EconS 503 - Microeconomic Theory II
Homework #6 - Due date: April 22nd, in class.

1. **Exercises from Espinola and Munoz's *Game Theory* (2023):**
   - Chapter 12, exercises 12.2 and 12.6.

2. **Moral hazard with multiple tasks.**[1] Let us consider a moral hazard problem between a principal and an agent. However, let us now allow the agent to take two effort levels e_1 and e_2. This represents, for instance, a salesman choosing how much effort to exert visiting potential customers, how much time to spend creating a more attractive website for online sales, investigating new sales strategies, etc. In this exercise we seek to understand how the multidimensionality in the agent's effort affects our results in the standard moral hazard problem analyzed in this chapter.

   Assume that the cost of exerting effort levels e_1 and e_2 is

$$c(e_1, e_2) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2$$

   These effort levels produce output y with output function

$$y = f_1 e_1 + f_2 e_2 + \varepsilon$$

   with performance $p = g_1 e_1 + g_2 e_2 + \phi$. Random shocks in output, ε, and performance, ϕ, follow distributions of $G(\phi)$ and $H(\varepsilon)$, respectively, with zero expectations, that is, $E(\varepsilon) = E(\phi) = 0$.

   For simplicity, assume that both principal and agent are risk neutral with payoff functions of $\pi = y - w$ for the principal (e.g., firm), where w denotes the salary she pays to the agent; and $U = w - c(e_1, e_2)$ for the agent (e.g., worker). Consider that the principal offers a salary $w = F + bp$ where F is fixed component of the contract and b is the bonus which provides a higher salary to the agent as his performance p increases. In particular, the timing of the game is as follows:

   - The principal and agent sign a contract $w = F + bp$.
   - The agent takes effort levels e_1 and e_2 which are unobservable to the principal.
   - Random shocks ε and ϕ, are realized, affecting the agent's output and performance, respectively.
   - Output y and performance p are observed by the principal and agent.
   - The agent receives wage $w = F + bp$.

   Answer the following questions.

[1] For a more general presentation, see Bolton and Dewatripont (2005), pp. 216-28.

(a) Find the agent's optimal efforts and indirect utility as a function of the bonus parameter b.

(b) Find the principal's optimal contract w^* and his equilibrium profits.

(c) *Comparative Statics*. How is the optimal contract you found in part (b) affected by the output rates f_1 and f_2? How is it affected by the performance rates g_1 and g_2? Explain.

(d) Given the optimal contract found above, what are the principal's expected payoff, the agent's expected utility, and the expected social welfare in equilibrium?

(e) What is the socially optimal contract? Compare it against the contract that emerges in the subgame perfect equilibrium of the game you found in part (b).

(f) What is the deadweight loss in this contractual setting?

(g) *Numerical example*. Consider output rates $f_1 = \frac{1}{2}$ and $f_2 = \frac{1}{3}$, and performance rates $g_1 = \frac{2}{3}$ and $g_2 = \frac{1}{4}$. In this context, evaluate the equilibrium bonus b^*, efforts e_1^* and e_2^*, wage w^*, the agent's expected utility $U(w^*)$, the principal's expected profit $\pi(w^*)$, and social welfare SW^*. Then, evaluate the socially optimal bonus b^{**}, social welfare SW^{**} at b^{**}, and deadweight loss due to unobservability of effort.

3. **Stone-Geary utility function in a pure exchange economy.** Consider a pure exchange economy with two individuals, A and B, whose utility functions are

$$u^A\left(x_1^A, x_2^A\right) = \left(x_1^A - b_1\right)^{\frac{1}{2}}\left(x_2^A - b_2\right)^{\frac{1}{2}}$$
$$u^B\left(x_1^B, x_2^B\right) = x_1^B x_2^B$$

where $b_1, b_2 > 0$ represent the minimal amounts of goods 1 and 2 that individual A must consume in order to remain alive (such as water and shelter). Individuals A and B have endowments of $\omega^A = \left(\omega_1^A, \omega_2^A\right) = (4, 2)$ and $\omega^B = \left(\omega_1^B, \omega_2^B\right) = (2, 4)$, respectively.

(a) Set up the Lagrangian and find the individuals' Walrasian demand functions.

(b) Find the set of Pareto efficient allocations (PEAs). (*Hint*: Your answer should be in terms of b_1 and b_2).

(c) Find the Walrasian equilibrium allocation (WEA). (*Hint*: Your answer should be in terms of b_1 and b_2).

(d) Evaluate the contract curve and WEA at the following three different subsistence levels: (i) $(b_1, b_2) = (4, 2)$, (ii) $(b_1, b_2) = (3, 3)$, and (iii) $(b_1, b_2) = (2, 4)$. In which case(s) is individual A unable to survive?

(e) Consider now a tax transfer so individual A survives in the case(s) you identify in part (b) where he suffers from a negative utility at the WEA. Identify the tax/transfer that the government can impose, and the resulting WEA. (For compactness, let us normalize $p_2 = 1$ so that $p \equiv p_1 = \frac{p_1}{p_2}$.)