delta^n} \Leftrightarrow \delta^{m-2} < 1,$$

which is true since $m > 2$ and $0 < \delta < 1$.

The final case for household 1 is for $n + 1'$ histories. As the strategy prescribes the self-sufficiency rule, this is clearly optimal. Given the other players are not going to share their savings, it is not optimal for household 1 to do so.

We conclude that the strategy is optimal for household 1 and this is independent of the value of δ.

In turning to household 2, note that the situation it faces for histories in $1'$ is exactly the same as that faced by household 1 for histories in n' as, in both cases, each household is to give its savings to another household and its turn to buy an appliance is in the next period. Similarly, the situation household 2 faces for histories in $2'$ is exactly the same as that faced by household 1 for histories in $1'$ and, more generally, the situation household 2 faces for histories in m' is exactly the same as that faced by household 1 for histories in $m - 1'$ for $m \in \{2, 3, \ldots, n\}$. All households face the same situation for $n + 1'$ histories. Thus, if the conditions hold for household 1 then they hold for household 2. A similar logic applies when we consider households 3, 4, . . . , n.

In conclusion, this strategy profile is an SPNE for any discount factor.

12. Consider the infinitely repeated version of this symmetric two-player stage game. Each player's discount factor is δ where $1/2 < \delta < 1$. Find a symmetric SPNE.

		Player 2		
		a	***b***	***c***
Player 1	***a***	1,1	2,2	0,2
	b	2,2	3,3	2,4
	c	2,0	4,2	1,1

ANSWER: Note that there is no symmetric Nash equilibrium for the stage game. Consider a symmetric strategy that has a player choose action b in period 1. In period t, a player chooses b if, in the previous period, both players chose b or both chose a; otherwise, a player chooses a. In deriving the conditions for this strategy pair to be an SPNE, consider either period 1 or period t where, in the previous period, both chose b or both chose a:

$$\frac{3}{1-\delta} \geq 4 + \delta \times 1 + \delta^2\left(\frac{3}{1-\delta}\right) \Leftrightarrow 3 + \delta \times 3 \geq 4 + \delta \times 1 \Leftrightarrow \delta \geq \frac{1}{2}$$

Suppose instead that in the previous period either both did not choose a or both did not choose b:

$$1 + \delta\left(\frac{3}{1-\delta}\right) \geq 2 + \delta \times 1 + \delta^2\left(\frac{3}{1-\delta}\right) \Leftrightarrow 1 + \delta \times 3 \geq 2 + \delta \times 1 \Leftrightarrow \delta \geq \frac{1}{2}$$

Given the assumption $\frac{1}{2} < \delta < 1$ then these conditions are satisfied and it is a SPNE. Similarly, it is a SPNE to replace action a in that strategy with action c.

13. Consider this infinitely repeated version of the symmetric two-player stage game below. Each player's discount factor is δ where $0 < \delta < 1$.

		Player 2			
		a	***b***	***c***	***d***
Player 1	***a***	−1,−1	2,2	0,2	3,2
	b	2,2	4,4	4,3	13,0
	c	2,0	3,4	6,6	7,7
	d	2,3	0,13	7,7	9,9

14. Consider the infinitely repeated version of the symmetric two-player stage game shown here. Each player's discount factor is δ where $0 < \delta < 1$.

		Player 2			
		a	***b***	***c***	***d***
Player 1	***a***	0,0	4,2	14,1	0,−3
	b	2,4	5,5	8,3	7,1
	c	1,14	3,8	10,10	4,4
	d	−3,0	1,7	4,4	2,2

a. Consider a symmetric strategy pair that has a player choose action *c* in period 1 and in period *t* as long as action *c* was chosen by both players in the previous period; otherwise, a player chooses action *b*. Derive the conditions for it to be an SPNE.

ANSWER: Consider either period 1 or period *t* where, in the previous period, (*c*,*c*) has been always played. It is optimal for player 1 (or player 2) to choose *c* when

$$\frac{10}{1-\delta} \geq 14 + \delta\left(\frac{5}{1-\delta}\right) \Rightarrow \delta \geq \frac{4}{9}.$$

For the other periods, the profile specifies a forever play of (*b*,*b*) which is a Nash equilibrium of the one-shot game, so no player wants to deviate.

b. Consider a symmetric strategy pair that has a player choose action *c* in period 1. In period *t*, action *c* is chosen if either both players chose *c* or both chose *d* in the previous period; otherwise, action *d* is chosen. Derive the conditions for it to be an SPNE.

ANSWER: Consider either period 1 or period *t* where, in the previous period, (*c*,*c*) or (*d*,*d*) has been just played. For it to be optimal for player 1 (or player 2) to choose *c*, we must have

$$\frac{10}{1-\delta} \geq 14 + \delta \times 2 + \delta^2\left(\frac{10}{1-\delta}\right) \Rightarrow 10 + \delta \times 10 \geq 14 + \delta \times 2 \Rightarrow \delta \geq \frac{1}{2}$$

For any other periods, the profile specifies (*d*,*d*) to be played. For a player to find *d* optimal, we must have

$$2 + \delta\left(\frac{10}{1-\delta}\right) \geq 7 + \delta \times 2 + \delta^2\left(\frac{10}{1-\delta}\right) \Rightarrow 2 + \delta \times 10 \geq 7 + \delta \times 2 \Rightarrow \delta \geq \frac{5}{8}.$$

It is an SPNE if and only if $\delta \geq \frac{1}{2}$ and $\delta \geq \frac{5}{8}$ or, equivalently, $\delta \geq \frac{5}{8}$.

c. Let (*x*,*y*) refer to a pair of actions for players 1 and 2 where *x* is player 1's action and *y* is player 2's action. Consider a symmetric strategy pair that has players play (*c*,*c*) in period 1. Actions for future periods are prescribed as follows. If players have always chosen (*c*,*c*), then they are to play (*c*,*c*). If players were supposed to play (*c*,*c*) in the previous period and: 1) they chose (*c*,*c*) then they are to play (*c*,*c*); 2) player 1 did not choose *c* and player 2 did choose *c*, then they are to play (*d*,*a*); 3) player 1 did choose *c* and player 2 did not choose *c*, then they are to play (*a*,*d*); and 4) both players did not choose *c*, then they are to play (*c*,*c*). If players were supposed to play (*d*,*a*) in the previous period and: 1) they chose (*d*,*a*) then they are to play (*c*,*c*); 2) player 1 did not choose *d* and player 2 did choose *a*, then they are to play (*d*,*a*); 3) player 1 did choose *d* and player 2 did not choose *a*, then they are to play (*a*,*d*); and 4) both players did not (player 1 did not choose *d* and player 2 did not choose *a*) then they are to play (*c*,*c*). If players were supposed to play (*a*,*d*) in the previous period and: 1) they chose (*a*,*d*) then they are to play (*c*,*c*); 2) player 1 did not choose *a* and player 2 did choose *d* then they are to play (*d*,*a*); 3) player 1 did choose *a* and player 2 did not choose *d* then they are to play (*a*,*d*); and 4) both players did not (player 1 did not choose *a* and player 2 did not choose *d*) then they are to play (*c*,*c*).

ANSWER: According to this strategy pair, players are either supposed to play (c,c), (d,a), or (a,d) depending on the history. Suppose they are to play (c,c). The equilibrium condition is

$$\frac{10}{1-\delta} \geq 14 + \delta \times (-3) + \delta^2\left(\frac{10}{1-\delta}\right) \Rightarrow 10 + \delta \times 10 \geq 14 - \delta \times 3 \Rightarrow \delta \geq \frac{4}{13}.$$

Suppose they are to play (d,a). The equilibrium condition for player 1 is

$$-3 + \delta\left(\frac{10}{1-\delta}\right) \geq 2 + \delta \times (-3) + \delta^2\left(\frac{10}{1-\delta}\right) \Rightarrow -3 + \delta \times 10 \geq 2 - \delta \times 3 \Rightarrow \delta \geq \frac{5}{13}$$

and for player 2 is

$$0 + \delta\left(\frac{10}{1-\delta}\right) \geq 7 + \delta \times (-3) + \delta^2\left(\frac{10}{1-\delta}\right) \Rightarrow \delta \times 10 \geq 7 - \delta \times 3 \Rightarrow \delta \geq \frac{7}{13}.$$

Suppose they are to play (a,d). The equilibrium condition for player 1 is

$$0 + \delta\left(\frac{10}{1-\delta}\right) \geq 7 + \delta \times (-3) + \delta^2\left(\frac{10}{1-\delta}\right) \Rightarrow \delta \times 10 \geq 7 - \delta \times 3 \Rightarrow \delta \geq \frac{7}{13}$$

and for player 2 is

$$-3 + \delta\left(\frac{10}{1-\delta}\right) \geq 2 + \delta \times (-3) + \delta^2\left(\frac{10}{1-\delta}\right) \Rightarrow -3 + \delta \times 10 \geq 2 - \delta \times 3 \Rightarrow \delta \geq \frac{5}{13}.$$

It is an SPNE if and only if $\delta \geq \frac{5}{13}$ and $\delta \geq \frac{7}{13}$ or, equivalently, $\delta \geq \frac{7}{13}$.

15. Consider a market with $n \geq 2$ stores. Market demand is $10 - P$, which means that if the price faced by consumers is P, then the total number of units demanded is $10 - P$. Stores offer an identical product, so consumers buy from the store with the lowest price. If two or more stores set the lowest price, then those stores equally divide up the demand. For example, if store 1 prices at 7 and all other stores price above 7, then store 1 sells $10 - 7 = 3$ units and the other stores sell nothing. If stores 1 and 2 both price at 7 and the other stores price above 7, then stores 1 and 2 each sell 1.5 units and the other stores sell nothing. A store's payoff equals its profit, which is its revenue (the price it charges multiplied by the number of units it sells) minus its cost (which equals 2 multiplied by the number of units it sells). For example, if store 1 prices at 4 and all other stores price above 4, then store 1's payoff is $(10 - 4) \times 4 - (10 - 4) \times 2 = 12$, and all other stores have a payoff of 0. Assume a store can choose any price in $\{0,1,2, \ldots, 10\}$. There are an infinite number of periods and, in each period, stores simultaneously choose price and earn profit. In any period, all stores' past prices are observed by all stores.

a. For the stage game, find the symmetric Nash equilibria.

ANSWER: Suppose all stores set a price $p \geq 4$. The profit to each store is $(10 - p) \times (p - 2)/n$. However, if one store deviates by lowing its price by 1, its profit will be $(10 - p + 1) \times (p - 1 - 2)$. Note that

$$(10 - p + 1) \times (p - 1 - 2) = (11 - p) \times (p - 3) > (10 - p) \times (p - 3)$$
$$\geq (10 - p) \times \frac{(p-2)}{n}$$

where the last inequality holds as long as $p \geq 4$ and $n \geq 2$. Hence, the store has incentive to deviate. Next note that it is not an equilibrium for all stores to price $p \leq 1$ because every store will make a negative expected profit. In terms of candidates for symmetric Nash equilibrium, this leaves us with $p = 2$ or $p = 3$. At $p = 2$, each store makes zero profit, but lowering one's price means negative profit while increasing the price also means zero profit. So it is a Nash equilibrium. At $p = 3$,