ECONS 424 – STRATEGY AND GAME THEORY

HOMEWORK #6 – *ANSWER KEY*

<u>Exercise 5-Chapter 28-Watson (*Signaling between a judge and a defendant*)</u>

a. This game has a unique PBE. Find and report it.

After E^1, the judge chooses $\bar{y}$ such that:

$$\underset{\bar{y}}{Max} -(\bar{y}-1)^2$$

Taking FOCs with respect to $\bar{y}$, we obtain:

$$-2(\bar{y}-1)=0$$
$$\rightarrow \bar{y}=1$$

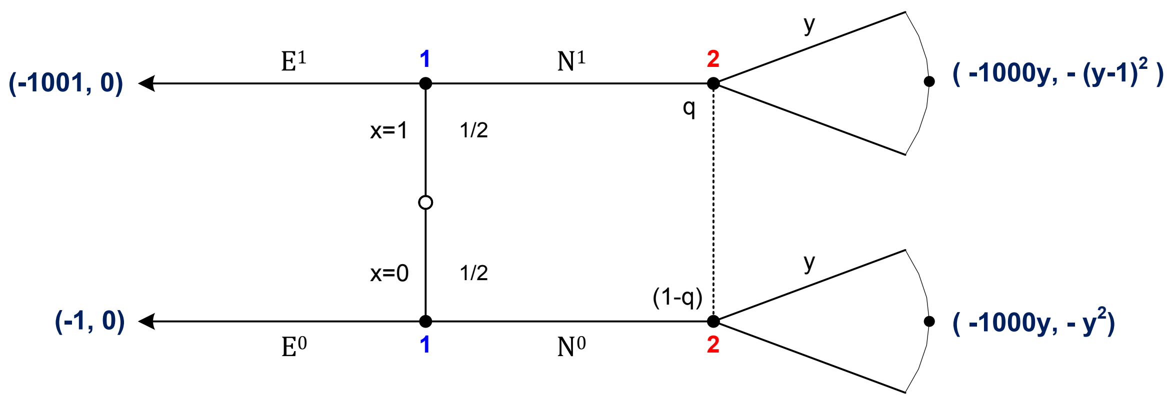
Similarly, after E^0, the judge chooses $\underline{y}$ such that:

$$\underset{\underline{y}}{Max} -\underline{y}^2$$

Taking FOCs with respect to $\underline{y}$, we obtain:

$$-2\underline{y} \leq 0 \quad \rightarrow \quad \underline{y}=0$$

Hence, the game becomes:



- ***Let us first check for the existence of a separating PBE where E^0 and N^1:***

1. *Belief*: q=1 since N only comes from x=1

2. *Judge (second mover)*: After observing N, the judge selects y assigning full probability to being in the open node of his information set (see figure below)

Hence,

$$\underset{y}{Max} -(y-1)^2$$

Taking FOCs with respect to y, we obtain:

$$-2(y-1)=0, \text{which implies } y=1$$

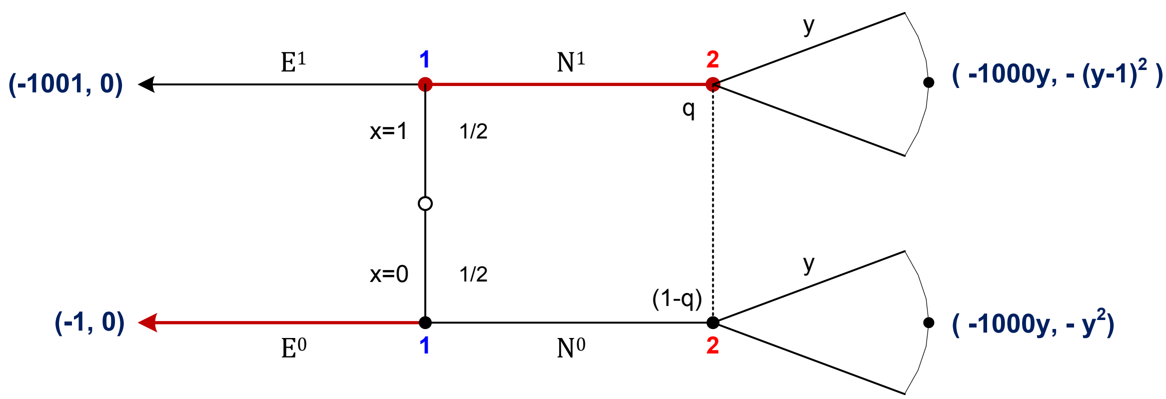
3. *Defendant (first mover)*:
   - If $x=1$, the defendant compares:

     -1001 if he chooses E^1

     -1000 if he chooses N^1

     So, N^1 is better.

- If $x = 0$, the defendant compares:
  - -1 if he chooses E^0
  - -1000 if he chooses N^0
  - So, E^0 is better.



Hence, this separating PBE can be supported.

- ***Let us now check the separating N^0E^1***

1. *Beliefs*: $q = 0$ since N only comes from $x = 0$
2. *Judge*: After observing N, the judge assigns full probability to lower node of his information set. Then, he selects y such that:

$$\underset{y}{Max} -y^2$$

Taking FOCs with respect to y, we obtain:

$$-2y \leq 0, \text{ which implies } y = 0$$

3. *Defendant*:
   - If $x = 1$, the defendant compares:
     -1001 if he chooses E^1
     0 if he chooses N^1
     So, N^1 is better $\leftarrow$ Deviation from the prescribed separating N^0E^1.
   - If $x = 0$, the defendant compares:
     -1 if he chooses E^0
     0 if he chooses N^0
     So, N^0 is better. Hence, the separating N^0E^1 cannot be supported as PBE.

- ***Let us now check if a pooling PBE where N^0N^1 can be sustained***

  1. *Beliefs*:

$$q = \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + \frac{1}{2} \times 1} = \frac{1}{2}$$

  2. *Judge*: After observing N, given his beliefs q=1/2, he must choose y in order to maximize his expected utility:

$$\underset{y}{Max} \ \frac{1}{2}[-(y-1)^2] + \frac{1}{2}[-y^2]$$

Taking FOCs with respect to y, we obtain:

$$-\frac{1}{2}\times 2(y-1)-\frac{1}{2}\times 2y=0, \text{ which implies } y=\frac{1}{2}$$

3. *Defendant*:
   - If $x=1$, the defendant compares:
     -1001 if he chooses E^1
     $-1000\times\frac{1}{2}=-500$ if he chooses N^1
     So, N^1 is better
   - If $x=0$, the defendant compares:
     -1 if he chooses E^0
     $-1000\times\frac{1}{2}=-500$ if he chooses N^0
     So, N^0 is better → Deviation from the prescribed pooling. Hence, the pooling N^0N^1 cannot be sustained.

- ***Let us now check if a pooling PBE where $\mathbf{E^0E^1}$ can be sustained.***

1. *Beliefs*: $q\in[0,1]$ since N is only observed off-the-equilibrium
2. *Judge*: From his beliefs, he chooses y in order to maximize his expected utility:

$$\underset{y}{Max}\ \ q[-(y-1)^2]+(1-q)[-y^2]$$

Taking FOCs with respect to y, we obtain:

$$-q\times 2(y-1)-(1-q)\times 2y=0, \text{ which implies } y=q$$

3. *Defendant*:
   - If $x=1$, the defendant compares:
     -1001 if he chooses E^1
     $-1000q$ if he chooses N^1
     So, N^1 is better for any $q<1$ → Deviation from the prescribed pooling.
   - If $x=0$, the defendant compares:
     -1 if he chooses E^0
     $-1000q$ if he chooses N^0
     So, E^0 is better for any $q>1/1000$ → the pooling E^0E^1 cannot be supported as PBE either.

b. Explain why the result of part (a) is interesting from an economic standpoint?

The only equilibrium that we can support in this game is the separating equilibrium in which the innocent defendant provides evidence of his innocence, whereas the guilty defendant does not provide such evidence. This is something desirable, since the judge can perfectly infer the true innocence of a defendant by simply observing whether he/she presented evidences.

c. When $\mathbf{x\in[0,K]}$ with each value equally likely, compute the PBE.

We are going to test the equilibrium where all types of x={0,…,K-1} present evidence (E), but the last type x=K presents no evidence (N).

1) <u>Beliefs</u>

   After observing the evidence presented by the defendant, the judge can perfectly observe his type 0,1,2,…,K-1. In these cases we don't need to specify beliefs. When no evidence(N) is presented, the judge's beliefs are:

$$\mu(t_j|N) = 0 \;\; \forall \;\; j = \{0, \dots, K-1\}$$

$$\mu(t_K|N) = 1$$

Which implies that after receiving no evidence, the judge assigns full probability to the K-type, and therefore no probability to any of the 0,1,2,…,K-1 types.

2) Judge's Best Response:

Given N:

$$\max_y -(y-K)^2 \quad \rightarrow \quad y^N = K$$

Given E(where there is no information set and the judge knows what type has played E):

$$\max_y [-(y-x)^2]$$

where x is the specific type of the defendant that presented evidence (a type that is observed by the judge thanks to the presentation of evidence). Taking FOCs with respect to y, we obtain

$$-2y + 2x = 0$$

$$\rightarrow y^E = x$$

3) Defendant's Best Response:

For types 0,…,K-1 : if he provides evidence, E, then they get y^E from the judge, providing:

$$-1000y^E - 1$$

which must exceed his payoff from not presenting evidence: $-1000y^N = -1000K$

(in this case the judge interprets that the defendant is a K-type and chooses a sentence $y^N = K$)

- Note, type x=0 prefers the payoff he obtains by presenting evidence, $-1000y^E - 1 = -1$, than his payoff from not presenting evidence, $-1000K$ (since K > 2 given that there are more than two types of defendants).
- Similarly for type x=1, where $-1000y^E - 1 = -1001 > -1000 * K$; and for all other types x=2,3,…
- The defendant who obtains the lowest equilibrium payoff from providing evidence is x=K-1, who obtains $-1000y^E - 1 = -1000(K-1) - 1$. Let us check if his equilibrium payoff from providing evidence is larger than from deviating, that is:

$$-1000K + 1000 - 1 > -1000K$$

$$-1000K + 999 > -1000K$$

$$999 > 0$$

This obviously holds, so the defendant behaves as prescribed when his type is x=0,…,K-1

For type K: if he doesn't provide evidence, N, (as initially prescribed) then he gets a sentence $y^N = K$ from the judge, providing a payoff of:

$$-1000K$$

This must exceed his alternative payoff from providing evidence (E):

$$-1000K - 1$$

The condition reduces to:

$$-1000K > -1000K - 1$$

$$0 > -1$$

This as well holds, showing the initially stated strategy, where types x=0,1,2,…,K-1 present evidence but type x=K does not, can be sustained as a PBE.

ANSWER: Consider the following beliefs for Bush: (1) if Saddam allows inspections and WMD are found, then Saddam has WMD with probability 1; (2) if Saddam allows inspections and WMD are not found, then Saddam has WMD with probability zero; and (3) if Saddam does not allow inspections, then Saddam has WMD with probability $\frac{w}{w + (1-w)(1-h)}$. Let us begin by showing the consistency of Bush's beliefs. When inspections are allowed, it is trivial that beliefs are consistent. (They are actually consistent with the truth, not just Saddam's strategy.) When inspections are not allowed, then the posterior probability of Saddam's having WMD is given by Bayes's rule to be

$$\frac{w \times 1}{w \times 1 + (1-w) \times (1-h)}$$

as with probability w, Saddam has WMD and, in that event, he does not allow inspections with probability 1; while with probability $1 - w$, Saddam has WMD and, in that event, he does not allow inspections with probability $1 - h$. Turning to Bush's strategy, its optimality is clear when there are inspections, whether WMD are found or not. When inspections are not allowed, Bush is content to randomize (that is, $0 < b < 1$) if and only if

$$\left(\frac{w \times 1}{w \times 1 + (1-w) \times (1-h)}\right) \times 3 + \left(\frac{(1-w) \times (1-h)}{w \times 1 + (1-w) \times (1-h)}\right) \times 6$$

$$= \left(\frac{w \times 1}{w \times 1 + (1-w) \times (1-h)}\right) \times 1 + \left(\frac{(1-w) \times (1-h)}{w \times 1 + (1-w) \times (1-h)}\right) \times 9.$$

The left-hand expression is the expected payoff from invading, and the right-hand expression is the expected payoff from not invading. Solving this equation for h yields $h = \frac{3-5w}{3(1-w)}$. Note that $\frac{3-5w}{3(1-w)} > 0$ and $\frac{3-5w}{3(1-w)} < 1$ if and only if $0 < w < \frac{3}{5}$. The latter condition was assumed.

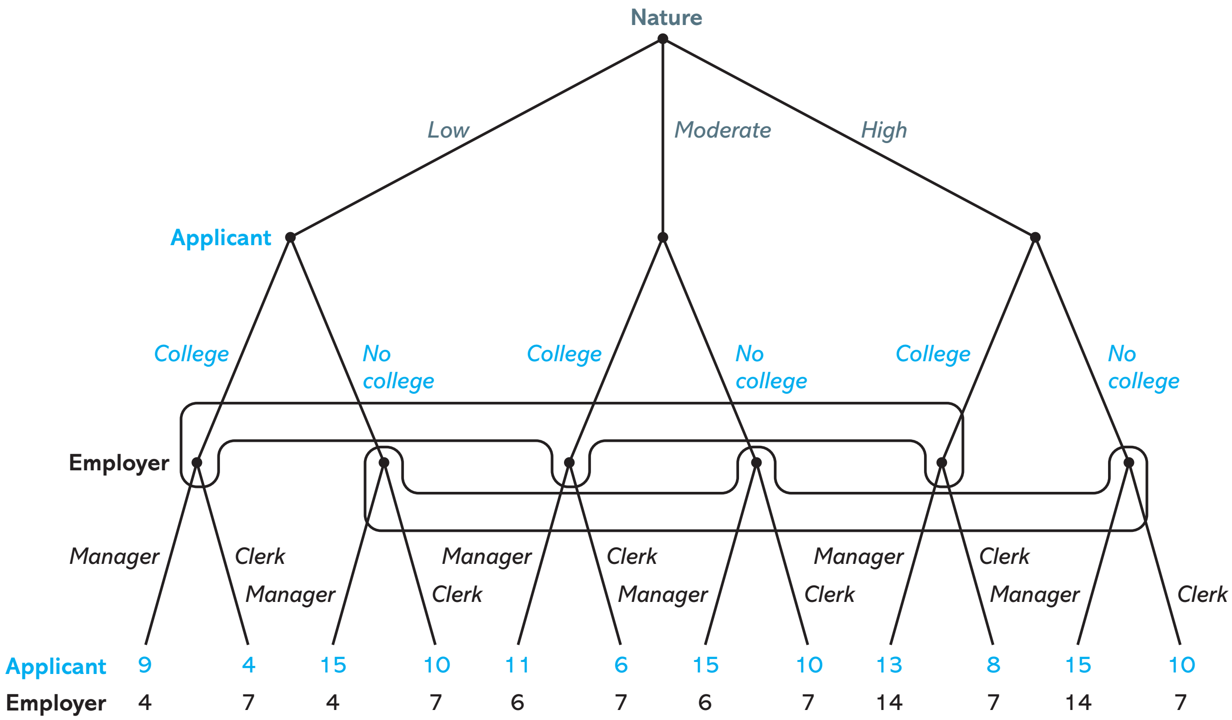
When Saddam has WMD, it is clearly optimal for him to not allow inspections. When he does not have WMD, it is optimal to randomize if and only if

$$b \times 2 + (1-b) \times 8 = 4,$$

where he earns a payoff of 4 by allowing inspections—in which case there is no invasion—and gets an expected payoff of $b \times 2 + (1-b) \times 8$ from not allowing inspections (where there is an invasion with probability b). Solving this equation, we find $b = \frac{2}{3}$.

5. We'll now show how a college degree can get you a better job even if it doesn't make you a better worker. Consider a two-player game between a prospective employee, whom we'll refer to as the applicant, and an employer. The applicant's type is her intellect, which may be *low*, *moderate*, or *high*, with probability $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{1}{6}$, respectively. After the applicant learns her type, she decides whether or not to go to college. The personal cost in gaining a college degree is higher when the applicant is less intelligent, because a less smart student has to work harder if she is to graduate. Assume that the cost of gaining a college degree is 2, 4, and 6 for an applicant who is of high, moderate, and low intelligence, respectively. The employer decides whether to offer the applicant a job as a manager or as a clerk. The applicant's payoff to being hired as a manager is 15, while the payoff to being a clerk is 10. These payoffs are independent of the applicant's type. The employer's payoff from hiring someone as a clerk is 7 (and is the same regardless of intelligence and whether or not the person has a college degree). If the applicant is hired as a manager, then the employer's payoff increases with the applicant's intellect, from 4, to 6, to 14, depending on whether the applicant has low, moderate, or high intellect, respectively. Note that the employer's payoff does not depend on whether or not the applicant has a college degree. The extensive form of this game is shown in the accompanying figure. Find a PBNE in which students of low intellect do not go to college and those of moderate and high intellect do.

The College Signaling Game



ANSWER:

Student's strategy:

If of low intellect, do not go to college.

If of moderate or high intellect, go to college.

Employer's strategy:

If the student did not go to college, then hire him as a clerk.

If the student did go to college, then hire him as a manager.

Employer's beliefs:

If the student did not go to college, then he is of low intellect with probability one.

If the student did go to college, then he is of low intellect with probability zero, moderate intellect with probability $\frac{3}{4}$, and high intellect with probability $\frac{1}{4}$.

Consider the student's strategy. If he is of low intellect, his payoff from not going to college and getting a job as a clerk is 10, which exceeds his payoff of $9(=15-6)$ from going to college and getting a job as manager. If he is of moderate intellect, his payoff from not going to college and getting a job as a clerk is again 10, which is less than his payoff of $11(=15-4)$ from going to college and getting a job as manager. If he is of high intellect, his payoff from going to college is $13(=15-2)$, which exceeds that from not going to college. Thus, the student's strategy is optimal for each type. The employer's beliefs are consistent with respect to the student having not gone to college, since only when his type is low intellect does he not go to college. The student goes to college when he is either moderate or high intellect and, by Bayes's rule, the posterior probability that he is of moderate intellect is

$$\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{6}}=\frac{3}{4}.$$

Thus, beliefs are consistent. As to the employer's strategy, he clearly prefers to hire the student as a clerk when he did not go to college. Since she believes the student is of low intellect, the payoff from hiring him as a clerk, which is 7, exceeds the payoff of 4 from hiring him as a manager. When the student went to college, the expected payoff from hiring him as a manager is

$$\frac{3}{4} \times 6 + \frac{1}{4} \times 14 = 8,$$

and this exceeds the payoff of 7 from hiring him as a clerk. Thus, this strategy profile is a perfect Bayes-Nash equilibrium.

6. The owner of a new restaurant is planning to advertise to attract customers. In the Bayesian game, Nature determines the restaurant's quality, which is either *high* or *low*. Assume that each quality occurs with equal probability. After the owner learns about quality, he decides how much to advertise. Let A denote the amount of advertising expenditure. For simplicity, assume that there is a single consumer. The consumer observes how much advertising is conducted, updates her beliefs about the quality of the restaurant, and then decides whether or not to go to the restaurant. (One can imagine that A is observed by noticing how many commercial spots are on local television and how many ads are in the newspaper and on billboards.) Assume that the price of a meal is fixed at \$50. The value of a high-quality meal to a consumer is \$85 and of a low-quality meal is \$30. A consumer who goes to the restaurant and finds out that the food is of low quality ends up with a payoff of −\$20, which is the value of a low-quality meal, \$30, less the price paid, \$50. If the food is of high quality, then the consumer receives a value of \$35 (= \$85 − \$50). Furthermore, upon learning of the high quality, a consumer anticipates going to the restaurant a second time. Thus, the payoff to a consumer from visiting a high-quality restaurant is actually \$70 (= 2 × \$35). For the restaurant owner, assume that the cost of providing a meal is \$35 whether it is of low or high quality. If the restaurant is of high quality, the consumer goes to the restaurant, and the restaurant spends A in advertising, then its profit (and payoff) is 2 × (\$50 − \$35) − A = \$30 − A. If the restaurant is of low quality, the consumer goes to the restaurant, and the restaurant spends A in advertising, then its profit is (\$50 − \$35) − A = \$15 − A. These payoffs are summarized in the following table. If the consumer does not go to the restaurant, then her payoff is zero and the owner's payoff is −A.

Restaurant Game: Payoffs When a Consumer Goes to the Restaurant

Restaurant Quality	Owner's Payoff	Customer's Payoff
Low	\$15 − A	−\$20
High	\$30 − A	\$70

a. Find a separating PBNE.

ANSWER: Consider the following separating strategy profile where $A' > A''$:

Owner's strategy:

If of high quality, then spend A' on advertising.

If of low quality, then spend A'' on advertising.

Customer's strategy:

If advertising is at least A', then go to the restaurant.

If advertising is less than A', then do not go to the restaurant.

Customer's beliefs:

If advertising is at least A', then the restaurant is high quality with probability 1.

If advertising is less than A', then the restaurant is low quality with probability 1.

Starting with the customer's beliefs, consistency requires that the customer believe the restaurant is high quality when she observes advertising of A' and is low quality when she observes advertising of A''. Thus, these beliefs are consistent. Turning to her strategy, first note that the customer finds it optimal to go to the restaurant if she believes it is high quality—realizing a payoff of 70 versus 0 from not going—and optimal not to go if she believes it is low quality (realizing a payoff of –20 versus 0). According to her beliefs, the restaurant is high quality if advertising is at least A'. As her strategy has her go to the restaurant if advertising is at least A', prescribed behavior is optimal. She believes it is of low quality when advertising falls below A'; again her strategy is optimal.

Finally, consider the owner's strategy. If the restaurant is of high quality, the payoff from advertising A is $30 - A$ when $A \geq A'$ and zero when $A < A'$. Hence, advertising A' is optimal if and only if $A' \leq 30$. When the restaurant is of low quality, the payoff from advertising A is $15 - A$ when $A \geq A'$ and zero when $A < A'$. Hence, advertising A'', when $A'' < A'$, is optimal if and only if $A'' = 0$ and $A' \geq 15$. That is, if $A' < 15$, then the restaurant earns a higher payoff (of $15 - A'$) by advertising A' than by advertising A'' (which results in a payoff of $-A''$). And if advertising A'' means no customers are going to come—so the payoff is $-A''$—then optimality requires $A'' = 0$. There is no point in advertising if it doesn't deliver any customers. In sum, this is a separating perfect Bayes-Nash equilibrium if and only if $A'' = 0$ and $15 \leq A' \leq 30$. Note that advertising provides no direct information. Rather it is a signal of a restaurant's quality. A high-quality restaurant is willing to advertise more to induce a customer to try its food because it knows the customer will return in the future.

b. At a separating PBNE, what is the maximum amount of advertising that a restaurant conducts? What is the minimum amount?

ANSWER: With this separating equilibrium, advertising cannot exceed 30 and cannot be less than 15. If it exceeds 30, then a high-quality restaurant would prefer not to advertise at all. If it is less than 15, then the low-quality restaurant would imitate the high-quality restaurant.

c. Find a pooling PBNE.

ANSWER: Consider the following separating strategy profile.

Owner's strategy:

If restaurant is of high quality, then spend A^o on advertising.

If restaurant is of low quality, then spend A^o on advertising.

Customer's strategy:

If advertising is at least A^o, then go to the restaurant.

If advertising is less than A^o, then do not go to the restaurant.

Customer's beliefs:

If advertising is at least A^o, then the restaurant is high quality with probability $\frac{1}{2}$.

If advertising is less than A^o, then the restaurant is high quality with probability p.

Since the customer's beliefs are the same as her prior beliefs when $A = A^o$, beliefs are consistent. Given those beliefs, it is optimal to go to the restaurant when $A \geq A^o$ if and only if:

$$\frac{1}{2} \times 70 + \frac{1}{2} \times (-20) \geq 0,$$

where the left-hand expression is the expected payoff from going, which equals 25, and the right-hand expression is the payoff from not going. This inequality holds. It is optimal not to go to the restaurant when $A < A^o$ if and only if

$$p \times 70 + (1 - p) \times (-20) \leq 0,$$
$$p \leq \frac{2}{9}.$$

Thus, we must have $p \leq \frac{2}{9}$. For the restaurant, it either wants to advertise A^o (the minimum amount necessary to induce a customer to come) or zero. The former yields a higher expected payoff when:

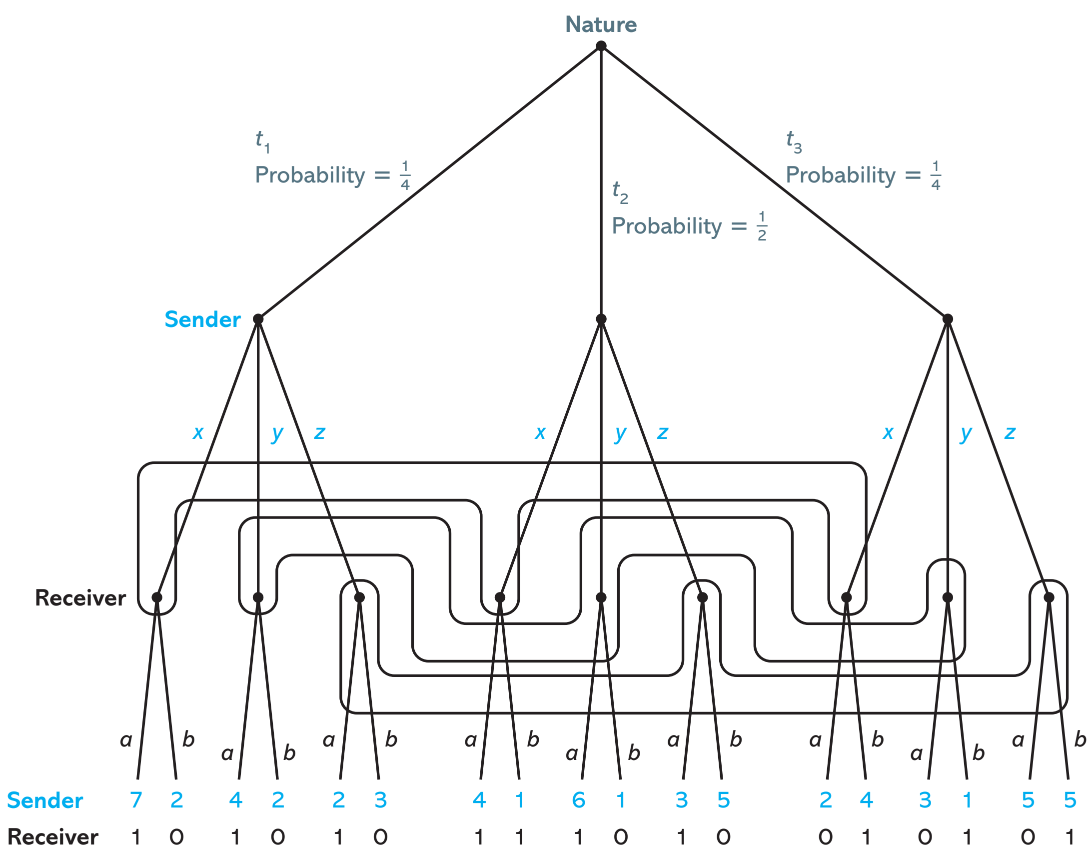
High-quality restaurant: $30 - A^o \geq 0$ or $A^o \leq 30$
Low-quality restaurant: $15 - A^o \geq 0$ or $A^o \leq 15$.

Thus, we need $A^o \leq 15$.

d. At a pooling PBNE, what is the maximum amount of advertising?

ANSWER: The maximum amount of advertising at a pooling equilibrium is 15. If it exceeds 15, then the low-quality restaurant owner would prefer to advertise zero.

7. Consider this signaling game. Nature chooses one of three types for the sender, and after learning her type, the sender chooses one of three actions. The receiver observes the sender's action, but not her type, and then chooses one of two actions. Find a semiseparating PBNE in which the sender chooses the same action when her type is t_1 or t_2 and chooses a different action when her type is t_3.



b. Find a separating perfect Bayes–Nash equilibrium.

ANSWER: There is no separating perfect Bayes-Nash equilibrium. Consider a separating strategy for the sender whereby he chooses message m_1 when his type is x, m_2 when his type is y, and m_3 when his type is z. (All that is important is that the sender chooses a distinct message for each type.) Given consistent beliefs for the receiver, her optimal strategy is then to choose action a in response to message m_1 (as m_1 signals that the sender's type is x and a is the best action in that situation), action c in response to message m_2, and action a in response to message m_3.

Now consider a type y sender. His payoff is 4 from sending message m_1 (as this induces the receiver to choose action a) and 3 from message m_2 (as this induces the receiver to choose action c). He would then prefer to send m_1 and mislead the receiver into thinking his type is x. We conclude there is no separating perfect Bayes-Nash equilibrium.

4. Return to the Stock Recommendation game, and consider again the semiseparating strategy profile, but let the analyst's strategy now be as follows: Recommend *buy* when the stock will *outperform* or be *neutral*; and recommend *sell* when the stock will *underperform*. Show that if $a + b \leq 2$, then this strategy pair is part of a perfect Bayes–Nash equilibrium.

ANSWER: The only difference is that the analyst now puts out a sell recommendation when the stock is an underperform rather than a hold recommendation. Since these are costless actions, all that matters is what a message signals and how it influences other players' behaviors. In Chapter 12 in the text, a hold recommendation signaled the stock was an underperform, and now it is a sell recommendation that signals it is an underperform. More specifically, note that the investor's beliefs are still consistent, which implies that her strategy is still optimal (as a strategy is required to be best given beliefs). For the analyst, the associated payoffs are the same since before either a hold or sell recommendation induced the investor to sell, and that is still the case here.

5. The accompanying figure is a cheap talk game in which the sender has two possible types—denoted t_1 and t_2—and can choose one of two possible messages—denoted m_1 and m_2. After observing the sender's message, the receiver chooses from amongst three possible actions: a, b, and c.