Chapter 12: Signaling Games with Continuous Messages

Game Theory:

An Introduction with Step-by-Step Examples

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Introduction

- We extend Spence's Labor Market signaling game to a setting with continuous messages and responses.
- Outline:
  - Identify equilibrium behavior under complete information
  - Find separating and pooling PBEs of the game and apply refinement criteria discussed in previous lectures.
  - Compare equilibrium results under complete and incomplete information
  - Fundamental question:
    - Are workers better off when firms observe their productivity or…
    - when they need to use education to signal their type to the employers?

Utility functions with Continuous Actions

- Consider a worker with utility function

$$u(e, w|\theta_K) = w - c(e, \theta_K)$$

where

- $w \geq 0$ represents the salary that she receives from the firm,
- $c(e, \theta_K)$ denotes her cost of acquiring e units (e.g., years) of education given that her productivity is θ_K,
  - where $K = \{H, L\}$ is worker's type (high or low productivity).

Utility functions with Continuous Actions

- For simplicity, function $c(e, \theta_K)$ often satisfies the following four assumptions:
  1. *Zero cost of no education.* When the worker acquires no education, she incurs no cost, that is, $c(0, \theta_K) = 0$.
  2. *Cost is increasing and convex.* Cost function $c(e, \theta_K)$ is strictly increasing and convex in education that the worker acquires, i.e., $c_e > 0$ and $c_{ee} > 0$.
     - indicating that additional years of education become progressively more costly.
  3. *Cost is decreasing in productivity.* Cost function $c(e, \theta_K)$ is decreasing in the worker's productivity, θ_K, i.e., $c(e, \theta_H) < c(e, \theta_L)$.
     - implying that a given education e (e.g., a university degree) is easier to complete for the high-productivity than for the low-productivity worker.

Utility functions with Continuous Actions

4. *Marginal cost is decreasing in productivity (single-crossing property).* A similar property applies to the marginal cost of education, $c_e(e, \theta_K)$ which is also lower for the high-productivity worker than the low-productivity worker,

$$c_e\,(e, \theta_H) < c_e(e, \theta_L).$$

Remark:

- When messages are discrete, the fourth assumption can be expressed as "cost increments", as follows:

$$c(e', \theta_H) - c(e, \theta_H) < c(e', \theta_L) - c(e, \theta_L)$$

Example 12.1 Cost of Acquiring Education

- Consider a cost function such as

$$c(e, \theta_K) = \frac{Ae^2}{\theta_K}$$

where $A > 0$ is a constant.

As a practice, show that it satisfies the above four assumptions.

Example 12.1 Cost of Acquiring Education

- Graphically, we can depict the worker's indifference curve in the $(e, w) -$ quadrant by, first, solving for w, which yields

$$w = u + c(e, \theta_K) \Rightarrow w = u + \frac{Ae^2}{\theta_K},$$

$$\text{if } c(e, \theta_K) = \frac{Ae^2}{\theta_K}$$

- Since the cost of education is strictly increasing and convex in e,
  - indifference curves are also monotonically increasing and convex in e, as depicted in Figure 12.1.

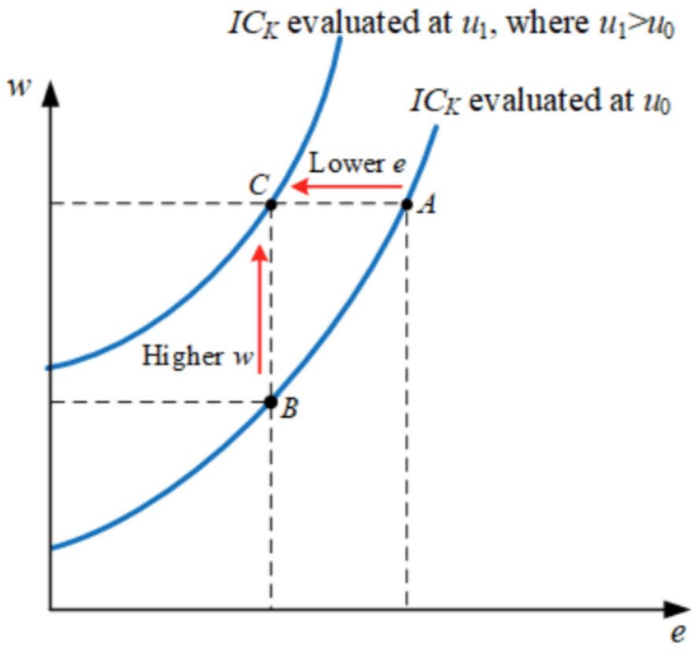


Figure 12.1 Indifference curves for a representative worker.

Example 12.1 Cost of Acquiring Education

- Intuitively, a higher education must be accompanied by an increase in her salary for the worker's utility to remain unchanged.
- Indifference curves to the northwest are associated with a higher utility:
  - for a given education e, the worker receives a higher wage (movement from point A to C in Figure 12.1), or
  - for a given wage w, the worker acquires less education (movement from B to C).

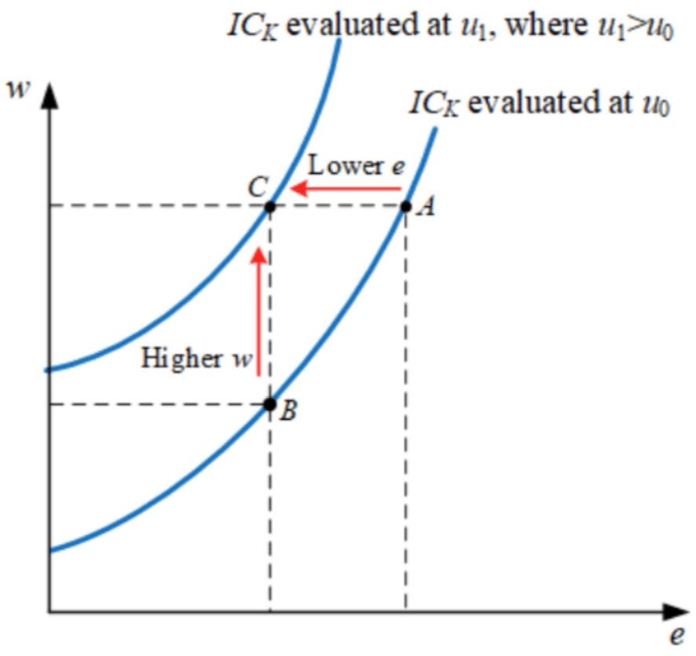


Figure 12.1 Indifference curves for a representative worker.

Example 12.1 Cost of Acquiring Education

- If $A = \theta_L = 1$ and $\theta_H = 2$, then the high- and low-type indifference curves cross at

$$u_H + \frac{e^2}{2} = u_L + e^2$$

Or, after solving for *e*,

$$e = [2(u_H - u_L)]^{1/2}$$

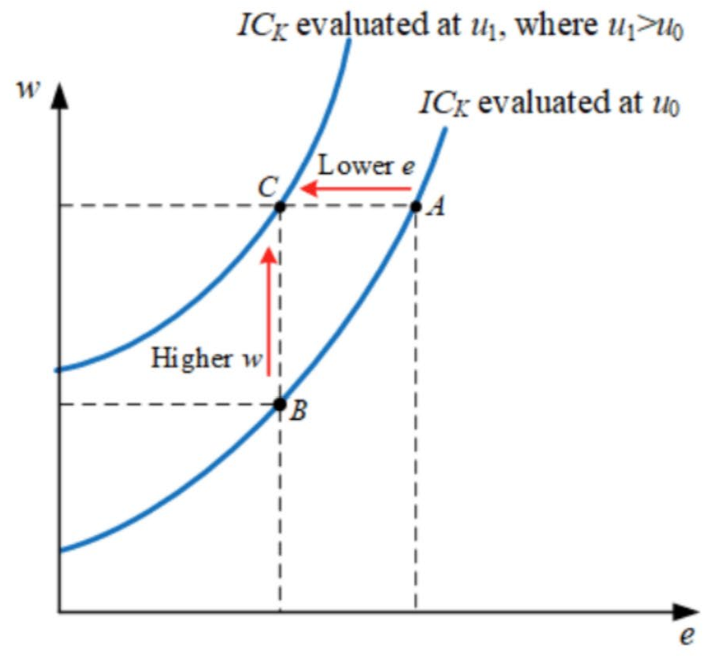


Figure 12.1 Indifference curves for a representative worker.

Complete Information

- We assume the labor market is perfectly competitive.
- Firm observes the worker's type and pays a salary equal to the worker's productivity $w = \theta_K$
  - If $w < \theta_K$, other firms could offer a slightly higher salary w' such that $w < w' \leq \theta_K$, making a weakly positive profit
  - If $w > \theta_K$, it would attract the worker but make a negative profit
- A similar argument applies when the firm does not observe the worker's type and pays a salary equal to her expected productivity, $w = E[\theta_K]$,
  - where this expectation is based on the firm's beliefs about the worker's type.

Complete Information

- As a benchmark, we consider a complete information context where the firm observes the worker's productivity, θ_K.
- In this setting, the only subgame perfect equilibrium has the worker acquiring zero education regardless of her productivity, $e_H = e_L = 0$.
- The firm responds with salary $w(\theta_H) = \theta_H$ when the worker's productivity is high, and $w(\theta_L) = \theta_L$ when her productivity is low.
- Intuitively, the worker cannot use education as a signal about her type since the firm observes her productivity.

Incomplete Information

- Under incomplete information, education can become an informative (although costly) signal.
- Assume that the worker privately observes her type θ_K before choosing her education level e.
- Firm observes education e but does not know the worker's type θ_K.
- However, it assigned a prior probability $p \in [0,1]$ to the worker's productivity being high and $1 - p$ to her productivity being low.
    - This probability distribution is common knowledge among players.

Separating PBE

- We check if a separating strategy profile where each type of worker acquires a different education level can be sustained as a PBE.
- Follow the same four steps as in the discrete version of the game.

First step - strategy profile

- High-productivity worker chooses education level e_H
- Low-productivity worker chooses e_L, where $e_H \neq e_L$.

Second step – updating beliefs

- Upon observing e_H, firm assigns full probability to facing a high-productivity worker, i.e., $\mu(\theta_H|e_H) = 1$.
- In contrast, after observing e_L, firm assigns no probability to facing a high-productivity worker, i.e., $\mu(\theta_H|e_L) = 0$.
- If the firm observes the worker acquiring an off-the-equilibrium education level, i.e., an education e different than e_H and e_L,
  - it cannot update its beliefs using Bayes' rule, leaving them unrestricted, i.e., $\mu(\theta_H|e) \in [0,1]$ for all $e \neq e_H \neq e_L$.

Separating PBE

Third step – optimal responses

- If firm observes e_H, it pays $w(e_H) = \theta_H$, and if it observes e_L**,** it pays $w(e_L) = \theta_L$
- Salaries coincide with the worker's productivity, thus being the same as under complete information.
- Yet, education levels in the separating PBE do not coincide with those under complete information.
  - Upon observing $e \neq e_H \neq e_L$, a firm's beliefs are $\mu(\theta_H|e) \in [0,1]$
  - Therefore, the firm pays a salary between
    - $w(e) = \theta_L$ when $\mu(\theta_H|e) = 0$ and
    - $w(e) = \theta_H$ when $\mu(\theta_H|e) = 1$
  - That is, $w(e) \in [\theta_L, \theta_H]$

Separating PBE

- As illustrated in Figure 12.2, this wage schedule means that $w(e_L) = \theta_L$ at education level e_L, and $w(e_H) = \theta_H$ at e_H.
- However, for all $e \neq e_H \neq e_L$, the wage can lie strictly above θ_L or below θ_H (but of course bounded between θ_L and θ_H)

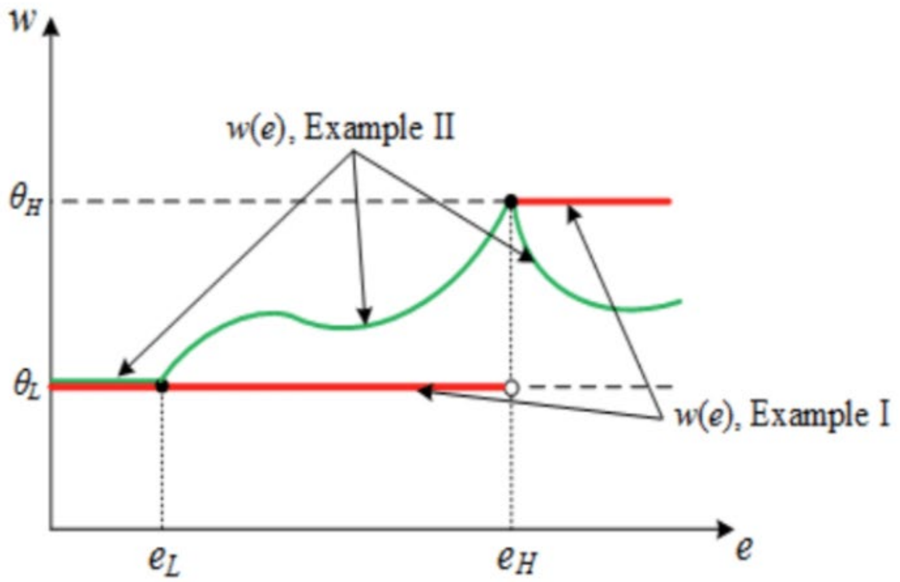


Figure 12.2. Separating PBE - Two wage schedules.

Separating PBE

Fourth step – optimal messages

Let's identify conditions under which worker of type θ_L has the incentives to choose e_L while θ_H type chooses e_H.

Low productivity worker: Let's find when e_L satisfies $e_L = 0$

- Figure 12.3 depicts the indifference curve of the low-productivity worker, IC_L, that passes through point $(w, e) = (\theta_L, 0)$.

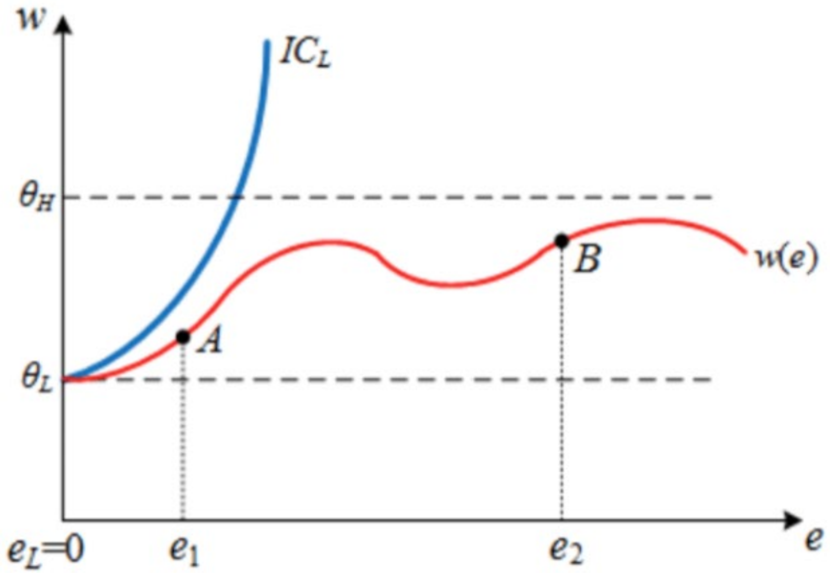


Figure 12.3 Separating PBE - Low productivity worker.

Separating PBE

- At $e_L = 0$, the firm is convinced to deal with a low-productivity worker, paying a wage $w(e_L) = \theta_L$.
- Wage schedule $w(e)$ guarantees that this type of worker has no incentives to deviate from education level $e_L = 0$.
- Consider point A:
  - At e_1, the salary the worker receives is above θ_L,
  - but the indifference curve passing through A, IC_L^A is associated to a lower utility level than IC_L

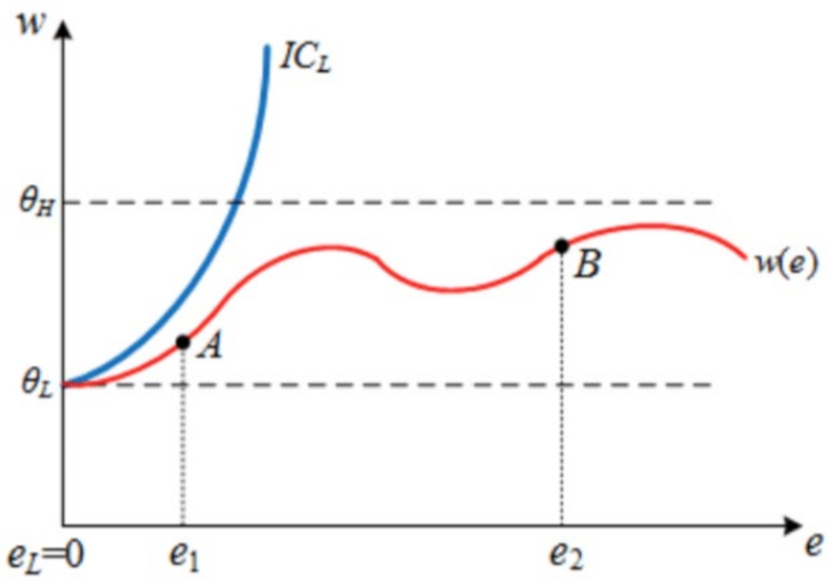


Figure 12.3 Separating PBE - Low productivity worker.

Separating PBE

- Additional cost of education she incurs when deviating from $e_L = 0$ to $e_1 > 0$ offsets the extra salary she receives.
- A similar argument applies for any education levels $e > e_L$.
- More generally, for the low-productivity worker to stick to $e_L = 0$, we need that the firm's wage schedule to lie below IC_L

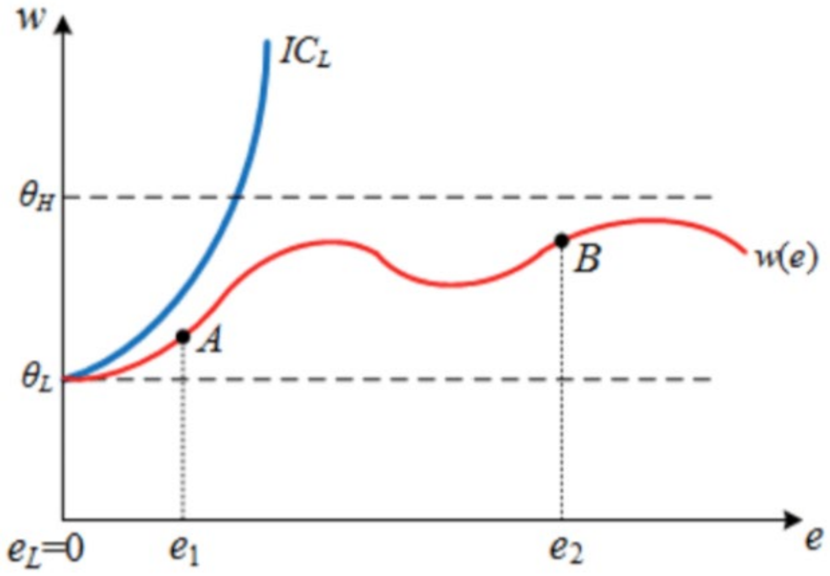


Figure 12.3 Separating PBE - Low productivity worker.

Separating PBE

High productivity worker chooses the education level prescribed in this separating strategy profile, e_H, rather than deviating to the low-type's education, $e_H = 0$ if

$$\theta_H - c(e_H, \theta_H) \geq \theta_L - c(0, \theta_H)$$

since she can anticipate that acquiring e_H identifies her as a high-productivity worker, yielding a salary $w(e_H) = \theta_H$,

while acquiring $e_L = 0$ identifies her as a low-productivity worker, receiving $w(e_L) = \theta_L$

Separating PBE

- Since cost $c(0, \theta_H) = 0$ by assumption, we can rearrange the above inequality as

$$\underbrace{\theta_H - \theta_L}_{\substack{Wage \\ Differential}} \geq \underbrace{c(e_H, \theta_H)}_{\substack{Additional\ cost \\ of\ education}}$$

- Similarly, her incentive to not deviate off the equilibrium education level $e \neq e_H \neq e_L$

$$\theta_H - c(e_H, \theta_H) \geq w(e) - c(e, \theta_H)$$

Separating PBE

- Figure 12.4 depicts a wage schedule $w(e)$ that provides the high-productivity worker with incentives to choose education level e_H, where she reaches IC_H, rather than deviating towards any other education $e \neq e_H$.
- High productivity worker does not have incentives to deviate from e_H to e_1 (lower utility level), and this applies to $e < e_H$ (including $e_L = 0$).
- No incentive either to deviate above e_H such as e_2 where her salary is represented by the height of point B.

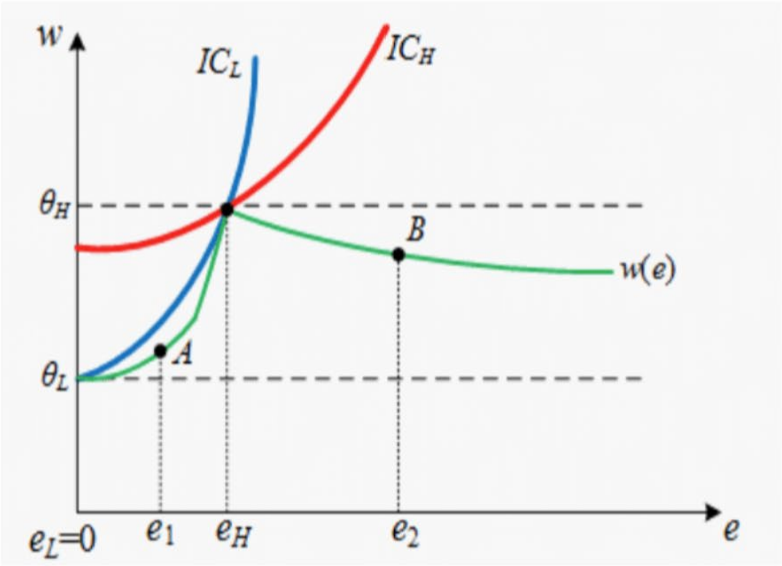


Figure 12.4 Separating PBE - Low and high-productivity worker.

Separating PBE

- Other wage schedules also induce the high-productivity worker to choose education e_H, such as that of Figure 12.5a.
- Intuitively, this wage schedule indicates that the firm pays:
  - the lowest salary $w(e) = \theta_L$ upon observing $e < e_H$,
  - but pays the highest salary otherwise.
- The figure also depicts education levels e_1 and e_2, confirming that the worker does not have incentives to deviate from e_H.

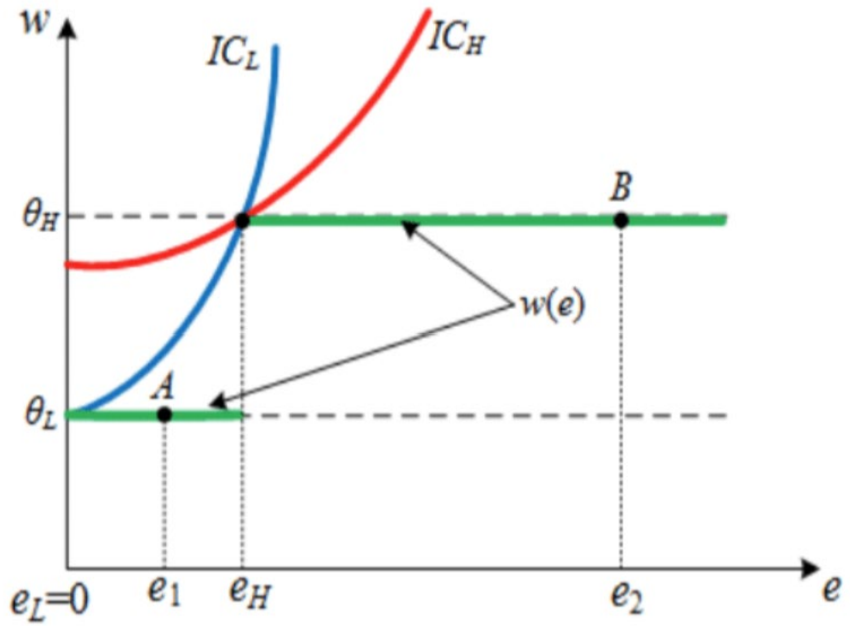


Figure 12.5a. Separating PBE – Case I

Separating PBE

- Different wage schedules help us support different education levels e_H for the high-productivity worker, such as that in Figure 12.5b, where the firm only pays the high salary θ_H when e_H is extremely high, that is, $e_H = e_4$.
- In this case, the high-productivity worker, despite receiving salary $w(e_4) = \theta_H$, is indifferent between acquiring education e_4 or deviating to $e_L = 0$.

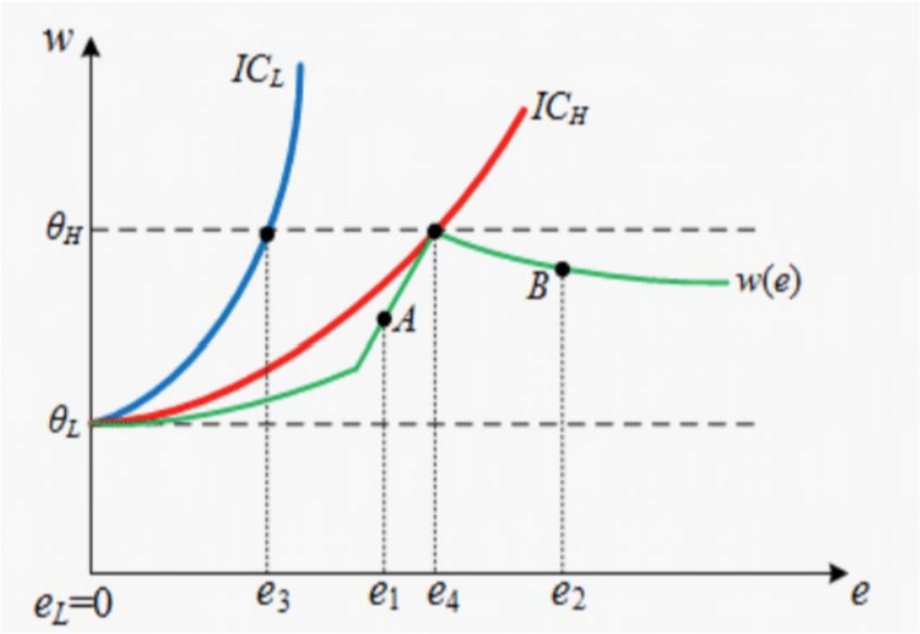


Figure 12.5b. Separating PBE – Case II

Separating PBE

- Specifically, e_4 solves
$$\theta_H - \theta_L = c(e_4, \theta_H)$$
- The least-costly separating PBE, e_3, cannot be supported with the wage schedule depicted in Figure 12.5b.
- It could be sustained with wage schedules such as those in Figures 12.4 or 12.5a.
- Overall, a wage schedule may help us sustain one equilibrium but not another one.

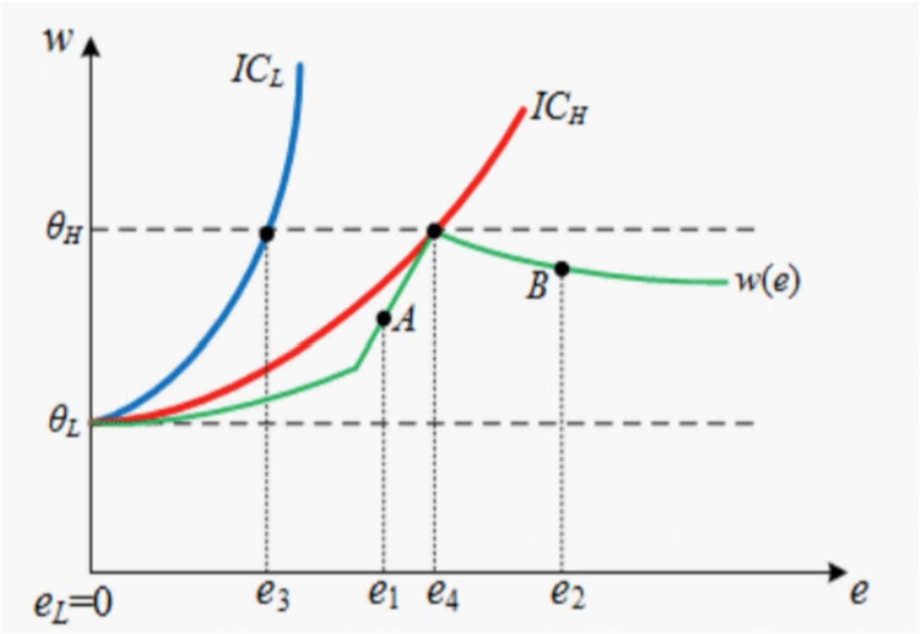


Figure 12.5b. Separating PBE – Case II

Example 12.2. Identifying the highest education that achieves separation

Following the setting in Example 12.1.

- Consider that $\theta_H = 2, \theta_L = 1$
- Cost of education is $c(e, \theta_K) = \frac{e^2}{\theta_K}$
- Solving for e_4 in $\theta_H - \theta_L = c(e, \theta_K)$

$$\Rightarrow \theta_H - \theta_L = \frac{e^2}{\theta_H}$$

$$\Rightarrow e_4 = \sqrt{2} \approx 1.41.$$

Example 12.2. Identifying the highest education that achieves separation

- More generally, in separating PBEs of the labor market signaling game,
  - Low-productivity worker chooses $e_L = 0$, the high-productivity worker selects an education level e_H in the range $[e_3, e_4]$
  - $e_H = e_4$ represents the "most-costly separating PBE"
  - $e_H = e_3$ represents the "least-costly separating PBE", since the high type conveys her type to the firm acquiring the lowest education level
- The least-costly separating education level $e_H = e_3$ solves

$$\theta_H - c(e_3, \theta_L) = \theta_L - c(0, \theta_L)$$

  - Intuitively, the low-productivity worker is indifferent between her equilibrium strategy $e_L = 0$, receiving a wage of θ_L, and deviating to education level e_3 which provides her with wage θ_H.

Example 12.3. Identifying the highest education that achieves separation

- Since $c(0, \theta_L) = 0$ by assumption, the above equation simplifies to

$$\theta_H - \theta_L = c(e_3, \theta_L)$$

- where e_3 solves $\theta_H - \theta_L = \frac{e^2}{\theta_L}$
- Since, $\theta_H = 2$ and $\theta_L = 1$, we obtain $e_3 = 1$.
- Therefore, $e_L = 0$ but e_H lies between $e_3 = 1$ and $e_4 = \sqrt{2} \approx 1.41$.

Separating PBE- Applying the Intuitive Criterion

- The most-costly separating PBE where the high-productivity worker chooses $e_H = e_4$ can only be sustained if:
  - the firm, upon observing off-the-equilibrium education levels in the interval $e_H \in [e_3, e_4)$, strictly below e_4, believes it doesn't face a high-productivity worker, i.e., $\mu(\theta_H | e_H) < 1$, and thus pays her strictly less than θ_H.
- But are these off-the-equilibrium beliefs sensible? No!
- We show this using the **Cho and Kreps' (1987) Intuitive Criterion** following the six-step approach.

Cho and Kreps' (1987) Intuitive Criterion

Step 1 – Identify a PBE to test. Consider a specific PBE, such as the most-costly separating PBE, where $(e_L, e_H) = (0, e_4)$.

Step 2 – Off-the-equilibrium message. Identify an off-the-equilibrium message for the worker, such as $e' \in [e_3, e_4)$, as depicted in Figure 12.6.

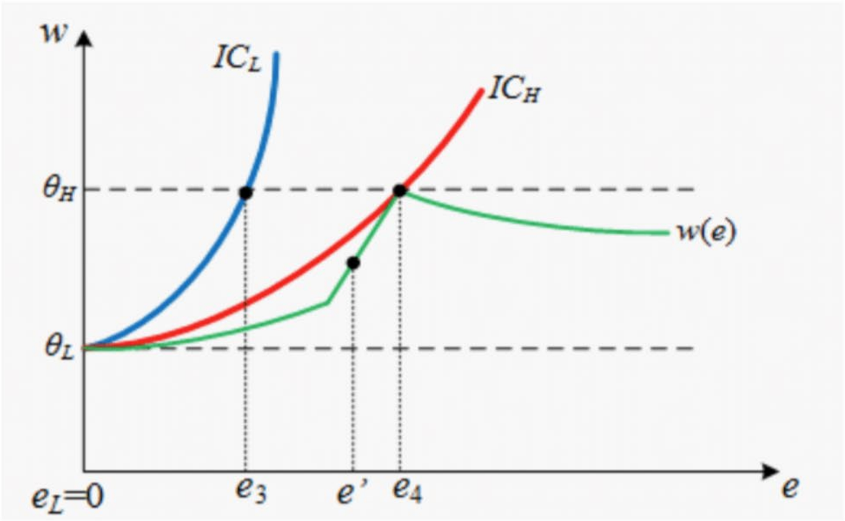


Figure 12.6. Applying the Intuitive Criterion to the separating PBEs.

Cho and Kreps' (1987) Intuitive Criterion

Step 3 – Types who profitably deviate. Find which types of worker can benefit by deviating to e'.

Low productivity worker has no incentive to deviate, because the cost of effort is too large for this type of worker even when the firm responds paying her the highest salary, $w(e') = \theta_H$. Formally,

$$\theta_L - 0 \geq \theta_H - c(e', \theta_L) \Rightarrow c(e', \theta_L) \geq \theta_H - \theta_L$$

Cho and Kreps' (1987) Intuitive Criterion

Step 3 – Types who profitably deviate. Find which types of worker can benefit by deviating to e'**.**

High productivity worker can benefit from choosing e'.

- Consider that firm keeps paying her highest salary $w(e') = \theta_H$
- If $e' < e_4$, she will deviate to e'. Formally,

$$\theta_H - c(e', \theta_H) \geq \theta_H - c(e_4, \theta_H) \Rightarrow c(e_4, \theta_H) \geq c(e', \theta_H)$$

which holds given $e_4 > e'$.

- Overall, this means that education level e' can only originate from the high-productivity worker.

Cho and Kreps' (1987) Intuitive Criterion

Step 4 – Restricting off-the-equilibrium beliefs.

- We can now restrict the off-the-equilibrium beliefs of the firm. If e' is observed, it can only originate from the high-productivity worker, i.e., $\mu(\theta_H|e') = 1$.

Step 5 – Updated Responses.

- Let us find the optimal response given the restricted belief $\mu(\theta_H|e') = 1$.
- As the firm is convinced of dealing with a high-productivity worker, it optimally responds paying $w(e') = \theta_H$.

Step 6 – Conclusion.

- Given the optimal response in step 5, high productivity worker has incentive to deviate from e_4 to e'.
- Therefore, the most-costly separating PBE $(e_L, e_H) = (0, e_4)$ violates the Intuitive Criterion.

Summary: Cho and Kreps' (1987) Intuitive Criterion

- A similar argument applies to any other separating PBE where $e_H \in (e_3, e_4]$ since only the high-productivity worker has incentives to deviate to $e' < e_4$.
- At the least-costly separating PBE, her choice is $e_H = e_3$, and it survives the Intuitive Criterion. How?
  - Both types of worker can benefit from deviating to education levels strictly below e_3, implying that the firm cannot restrict its beliefs upon observing off-the-equilibrium education e' satisfying $e' < e_3$, keeping its beliefs unaltered relative to those in the separating PBE.
  - Only one separating PBE survives the Intuitive Criterion, namely, the least-costly separating PBE where $(e_L, e_H) = (0, e_3)$.

Separating PBE – Applying the D1 Criterion

- From section 11.3, we know that, in signaling games with only two sender types, the Intuitive and D1 Criterion have the same refinement power.
  - Same set of PBEs survives both refinement criteria.
- As a practice, show that the separating PBE in Figure 12.7, where $e_L = 0$ and $e_H = e_2$, violates the Intuitive Criterion.
- We next show that it also violates the D1 criterion.

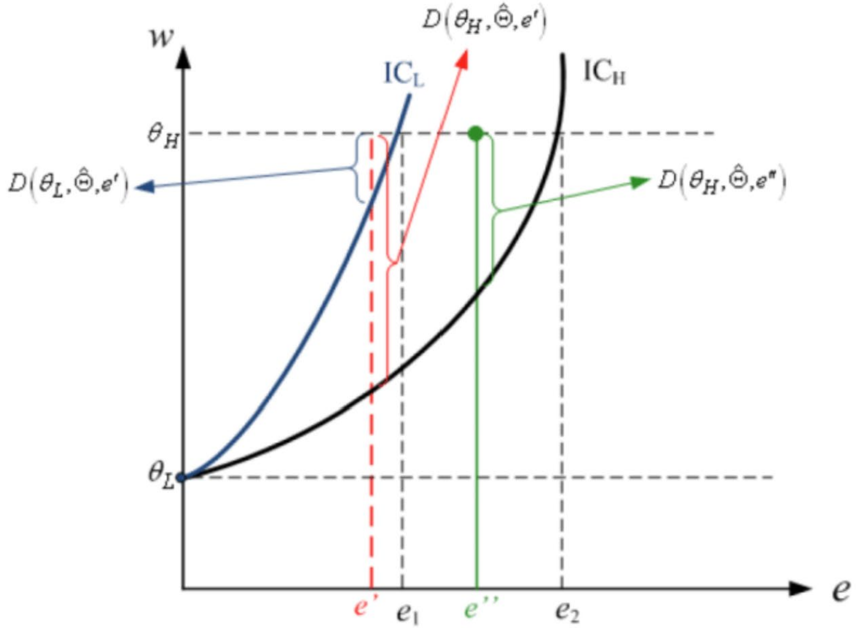


Figure 12.7. Applying the D1 Criterion to the separating PBEs.

Separating PBE – Applying the D1 Criterion

Step 1 —Identify a PBE to test.

- Consider a specific PBE, such as the separating PBE where $(e_L, e_H) = (0, e_2)$.

Step 2 —Off-the-equilibrium message.

- Identify an off-the-equilibrium message for the worker, such as $e' \in [0, e_1)$, as depicted in Figure 12.7.

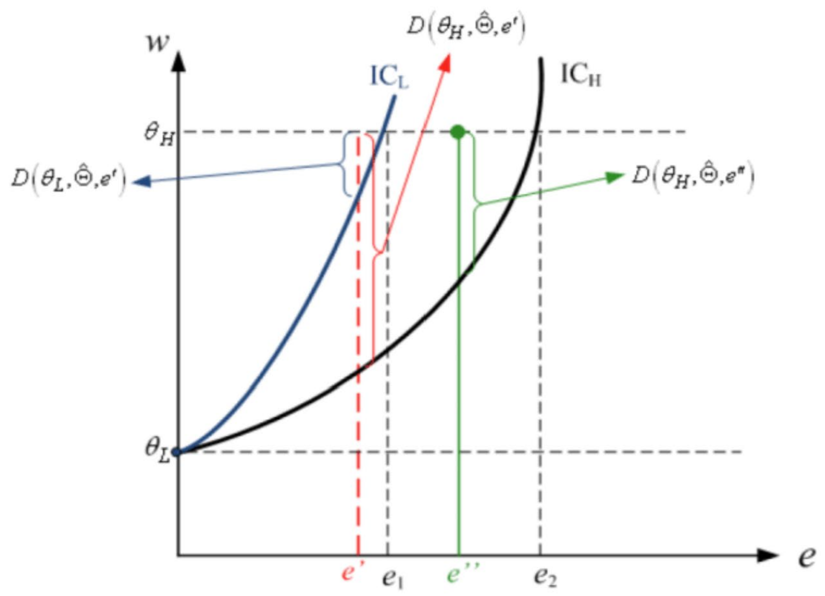


Figure 12.7. Applying the D1 Criterion to the separating PBEs.

Separating PBE – Applying the D1 Criterion

Step 3 — Type most likely to deviate.

- Find which worker type is most likely to benefit from deviating to e'
- $D\left(\theta_L, \widehat{\Theta}, e'\right)$ and $D\left(\theta_H, \widehat{\Theta}, e'\right)$ respectively denotes set of wages that make low productivity and high productivity worker better off after deviating to e'.
- Comparing $D\left(\theta_L, \widehat{\Theta}, e'\right)$ and $D\left(\theta_H, \widehat{\Theta}, e'\right)$, we see that $D\left(\theta_L, \widehat{\Theta}, e'\right)$ is a subset of $D\left(\theta_H, \widehat{\Theta}, e'\right)$, indicating that:
  - after deviating to e', there are more wage offers that improve the equilibrium payoff of the high-productivity worker than those improving the low-productivity worker's.

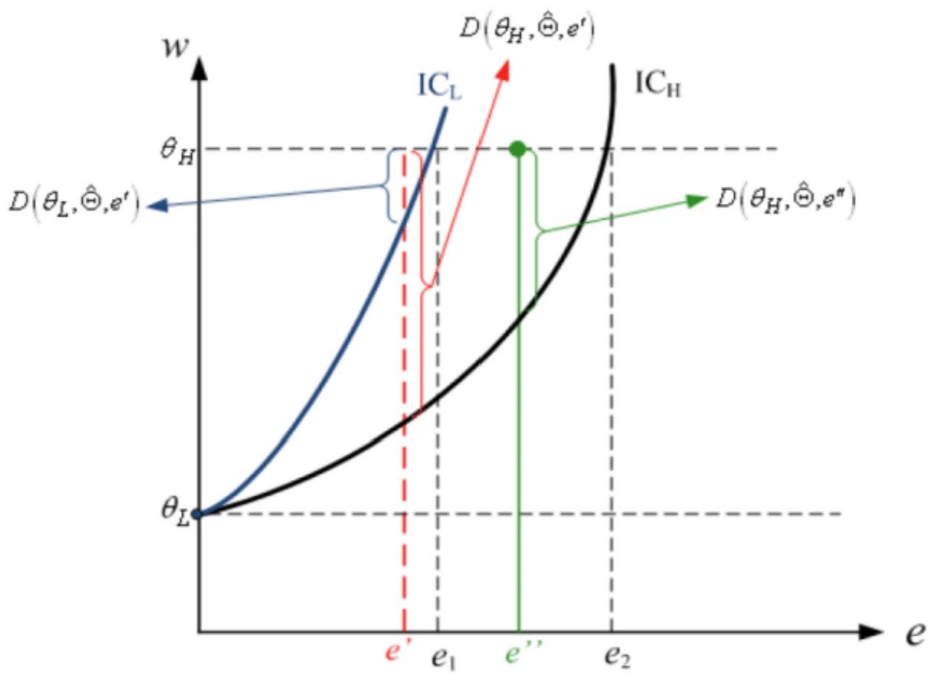


Figure 12.7. Applying the D1 Criterion to the separating PBEs.

Separating PBE – Applying the D1 Criterion

Step 3 — Type most likely to deviate.

- We can, then, conclude that the high-productivity worker is more likely to send message e',
- which helps the firm update its off-the-equilibrium beliefs:
  - upon observing e', $\mu(\theta_H|e') = 1$.

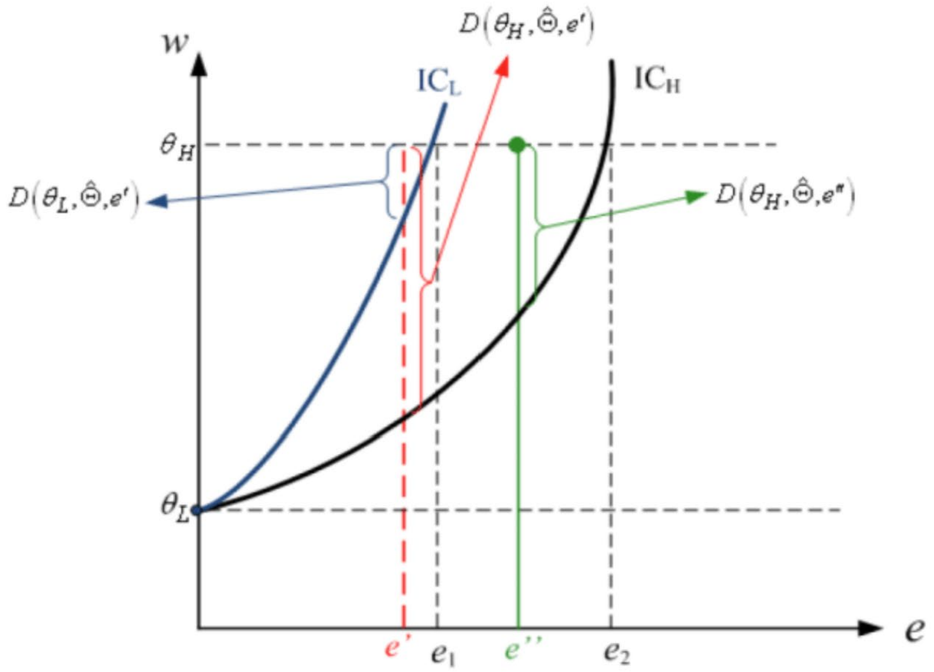


Figure 12.7. Applying the D1 Criterion to the separating PBEs.

Separating PBE – Applying the D1 Criterion

Step 3 — Type most likely to deviate.

- For completeness, the figure also includes deviations to off-the-equilibrium message e'', where $e'' \in (e_1, e_2)$.
- The low-productivity worker would never benefit from deviating to e'', entailing that the set of wage offers improving her equilibrium utility $D\left(\theta_L, \widehat{\Theta}, e''\right)$, is in this case nil.

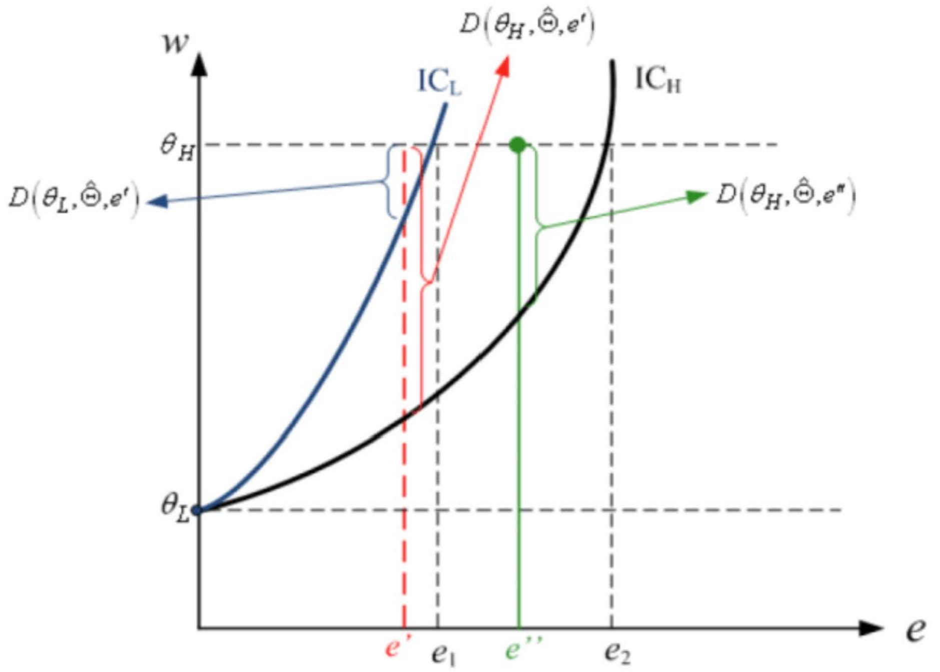


Figure 12.7. Applying the D1 Criterion to the separating PBEs.

Separating PBE – Applying the D1 Criterion

Step 3 — Type most likely to deviate.

- In contrast, sending e'' might be profitable for the high-productivity worker, as depicted in the figure.
- Therefore, upon observing e'', the firm's updated beliefs are also $\mu(\theta_H|e'') = 1$.

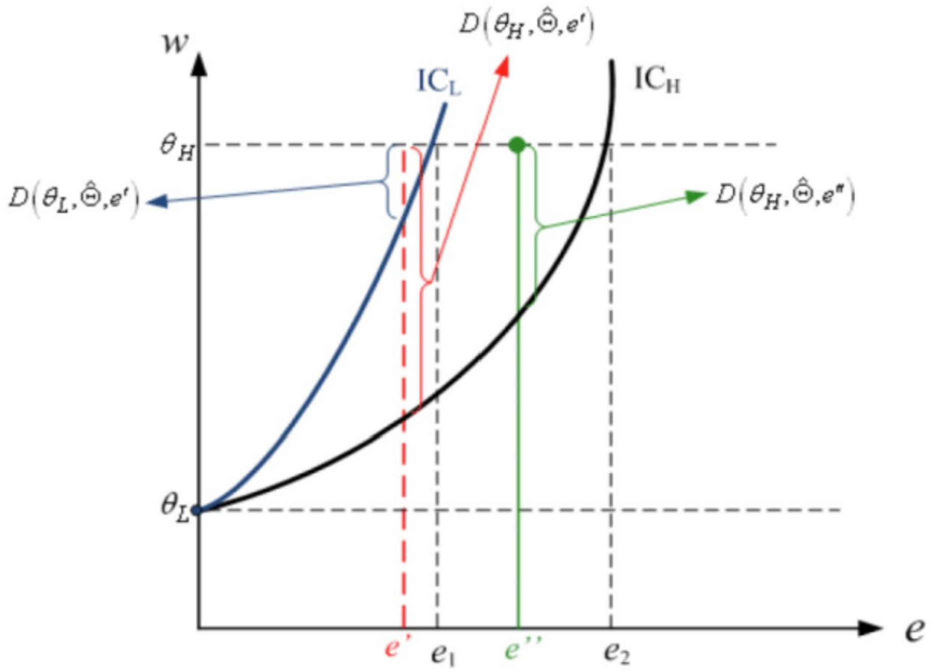


Figure 12.7. Applying the D1 Criterion to the separating PBEs.

Separating PBE – Applying the D1 Criterion

Step 4 — Restricting off-the-equilibrium beliefs.

- We can now restrict the off-the-equilibrium beliefs of the firm.
- From Step 3, we found that, any deviation from the separating PBE $(e_L, e_H) = (0, e_2)$, where $e' < e_2$, is more likely to originate from the high- than the low-productivity worker,
  - so the firm's beliefs become $\mu(\theta_H | e') = 1$.

Step 5 — Updated Responses.

- Because the firm is convinced of dealing with a high-productivity worker, it optimally responds offering a salary $w(e') = \theta_H$.

Separating PBE – Applying the D1 Criterion

Step 6 — Conclusion.

- Given the optimal response found in Step 5, the high-productivity worker deviate from her equilibrium strategy of e_2 to e'.
- Therefore, separating PBE $(e_L, e_H) = (0, e_2)$ violates the D1 Criterion.

- Like in the Intuitive Criterion, one can show that all separating PBEs in this game can be eliminated using the D1 Criterion, except for the efficient (Riley) outcome, where:
  - the low-productivity worker acquires no education, $e_L = 0$, and
  - the high-productivity worker acquires the lowest education level that helps convey her type to the firm, $e_H = e_1$.

Pooling PBE

First Step – Strategy Profile.

- In a pooling strategy profile, both the high- and low-productivity worker choose education level e_P, where the subscript P denotes pooling equilibrium.

Second Step – Updating Beliefs.

- Upon observing education level e_P:
  - The firm's posterior beliefs coincide with its prior, i.e., $\mu(\theta_H|e_P) = p$.
  - Intuitively, the observation of education level e^P provides the firm with no additional information about the worker's productivity since all worker types acquire the same education.
- Upon observing the off-the-equilibrium education level $e \neq e_P$:
  - The firm cannot update its beliefs using Bayes' rule,
  - leaving its off-the-equilibrium beliefs unrestricted, that is $\mu(\theta_H|e) \in [0,1]$.

Pooling PBE

Third step – Optimal responses.

- Upon observing the pooling education level e^P, the firm optimally responds with salary

$$w(e^P) = \underbrace{p\theta_H + (1-p)\theta_L}_{\substack{Worker's\ expected \\ productivity \equiv E[\theta]}}$$

- After observing any off-the-equilibrium education $e \neq e_P$, the firm responds with

$$w(e) \in [\theta_L, \theta_H]$$

Technically,

$$w(e) = \mu(\theta_H|e)\theta_H + \big(1 - \mu(\theta_H|e)\big)\theta_L$$

since its off-the-equilibrium beliefs are $\mu(\theta_H|e) \in [0,1]$.

Pooling PBE

Third step – Optimal responses.

- Figure 12.8 depicts, as an example, a wage schedule that satisfies the above two properties:
  - at the pooling education level e^P, the salary is $w(e^P) = E[\theta]$, and
  - for all other education levels $e \neq e_P$, the firm pays salaries that are bounded between θ_L and θ_H.

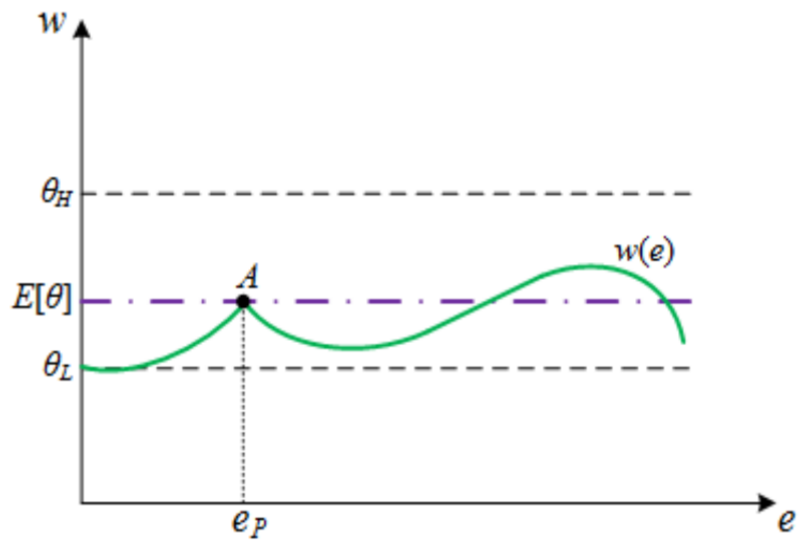


Figure 12.8. Pooling PBE - Example of wage schedule.

Pooling PBE

Fourth step – Optimal messages.

- Identify under which conditions both types of workers choose the same education level e^P

<u>Low-productivity worker</u>

- IC_L originates at $(\theta_L, 0)$, where the worker acquires no education and receives the lowest salary θ_L, and passes through point A, which represents $(w, e) = (E[\theta], e^P)$.

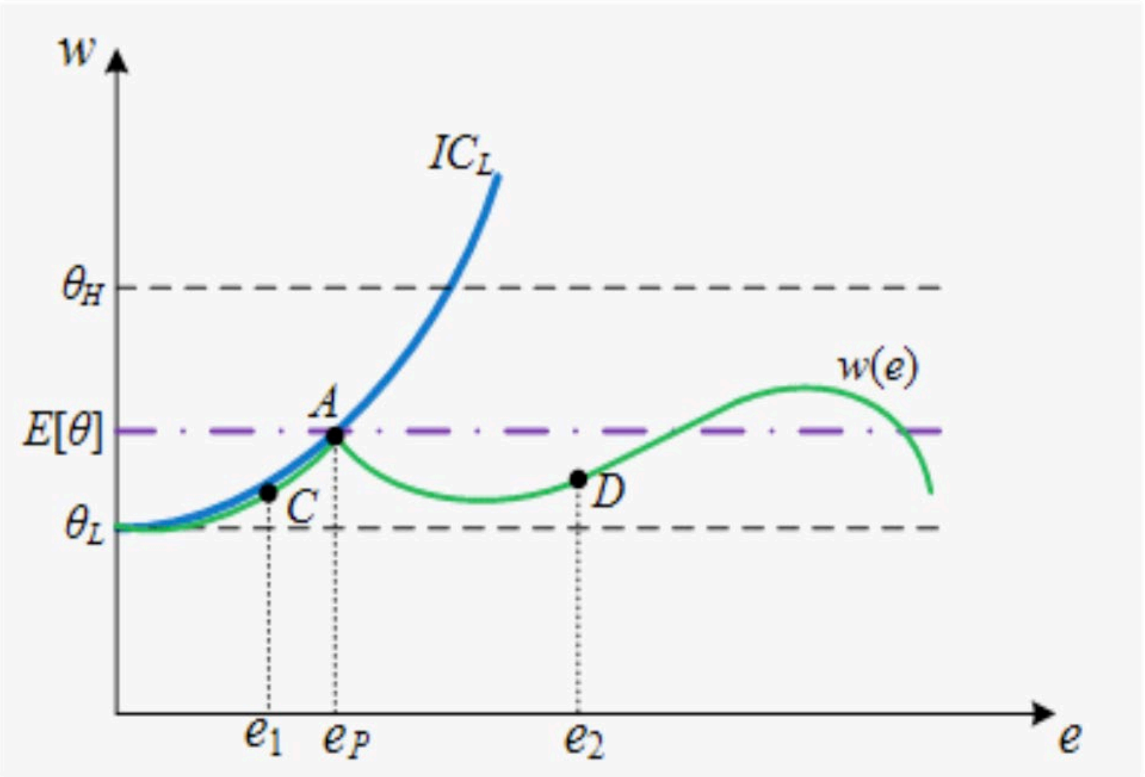


Figure 12.9. Pooling PBE – Low-productivity worker

Pooling PBE

Fourth step – Optimal messages.

<u>Low-productivity worker</u>

- This type of worker is, then, indifferent between:
  - identifying herself as a low-productivity worker (i.e., acquiring no education and receiving salary θ_L) and
  - acquiring the pooling level e^P.

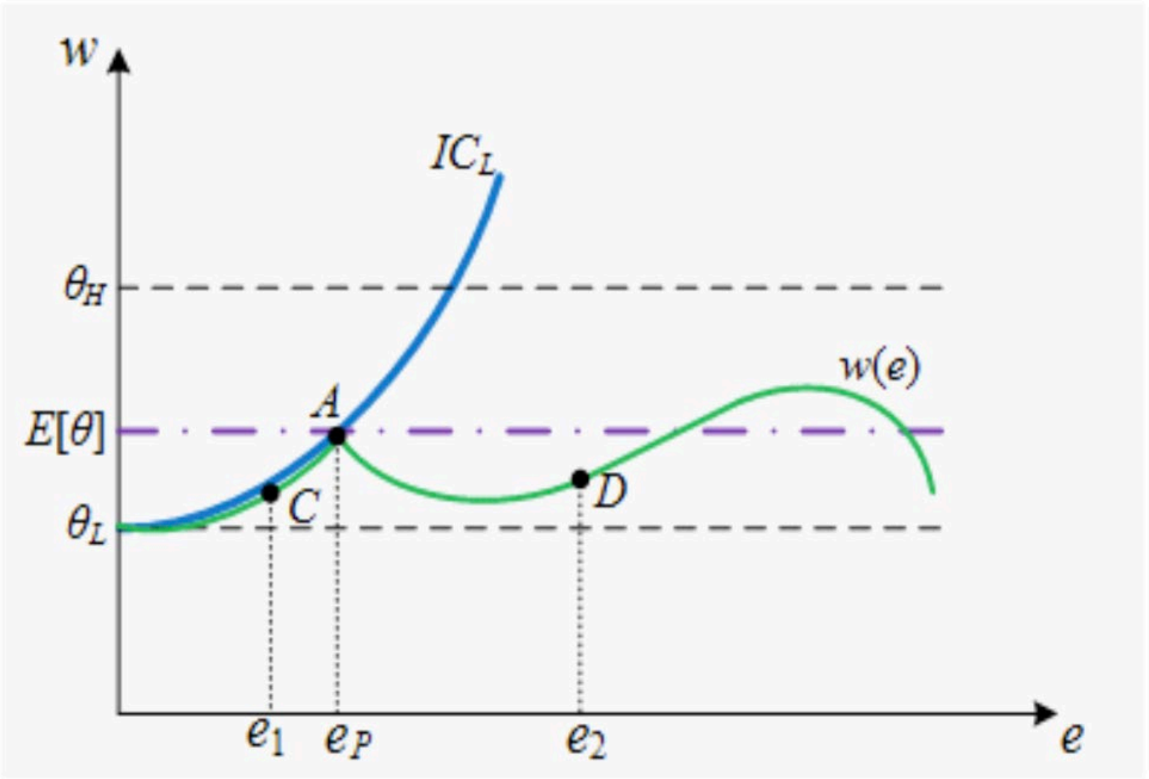


Figure 12.9. Pooling PBE – Low-productivity worker

Pooling PBE

Fourth step – Optimal messages.

<u>Low-productivity worker</u>

- The IC_L curve in Figure 12.9 must satisfy

$$\theta_L - c(0, \theta_L) = E[\theta] - c(e^P, \theta_L)$$

and, given that $c(0, \theta_L) = 0$ by definition,

$$c(e^P, \theta_L) = E[\theta] - \theta_L$$

where $E[\theta] > \theta_L$ since $p > 0$.

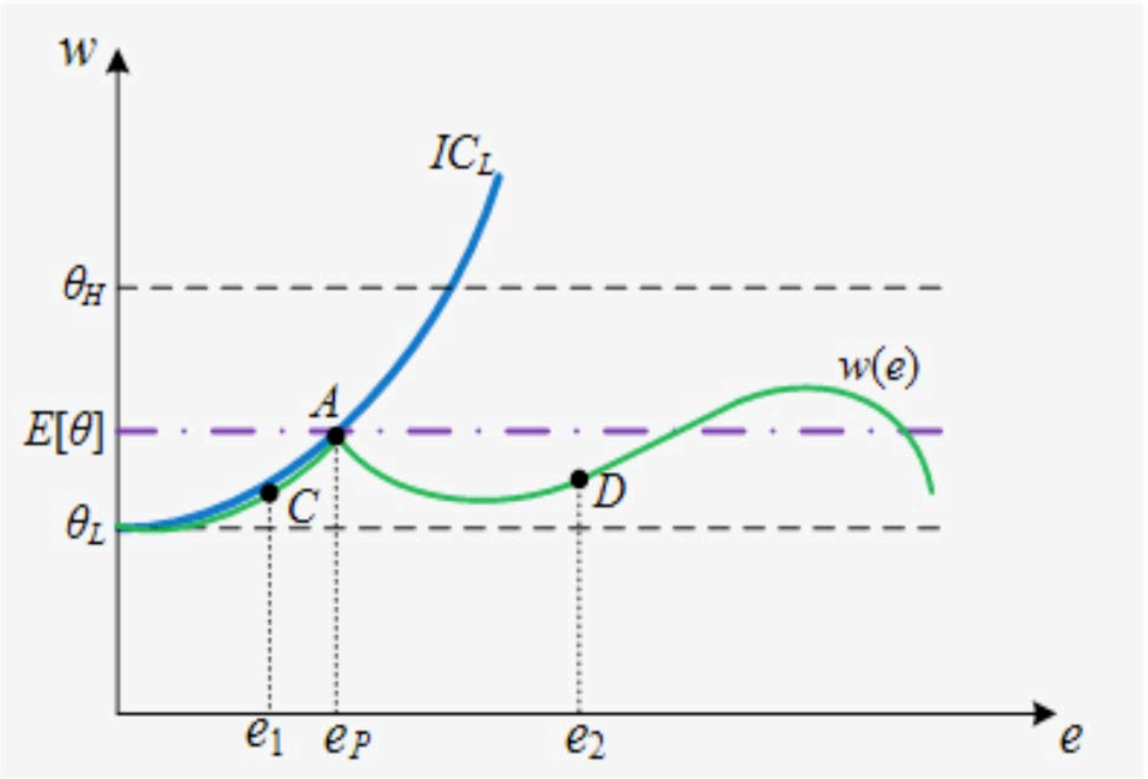


Figure 12.9. Pooling PBE – Low-productivity worker

Pooling PBE

Fourth step – Optimal messages.

<u>Low-productivity worker</u>

- For this to be the case, we need the wage schedule $w(e)$ to lie weakly below IC_L:
  - since that implies that, a deviation towards any education level $e \neq e_P$ produces an overall utility lower than that at e^P for the low-productivity worker.

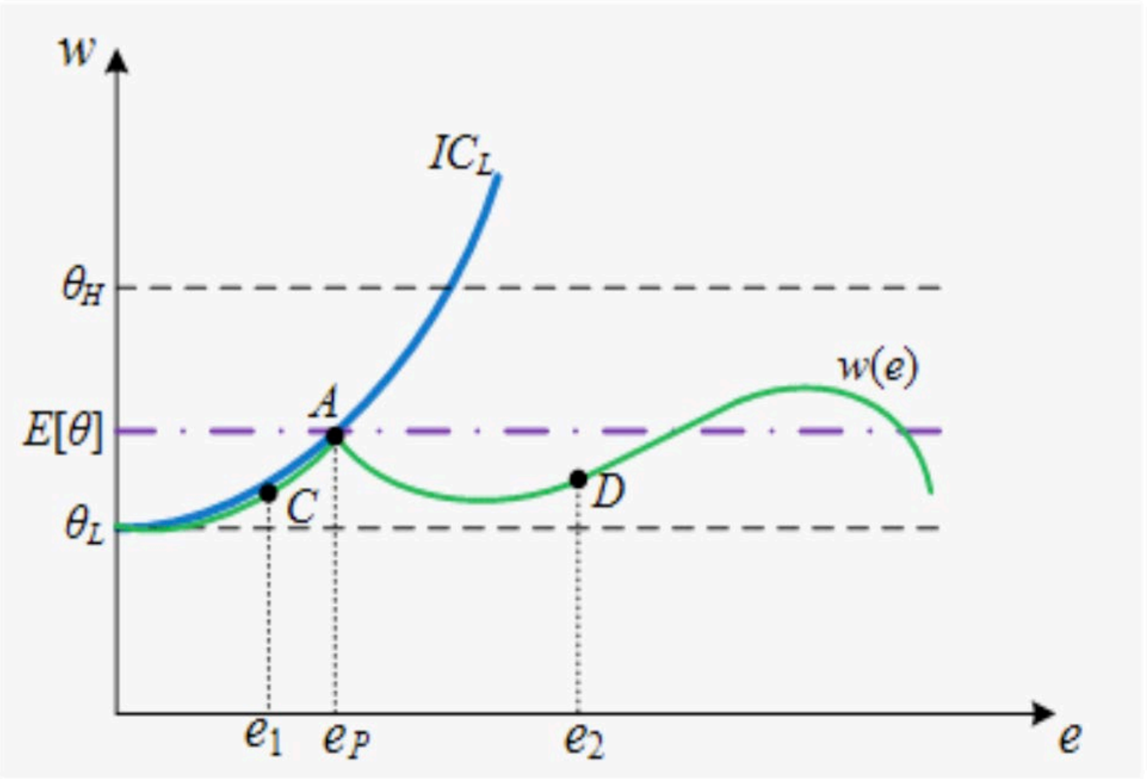


Figure 12.9. Pooling PBE – Low-productivity worker

Example 12.4. Finding a pooling education level

- Consider the parametric example of Example 12.1, where $\theta_H = 2$, $\theta_L = 1$, and the cost of education is $c(e, \theta_K) = \frac{e^2}{\theta_K}$.
- If we assume $p = \frac{1}{3}$, we obtain an expected productivity of

$$E[\theta] = \frac{1}{3}2 + \frac{2}{3}1 = \frac{4}{3}.$$

- The above equation becomes:

$$c(e^P, \theta_L) = E[\theta] - \theta_L \;\Rightarrow\; (e^P)^2 = \frac{4}{3} - 1$$

which yields a pooling education level of $e^P = 0.58$.

Pooling PBE

<u>High-productivity worker</u>

- Must have incentives to choose e^P rather than deviating to any other education level $e \neq e_P$.
- Figure 12.10a illustrates a wage schedule $w(e)$ that leaves no incentives to deviate from e^P to this type of worker.
- Deviation to e_2 yields a lower salary than $w(e^P) = E[\theta]$ and a higher education cost!
- Deviation to e_1 yields a lower salary and lower education cost, but it's not worthy (point *C* lies to the southeast of *A*).

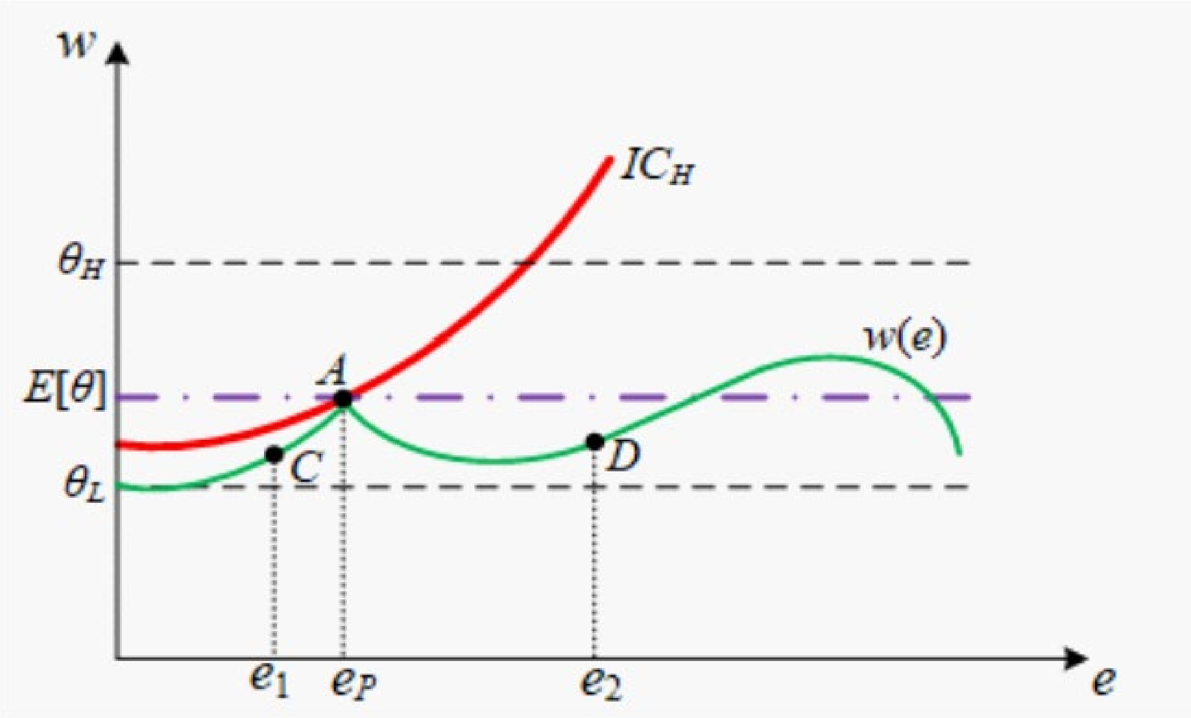


Figure 12.10a. Pooling PBE – High-productivity worker

Pooling PBE

<u>High-productivity worker</u>

- Figure 12.10b superimposes:
  - Figure 12.9 from our analysis of the low-productivity worker, and
  - Figure 12.10a, from our discussion of the high-productivity worker.
- Wage schedule $w(e)$ must lie weakly below both IC_L and IC_H
  - for both types of workers to have incentives to choose e^P instead of deviating,
- The pooling education level lies at the point where IC_L and IC_H cross each other.
  - Alternatively, at the point where IC_L that originates at $(\theta_L, 0)$ crosses $E[\theta]$.

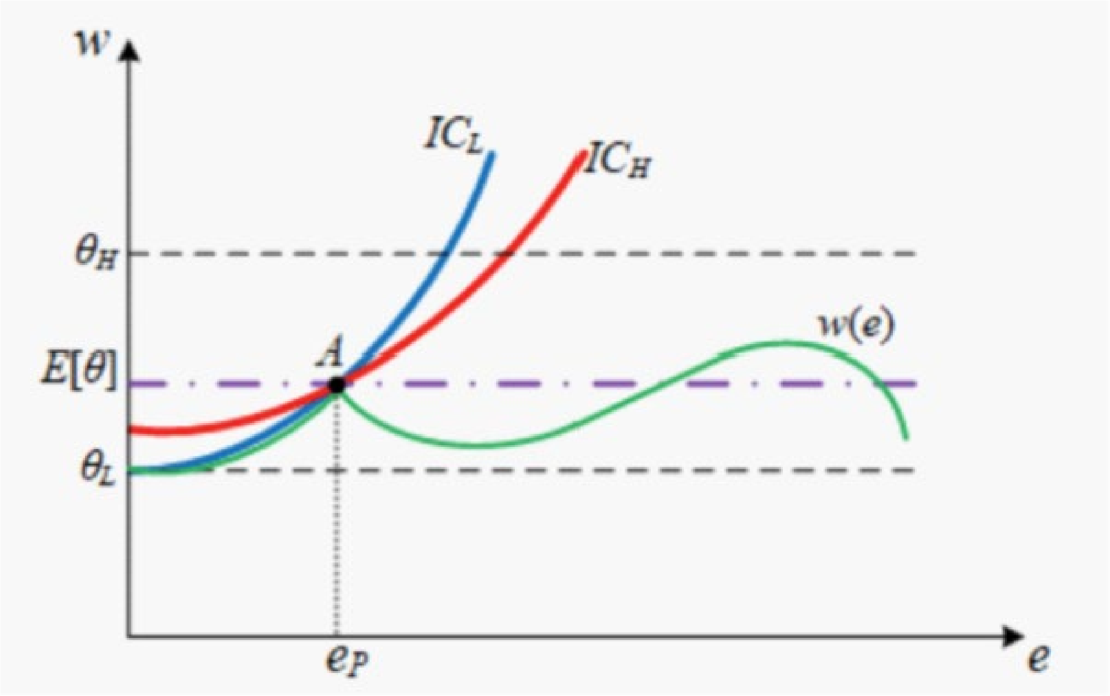


Figure 12.10b. Pooling PBE – Both worker types

Other Pooling PBE

- Are there any other pooling PBEs? Yes!
- If IC_L originates strictly above θ_L:
  - the crossing point of IC_L and IC_H happens closer to the origin, as depicted in Figure 12.11.
- A similar argument applies if we keep increasing the origin of IC_L until we reach a crossing point at $e^P = 0$.
- In this case,
  - IC_L originates at a height of $E[\theta]$.
  - Both types of workers acquire zero education, and they receive a salary equal to their expected productivity $E[\theta]$.
  - Same as in incomplete information game where workers cannot acquire education to signal their type.

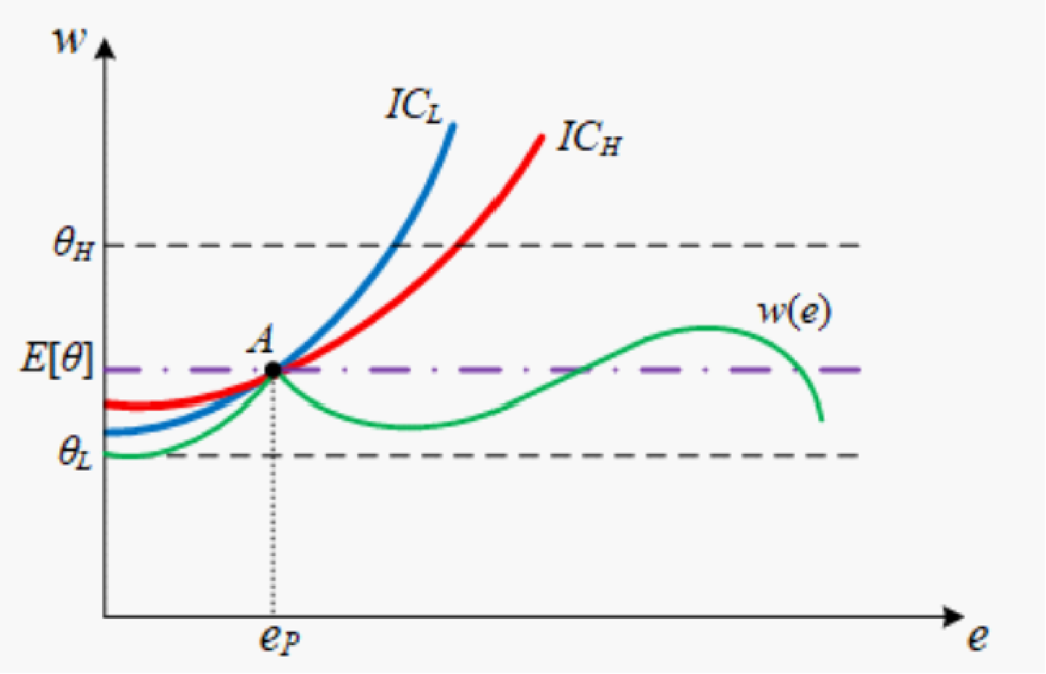


Figure 12.11. Pooling PBE - Other equilibria.

Other Pooling PBE

- Therefore, we can summarize all pooling PBEs as (w, e)-pairs where both workers choose $e^P \in [0, e_A]$, where education e_A is the "most costly pooling PBE" and solves

$$\theta_L - c(0, \theta_L) = E[\theta] - c(e^P, \theta_L)$$

or, given that $c(0, \theta_L) = 0$ by definition,

$$c(e^P, \theta_L) = E[\theta] - \theta_L$$

- This equation just identifies the education level where the IC_L starting at a height of θ_L crosses the horizontal line $E[\theta]$ in Figure 12.10b.

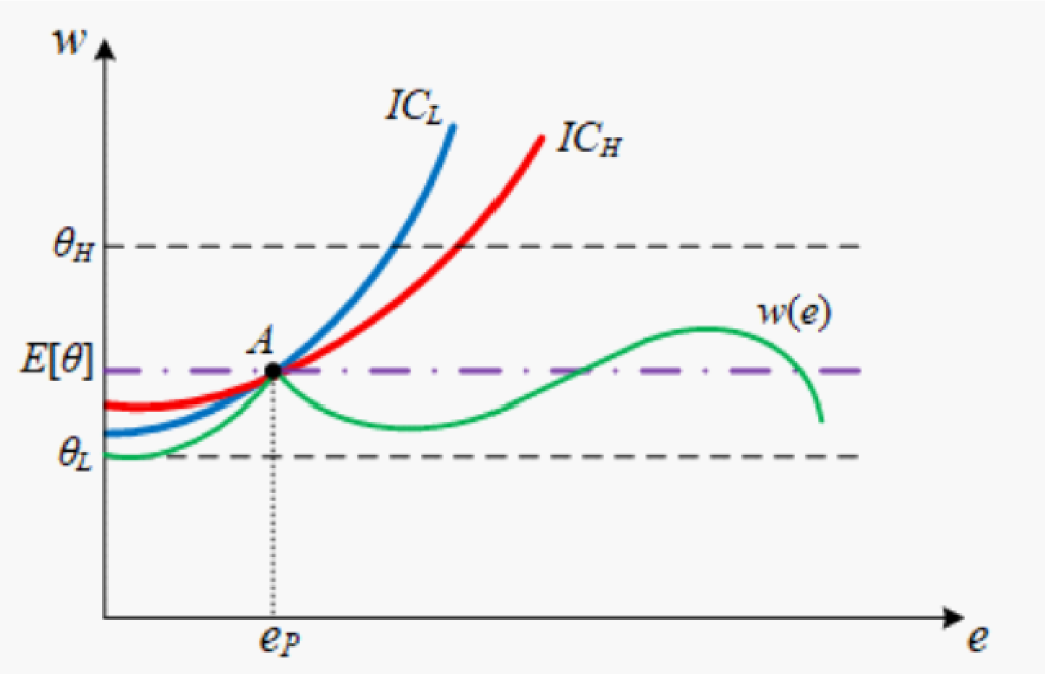


Figure 12.11. Pooling PBE - Other equilibria.

Other Pooling PBE

- Intuitively, at this education level, the low-productivity worker is indifferent between:
  - the pooling education e^P, earning a salary $w(e^P) = E[\theta]$, and
  - a zero-education level receiving a salary of θ_L.
- In all these pooling PBEs, the firm responds with:
  - a salary $w(e^P) = E[\theta]$ upon observing e^P, and
  - a wage schedule $w(e)$ that lies below both IC_L and IC_H for all $e \neq e_P$.

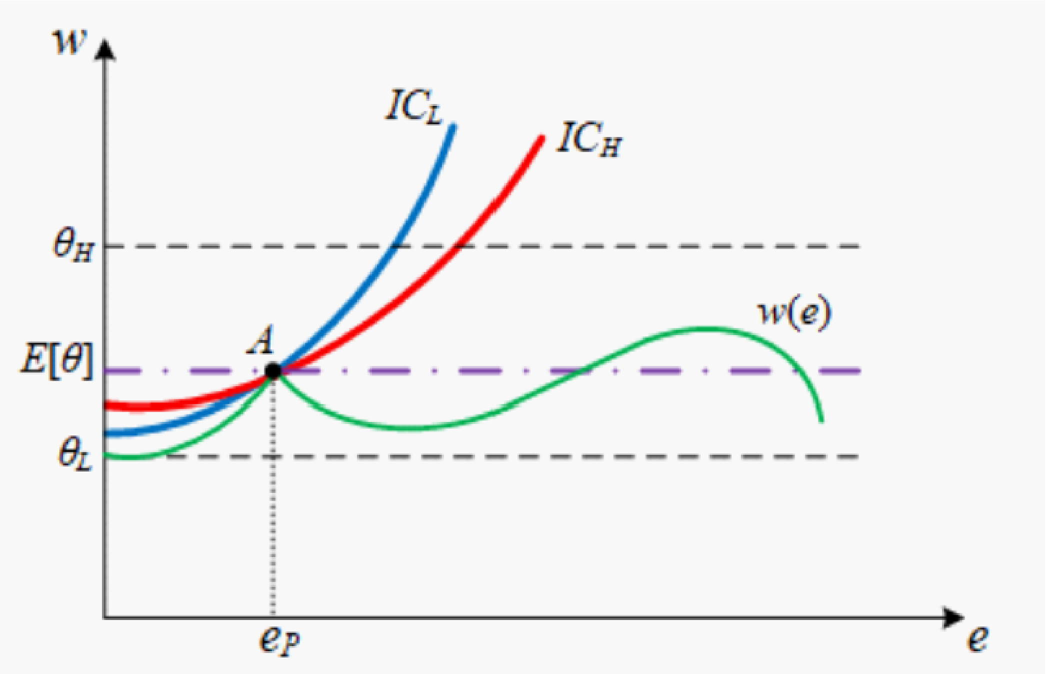


Figure 12.11. Pooling PBE - Other equilibria.

Other Pooling PBE

- Therefore, we have a range of pooling PBEs:
  - from the least-costly pooling PBE where both types of workers acquire zero education, $e^P = 0$, to…
  - the most-costly pooling PBE where both acquire the highest education $e^P = e_A$.

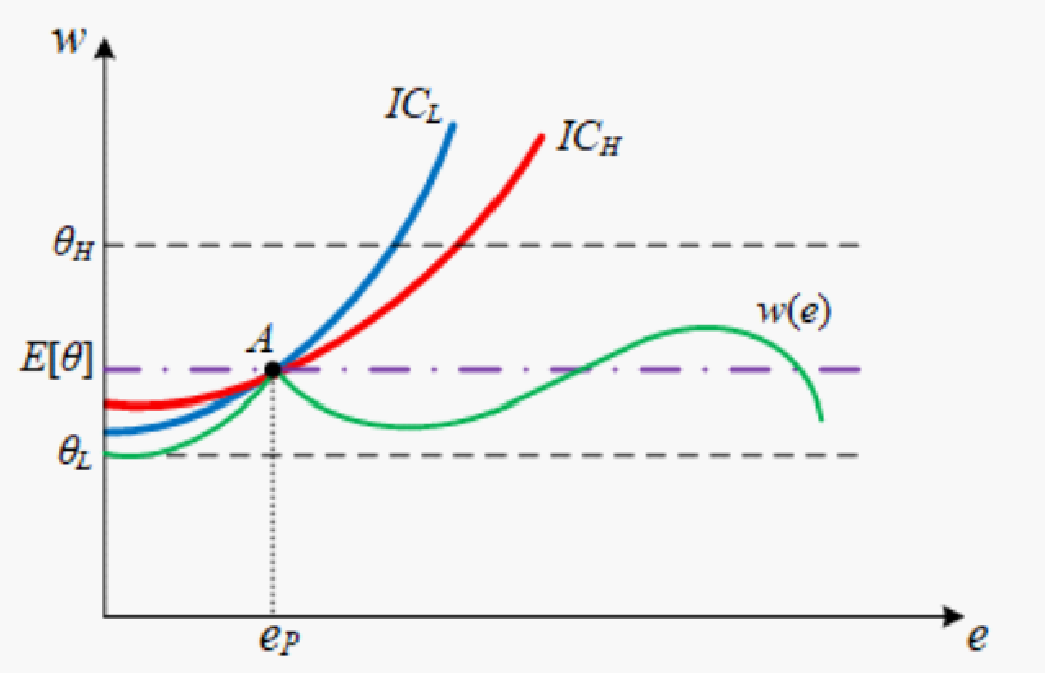


Figure 12.11. Pooling PBE - Other equilibria.

Pooling PBE – Applying the Intuitive Criterion

- We identified under which conditions we can sustain pooling PBEs, but…
  - Do they survive the Intuitive Criterion? No.
- Consider the off-the-equilibrium beliefs for any education level $e \neq e_P$.
- The condition that $w(e)$ lies below IC_L and IC_H for all $e \neq e_P$ means that:
  - upon observing a deviation from e^P, the firm infers that such a deviation is not likely originating from the high-productivity worker
  - thus, paying a relatively low wage as depicted by the height of $w(e)$ in the right-hand side of Figure 12.12.

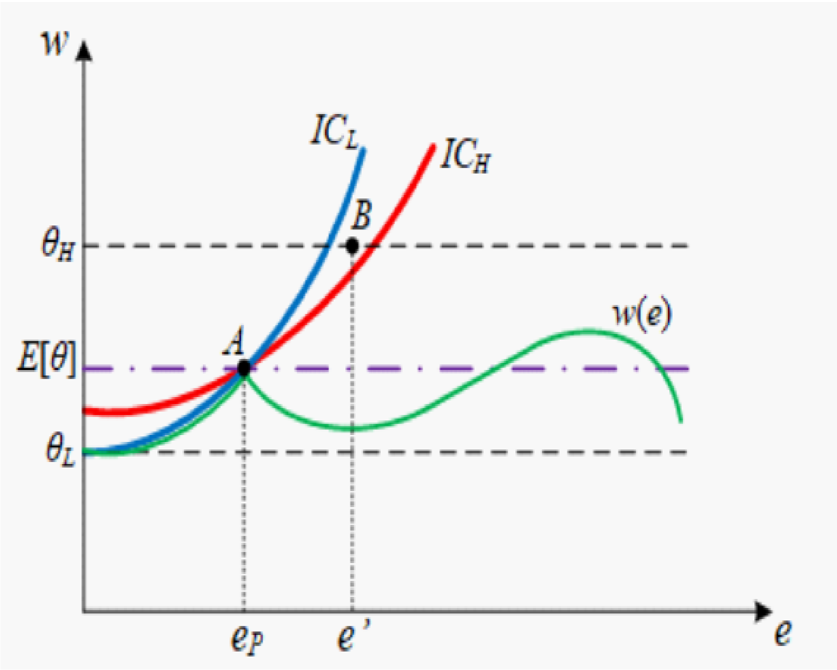


Figure 12.12. Applying the Intuitive Criterion to Pooling PBEs.

Pooling PBE – Applying the Intuitive Criterion

- This off-the-equilibrium beliefs are, of course, not sensible:
  - one could argue that deviations towards high education levels (higher than e_P in figure 12.12)
  - are more likely to stem from the high-type worker.
- We next show this point, following the six-steps approach to the Intuitive Criterion.

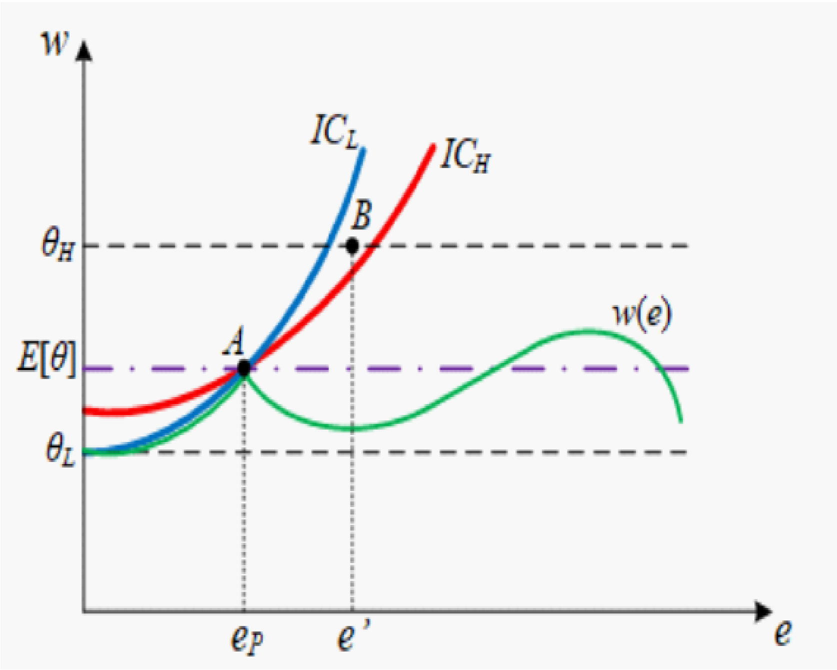


Figure 12.12. Applying the Intuitive Criterion to Pooling PBEs.

Pooling PBE – Applying the Intuitive Criterion

Step 1 – Identify a PBE to test.

Consider a specific PBE, such as the most-costly pooling PBE where $e^P = e_A$.

Step 2 – Off-the-equilibrium message.

Identify an off-the-equilibrium education, such as $e' > e_A$ in figure 12.12.

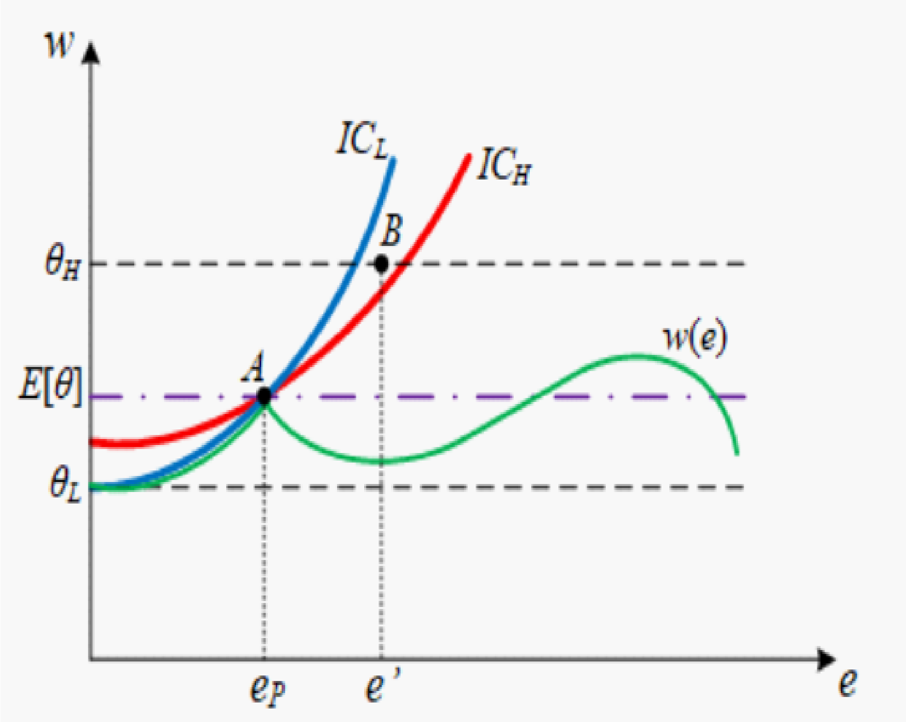


Figure 12.12. Applying the Intuitive Criterion to Pooling PBEs.

Pooling PBE – Applying the Intuitive Criterion

Step 3 – Types who profitably deviate.

Low-productivity worker: cannot benefit.

Even if the firm responds paying her the highest salary, $w(e') = \theta_H$, the cost of effort is too large for this type of worker.

Formally,

$$E[\theta] - c(e_A, \theta_L) \geq \theta_H - c(e', \theta_L)$$

$$\Rightarrow c(e', \theta_L) - c(e_A, \theta_L) \geq \theta_H - E[\theta]$$

Intuitively, the additional cost that the worker must incur offsets the wage increase she experiences.

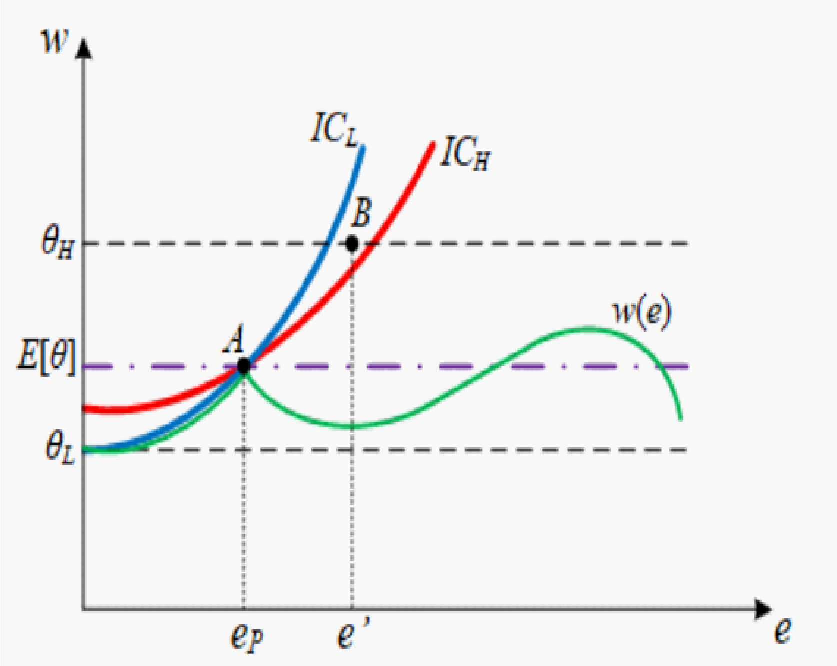


Figure 12.12. Applying the Intuitive Criterion to Pooling PBEs.

Pooling PBE – Applying the Intuitive Criterion

Step 3 – Types who profitably deviate.

High-productivity worker: can benefit from choosing e' if

$$E[\theta] - c(e_A, \theta_H) \leq \theta_H - c(e', \theta_L)$$

$$\Rightarrow c(e', \theta_H) - c(e_A, \theta_H) \leq \theta_H - E[\theta]$$

- Intuitively, if the firm responds to the higher education level e' by paying her the highest salary $w(e') = \theta_H$...
  - her wage increase (right-hand side) would offset the additional education cost (left-hand side).

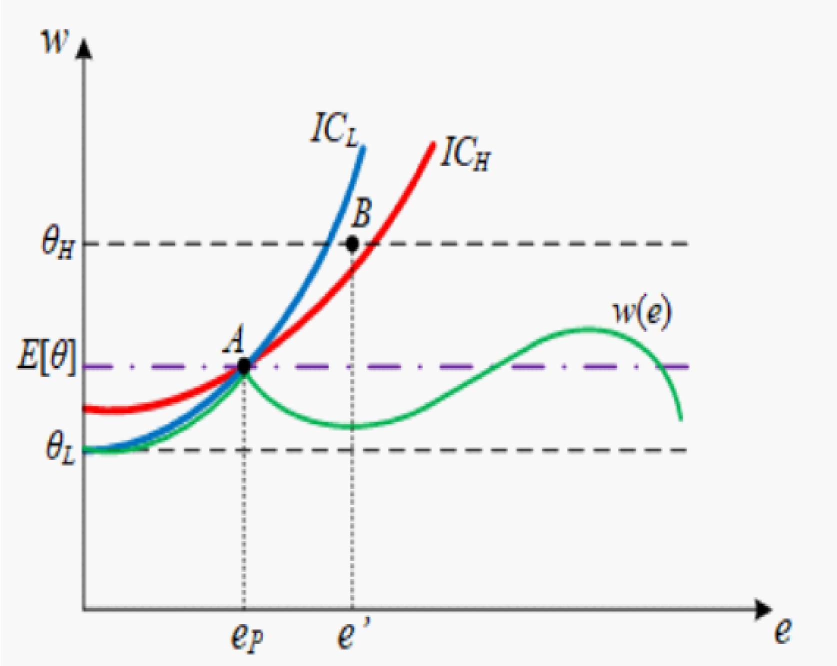


Figure 12.12. Applying the Intuitive Criterion to Pooling PBEs.

Pooling PBE – Applying the Intuitive Criterion

Step 3 – Types who profitably deviate.

- At this point, we can combine the above two inequalities to obtain

$$c(e',\theta_H) - c(e_A,\theta_H) \leq \theta_H - E[\theta] \leq c(e',\theta_L) - c(e_A,\theta_L)$$

$$\Rightarrow c(e',\theta_H) - c(e_A,\theta_H) \leq c(e',\theta_L) - c(e_A,\theta_L)$$

- which is true by definition (single-crossing property in discrete settings).

- Overall, this means that education level e' can only originate from the high-productivity worker.

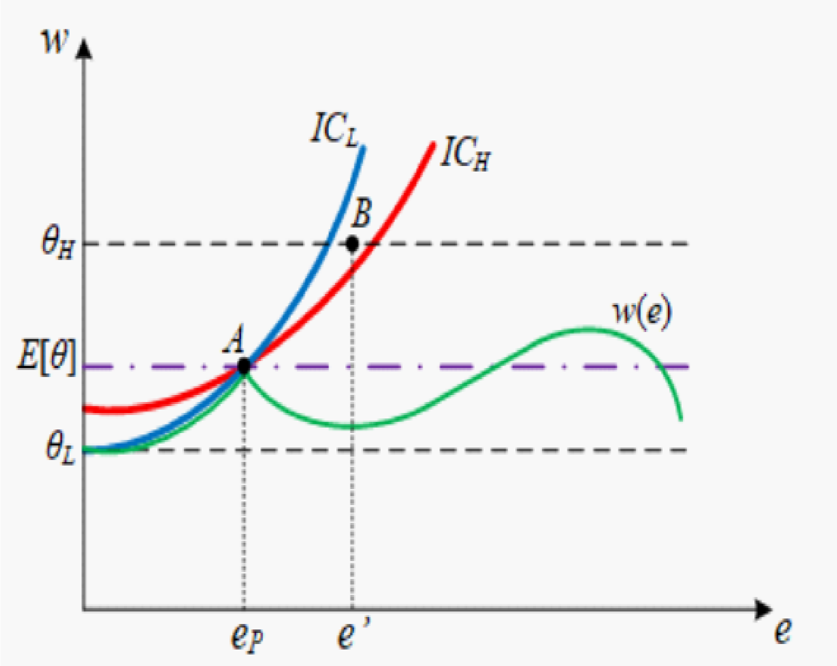


Figure 12.12. Applying the Intuitive Criterion to Pooling PBEs.

Pooling PBE – Applying the Intuitive Criterion

Step 4 – Restricting off-the-equilibrium beliefs.

- We can now restrict the firm's off-the-equilibrium beliefs as follows:
  - If education level e' is observed, it can only originate from the high-productivity worker, i.e., $\mu(\theta_H|e') = 1$.

Step 5 – Updated Responses.

- As the firm is convinced of dealing with a high-productivity worker, it optimally responds paying $w(e') = \theta_H$.

Pooling PBE – Applying the Intuitive Criterion

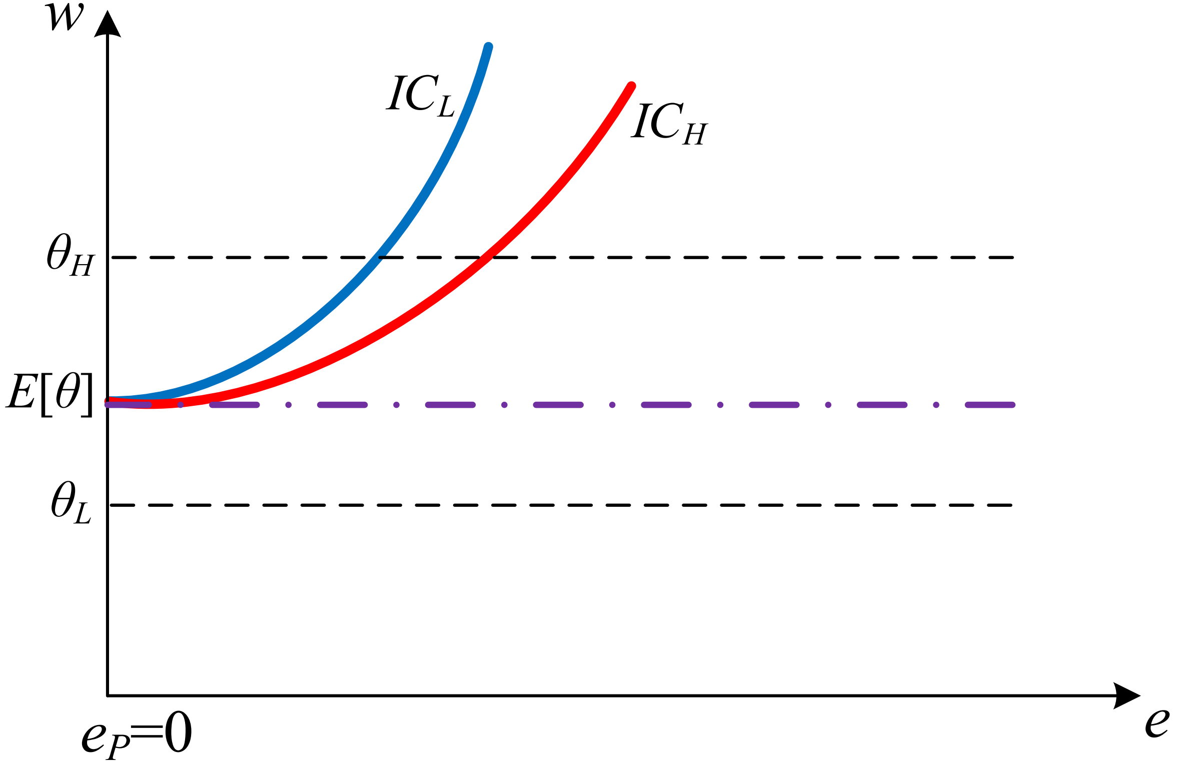
Step 6 – Conclusion.

- Given the optimal response found in Step 5, we can see that the high-productivity worker has incentives to deviate from her equilibrium strategy of e_A to e'.
- Therefore, the most-costly pooling PBE $e^P = e_A$ *violates* the Intuitive Criterion.
- A similar argument applies to all other pooling PBE where $e^P < e_A$.

Pooling PBE – Applying the Intuitive Criterion

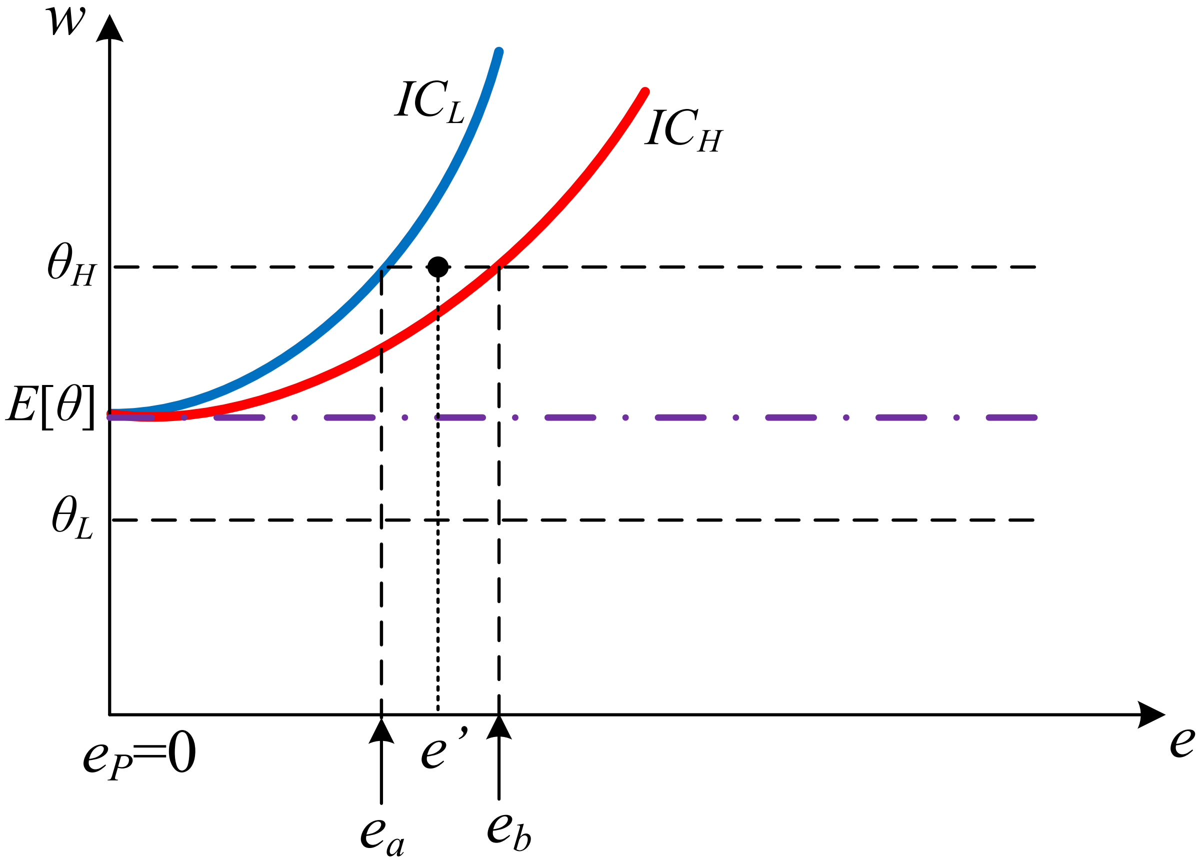
Step 6 – Conclusion.

- The argument also applies the least-costly pooling PBE, where $e^P = 0$.
- First, recall this pooling PBE.



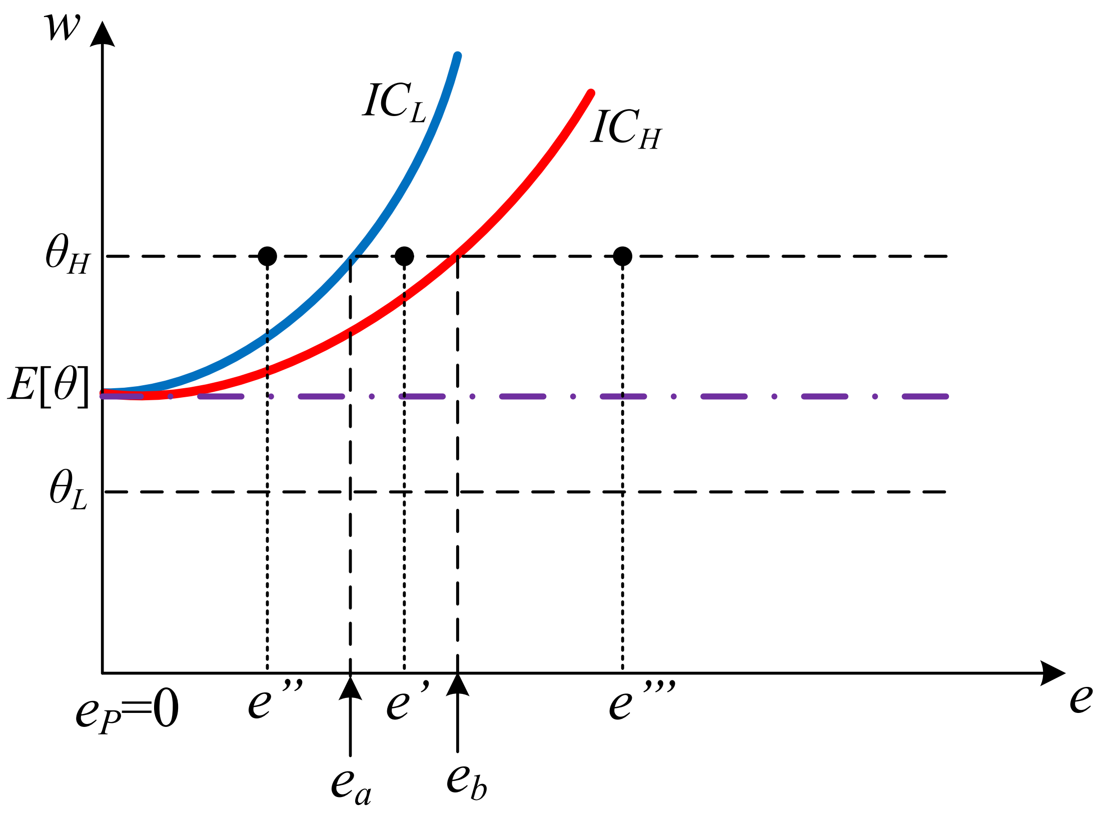
Pooling PBE – Applying the Intuitive Criterion

- Any deviation towards an education level e' can only originate from the high-productivity worker.
  - The firm updates its off-the-equilibrium beliefs, responding with a wage of $\mu(\theta_H|e') = 1$.
  - And updates its response, $w(e') = \theta_H$
  - In conclusion, $e^P = 0$ violates the Intuitive Criterion.



Pooling PBE – Applying the Intuitive Criterion

- Not all deviations work out.
  - e' works, as shown before.
  - But e'' or e''' do not.
- It's ok.
  - The Intuitive Criterion only requires us to find at least one deviation that helps the receiver update its off-the-equilibrium beliefs and responses…
  - ultimately leading one type of sender to change its behavior in the PBE that we are trying to destroy.



Pooling PBE – Applying the Intuitive Criterion

Step 6 – Conclusion.

- Therefore, no pooling PBE in the labor market signaling game survives the Intuitive Criterion,
  - implying that the only PBE surviving the Intuitive Criterion is the least-costly *separating* PBE where $(e_L, e_H) = (0, e_3)$.

Can signaling be welfare improving?

- A natural question is whether the worker is better off when:
  - she uses education to signal her type (in the least-costly separating PBE found above) than
  - when such signal is not available.
- Figure 12.13 compares the indifference curve that each type of worker reaches in these two information settings.

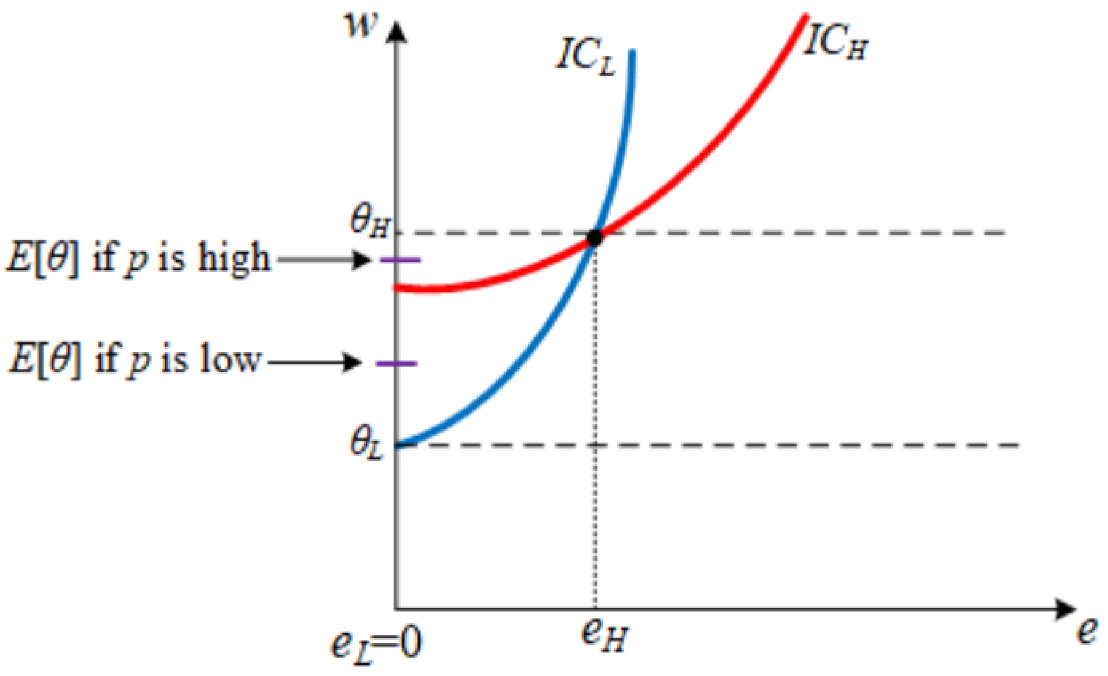


Figure 12.13. Utility comparison across information settings.

Can signaling be welfare improving?

<u>Low-productivity worker</u>

- Unambiguously worse off with signaling, where she acquires zero education but receives the lowest salary θ_L, than…
  - without signaling, where she still acquires no education and earns a higher salary $E[\theta]$.
- Graphically, the indifference curve passing through point $(0, E[\theta])$ reaches a higher utility level than IC_L.
- This result holds regardless of the specific probability that the worker is of high-productivity, p,
  - which graphically means regardless of the height of $E[\theta]$ in the (θ_L, θ_H) interval.

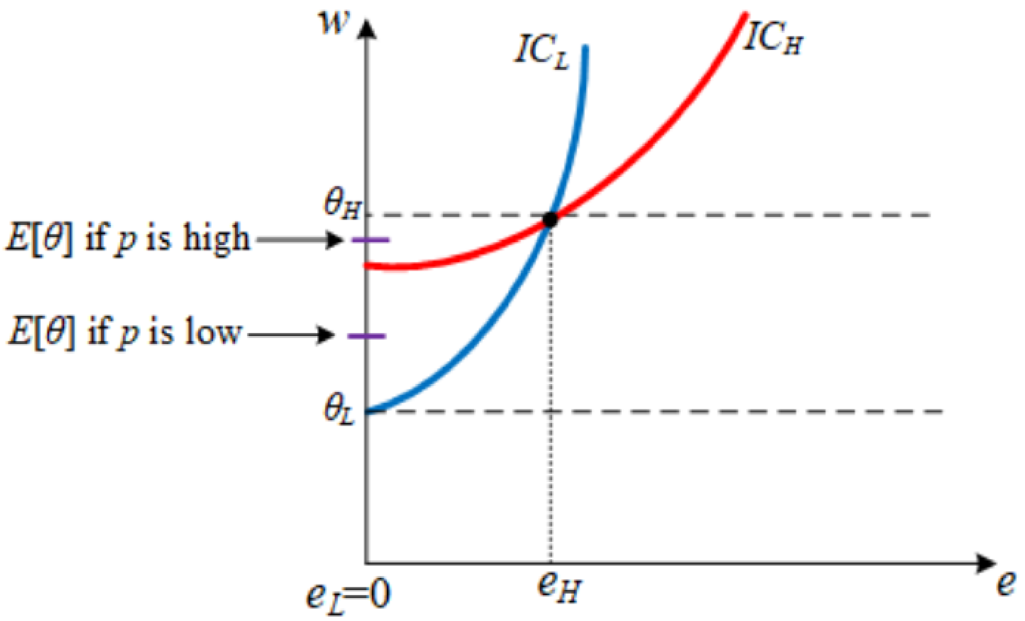


Figure 12.13. Utility comparison across information settings.

Can signaling be welfare improving?

<u>High-productivity worker</u>

- In contrast, the high-type worker is better off with signaling, where she reaches IC_H, than…
  - without signaling, where she acquires no education and earns a salary $E[\theta]$,
  - only if $E[\theta]$ is sufficiently low, which occurs when p is relatively low.
- Intuitively, this type of worker is better off acquiring education, despite its cost, and earning the highest wage θ_H, than
  - not investing in education and receiving $E[\theta]$ when this wage is sufficiently low,
  - which occurs when the firm believes that the high-productivity worker is relatively unlikely.

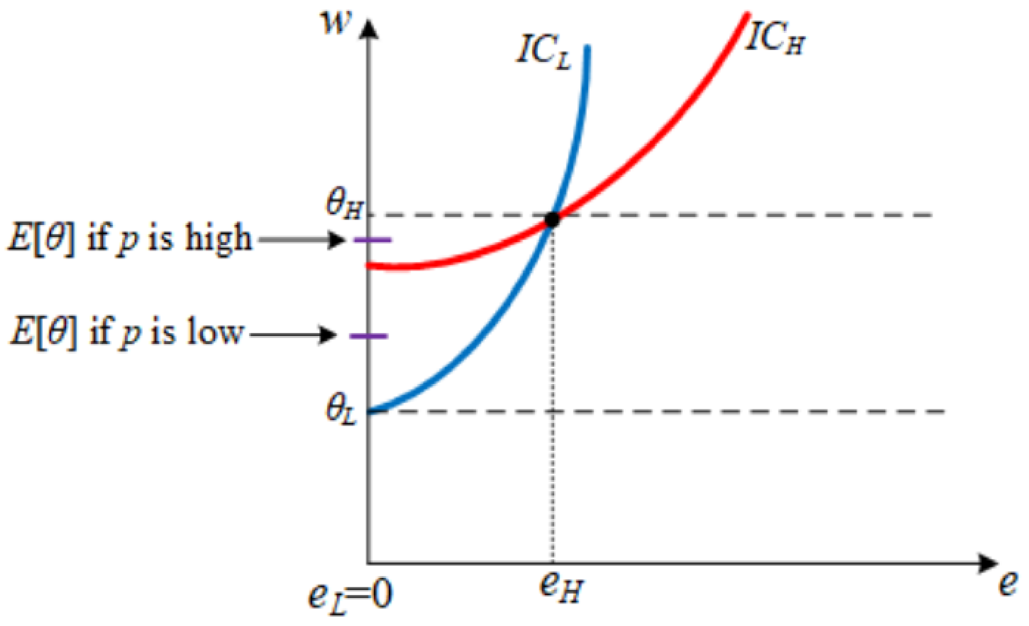


Figure 12.13. Utility comparison across information settings.

Can signaling be welfare improving?

High-productivity worker

- If, instead, probability p (as then $E[\theta]$) is sufficiently high,
  - the high-type worker is better off in the setting where education cannot be used as a signal to firms.

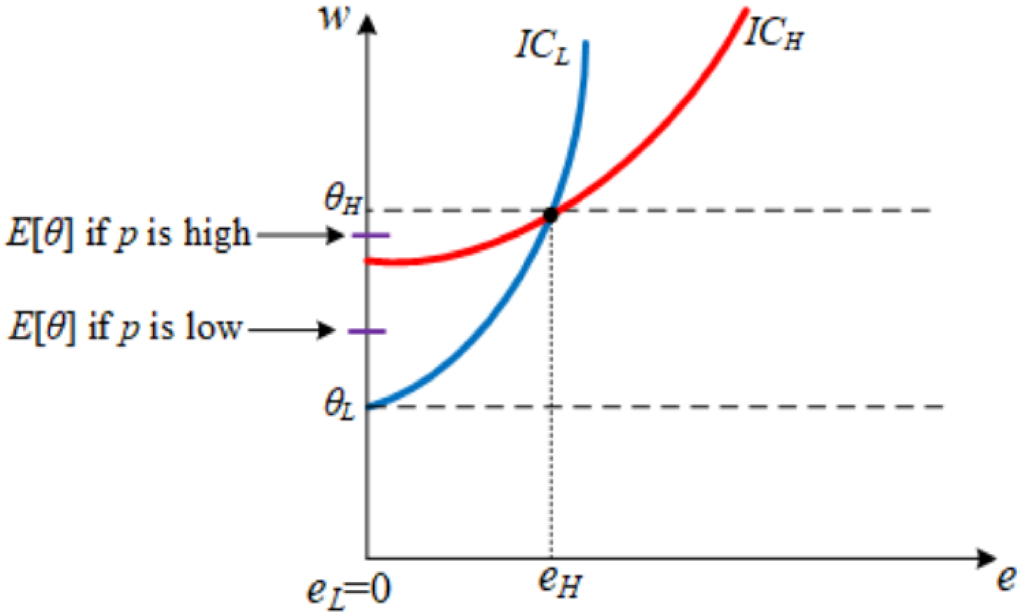


Figure 12.13. Utility comparison across information settings.

Example 12.5 Firms left in the dark?

- Consider again the setting in examples 12.1-12.3, where $\theta_H = 2$ and $\theta_L = 1$.
- Recall, in the least-costly separating PBE, the high-productivity worker chooses education $e_3 = 1$. Therefore, her utility in equilibrium is:

$$u_H = w - \frac{e^2}{\theta_H} = \theta_H - \frac{1^2}{\theta_H} = 2 - \frac{1}{2} = \frac{3}{2}$$

- However, when signaling is not available, she earns a salary

$$E[\theta] = p2 + (1-p)1 = 1 + p$$

yielding a utility of $1 + p$ since in this setting she acquires zero education.

- Therefore, this type of worker is better off in the environment where education cannot be used a signal to firms if $1 + p > \frac{3}{2}$, or solving for p, when $p > \frac{1}{2}$.
- Intuitively, the frequency of high types is so high that the low-productivity worker benefits from firms being "in the dark" about her type.

What if the sender has three types?

- Consider an extension of the previous signaling game, where the sender can now have three equally likely types, θ_L, θ_M, and θ_H, where $\theta_L < \theta_M < \theta_H$.

- Outline:
  1. Identify the set of separating PBEs
  2. Show that applying the Intuitive Criterion in this setting would have no bite, meaning that no separating PBE violates the Intuitive Criterion
  3. Show that the D1 Criterion helps us delete some separating PBEs in this signaling game with three sender types. Indeed, only the least-costly separating PBE (Riley outcome) survives the application of the D1 Criterion.

Separating PBEs

- For compactness, we describe here the set of separating PBEs.
- The set of all separating PBEs satisfies

$$(e_L, e_M, e_H) = (0, e_M^*, e_H^*)$$

where $e_M^* \in [e_1^M, e_2^M], e_M^* \in [e_1^H, e_2^H]$.

- Wages satisfy $w(e_K) = \theta_i$ for every $i = \{L, M, H\}$ after observing equilibrium education levels $(0, e_M^*, e_H^*)$ and $w(e)$ after observing the off-the-equilibrium education levels $e \neq e_i$, where $w(e)$ lies below the indifference curve of all worker types, as depicted in Figure 12.14.

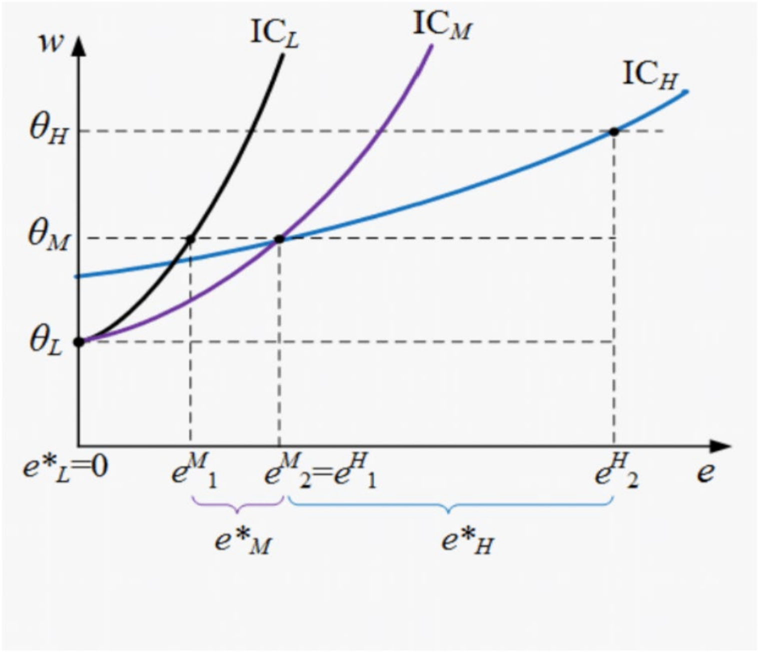


Figure 12.14. Separating PBEs with three worker types.

Separating PBEs

- Indifference curves in Figure 12.14 correspond to a specific separating PBE:
  - $(0, e_2^M, e_2^H)$,
  - where both the medium- and high-type worker choose the most-costly education level in their corresponding intervals.
- The high-type worker has no incentives to deviate to the medium-type worker's education level, e_2^M, which would yield a wage of $w = \theta_M$.
- This wage and education pair, (θ_M, e_2^M), would provide her with the same utility as her equilibrium education, e_2^H, as depicted IC_H in the figure.

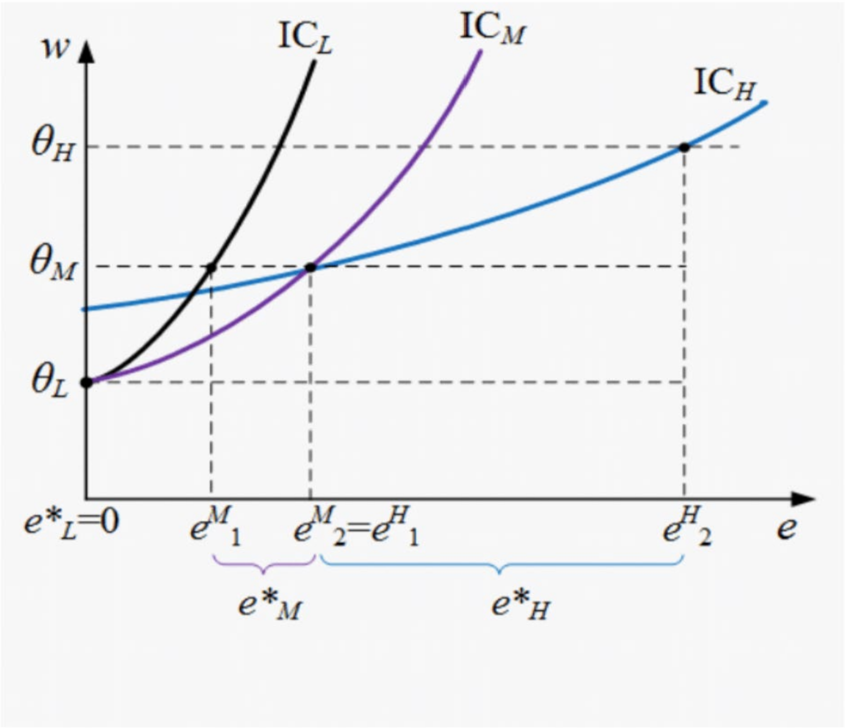


Figure 12.14. Separating PBEs with three worker types.

Separating PBE – Applying the Intuitive Criterion

Step 1 – Identify a PBE to test.

Consider a specific PBE, such as the separating equilibrium described above $(e_L, e_M, e_H) = (0, e_M^*, e_H^*)$

Step 2 – Off-the-equilibrium message.

Identify an off-the-equilibrium education, e, where $e \in (\hat{e}, e_H^*)$ as depicted in Figure 12.15.

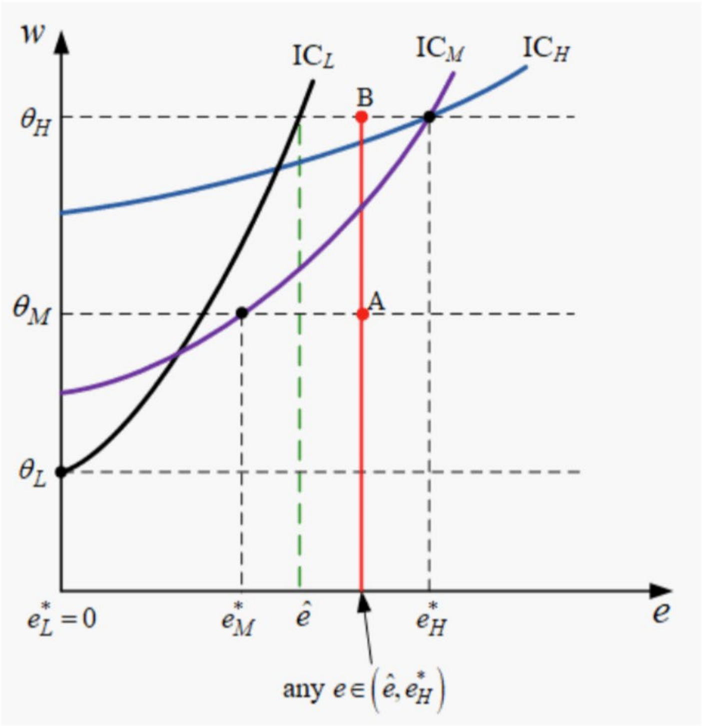


Figure 12.15. Applying the Intuitive Criterion with three worker types.

Separating PBE – Applying the Intuitive Criterion

Step 3 – Types who profitable deviate.

Find which types of worker can benefit by deviating to e:

<u>*Low type*</u>

- Utility in equilibrium is

$$u_L^* = \theta_L - c(0, \theta_L) = \theta_L$$

given that $c(0, \theta_L) = 0$.

- Highest utility that she can earn by deviating to e is

$$\theta_H - c(e, \theta_L)$$

- Therefore, $\theta_H - c(e, \theta_L) < \theta_L$ holds if $c(e, \theta_L) > \theta_H - \theta_L$, meaning that the cost of education exceeds highest possible salary gain.
- Then, θ_L has no incentive to deviate to $e \in (\hat{e}, e_H)$.

Separating PBE – Applying the Intuitive Criterion

Step 3 – Types who profitable deviate.

Find which types of worker can benefit by deviating to e:

Medium type

- Medium-productivity workers could send such a message $e \in (\hat{e}, e_H)$ because

$$\theta_M - c(e_M^*, \theta_M) < \theta_H - c(e, \theta_M)$$

or alternatively, $c(e, \theta_M) - c(e_M^*, \theta_M) < \theta_H - \theta_M$, reflecting that the cost of acquiring $e - e_M^*$ additional years of education is offset by her salary increase when the firm offers her a high-productivity wage $w(e) = \theta_H$.

Separating PBE – Applying the Intuitive Criterion

Step 3 – Types who profitable deviate.

Find which types of worker can benefit by deviating to e:

<u>*High type*</u>

- A similar argument applies to the high-productivity worker, who can benefit by deviating to e since

$$\theta_H - c(e_H^*, \theta_H) < \theta_H - c(e, \theta_H)$$

given that $c(e_H^*, \theta_H) > c(e, \theta_H)$.

- Intuitively, by deviating towards e, she does not modify her salary, but saves on education costs.

Separating PBE – Applying the Intuitive Criterion

Step 4 – Restricting Off-the-equilibrium beliefs.

- If $e \in (\hat{e}, e_H)$ is observed, it can only originate from the high- or medium-productivity worker, but not from the low-types.
- Technically, we say that deviations towards $e \in (\hat{e}, e_H)$ are:
  - *not equilibrium dominated* for the high-and medium-productivity workers,
  - but equilibrium dominated for the low-productivity worker.
- Therefore, upon observing $e \in (\hat{e}, e_H)$, the firm concentrates its beliefs on these two worker types:

$$\Theta^{**}(e) = \left\{\theta_{M,} \theta_H\right\} \quad \text{for all } e \in (\hat{e}, e_H)$$

Separating PBE – Applying the Intuitive Criterion

Step 5 – Updating Responses.

- Since $\Theta^{**}(e) = \{\theta_{M,}\theta_H\}$ for all $e \in (\hat{e}, e_H)$, the firm's best response to this education level will be a wage offer somewhere in between $w(e) = \theta_M$ and $w(e) = \theta_H$.
- This is different from what we saw in models with two worker types, where we could put full probability weight on a single worker type.

Separating PBE – Applying the Intuitive Criterion

Step 5 – Updating Responses.

<u>Medium-productivity worker</u>

- Lowest wage she can receive is $w(e) = \theta_M$, implying that her indifference curve will pass through point A,
  - being *below* (to the southeast) of her indifference curve in equilibrium, IC_M.
- Alternatively, even after the firm has restricted its beliefs so that deviations to e must originate from the high- and medium-type…
  - the medium-prod. worker cannot guarantee that her equilibrium utility will be strictly improved when deviating to e.

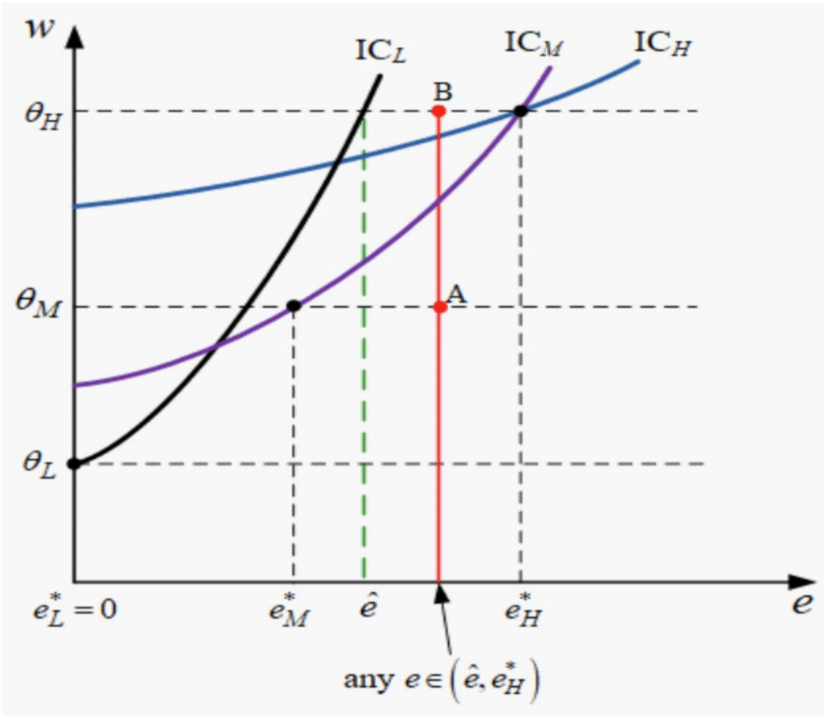


Figure 12.15 Applying the Intuitive Criterion with there worker types

Separating PBE – Applying the Intuitive Criterion

Step 5 – Updating Responses.

High-productivity worker

- Lowest wage she can earn after deviating to e is $w(e) = \theta_M$, yielding an indifference curve that passes through point A.
  - This indifference curve is also *below* her indifference curve in equilibrium payoff, IC_H, providing her with a lower utility level.
- Then, she cannot guarantee that her equilibrium utility will improve when deviating to e.

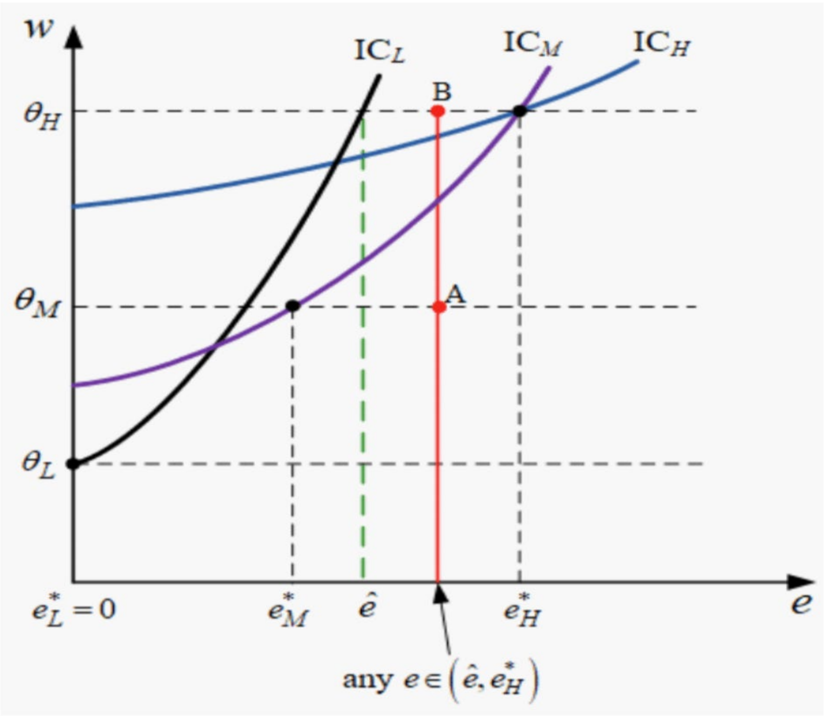


Figure 12.15 Applying the Intuitive Criterion with there worker types

Separating PBE – Applying the Intuitive Criterion

Step 6 – Conclusion.

- Given the optimal responses found in Step 5, we see that no worker type has incentives to deviate from her equilibrium strategy to e.
- In conclusion, the separating PBE specified in Figure 12.15 survives the Intuitive Criterion.
- Take-away lesson:
  - The application of the Intuitive Criterion does not necessarily eliminate separating PBEs in models with more than two sender types (e.g., workers).

Separating PBE – Applying the D1 Criterion

- D1 Criterion to the rescue!
- We now show that the separating PBE examined in the previous section violates D1 Criterion.
- For compactness, we do not reproduce all steps here, as they all coincide for Step 3.

- When deviating to e' the worker with the largest set of wage offers that can improve her equilibrium utility is the medium-productivity worker,
  - i.e., $D(\theta_M, \widehat{\Theta}, e')$ is longer than that of the other worker types, $D(\theta_L, \widehat{\Theta}, e')$ and $D(\theta_H, \widehat{\Theta}, e')$

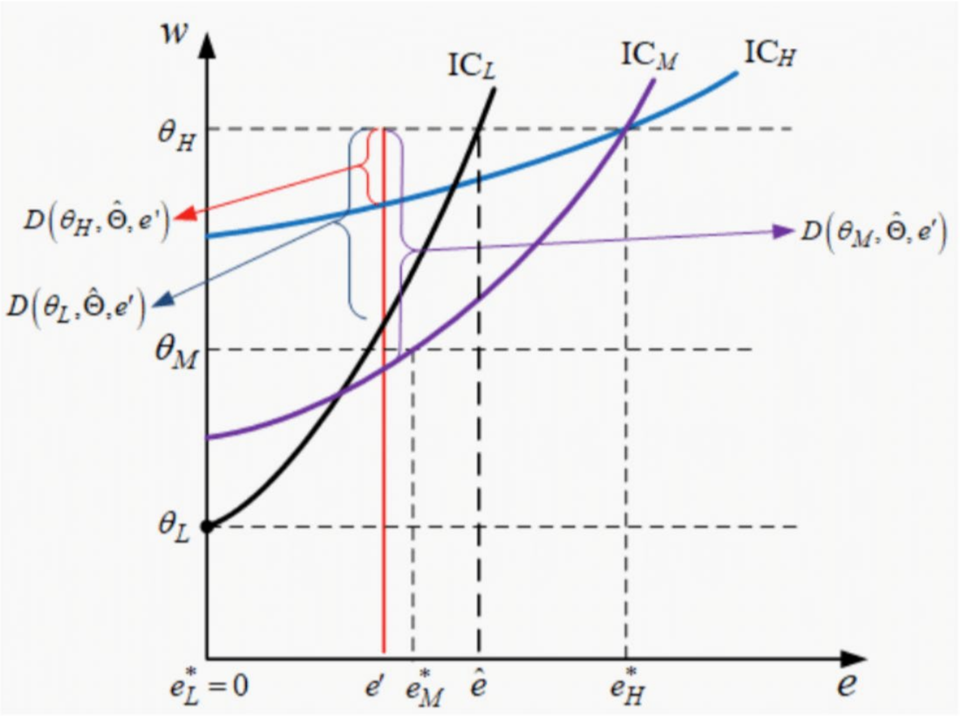


Figure 12.16. Applying the D1 Criterion with three worker types.

Separating PBE – Applying the D1 Criterion

- **Step 4.** Given our results in Step 3, upon observing e, the firm believes that this education level must originate from the medium-productivity worker alone.
- **Step 5.** Given its restricted beliefs, the firm offers a wage $w(e) = \theta_M$ after observing e.
- **Step 6.** Finally, we obtain that the medium-productivity worker's deviation payoff is

$$w(e) - c(e, \theta_M) = \theta_M - c(e, \theta_M)$$

while her equilibrium payoff, $u_M^* = \theta_M - c(e_M, \theta_M)$.

- And given that $e < e_M$ and $c_e(e, 0) > 0$,we have that $c(e, \theta_M) < c(e_M, \theta_M)$; which implies

$$\theta_M - c(e, \theta_M) > \theta_M - c(e_M, \theta_M)$$

Separating PBE – Applying the D1 Criterion

- Hence, we have found that the medium-productivity worker *has incentives to deviate* towards e,
  - entailing that the separating PBE $(e_L, e_M, e_H) = (0, e_M^*, e_H^*)$ *violates* the D1 Criterion.
- Repeating this process for all off-the-equilibrium messages…
  - we can delete all separating PBEs, except for the efficient (Riley) equilibrium outcome depicted in Figure 12.17.

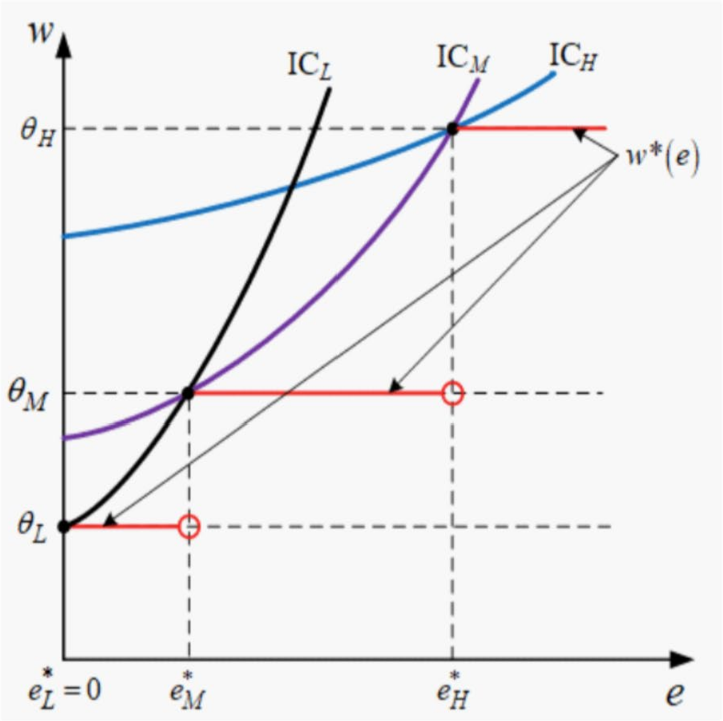


Figure 12.17. Unique separating PBE surviving the D1 Criterion with three types.

Appendix

Appendix - Equilibrium Refinements

- We present a more formal definition of the Intuitive and D1 Criterion, following Fudenberg and Tirole (2002, pp. 452-453).

Intuitive Criterion

- The first step focuses on those types of senders who can improve their equilibrium utility level by deviating.
- Formally, for any off-the-equilibrium message, m, we construct a subset of types $\Theta^{**}(m) \subset \Theta$ for which sending m is not equilibrium dominated. That is,

$$\Theta^{**}(m) = \left\{\theta \in \Theta \mid u_i^*(\theta) \leq \max_{a \in A^*(\Theta, m)} u_i(m, a, \theta)\right\}$$

- Intuitively, this expression states that, from all types in Θ, we restrict our attention to those types who *could* improve their equilibrium utility level, $u_i^*(\theta)$.
  - Note that emphasis on "could" since the term $\max_{a \in A^*(\Theta, m)} u_i(m, a, \theta)$ represents the highest payoff that θ-type can achieve by deviating to the off-the-equilibrium message m.

Intuitive Criterion

- In short, we can interpret $\Theta^{**}(m)$ as the subset of senders who *could improve* their equilibrium utility by deviating from m^* to m.
- Given $\Theta^{**}(m)$, we now check if the following inequality holds for some type $\theta \in \Theta^{**}(m)$,

$$\min_{a \in A^*(\Theta^{**}(m),m)} u_i(m,a,\theta) > u_i^*(\theta)$$

- Intuitively, this inequality states that:
  - once beliefs are restricted to $\Theta^{**}(m)$,
  - there is at least one sender type who prefers to deviate to a message m,
  - since this deviation provides her with a strictly higher utility level than her equilibrium message, m^*, *regardless* of the response of the receiver,
  - i.e., even if the receiver responds in the least beneficial way for the sender, as captured by $\min_{a \in A^*(\Theta^{**}(m),m)} u_i(m,a,\theta)$.

Intuitive Criterion

- In settings where the first step of the Intuitive Criterion helps us restrict off-the-equilibrium beliefs to a single type:
  - the application of the second step is unnecessary.
  - specific response from the sender.
  - as the sender type has incentives to deviate from her equilibrium message.
- However, in contexts where the first step leaves us with two or more sender types in $\Theta^{**}(m)$:
  - off-the-equilibrium beliefs for these types cannot be restricted to a single type,
  - not a specific response from the sender,
  - and the application of the second step is still necessary.
- Formally, an equilibrium strategy profile (m^*, a^*) *violates* the Intuitive Criterion if:
  - there is a type of agent θ and an action m such that condition (2) is satisfied.
  - Otherwise, we say that the equilibrium strategy profile *survives* the Intuitive Criterion.

D1 Criterion

- The D1 Criterion considers that, among all potential deviators, the receiver restricts her beliefs to only those types of senders who *most likely* send the off-the-equilibrium message.
- In particular, we now identify:
  - the sender for whom most of the responder's actions (in a setting with discrete actions) or
  - the sender with the largest set of responses (in a context with continuous actions)
  - provide a payoff strictly higher than her equilibrium payoff.

- Formally, for any off-the-equilibrium message m, let us define

$$D\left(\theta, \widehat{\Theta}, m\right) = \bigcup_{\mu:\mu\left(\widehat{\Theta}|m\right)=1} \{a \in MBR(\mu, m) |\ u_i^*(\theta) < u_i(m, a, \theta)\}$$

as the set of mixed best responses (MBR) of the receiver for which the $\theta - type$ of sender is *strictly better-off* deviating towards message m than sending her equilibrium message m^*.

D1 Criterion

- Note that $\mu(\widehat{\Theta}|m) = 1$ in the previous expression represents that the receiver believes that message m only comes from types in the subset $\widehat{\Theta} \in \Theta$.
- Let us also define

$$D^{\circ}(\theta, \widehat{\Theta}, m) = \bigcup_{\mu:\mu(\widehat{\Theta}|m)=1} \{a \in MBR(\mu, m) |\, u_i^*(\theta) = u_i(m, a, \theta)\}$$

as the set of MBR of the receiver that make the $\theta - type\ indifferent$ between deviating towards m and sending her equilibrium message m^*.

D1 Criterion

- First, we say that $\theta - type$ can be *deleted* if there is another $\theta' - type$ such that, when the off-the-equilibrium message m is observed

$$\left[D\left(\theta, \widehat{\Theta}, m\right) \cup D^{\circ}\left(\theta, \widehat{\Theta}, m\right)\right] \subset D\left(\theta', \widehat{\Theta}, m\right)$$

- That is, for a given message m:
  - the set of receiver's actions which make the $\theta' - type$ of sender strictly better off, $D\left(\theta', \widehat{\Theta}, m\right)$, is *larger* than
  - the set of actions making the $\theta - type$ of sender strictly better off, $D\left(\theta, \widehat{\Theta}, m\right)$, or indifferent, $D^{\circ}\left(\theta, \widehat{\Theta}, m\right)$.
- Intuitively, after receiving message m,
  - there are more best responses of the receiver that improve the $\theta' - type$'s equilibrium payoff than there are for the $\theta - type$.

D1 Criterion

- As a consequence, the $\theta' - type$ is the sender who is the most likely to deviate from her equilibrium message m^* to the off-the-equilibrium message m.
- We continue this comparison for all types of senders, deleting those for which there is another type of sender who is more likely to deviate towards m.
- Finally, the set of types that cannot be deleted after using this procedure is denoted by $\Theta^{**}(m)$.

D1 Criterion

- Once the set of types has been restricted to $\Theta^{**}(m)$, the Intuitive and the D1 Criterion proceed similarly:
  - seeking to find at least a sender type $\theta \in \Theta^{**}(m)$ with incentives to deviate even if the receiver responds in the least beneficial way for the sender,
  - that is,

$$\min_{a \in A^*(\Theta^{**}(m), m)} u_i(m, a, \theta) > u_i^*(\theta)$$

which coincides with the second step in our application of the Intuitive Criterion.