# EconS 503 - Microeconomic Theory II
# Homework #6 - Answer key

12.2 **Labor market signaling with productivity enhancing education.** Consider the setting in chapter 12, but assume that education improves the worker's productivity. In particular, a worker with type $\theta_i$ who acquires $e$ years of education, has productivity $\theta_i(1+\alpha e)$, where $\alpha \geq 0$ denotes the productivity enhancing effect of education. When $\alpha = 0$, the worker's productivity is $\theta_i$, thus being unaffected by $e$, as in chapter 12. When $\alpha > 0$, however, education improves the worker's productivity. For simplicity, let us assume that $\alpha = 1$.

(a) *Complete information.* As a benchmark, find the education level that the high- and low-type workers choose when the firm observes his type, $\theta_H$ or $\theta_L$, respectively.

- *Second stage.* Operating by backward induction, in the second stage, upon observing a worker with type $\theta_i$ and education level $e$, the firm pays a wage equal to this worker's productivity, $w_i(e) = \theta_i(1+e)$.
- *First stage.* Anticipating wage $w_i(e)$, the worker chooses his education level $e$ to solve

$$\max_{e \geq 0} \; w_i(e) - c(e, \theta_i) \quad \text{subject to} \quad w_i(e) = \theta_i(1+e)$$

Inserting $w_i(e) = \theta_i(1+e)$ into the worker's objective function, the above problem simplifies to the following (unconstrained) problem

$$\max_{e \geq 0} \; \theta_i(1+e) - c(e, \theta_i).$$

Differentiating with respect to $e$, yields

$$\theta_i = \frac{\partial c(e, \theta_i)}{\partial e}$$

implying that the worker increases his education until the point where the marginal wage increase, $\theta_i$ (because education is productivity enhancing), coincides with the marginal cost of acquiring education, $\frac{\partial c(e,\theta_i)}{\partial e}$. This is a Pareto optimal result.

- Recall that, when education is not productivity enhancing, the left-hand side of this equality is zero, entailing that the worker has no incentives to acquire education under complete information, regardless of his type.

(b) *Separating PBEs.* Assume for the remainder of the exercise that the firm cannot observe the worker's type. Find the set of separating PBEs in this game, where each worker type chooses a different education level.

- For easier reference, we follow the same steps as in section 12.4.

- *First step, strategy profile.* In a separating strategy profile, the high-productivity worker chooses education level $e_H$ while the low-productivity worker chooses $e_L \neq e_H$.
- *Second step, updating beliefs.* Upon observing education level $e_H$, the firm assigns full probability to facing a high-productivity worker, $\mu(\theta_H|e_H) = 1$; while after observing education $e_L$, the firm's beliefs are $\mu(\theta_H|e_L) = 0$. For all other education levels, $e \neq e_H \neq e_L$, the firm's beliefs cannot be updated by Bayes' rule and are left unrestricted, $\mu(\theta_H|e) \in [0,1]$.
- *Third step, optimal responses.* Given the above beliefs, the firm responds with wage $w(e_H) = \theta_H(1+e_H)$ upon observing education $e_H$, $w(e_L) = \theta_L(1+e_L)$ upon observing $e_L$, and

$$w(e) = \underbrace{[\mu(\theta_H|e)\,\theta_H + (1-\mu(\theta_H|e))\,\theta_L]}_{\text{Expected productivity}}(1+e)$$

  upon observing $e \neq e_H \neq e_L$. As in section 12.4, off-the-equilibrium belief $\mu(\theta_H|e) \in [0,1]$ allows for the wage $w(e)$ to lie between $w(e_H)$ and $w(e_L)$. For simplicity, we focus here in the case where $\mu(\theta_H|e) = 0$, entailing that $w(e) = w(e_L)$.
- *Fourth step, optimal messages.* Given the above wage schedule, we must now identify under which conditions each worker has no incentives to deviate from acquiring his education level.
  - *Low-productivity worker.* When acquiring education level $e_L$, the low-productivity worker's utility is $\theta_L(1+e_L) - c(e_L,\theta_L)$. If, instead, he deviates to the high-type's education level, $e_H$, his utility becomes $\theta_H(1+e_H) - c(e_H,\theta_L)$, implying that he has no incentives to deviate to $e_H$ if

$$\theta_L(1+e_L) - c(e_L,\theta_L) \geq \theta_H(1+e_H) - c(e_H,\theta_L)$$

    or

$$c(e_H,\theta_L) - c(e_L,\theta_L) \geq \theta_H(1+e_H) - \theta_L(1+e_L)$$

    which indicates that the cost increase that the low-productivity worker experiences when increasing his education from $e_L$ to $e_H$ offsets his associated wage increase.
  - *High-productivity worker.* When acquiring education level $e_H$, the high-productivity worker's utility is $\theta_H(1+e_H) - c(e_H,\theta_H)$. If, instead, he deviates to the low-type's education level, $e_L$, his utility becomes $\theta_L(1+e_L) - c(e_L,\theta_H)$, implying that he has no incentives to deviate to $e_L$ if

$$\theta_H(1+e_H) - c(e_H,\theta_H) \geq \theta_L(1+e_L) - c(e_L,\theta_H)$$

    or

$$\theta_H(1+e_H) - \theta_L(1+e_L) \geq c(e_H,\theta_H) - c(e_L,\theta_H)$$

    which indicates that the wage reduction that the high-productivity worker experiences when deviating from $e_H$ to $e_L$ offsets his cost savings from acquiring less education.

- *Summary.* Combining the above inequalities, we obtain that a continuum of separating PBEs where:
  - The low-productivity worker chooses an education level $e_L$ that solves $\theta_L(1 + e_L) - c(e_L, \theta_L) = 0$.
  - The high-productivity worker chooses an education level $e_H \in [e'_H, e''_H]$, where $e'_H$ solves $c(e_H, \theta_L) - c(e_L, \theta_L) = \theta_H(1 + e_H) - \theta_L(1 + e_L)$, and $e''_H$ solves $\theta_H(1 + e_H) - \theta_L(1 + e_L) = c(e_H, \theta_H) - c(e_L, \theta_H)$.
  - Upon observing $e_H$, the firm pays a wage $w(e_H) = \theta_H(1 + e_H)$, and otherwise the firm pays a wage $w(e) = \theta_L(1 + e)$ for all $e \neq e_H$, given beliefs $\mu(\theta_H|e_H) = 1$ and $\mu(\theta_H|e) = 0$ for all $e \neq e_H$.

(c) *Pooling PBEs.* Find the set of pooling PBEs where both worker types choose the same education level.

- For easier reference, we follow the same steps as in section 12.5.
- *First step, strategy profile.* In a pooling strategy profile, both the high- and low-productivity worker chooses education level $e_P$.
- *Second step, updating beliefs.* Upon observing education level $e_P$, the firm keeps its prior belief, $p$, unaffected, i.e., $\mu(\theta_H|e_P) = p$. For all other education levels, $e \neq e_P$, the firm's beliefs cannot be updated by Bayes' rule and are left unrestricted, $\mu(\theta_H|e) \in [0, 1]$.
- *Third step, optimal responses.* Given the above beliefs, upon observing education $e_P$ the firm responds with wage

$$\begin{aligned} w(e_P) &= p[\theta_H(1 + e_P)] + (1 - p)[\theta_L(1 + e_P)] \\ &= [p\theta_H + (1 - p)\theta_L](1 + e_P) \end{aligned}$$

  where $E[\theta] = p\theta_H + (1 - p)\theta_L$ denotes the worker's expected type. Upon observing any other education level, $e \neq e_P$, the firm responds with wage

$$\begin{aligned} w(e_P) &= \mu(\theta_H|e)[\theta_H(1 + e_P)] + (1 - \mu(\theta_H|e))[\theta_L(1 + e_P)] \\ &= [\mu(\theta_H|e)\theta_H + (1 - \mu(\theta_H|e))\theta_H](1 + e_P) \end{aligned}$$

  where $\mu(\theta_H|e)\theta_H + (1 - \mu(\theta_H|e))\theta_H$ represents the worker's expected type, given off-the-equilibrium belief $\mu(\theta_H|e)$. As in section 12.5, off-the-equilibrium belief $\mu(\theta_H|e) \in [0, 1]$ allows for the wage $w(e)$ to lie between $\theta_L(1 + e_P)$ and $\theta_H(1 + e_P)$. For simplicity, we focus here in the case where $\mu(\theta_H|e) = 0$, entailing that $w(e) = \theta_L(1 + e_P)$.
- *Fourth step, optimal messages.* Given the above wage schedule, we must now identify under which conditions each worker has no incentives to deviate from acquiring his education level.
  - *Low-productivity worker.* When acquiring education level $e_P$, the low-productivity worker's utility is $E[\theta](1 + e_P) - c(e_P, \theta_L)$. If, instead, he deviates to any other education level, $e \neq e_P$, his utility becomes $\theta_L(1 + e) - c(e, \theta_L)$, implying that he has no incentives to deviate to $e$ if

$$E[\theta](1 + e_P) - c(e_P, \theta_L) \geq \theta_L(1 + e) - c(e, \theta_L)$$

Therefore, his most profitable deviation is to $e = 0$, which yields a utility $\theta_L(1+0) - c(0, \theta_L) = \theta_L$. The low-productivity worker does not want to deviate from $e_P$ if

$$E[\theta](1 + e_P) - c(e_P, \theta_L) \geq \theta_L$$

- *High-productivity worker.* When acquiring education level $e_P$, the high-productivity worker's utility is $E[\theta](1 + e_P) - c(e_P, \theta_H)$. If, instead, he deviates any other education level, $e \neq e_P$, his utility becomes $\theta_L(1+e) - c(e, \theta_H)$, implying that he has no incentives to deviate to $e$ if

$$E[\theta](1 + e_P) - c(e_P, \theta_H) \geq \theta_L(1 + e) - c(e, \theta_H)$$

Therefore, his most profitable deviation is to $e = 0$, which yields a utility $\theta_L(1+0) - c(0, \theta_L) = \theta_L$. The high-productivity worker does not want to deviate from $e_P$ if

$$E[\theta](1 + e_P) - c(e_P, \theta_H) \geq \theta_H$$

- Combining the above inequalities, we obtain that a continuum of separating PBEs where:
  - Both worker types choose education level $e_P \in [e'_P, e''_P]$, where $e'_P$ solves $E[\theta](1+e_P) - c(e_P, \theta_L) = \theta_L$, and $e''_P$ solves $E[\theta](1+e_P) - c(e_P, \theta_H) = \theta_H$.
  - Upon observing $e_p$, the firm pays a wage $w(e_p) = [p\theta_H + (1-p)\theta_L](1 + e_P)$, and otherwise the firm pays a wage $w(e) = \theta_L(1+e)$ for all $e \neq e_p$, given beliefs $\mu(\theta_H | e_P) = p$ and $\mu(\theta_H | e) = 0$ for all $e \neq e_P$.

12.6 **Finding separating PBEs in limit pricing.** Consider a market with inverse demand function $p(Q) = 1 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. Let us analyze an entry game with an incumbent monopolist (Firm 1) and an entrant (Firm 2) who analyzes whether or not to join the market. The incumbent's marginal costs are either high $H$ or low $L$, i.e., $c_1^H = \frac{1}{2} > c_1^L = \frac{1}{3}$, while it is common knowledge that the entrant's marginal costs are high, i.e., $c_2 = \frac{1}{2}$. To make the entry decision interesting, assume that when the incumbent's costs are low, entry is unprofitable; whereas when the incumbent's costs are high, entry is profitable. (Otherwise, the entrant would enter regardless of the incumbent's cost, or stay out regardless of the incumbent's cost.) For simplicity, assume no discounting of future payoffs throughout all the exercise.

(a) *Complete information.* Let us first examine the case in which the entrant and the incumbent are informed about each others' marginal costs. Consider a two-stage game where, in the first stage, the incumbent has monopoly power and selects an output level, $q$. In the second stage, a potential entrant decides whether or not to enter. If entry occurs, agents compete as Cournot duopolists, simultaneously and independently selecting production levels, $x_1$ and $x_2$. If entry does not occur, the incumbent maintains its monopoly power in both periods (producing $q$ in the first period and $x$ in the second period). Find the subgame perfect equilibrium (SPNE) of this complete information game.

- We next apply backward induction, starting from the second-period game.
- *Second period.* When no entry occurs, the incumbent solves

$$\max_{x_1 \geq 0} (1 - x_1)x_1 - c_1^K x_1$$

thus selecting monopoly output $x_1^{K,m} = \frac{1-c_1^K}{2}$ for every incumbent type $K = \{H, L\}$. If entry occurs, every firm $i = \{1, 2\}$ solves

$$\max_{x_i \geq 0} (1 - x_i - x_j)x_i - c_i^K x_i$$

which, after finding best response functions and simultaneously solving for the incumbent and the entrant's outputs, yields equilibrium output $x_1^{K,d} = \frac{1+c_2-2c_1^K}{3}$ for the incumbent and $x_2^{K,d} = \frac{1-2c_2+c_1^K}{3}$ for the entrant.
- *First period.* Regardless of the entrant's entry decision in the second period, the incumbent selects the standard monopoly output $q^{K,Info} = \frac{1-c_1^K}{2}$ in the first period. This is because the incumbent's output choice in this complete information setting does not affect the entrant's entry decision.

(b) *Incomplete information.* In this section we investigate the case where the incumbent is privately informed about its marginal costs, while the entrant only observes the incumbent's first-period output which the entrant uses as a signal to infer the incumbent's cost. The time structure of this signaling game is as follows:

1. Nature decides the realization of the incumbent's marginal costs, either high or low, with probabilities $p \in (0, 1)$ and $1 - p$, respectively. The incumbent privately observes this realization but the entrant does not.
2. The incumbent chooses its first-period output level, $q$.
3. Observing the incumbent's output decision, the entrant forms beliefs about the incumbent's initial marginal costs. Let $\mu(c_1^H|q)$ denote the entrant's posterior belief about the initial costs being high after observing a particular first-period output from the incumbent $q$.
4. Given the above beliefs, the entrant decides whether or not to enter the industry.
5. If entry does not occur, the incumbent maintains its monopoly power; whereas if entry occurs, both agents compete as Cournot duopolists and the entrant observes the incumbent's type.

(c) Write down the incentive compatibility conditions that must hold for a separating Perfect Bayesian Equilibrium (PBE) to be sustained. Then find the set of separating PBEs.

- In a separating equilibrium in which the high-cost firm selects $q^H$ while the low-cost firm chooses $q^L$ information about the incumbent's type is conveyed to the potential entrant, who responds entering after observing the incumbent producing $q^H$, and does not enter after observing $q^L$. For simplicity, we assume that all other output levels $q \neq q^H \neq q^L$ (i.e., off-the-equilibrium outputs) also lead the entrant to enter the industry. Let us next separately analyze each type of incumbent.

- *High-cost incumbent.* Since, by selecting $q^H$ this type of incumbent attracts entry, this firm selects the output that maximizes its first-period (monopoly) profits, that is, $q^H$ coincides with its output under complete information $q^{H,Info} = \frac{1-c_1^H}{2}$. If, instead, the incumbent deviates towards the low-cost incumbent's output $q^L$, it conceals its type from the entrant and deters entry. Hence, the high-cost incumbent selects its equilibrium output $q^H$ rather than deviating if $M_1^H(q^{H,Info}) + \delta D_1^H \geq M_1^H(q^L) + \delta \overline{M}_1^H$, where

$$M_1^H(q) = (1-q)q - c^H q \quad \text{for every output } q$$

denotes the incumbent's first-period monopoly profits, $D_1^H$ represents second-period duopoly profits when the incumbent's costs are high, and $\overline{M}_1^H$ indicates the second-period monopoly profits for the incumbent (in the case of no entry) when its costs are high. We can now rewrite the above incentive compatibility condition as follows

$$M_1^H(q^{H,Info}) - M_1^H(q^L) \geq \delta \left[ \overline{M}_1^H - D_1^H \right] \qquad (IC_H)$$

(where we grouped first-period profits on the left-hand side, and discounted second-period profits on the right-hand side). For our parameter values, we obtain profits of $M_1^H(q^{H,Info}) = \overline{M}_1^H = \frac{1}{16}$ since $c_1^H = 1/2$, and $D_1^H = \frac{1}{36}$ given that $c_1^H = c_2 = \frac{1}{2}$. Hence, condition $IC_H$ reduces to

$$\frac{1}{16} - \left[ (1-q^L)q^L - \frac{1}{2}q^L \right] \geq \delta \left[ \frac{1}{16} - \frac{1}{36} \right]$$

The difference in first-period profits, $M_1^H(q^{H,Info}) - M_1^H(q^L)$, becomes zero at $q^L = q^{H,Info}$ since at that point $M_1^H(q^{H,Info}) = M_1^H(q^L)$, but otherwise is positive since $M_1^H(q^{H,Info}) > M_1^H(q^L)$ for all $q^L \neq q^{H,Info}$. In contrast, the difference in discounted second-period profits, $\delta \left[ \overline{M}_1^H - D_1^H \right]$, is constant in first-period output $q^L$. Hence, $IC_H$ holds if output $q^L$ lies in the range depicted in the horizontal axis of figure 2.

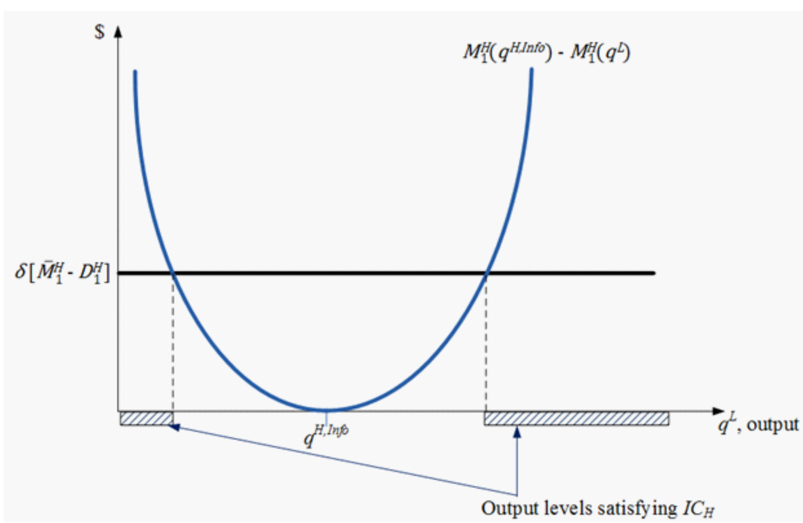


Fig 2. Incentive compatibility condition $IC_H$.

- *Low-cost incumbent.* If the low-cost incumbent chooses the equilibrium output $q^L$, it deters entry. If instead the incumbent deviates towards the high-cost incumbent's output, $q^H$, it attracts entry. Conditional on attracting entry, the low-cost incumbent would select output $q^{L,Info}$, since such output maximizes its first-period profits, yielding $M_1^L(q^{L,Info}) + \delta D_1^L$. Thus, the low-cost incumbent selects its equilibrium output of $q^L$ if $M_1^L(q^{L,Info}) + \delta D_1^L \leq M_1^L(q^L) + \delta \overline{M}_1^L$, or equivalently,

$$M_1^L(q^{L,Info}) - M_1^L(q^L) \leq \delta \left[\overline{M}_1^L - D_1^L\right] \qquad (IC_L)$$

which, for our parameter values, yields $M_1^L(q^{L,Info}) = \overline{M}_1^L = \frac{1}{9}$ and $D_1^L = \frac{25}{324}$ given that $c_1^L = 1/3$ and $c_2 = 1/2$. Hence, condition $IC_L$ reduces to

$$\frac{1}{9} - \left[(1-q^L)q^L - \frac{1}{3}q^L\right] \leq \delta \left[\frac{1}{9} - \frac{25}{324}\right]$$

A similar argument as for $IC_H$ applies to the graphical representation of $IC_L$. As figure 3 illustrates, the curve depicting the difference in first-period profits, $M_1^L(q^{L,Info}) - M_1^L(q^L)$, becomes zero at $q^L = q^{L,Info}$ since at that point $M_1^L(q^{L,Info}) = M_1^L(q^L)$, but otherwise is positive since $M_1^L(q^{L,Info}) > M_1^L(q^L)$ for all $q^L \neq q^{L,Info}$. In contrast, the difference in discounted second-period profits, $\delta\left[\overline{M}_1^L - D_1^L\right]$, is constant in first-period output $q^L$. Hence, $IC_L$ holds if output $q^L$ lies in the range depicted in the horizontal axis of figure 3.

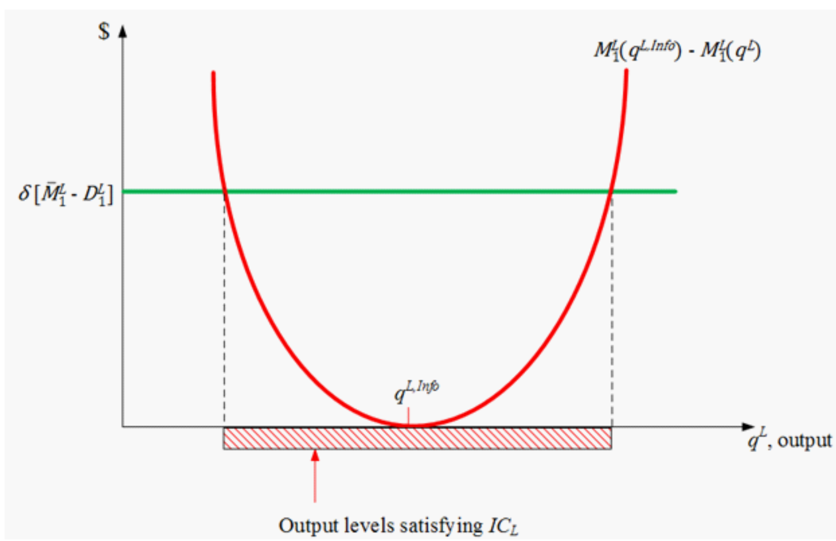


Fig 3. Incentive compatibility condition $IC_L$.

- *Combining both ICs.* Superimposing figures 2 and 3, we can examine the set of output levels that simultaneously satisfy condition $IC_H$ and $IC_L$, as depicted in figure 4. In particular, the overlap between the range of outputs identified in figures 2 and 3 provides us with the set of output levels that constitute a separating PBE, $q^L \in [q^A, q^B]$. The low-cost incumbent increases its first-period output in order to communicate its efficient costs to the potential entrant, deterring entry as a result.

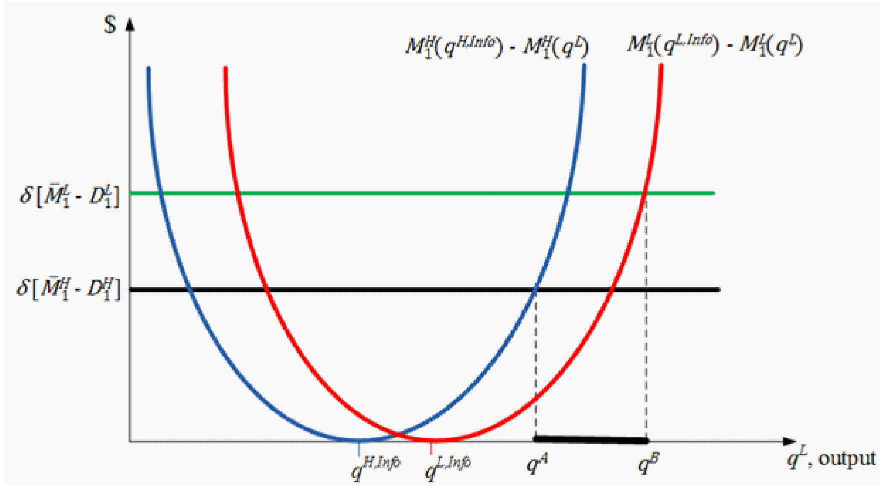

Fig 4. Separating equilibria in the limit pricing model.

In particular, the lower-bound output $q^A$ solves condition $IC_H$ with equality, and the upper-bound output $q^B$ solves $IC_L$ with equality. Rearranging condition $IC_H$, and assuming that there is no discounting, $\delta = 1$, we obtain

$$\frac{1}{36} = (1 - q^L)q^L - \frac{1}{2}q^L$$

which we rearrange to

$$36(q^L)^2 - 18q^L + 1 = 0$$

that simplifies to

$$q^L = \frac{3 - \sqrt{5}}{12} \simeq 0.06 \quad \text{or} \quad q^L = \frac{3 + \sqrt{5}}{12} \simeq 0.44$$

Similarly operating with condition $IC_L$ in order to obtain the upper bound $q^B$, and since there is no discounting, $\delta = 1$, we have

$$\frac{25}{324} = (1 - q^L)q^L - \frac{1}{3}q^L$$

which we rearrange to

$$324(q^L)^2 - 216q^L + 25 = 0$$

that simplifies to

$$q^L = \frac{6 - \sqrt{11}}{18} \simeq 0.15 \quad \text{or} \quad q^L = \frac{6 + \sqrt{11}}{18} \simeq 0.52$$

Hence, the set of separating output levels for the low-cost firm must lie in the interval $q^L \in [0.44, 0.52]$.

(d) Which separating PBEs of those you found in part (b) survive the Cho and Kreps' Intuitive Criterion?

- Starting from the separating PBE in which the low-cost incumbent chooses the highest output level $q^L = q^B$, a deviation toward any output level in $q^L \in \left[q^A, q^B\right)$ can only be profitable for the low-cost incumbent (but not for the high-cost firm). Formally, deviating towards $q^L \in [q^A, q^B)$ is "equilibrium dominated" for the high-cost incumbent alone. Hence, the potential entrant would update its beliefs accordingly, making such a deviation profitable for the low-cost firm. A similar argument applies to all other separating PBEs in the interval $q^L \in (q^A, q^B)$ but not for $q^L = q^A$, the least-costly separating PBE (also known as the "Riley outcome").
- Summarizing, the low-cost incumbent raises its first-period output from $q^{L,Info} = \frac{1-c_1^L}{2} = \frac{1-\frac{1}{3}}{2} = 0.33$, under complete information, to $q_1^A = 0.44$, under the separating equilibrium. Hence, the "separating effort" that this firm must exert in order to reveal its type to the potential entrant (and thus deter entry) is measured by the distance $q^A - q^{L,Info} = 0.44 - 0.33 = 0.11$.

1. **Moral hazard with multiple tasks.**[1] Let us consider a moral hazard problem between a principal and an agent. However, let us now allow the agent to take two effort levels $e_1$ and $e_2$. This represents, for instance, a salesman choosing how much effort to exert visiting potential customers, how much time to spend creating a more attractive website for online sales, investigating new sales strategies, etc. In this exercise we seek to understand how the multidimensionality in the agent's effort affects our results in the standard moral hazard problem analyzed in this chapter.

   Assume that the cost of exerting effort levels $e_1$ and $e_2$ is

   $$c\left(e_1, e_2\right) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2$$

   These effort levels produce output $y$ with output function

   $$y = f_1 e_1 + f_2 e_2 + \varepsilon$$

   with performance $p = g_1 e_1 + g_2 e_2 + \phi$. Random shocks in output, $\varepsilon$, and performance, $\phi$, follow distributions of $G\left(\phi\right)$ and $H\left(\varepsilon\right)$, respectively, with zero expectations, that is, $E(\varepsilon) = E(\phi) = 0$.

   For simplicity, assume that both principal and agent are risk neutral with payoff functions of $\pi = y - w$ for the principal (e.g., firm), where $w$ denotes the salary she pays to the agent; and $U = w - c\left(e_1, e_2\right)$ for the agent (e.g., worker). Consider that the principal offers a salary $w = F + bp$ where $F$ is fixed component of the contract and $b$ is the bonus which provides a higher salary to the agent as his performance $p$ increases. In particular, the timing of the game is as follows:

   - The principal and agent sign a contract $w = F + bp$.
   - The agent takes effort levels $e_1$ and $e_2$ which are unobservable to the principal.

[1] For a more general presentation, see Bolton and Dewatripont (2005), pp. 216-28.

- Random shocks $\varepsilon$ and $\phi$, are realized, affecting the agent's output and performance, respectively.
- Output $y$ and performance $p$ are observed by the principal and agent.
- The agent receives wage $w = F + bp$.

Answer the following questions.

(a) Find the agent's optimal efforts and indirect utility as a function of the bonus parameter $b$.

- The agent solves the following expected utility maximization problem:

$$\begin{aligned}
\max_{e_1,e_2\geq 0} E_A\left[U\left(e_1,e_2\right)\right] &= E_A\left[w - c\left(e_1,e_2\right)\right] \\
&= \int \left[\underbrace{F+bp}_{w} - \underbrace{\left(\frac{1}{2}e_1^2+\frac{1}{2}e_2^2\right)}_{c(e_1,e_2)}\right] dG\left(\phi\right) \\
&= \int \left[F + b\underbrace{(g_1e_1+g_2e_2+\phi)}_{p} - \left(\frac{1}{2}e_1^2+\frac{1}{2}e_2^2\right)\right] dG\left(\phi\right) \\
&= F + b\left(g_1e_1+g_2e_2\right) - \frac{1}{2}e_1^2 - \frac{1}{2}e_2^2 + b\underbrace{\int \phi dG\left(\phi\right)}_{=0 \text{ since } E(\phi)=0} \\
&= F + bg_1e_1 + bg_2e_2 - \frac{1}{2}e_1^2 - \frac{1}{2}e_2^2
\end{aligned}$$

Taking first-order conditions with respect to $e_1$ and $e_2$, we obtain

$$\frac{\partial E_A\left[U\left(e_1,e_2\right)\right]}{\partial e_1} = bg_1 - e_1 = 0$$
$$\frac{\partial E_A\left[U\left(e_1,e_2\right)\right]}{\partial e_2} = bg_2 - e_2 = 0$$

Assuming interior solutions ($e_1 > 0$ and $e_2 > 0$), the agent's optimal efforts are

$$e_1\left(b\right) = bg_1$$
$$e_2\left(b\right) = bg_2$$

- *Indirect utility function.* Substituting the agent's optimal efforts back into his utility function, yields

$$\begin{aligned}
U\left(b,F\right) &= F + bg_1\cdot\underbrace{bg_1}_{e_1(b)} + bg_2\cdot\underbrace{bg_2}_{e_2(b)} - \frac{1}{2}\underbrace{[bg_1]}_{e_1(b)}^2 - \frac{1}{2}\underbrace{[bg_2]}_{e_2(b)}^2 \\
&= F + \left(bg_1\right)^2 + \left(bg_2\right)^2 - \frac{1}{2}\left(bg_1\right)^2 - \frac{1}{2}\left(bg_2\right)^2 \\
&= F + \frac{1}{2}\left(bg_1\right)^2 + \frac{1}{2}\left(bg_2\right)^2
\end{aligned}$$

(b) Find the principal's optimal contract $w^*$ and his equilibrium profits.

- Operating by backwards induction, the principal anticipates the equilibrium effort levels that the agent chooses in the second stage of the game. Then, the principal solves the following expected profit maximization problem:

$$\begin{aligned}
\max_{b \geq 0} E_P [\pi (b)] &= E_P [y - w] \\
&= E_P \left[ \underbrace{f_1 e_1 (b) + f_2 e_2 (b) + \varepsilon}_{y} - \underbrace{(F + bp)}_{w} \right] \\
&= (f_1 - bg_1) \underbrace{e_1 (b)}_{bg_1} + (f_2 - bg_2) \underbrace{e_2 (b)}_{bg_2} - F + \underbrace{\int \varepsilon dH (\varepsilon)}_{=0 \text{ since } E(\varepsilon)=0} \\
&= bg_1 (f_1 - bg_1) + bg_2 (f_2 - bg_2) - F
\end{aligned}$$

Taking first-order condition with respect to the bonus $b$, we find

$$\begin{aligned}
\frac{\partial E_P [\pi (b), F]}{\partial b} &= g_1 (f_1 - bg_1) - bg_1^2 + g_2 (f_2 - bg_2) - bg_2^2 \\
&= g_1 (f_1 - 2bg_1) + g_2 (f_2 - 2bg_2)
\end{aligned}$$

Assuming interior solutions, that is, $b > 0$, the bonus $b^*$ satisfies

$$b^* = \frac{f_1 g_1 + f_2 g_2}{2 (g_1^2 + g_2^2)}$$

- *Equilibrium profits.* Substituting the principal's optimal contract back into the profit function, we obtain

$$\begin{aligned}
\pi (b^*, F) &= b^* g_1 (f_1 - b^* g_1) + b^* g_2 (f_2 - b^* g_2) - F \\
&= \frac{f_1 g_1 + f_2 g_2}{2 (g_1^2 + g_2^2)} (f_1 g_1 + f_2 g_2) - \left( \frac{f_1 g_1 + f_2 g_2}{2 (g_1^2 + g_2^2)} \right)^2 (g_1^2 + g_2^2) - F \\
&= \frac{(f_1 g_1 + f_2 g_2)^2}{2 (g_1^2 + g_2^2)} - \frac{(f_1 g_1 + f_2 g_2)^2}{4 (g_1^2 + g_2^2)} - F \\
&= \frac{(f_1 g_1 + f_2 g_2)^2}{4 (g_1^2 + g_2^2)} - F
\end{aligned}$$

  Inspecting the principal's equilibrium profit above, we notice that any fixed wage $F > 0$ reduces her profit, without affecting the agent's optimal efforts; which do not depend on $F$, as shown in part (a) of the exercise. Therefore, the principal should set the optimal fixed wage at $F^* = 0$.

- As a result, the optimal contract becomes

$$\begin{aligned}
w^* &= F^* + b^* p \\
&= \frac{f_1 g_1 + f_2 g_2}{2 (g_1^2 + g_2^2)} p
\end{aligned}$$

which depends on the random shock $\phi$. Specifically, when a favorable shock $\phi > 0$ is realized, the agent outperforms so that the principal pays a higher wage to him. On the other hand, when an unfavorable shock $\phi < 0$ is realized, the agent underperforms so that the principal pays a lower wage to him. Therefore, in expectation (that is, at stage 1 of the game, before the shocks are realized), the expected wage is

$$
\begin{aligned}
E[w^*] &= b\frac{f_1 g_1 + f_2 g_2}{2(g_1^2 + g_2^2)}\left(g_1^2 + g_2^2\right) + \frac{f_1 g_1 + f_2 g_2}{2(g_1^2 + g_2^2)}\underbrace{E[p]}_{=0} \\
&= \frac{(f_1 g_1 + f_2 g_2)^2}{4(g_1^2 + g_2^2)}
\end{aligned}
$$

(c) *Comparative Statics.* How is the optimal contract you found in part (b) affected by the output rates $f_1$ and $f_2$? How is it affected by the performance rates $g_1$ and $g_2$? Explain.

- Differentiating the optimal wage with respect to $f_1$ and $f_2$, we find

$$
\begin{aligned}
\frac{\partial w^*}{\partial f_1} &= \frac{g_1(f_1 g_1 + f_2 g_2)}{2(g_1^2 + g_2^2)} > 0 \\
\frac{\partial w^*}{\partial f_2} &= \frac{g_2(f_1 g_1 + f_2 g_2)}{2(g_1^2 + g_2^2)} > 0
\end{aligned}
$$

  which means that as either effort level becomes more effective in producing output, wage payment increases.

- Differentiating the wage contract with respect to $g_1$ and $g_2$, yields

$$
\begin{aligned}
\frac{\partial w^*}{\partial g_1} &= \frac{f_1(f_1 g_1 + f_2 g_2)(g_1^2 + g_2^2) - g_1(f_1 g_1 + f_2 g_2)^2}{2(g_1^2 + g_2^2)^2} \\
&= \frac{g_2(f_1 g_1 + f_2 g_2)(f_1 g_2 - f_2 g_1)}{2(g_1^2 + g_2^2)^2} \\
\frac{\partial w^*}{\partial g_2} &= \frac{f_2(f_1 g_1 + f_2 g_2)(g_1^2 + g_2^2) - g_2(f_1 g_1 + f_2 g_2)^2}{2(g_1^2 + g_2^2)^2} \\
&= \frac{g_1(f_1 g_1 + f_2 g_2)(f_2 g_1 - f_1 g_2)}{2(g_1^2 + g_2^2)^2}
\end{aligned}
$$

  Therefore, if $\frac{f_1}{f_2} > \frac{g_1}{g_2}$, we obtain that $\frac{\partial w^*}{\partial g_1} > 0$ and $\frac{\partial w^*}{\partial g_2} < 0$. Intuitively, if effort 1 is more effective in generating output than in delivering performance, relatively to effort 2, the optimal wage increases in the performance rate of effort 1, $g_1$, but decrease in that of effort 2, $g_2$. The opposite holds if $\frac{f_1}{f_2} < \frac{g_1}{g_2}$ such that $\frac{\partial w^*}{\partial g_1} < 0$ and $\frac{\partial w^*}{\partial g_2} > 0$. In this case, the optimal wage increases in the performance rate of effort 2, $g_2$, but decreases in that of effort 1, $g_1$.

(d) Given the optimal contract found above, what are the principal's expected payoff, the agent's expected utility, and the expected social welfare in equilibrium?

- Substituting the optimal contract $w^*$ into the agent's indirect utility function, we find

$$\begin{aligned} U(w^*) &= F^* + \frac{1}{2}(b^* g_1)^2 + \frac{1}{2}(b^* g_2)^2 \\ &= 0 + \frac{1}{2}\left(\frac{f_1 g_1 + f_2 g_2}{2(g_1^2 + g_2^2)} g_1\right)^2 + \frac{1}{2}\left(\frac{f_1 g_1 + f_2 g_2}{2(g_1^2 + g_2^2)} g_2\right)^2 \\ &= \frac{1}{8}\left(\frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}\right)^2 (g_1^2 + g_2^2) \\ &= \frac{(f_1 g_1 + f_2 g_2)^2}{8(g_1^2 + g_2^2)} \end{aligned}$$

- Similarly, substituting the optimal contract $w^*$ into the principal's indirect utility function, we obtain

$$\pi(w^*) = \frac{(f_1 g_1 + f_2 g_2)^2}{4(g_1^2 + g_2^2)}$$

- Therefore, the expected social welfare is given by the sum of the principal's expected profit and the agent's expected utility in equilibrium.

$$\begin{aligned} SW^* &= \pi(w^*) + U(w^*) \\ &= \frac{(f_1 g_1 + f_2 g_2)^2}{4(g_1^2 + g_2^2)} + \frac{(f_1 g_1 + f_2 g_2)^2}{8(g_1^2 + g_2^2)} \\ &= \frac{3(f_1 g_1 + f_2 g_2)^2}{8(g_1^2 + g_2^2)} \end{aligned}$$

(e) What is the socially optimal contract? Compare it against the contract that emerges in the subgame perfect equilibrium of the game you found in part (b).

- The expected social welfare for both the principal and the agent is given by the sum

$$\begin{aligned} SW &= E[\pi(b,F) + U(b,F)] \\ &= \underbrace{[bg_1(f_1 - bg_1) + bg_2(f_2 - bg_2) - F]}_{E[\pi(b,F)]} + \underbrace{\left[F + \frac{1}{2}(bg_1)^2 + \frac{1}{2}(bg_2)^2\right]}_{E[U(b,F)]} \\ &= bf_1 g_1 - b^2 g_1^2 + bf_2 g_2 - b^2 g_2^2 + \frac{1}{2}b^2 g_1^2 + \frac{1}{2}b^2 g_2^2 \\ &= b(f_1 g_1 + f_2 g_2) - \frac{1}{2}(bg_1)^2 - \frac{1}{2}(bg_2)^2 \end{aligned}$$

  where the first line is operating by both parties maximizing joint payoffs *as if* efforts are observable; and because the fixed wage, $F$, which is an action-independent transfer from the principal to the agent, is cancelled out, we can set $F^{**} = 0$ without affecting the expected social welfare.

- Taking first-order condition with respect to $b$, yields

$$\frac{\partial SW}{\partial b} = f_1 g_1 + f_2 g_2 - b g_1^2 - b g_2^2$$

Assuming interior solutions, that is, $b > 0$, the socially optimal bonus $b^{**}$ becomes

$$b^{**} = \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2} \tag{1}$$

- Comparing with the equilibrium bonus found part (b), $b^* = \frac{f_1 g_1 + f_2 g_2}{2\left(g_1^2 + g_2^2\right)}$, we see that $b^{**} = 2b^*$. In words, this result indicates that effort unobservability (in part b) reduces the bonus $b$ by half. Intuitively, as the principal cannot observe the effort levels chosen by the agent, the principal believes that the observed performance can be a matter of luck (i.e., due to random shocks) other than the efforts exerted by the agent; and therefore, she reduces the variable wage (i.e., bonus) to the agent.

(f) What is the deadweight loss in this contractual setting?

- Substituting the socially optimal bonus found in part (e), $b^{**}$, into the social welfare function, we obtain

$$\begin{aligned}
SW^{**} &= E\left[\pi\left(b^{**}, F\right) + U\left(b^{**}, F\right)\right] \\
&= \underbrace{\left[b^{**} g_1 \left(f_1 - b g_1\right) + b^{**} g_2 \left(f_2 - b g_2\right) - F\right]}_{E[\pi(b,F)]} + \underbrace{\left[F + \frac{1}{2}\left(b^{**} g_1\right)^2 + \frac{1}{2}\left(b^{**} g_2\right)^2\right]}_{E[U(b,F)]} \\
&= b^{**} f_1 g_1 - \left(b^{**}\right)^2 g_1^2 + b^{**} f_2 g_2 - \left(b^{**}\right)^2 g_2^2 + \frac{1}{2}\left(b^{**}\right)^2 g_1^2 + \frac{1}{2}\left(b^{**}\right)^2 g_2^2 \\
&= b^{**}\left(f_1 g_1 + f_2 g_2\right) - \frac{1}{2}\left(b^{**} g_1\right)^2 - \frac{1}{2}\left(b^{**} g_2\right)^2 \\
&= \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}\left(f_1 g_1 + f_2 g_2\right) - \frac{1}{2}\left(\frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}\right)^2 \left(g_1^2 + g_2^2\right)^2 \\
&= \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{2\left(g_1^2 + g_2^2\right)}
\end{aligned}$$

Therefore, the deadweight loss is

$$\begin{aligned}
DWL &= SW^{**} - SW^* \\
&= \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{2\left(g_1^2 + g_2^2\right)} - \frac{3\left(f_1 g_1 + f_2 g_2\right)^2}{8\left(g_1^2 + g_2^2\right)} \\
&= \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{8\left(g_1^2 + g_2^2\right)}.
\end{aligned}$$

(g) *Numerical example.* Consider output rates $f_1 = \frac{1}{2}$ and $f_2 = \frac{1}{3}$, and performance rates $g_1 = \frac{2}{3}$ and $g_2 = \frac{1}{4}$. In this context, evaluate the equilibrium bonus $b^*$, efforts $e_1^*$ and $e_2^*$, wage $w^*$, the agent's expected utility $U\left(w^*\right)$, the principal's expected profit $\pi\left(w^*\right)$, and social welfare $SW^*$. Then, evaluate the socially optimal bonus $b^{**}$, social welfare $SW^{**}$ at $b^{**}$, and deadweight loss due to unobservability of effort.

- *Equilibrium outcomes.* Evaluating the equilibrium bonus, $b^* = \frac{f_1 g_1 + f_2 g_2}{2\left(g_1^2+g_2^2\right)}$, found in part (b) of the exercise at the above parameter values, we obtain

$$b^* = \frac{\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}}{2\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)} = \frac{30}{73}$$

Evaluating equilibrium efforts, $e_1(b^*) = b^* g_1$ and $e_2(b^*) = b^* g_2$, found in same part of the exercise, we obtain

$$e_1^* = \frac{20}{73}$$
$$e_2^* = \frac{15}{146}$$

Evaluating equilibrium wage, $w^* = F^* + b^* p = 0 + b^* p$, at the above parameter values, we obtain

$$w^* = \frac{30}{73} p$$

as a function of the performance $p$ which in turn depends on the random shock $\phi$, such that expected wage of the agent, $E[w^*] = \frac{(f_1 g_1 + f_2 g_2)^2}{4\left(g_1^2+g_2^2\right)}$, at stage 1 of the game (i.e., before the shocks are realized) found in part (b) of the exercise becomes

$$E[w^*] = \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{4\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)} = \frac{25}{292} \simeq 0.08$$

Therefore, expected utility of the agent, $U(w^*) = \frac{(f_1 g_1 + f_2 g_2)^2}{8\left(g_1^2+g_2^2\right)}$, found in part (d) of the exercise becomes

$$U^* = \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{8\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)} = \frac{25}{584} \simeq 0.04$$

and expected profit of the principal, $\pi(w^*) = \frac{(f_1 g_1 + f_2 g_2)^2}{4\left(g_1^2+g_2^2\right)}$, which is also found in part (d) becomes

$$\pi^* = \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{4\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)} = \frac{25}{292} \simeq 0.08$$

which implies a social welfare of $SW^* = \pi^* + U^* = \frac{75}{584} \simeq 0.12$.

- *Socially optimal outcomes.* Evaluating the socially optimal bonus $b^{**} = \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}$ we found in part (e) of the exercise at the above parameter values, we obtain

$$\begin{aligned} b^{**} &= \frac{\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}}{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2} \\ &= \frac{60}{73} \simeq 0.82 \end{aligned}$$

and social welfare, $SW^{**} = \frac{(f_1 g_1 + f_2 g_2)^2}{2\left(g_1^2 + g_2^2\right)}$, evaluated at bonus $b^{**}$, as found in part (e) of the exercise, is

$$\begin{aligned} SW^{**} &= \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{2\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)} \\ &= \frac{25}{146} \simeq 0.17 \end{aligned}$$

- *Comparison.* Deadweight loss, $DWL = \frac{(f_1 g_1 + f_2 g_2)^2}{8\left(g_1^2 + g_2^2\right)}$, as found in part (f) of the exercise, is

$$\begin{aligned} DWL &= \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{8\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)} \\ &= \frac{25}{584} \simeq 0.04 \end{aligned}$$

2. **Stone-Geary utility function in pure exchange economy.** Consider a pure exchange economy with two individuals, $A$ and $B$, whose utility functions are

$$\begin{aligned} u^A\left(x_1^A, x_2^A\right) &= \left(x_1^A - b_1\right)^{\frac{1}{2}}\left(x_2^A - b_2\right)^{\frac{1}{2}} \\ u^B\left(x_1^B, x_2^B\right) &= x_1^B x_2^B \end{aligned}$$

where $b_1, b_2 > 0$ represent the minimal amounts of goods 1 and 2 that individual $A$ must consume in order to remain alive (such as water and shelter). Individuals $A$ and $B$ have endowments of $\omega^A = \left(\omega_1^A, \omega_2^A\right) = (4, 2)$ and $\omega^B = \left(\omega_1^B, \omega_2^B\right) = (2, 4)$, respectively.

(a) Set up the Lagrangian and find the individuals' Walrasian demand functions.

- *UMP of individual $A$.* Individual $A$ chooses $x_1^A$ and $x_2^A$ to solve the following utility maximization problem,

$$\max_{x_1^A \geq b_1, x_2^A \geq b_2} u^A\left(x_1^A, x_2^A\right) = \left(x_1^A - b_1\right)^{\frac{1}{2}}\left(x_2^A - b_2\right)^{\frac{1}{2}}$$

$$\text{subject to } p_1 x_1^A + p_2 x_2^A = 4p_1 + 2p_2$$

Defining $\tilde{x}_1^A \equiv x_1^A - b_1$ and $\tilde{x}_2^A \equiv x_2^A - b_2$, which represent the above-subsistence consumption levels of individual $A$, we can rewrite his budget constraint as

$$p_1\tilde{x}_1^A + p_2\tilde{x}_2^A = p_1(4-b_1) + p_2(2-b_2)$$

Therefore, the Lagrangian function of individual $A$ becomes

$$L_A = \left(\tilde{x}_1^A\right)^{\frac{1}{2}}\left(\tilde{x}_2^A\right)^{\frac{1}{2}} + \lambda_A\left[p_1(4-b_1) + p_2(2-b_2) - p_1\tilde{x}_1^A - p_2\tilde{x}_2^A\right]$$

The first order conditions of individual $A$'s Lagrangian are

$$\begin{aligned}
\frac{\partial L_A}{\partial \tilde{x}_1^A} &= \frac{1}{2}\left(\frac{\tilde{x}_2^A}{\tilde{x}_1^A}\right)^{\frac{1}{2}} - \lambda_A p_1 \leq 0 \\
\frac{\partial L_A}{\partial \tilde{x}_2^A} &= \frac{1}{2}\left(\frac{\tilde{x}_1^A}{\tilde{x}_2^A}\right)^{\frac{1}{2}} - \lambda_A p_2 \leq 0 \\
\frac{\partial L_A}{\partial \lambda_A} &= p_1(4-b_1) + p_2(2-b_2) - p_1\tilde{x}_1^A - p_2\tilde{x}_2^A \geq 0
\end{aligned}$$

with the associated Kuhn-Tucker conditions of

$$\begin{aligned}
\tilde{x}_1^A\frac{\partial L_A}{\partial \tilde{x}_1^A} &= 0 \\
\tilde{x}_2^A\frac{\partial L_A}{\partial \tilde{x}_2^A} &= 0 \\
\lambda_A\frac{\partial L_A}{\partial \lambda_A} &= 0
\end{aligned}$$

Assuming interior solutions, the first order conditions hold with equality, so that by equating $\frac{\partial L_A}{\partial \tilde{x}_1^A} = \frac{\partial L_A}{\partial \tilde{x}_2^A} = 0$, we obtain

$$\frac{\frac{1}{2}\left(\frac{\tilde{x}_2^A}{\tilde{x}_1^A}\right)^{\frac{1}{2}}}{\frac{1}{2}\left(\frac{\tilde{x}_1^A}{\tilde{x}_2^A}\right)^{\frac{1}{2}}} = \frac{\lambda_A p_1}{\lambda_A p_2}$$

which, after rearranging, yields

$$p_1\tilde{x}_1^A = p_2\tilde{x}_2^A$$

Substituting $p_1\tilde{x}_1^A = p_2\tilde{x}_2^A$ into the budget constraint, we have

$$2p_1\tilde{x}_1^A = p_1(4-b_1) + p_2(2-b_2)$$

which is rearranged to give individual $A$'s Walrasian demand of good 1,

$$\tilde{x}_1^A = \frac{4-b_1}{2} + \frac{p_2(2-b_2)}{2p_1}$$

and, similarly, we can obtain individual $A$'s Walrasian demand of good 2,

$$\tilde{x}_2^A = \frac{p_1(4-b_1)}{2p_2} + \frac{2-b_2}{2}$$

- *UMP of individual B.* Individual $B$ chooses $x_1^B$ and $x_2^B$ to solve the following utility maximization problem,

$$\max_{x_1^B, x_2^B \geq 0} u^B\left(x_1^B, x_2^B\right) = x_1^B x_2^B$$

subject to

$$p_1 x_1^B + p_2 x_2^B = 2p_1 + 4p_2$$

The Lagrangian function of individual $B$ becomes

$$L_B = x_1^B x_2^B + \lambda_B\left[2p_1 + 4p_2 - p_1 x_1^B - p_2 x_2^B\right]$$

The first order conditions of individual $B$'s Lagrangian are

$$\frac{\partial L_B}{\partial x_1^B} = x_2^B - \lambda_B p_1 \leq 0$$
$$\frac{\partial L_B}{\partial x_2^B} = x_1^B - \lambda_B p_2 \leq 0$$
$$\frac{\partial L_B}{\partial \lambda_B} = 2p_1 + 4p_2 - p_1 x_1^B - p_2 x_2^B \geq 0$$

with the associated Kuhn-Tucker conditions of

$$x_1^B \frac{\partial L_B}{\partial x_1^B} = 0$$
$$x_2^B \frac{\partial L_B}{\partial x_2^B} = 0$$
$$\lambda_B \frac{\partial L_B}{\partial \lambda_B} = 0$$

Assuming interior solutions, the first order conditions hold with equality, so that by equating $\frac{\partial L_B}{\partial x_1^B} = \frac{\partial L_B}{\partial x_2^B} = 0$, we obtain

$$\frac{x_2^B}{x_1^B} = \frac{\lambda_B p_1}{\lambda_B p_2}$$

which, after rearranging, yields

$$p_1 x_1^B = p_2 x_2^B$$

Substituting $p_1 x_1^B = p_2 x_2^B$ into the budget constraint, we have

$$2p_1 x_1^B = 2p_1 + 4p_2$$

which is rearranged to give individual $B$'s Walrasian demand of good 1,

$$x_1^B = 1 + 2\frac{p_2}{p_1}$$

and, similarly, we can obtain individual $B$'s Walrasian demand of good 2,

$$x_2^B = \frac{p_1}{p_2} + 2$$

(b) Find the set of Pareto efficient allocations (PEAs). (*Hint*: Your answer should be in terms of $b_1$ and $b_2$).

- The feasibility constraints in this pure exchange economy are

$$\underbrace{\left(\tilde{x}_1^A + b_1\right)}_{=x_1^A} + x_1^B = 4 + 2$$
$$\underbrace{\left(\tilde{x}_2^A + b_2\right)}_{=x_2^A} + x_2^B = 2 + 4$$

which are rearranged to give

$$\tilde{x}_1^A = 6 - b_1 - x_1^B$$
$$\tilde{x}_2^A = 6 - b_2 - x_2^B$$

The contract curve, which defines the set of Pareto efficient allocations, is the locus of tangency of indifference curves between individuals $A$ and $B$, satisfying

$$MRS_{12}^A = \frac{MU_1^A}{MU_2^A} = \frac{MU_1^B}{MU_2^B} = MRS_{12}^B$$

which is rearranged to give

$$\frac{\tilde{x}_2^A}{\tilde{x}_1^A} = \frac{x_2^B}{x_1^B}$$

Substituting the feasibility constraints into the above expression, we obtain

$$\frac{6 - b_2 - x_2^B}{6 - b_1 - x_1^B} = \frac{x_2^B}{x_1^B}$$

which, after rearranging, yields the contract curve as follows,

$$x_2^B = \frac{6 - b_2}{6 - b_1} x_1^B$$

(c) Find the Walrasian equilibrium allocation (WEA). (*Hint*: Your answer should be in terms of $b_1$ and $b_2$).

- Substituting the Walrasian demands for good 1 into $\tilde{x}_1^A + x_1^B = 6 - b_1$, we obtain

$$\underbrace{\left(\frac{4 - b_1}{2} + \frac{p_2\left(2 - b_2\right)}{2p_1}\right)}_{=\tilde{x}_1^A} + \underbrace{\left(1 + 2\frac{p_2}{p_1}\right)}_{=x_1^B} = 6 - b_1$$

which is rearranged to yield the equilibrium price ratio, as follows.

$$\left(6 - b_2\right)\frac{p_2}{p_1} + \left(6 - b_1\right) = 2\left(6 - b_1\right)$$
$$\Rightarrow \frac{p_1}{p_2} = \frac{6 - b_2}{6 - b_1}$$

Substituting $\frac{p_1}{p_2} = \frac{6-b_2}{6-b_1}$ into the Walrasian demand functions of individual $A$, the Walrasian equilibrium allocation (WEA) of this individual becomes

$$\begin{aligned}
\tilde{x}_1^A &= \frac{4-b_1}{2} + \frac{(2-b_2)}{2} \cdot \frac{6-b_1}{6-b_2} \\
&= \frac{(4-b_1)(6-b_2) + (6-b_1)(2-b_2)}{2(6-b_2)} \\
&= \frac{18-4b_1-5b_2+b_1b_2}{6-b_2} \\
\tilde{x}_2^A &= \frac{(4-b_1)}{2} \cdot \frac{6-b_2}{6-b_1} + \frac{2-b_2}{2} \\
&= \frac{(4-b_1)(6-b_2) + (6-b_1)(2-b_2)}{2(6-b_1)} \\
&= \frac{18-4b_1-5b_2+b_1b_2}{6-b_1}
\end{aligned}$$

Given $\tilde{x}_1^A \equiv x_1^A - b_1$ and $\tilde{x}_2^A \equiv x_2^A - b_2$, we can rewrite the above expressions as

$$\begin{aligned}
x_1^A &= \tilde{x}_1^A + b_1 \\
&= \frac{18-4b_1-5b_2+b_1b_2}{6-b_2} + b_1 \\
&= \frac{18+2b_1-5b_2}{6-b_2} \\
x_2^A &= \tilde{x}_2^A + b_2 \\
&= \frac{18-4b_1-5b_2+b_1b_2}{6-b_1} + b_2 \\
&= \frac{18-4b_1+b_2}{6-b_1}
\end{aligned}$$

Similarly, the Walrasian equilibrium allocation (WEA) of individual $B$ is

$$\begin{aligned}
x_1^B &= 1 + 2 \cdot \frac{6-b_1}{6-b_2} \\
&= \frac{18-2b_1-b_2}{6-b_2} \\
x_2^B &= \frac{6-b_2}{6-b_1} + 2 \\
&= \frac{18-2b_1-b_2}{6-b_1}
\end{aligned}$$

(d) Evaluate the contract curve and WEA at the following three different subsistence levels: (i) $(b_1, b_2) = (4, 2)$, (ii) $(b_1, b_2) = (3, 3)$, and (iii) $(b_1, b_2) = (2, 4)$. In which case(s) is individual $A$ unable to survive?

- *First case.* Substituting $(b_1, b_2) = (4, 2)$ into the Walrasian equilibrium allocation,

$$\begin{aligned}
x_1^{A*} &= \frac{18 + 2 \cdot 4 - 5 \cdot 2}{6 - 2} = 4 \\
x_2^{A*} &= \frac{18 - 4 \cdot 4 + 2}{6 - 4} = 2 \\
x_1^{B*} &= \frac{18 - 2 \cdot 4 - 2}{6 - 2} = 2 \\
x_2^{B*} &= \frac{18 - 2 \cdot 4 - 2}{6 - 4} = 4 \\
\frac{p_1}{p_2} &= \frac{6 - 2}{6 - 4} = 2
\end{aligned}$$

  Summarizing, the WEA of

$$\left( x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2} \right) = (4, 2; 2, 4; 2)$$

  which means that individuals do not exchange their goods, and individual $A$ can survive by consuming endowment $\omega^A$. The contract curve in this context is

$$x_2^B = \frac{6 - 2}{6 - 4} x_1^B = 2x_1^B$$

- *Second case.* Substituting $(b_1, b_2) = (3, 3)$ into the Walrasian equilibrium allocation, we find

$$\begin{aligned}
x_1^{A*} &= \frac{18 + 2 \cdot 3 - 5 \cdot 3}{6 - 3} = 3 \\
x_2^{A*} &= \frac{18 - 4 \cdot 3 + 3}{6 - 3} = 3 \\
x_1^{B*} &= \frac{18 - 2 \cdot 3 - 3}{6 - 3} = 3 \\
x_2^{B*} &= \frac{18 - 2 \cdot 3 - 3}{6 - 3} = 3 \\
\frac{p_1}{p_2} &= \frac{6 - 3}{6 - 3} = 1
\end{aligned}$$

  Intuitively, individual $A$ ($B$) exchanges 1 unit of good 1 (2) for 1 unit of good 2 (1) to yield the WEA

$$\left( x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2} \right) = (3, 3; 3, 3; 1),$$

  such that individual $A$ can remain alive with this trade. The contract curve in this setting is

$$x_2^B = \frac{6 - 3}{6 - 3} x_1^B = x_1^B$$

- *Third case.* Substituting $(b_1, b_2) = (2, 4)$ into the Walrasian equilibrium allocation, we obtain

$$
\begin{aligned}
x_1^{A*} &= \frac{18 + 2 \cdot 2 - 5 \cdot 4}{6 - 4} = 1 \\
x_2^{A*} &= \frac{18 - 4 \cdot 2 + 4}{6 - 2} = \frac{7}{2} \\
x_1^{B*} &= \frac{18 - 2 \cdot 2 - 4}{6 - 4} = 5 \\
x_2^{B*} &= \frac{18 - 2 \cdot 2 - 4}{6 - 2} = \frac{5}{2} \\
\frac{p_1}{p_2} &= \frac{6 - 4}{6 - 2} = \frac{1}{2}
\end{aligned}
$$

  Summarizing, the WEA is

$$
\left( x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2} \right) = (1, 3.5; 5, 2.5; 0.5)
$$

  It is easy to check that, at this allocation, individual $A$'s utility is negative, entailing that he cannot survive. In part (e) of the exercise, we examine a wealth redistribution program to keep this individual alive.
  The contract curve in this context is

$$
x_2^B = \frac{6 - 4}{6 - 2} x_1^B = \frac{1}{2} x_1^B
$$

(e) Consider now a tax transfer so individual $A$ survives in the case(s) you identify in part (b) where he suffers from a negative utility at the WEA. Identify the tax/transfer that the government can impose, and the resulting WEA. (For compactness, let us normalize $p_2 = 1$ so that $p \equiv p_1 = \frac{p_1}{p_2}$.)

- Suppose the government levies a tax $t$ on individual $B$ to provide it to individual $A$ as a transfer. In this context, the budget constraint of individual $A$ becomes

$$
p\tilde{x}_1^A + \tilde{x}_2^A = p(4 - b_1) + (2 - b_2) + t
$$

  Substituting $p\tilde{x}_1^A = \tilde{x}_2^A$ and the price ratio $p = \frac{6 - b_2}{6 - b_1}$ into the budget constraint of individual $A$, we obtain

$$
\begin{aligned}
\tilde{x}_1^A &= \frac{4 - b_1}{2} + \frac{2 - b_2 + t}{2p} \\
&= \frac{4 - b_1}{2} + \frac{2 - b_2 + t}{2} \cdot \frac{6 - b_1}{6 - b_2} \\
&= \frac{2(18 - 4b_1 - 5b_2 + b_1 b_2) + (6 - b_1)t}{2(6 - b_2)} \\
\tilde{x}_2^A &= \frac{p(4 - b_1)}{2} + \frac{2 - b_2 + t}{2} \\
&= \frac{4 - b_1}{2} \cdot \frac{6 - b_2}{6 - b_1} + \frac{2 - b_2 + t}{2} \\
&= \frac{2(18 - 4b_1 - 5b_2 + b_1 b_2) + (6 - b_1)t}{2(6 - b_1)}
\end{aligned}
$$

Substituting the subsistence level of the third case we analyzed in part (d) of the exercise, $(b_1, b_2) = (2, 4)$, into the above expressions, yields

$$\tilde{x}_1^A = \frac{2 \cdot (-2) + 4t}{4} = t - 1$$
$$\tilde{x}_2^A = \frac{2 \cdot (-2) + 4t}{8} = \frac{t-1}{2}$$

Therefore, to ensure individual $A$ can remain alive, we need

$$\tilde{x}_1^A \geq 0$$
$$\tilde{x}_2^A \geq 0$$

which is equivalent to

$$t = 1$$

Therefore, the equilibrium allocation of individual $A$ is

$$x_1^{A*} = \tilde{x}_1^A + b_1 = 0 + 2 = 2$$
$$x_2^{A*} = \tilde{x}_2^A + b_2 = 0 + 4 = 4$$

- The budget constraint of individual $B$ becomes now

$$px_1^B + x_2^B = 2p_1 + 4p_2 - t$$

Substituting $px_1^B = x_2^B$ into the budget constraint of individual $B$, we have

$$x_1^B = 1 + \frac{4-t}{2p}$$
$$x_2^B = p + \frac{4-t}{2}$$

Further substituting $p = \frac{1}{2}$ and $t = 1$ into the above expressions, we obtain the equilibrium allocation of individual $B$, as follows

$$x_1^{B*} = 1 + \frac{4-1}{2 \cdot \frac{1}{2}} = 4$$
$$x_2^{B*} = \frac{1}{2} + \frac{4-1}{2} = 2$$

Therefore, the Walrasian equilibrium allocation (WEA) becomes

$$\left(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2}\right) = (2, 4; 4, 2; 0.5)$$

which is supported by a tax-transfer, $t^* = 1$, from individual $B$ to individual $A$.